ON SOLUTION OF A CLASS OF FUZZY BVPs

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Abstract. This paper investigates the existence and uniqueness of solutions to first-order nonlinear boundary value problems (BVPs) involving fuzzy differential equations and two-point boundary conditions. Some sufficient conditions are presented that guarantee the existence and uniqueness of solutions under the approach of Hukuhara differentiability.

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models [9]. Particularly, the use of fuzzy differential equations (FDEs) is a natural way to model the dynamical systems with embedded uncertainty. Due to the applicability of the FDEs for the analysis of phenomena where imprecision is inherent, this class of differential equations is a field of increasing interest. In fact, FDE is a very important topic from theoretical point of view (see [11, 15, 22, 24, 29, 35, 39, 2, 14]) and also its applications, for example, in population models [19, 20], civil engineering [32], particle systems [26, 27, 28, 38], medicine [1, 4, 21, 31], bioinformatics and computational biology [3, 8, 10].

Different points of view exist on the concept of differentiable fuzzy-valued functions. Historically, the first approach is to consider differentiability in the sense of Hukuhara. However, this approach suffers from some limitations: the solutions of a fuzzy differential equation have level sets whose diameter is necessarily a non-decreasing function with respect to time variable, i.e. the solutions are irreversible in possibilistic sense.

On the other hand, the study of periodic phenomena through fuzzy differential models is difficult to deal with. There are some approaches to study the problem of existence of periodic solutions for fuzzy differential equations. Rodríguez-López [36] gave the existence theorem for periodic solutions to fuzzy differential equations by considering an impulsive problem. Recently, Nieto and Rodríguez-López [30] have shown the existence of solutions of the fuzzy boundary value problem considered by applying the Banach and the Tarski fixed point theorems under continuity and Lipschitz conditions. In this study, we will establish the existence of (at least) a solution for fuzzy differential equations subject to boundary value conditions, under a nonlinear alternative theorem. The uniqueness result of the solutions is also obtained under appropriate conditions.

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2. Preliminaries

2.1. **Basic Concepts.** Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $v : \mathbb{R} \to [0, 1]$, satisfying the following properties:

(I) v is normal, i.e. there exists $x_0 \in \mathbb{R}$ with $v(x_0) = 1$;

(II) v is convex fuzzy set , i.e. $v(\kappa x+(1-\kappa)y) \geq \min\{v(x), v(y)\}$, for all $\kappa \in [0, 1]$, $x, y \in \mathbb{R}$;

(III) v is upper semicontinuous on $\mathbb{R}$;

(IV) $\overline{\{x \in \mathbb{R};\ v(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers (see e.g. [13]). Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_x;\ x \text{ is usual real number}\}$.

For $0 < r \leq 1$, the $r-$level set $[v]^r$ of a fuzzy set v on $\mathbb{R}$ is defined as

$$[v]^r = \{x \in \mathbb{R};\ v(x) \geq r\},$$

while its support $[v]^0$ is the closure in topology $\mathbb{R}$ of the union of all level sets, that is

$$[v]^0 = \overline{\bigcup_{r\in(0,1]} [v]^r} = \overline{\{x \in \mathbb{R}; v(x) > 0\}}.$$

Then, it is well-known (see e.g. [5, 6]) that for any $r \in [0, 1]$, $[v]^r$ is a bounded closed interval in $\mathbb{R}$, presented by $[v]^r = [\underline{v}^r, \overline{v}^r]$, where $\underline{v}^r$ and $\overline{v}^r$ denote respectively the left-hand endpoint and the right-hand endpoint of $[v]^r$.

For $u,\ v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $[u + v]^r = [u]^r + [v]^r$, $[\lambda.v]^r = \lambda[v]^r$, $\forall r \in [0, 1]$, where $[u]^r + [v]^r$ is the usual addition of two intervals of $\mathbb{R}$ and $\lambda[u]^r$ is the usual product between a scalar and a subset of $\mathbb{R}$.

Noting that a crisp number α is simply represented by $\underline{v}(r) = \overline{v}(r) = \alpha$, $0 \leq r \leq 1$.

Let A, B two nonempty bounded subsets of $\mathbb{R}$. The Hausdorff distance between A and B is

$$d_H(A, B) = \max\left[\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |a - b|\right].$$

The metric d_H on $\mathbb{R}_{\mathcal{F}}$ is as follows,

$$d_\infty(u, v) = \sup\left\{d_H([u]^r, [v]^r),\ r \in [0, 1]\right\},\ u, v \in \mathbb{R}_{\mathcal{F}}.$$

Definition 2.1. [37] Let I be a real interval. The mapping $f : I \to \mathbb{R}_{\mathcal{F}}$ is called a fuzzy function and its $r-$level set is denoted by

$$[f(t)]^r = \left[\underline{f}^r(t), \overline{f}^r(t)\right],\ t \in I,\ r \in [0, 1],$$

where $\underline{f}^r(t)$ and $\overline{f}^r(t)$ denote respectively the left-hand endpoint and the right-hand endpoint of $[f(t)]^r$; more precisely,

$$\underline{f}^r(t) = \min\{\tau;\ \tau \in [w]^r, w \in \mathbb{R}_{\mathcal{F}}, w = f(t)\}$$

and

$$\overline{f}^r(t) = \max\{\tau;\ \tau \in [w]^r, w \in \mathbb{R}_{\mathcal{F}}, w = f(t)\}.$$

Definition 2.2. [5, 6] Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and it is denoted by $x \ominus y$.

Let us remark that $x \ominus y \neq x + (-1)y$.

Definition 2.3. [5, 6] Let I be an open interval in $\mathbb{R}$. A fuzzy function $f : I \to \mathbb{R}_{\mathcal{F}}$ is called to be Hukuhara differentiable at $t_0 \in I$, if for some $h_0 > 0$ the Hukuhara difference $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ exist in $\mathbb{R}_{\mathcal{F}}$ for all $0 < h < h_0$ and if there exists an element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\lim_{h \to 0^+} d_\infty \left(\frac{f(t_0 + h) \ominus f(t_0)}{h}, f'(t_0) \right) = 0$$

and

$$\lim_{h \to 0^+} d_\infty \left(\frac{f(t_0) \ominus f(t_0 - h)}{h}, f'(t_0) \right) = 0.$$

The fuzzy set $f'(t_0)$ is called the Hukuhara derivative of f at t_0.

Theorem 2.4. [24] *Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be Hukuhara differentiable. For $t \in (a, b)$ denote $[f(t)]^r = [\underline{f}^r(t), \overline{f}^r(t)]$. Then, the boundary function $\underline{f}^r(t)$ and $\overline{f}^r(t)$ are differentiable and*

$$[\frac{d}{dt} f(t)]^r = [f'(t)]^r = [(\underline{f}^r)'(t), (\overline{f}^r)'(t)],\ r \in [0, 1]. \tag{1}$$

Definition 2.5. [7] Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be Hukuhara differentiable with $[f(t)]^r = [\underline{f}^r(t), \overline{f}^r(t)]$, for $t \in (a, b)$. Then f satisfies the *continuity condition* if $(\underline{f}^r)'(t)$ and $(\overline{f}^r)'(t)$ are continuous functions with respect to both t and r.

Theorem 2.6. [7] *Assuming the continuity condition holds, if one of the derivatives defined in* [12, 16, 34] *or* [35] *exists and it is a fuzzy number, then so do the others and they are all equal. Their value is provided from* (1).

The continuity condition is assumed to hold for all fuzzy functions in the rest of the paper.

Definition 2.7. [37] Let $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$. The integral of f in $[0, T]$, (denoted by $\int_{[0,T]} f(t)dt$ or $\int_0^T f(t)dt$) is defined levelwise as the set of integrals of the (real) measurable selections for $[f]^r$, for each $r \in (0, 1]$. We say that f is integrable over $[0, T]$ if $\int_{[0,T]} f(t)dt \in \mathbb{R}_{\mathcal{F}}$ and we have

$$\left[\int_0^T f(t)dt \right]^r = \left[\int_0^T \underline{f}^r(t)dt, \int_0^T \overline{f}^r(t)dt \right],$$

for each $r \in (0, 1]$.

Remark 2.8. It is obvious that a continuous function is integrable.

Remark 2.9. If $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$ is Hukuhara differentiable and its Hukuhara derivative f' is integrable over $[0, 1]$, then

$$f(t) = f(0) + \int_0^t f'(s)ds,$$

for all values of t where $0 \leq t \leq T$.

2.2. **Fuzzy Boundary Value Problem.** Let us consider the following fuzzy differential equation subject to boundary conditions

$$\begin{cases} y' = f(t, y), \quad t \in I = [0, T], \\ \lambda y(0) = y(T), \end{cases} \tag{2}$$

where $T > 0, \lambda > 0, f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$, and the derivative of y is considered in the sense of Hukuhara.

Set

$$C(I, \mathbb{R}_{\mathcal{F}}) = \{y : I \to \mathbb{R}_{\mathcal{F}} : \; y \text{ is continuous}\}$$

and

$$C^1(I, \mathbb{R}_{\mathcal{F}}) = \{y : I \to \mathbb{R}_{\mathcal{F}} : \; y, \; y' \text{ are continuous}\},$$

equipped with usual supremum norms.

In the present work, we are interested in finding a continuously differentiable function $y : I \to \mathbb{R}_{\mathcal{F}}$ (i.e., $y \in C^1(I, \mathbb{R}_{\mathcal{F}})$) satisfying (2).

To study problem (2), we use a topological tool which is the following simplified version of the Nonlinear Alternative [18, Theorem 4.1]. We say a map is compact if it is continuous with relatively compact range. Also, let J denote a convex subset of a normed, linear space E and let B_P denote an open ball in J with radius $P > 0$ and center 0.

Theorem 2.10. *Let $\mathcal{A} : \overline{B_P} \to J$ be a compact map and let $\gamma \in [0, 1]$. If*

$$u \neq \gamma \mathcal{A} u \text{ for all } u \in \partial B_P \text{ and all } \gamma \in (0, 1),$$

then, there exists at least one $u \in B_P$ such that $u = \mathcal{A}u$.

3. Existence and Uniqueness

In this section, the solvability of problem (2) shall be investigated. We consider the following three cases: $\lambda > 1$, $\lambda = 1$, and $0 < \lambda < 1$.

3.1. **Case $\lambda > 1$.** The following lemma gives us the integral form of (2).

Lemma 3.1. *Suppose $\lambda > 1$. The boundary value problem (2) is equivalent to the integral equation.*

$$y(t) = \frac{1}{\lambda - 1}\int_0^T f(s, y(s))ds + \int_0^t f(s, y(s))ds. \tag{3}$$

Proof. Let $y : I \to \mathbb{R}_{\mathcal{F}}$ satisfy (2). It is easy to see that

$$y(t) = y(0) + \int_0^t f(s, y(s))ds, \tag{4}$$

for $t \in [0, T]$ satisfying $\lambda y(0) = y(T)$. The boundary condition produces

$$\lambda y(0) = y(T) = y(0) + \int_0^T f(s, y(s))ds$$

which, in the ordinary case, is reduced to $y(0) = \frac{1}{\lambda-1}\int_0^T f(s, y(s))ds$.

In the fuzzy case, passing to the level sets, we get

$$\begin{aligned}\lambda\left[\underline{y}^r(0), \overline{y}^r(0)\right] &= \left[\lambda\underline{y}^r(0), \lambda\overline{y}^r(0)\right] = [y(T)]^r \\ &= \left[\underline{y}^r(0), \overline{y}^r(0)\right] + \left[\int_0^T f(s, y(s))ds\right]^r.\end{aligned}$$

Consequently,

$$\underline{y}^r(0) = \frac{1}{\lambda - 1}\left(\int_0^T \underline{f}^r(s, y(s))ds\right)$$

and

$$\overline{y}^r(0) = \frac{1}{\lambda - 1}\left(\int_0^T \overline{f}^r(s, y(s))ds\right);$$

which make sense, since $\lambda > 1$, producing the fuzzy number

$$y(0) = \frac{1}{\lambda - 1}\int_0^T f(s, y(s))ds. \tag{5}$$

So substituting (5) into (4) we obtain, for $t \in [0, T]$,

$$y(t) = \frac{1}{\lambda - 1}\int_0^T f(s, y(s))ds + \int_0^t f(s, y(s))ds.$$

If y is a solution of (3), then it is easy to show that (2) holds by a direct calculation. □

Theorem 3.2. (**Existence**) *Suppose $\lambda > 1$ holds and $f \in C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$. If there exists a non-negative constant M such that $d_\infty(f(t, y), \widetilde{0}) \leq M$ for all $(t, y) \in I \times \mathbb{R}_{\mathcal{F}}$, then the boundary value problem (2) has at least one solution.*

Proof. In view of Lemma 3.1, we want to show that there exists at least one solution for (3), which is equivalent to show that (2) has at least one solution. To do this, we use the Nonlinear Alternative.

Consider the operator $\mathcal{A} : C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}) \to C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$ defined for all $t \in I$ by

$$\mathcal{A}y(t) = \frac{1}{\lambda - 1} \int_0^T f(s, y(s))ds + \int_0^t f(s, y(s))ds. \tag{6}$$

Thus our problem is reduced to proving the existence of at least one ν such that

$$\nu = \mathcal{A}\nu. \tag{7}$$

Since f is continuous, $\mathcal{A}$ is also a continuous map. Next, we show that $\mathcal{A} : \overline{B_P} \to C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$ satisfies

$$y \neq \gamma \mathcal{A}y, \qquad \text{for all } \ y \in \partial B_P \quad \text{and all } \ \gamma \in (0, 1), \tag{8}$$

for some suitable ball $B_P \subset C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$ with radius $P > 0$. Let

$$B_P = \left\{ y \in C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}) : \max_{t \in I} d_\infty(y(t), \widetilde{0}) < P \right\},$$

where

$$P = \frac{\lambda M T}{\lambda - 1}.$$

See that the family $y = \gamma \mathcal{A}y$ is equivalent to the family of the boundary value problems

$$\begin{cases} y' = \gamma f(t, y), \ \ t \in I = [0, T], \ \gamma \in [0, 1], \\ \lambda y(0) = y(T). \end{cases} \tag{9}$$

Let $H := \frac{\lambda}{\lambda - 1}$. All solutions of $y = \gamma \mathcal{A}y$ must satisfy:

$$\begin{aligned}
d_\infty(y(t), \widetilde{0}) = d_\infty(\gamma \mathcal{A}y, \widetilde{0}) &= d_\infty(\frac{\gamma}{\lambda - 1} \int_0^T f(s, y(s))ds + \gamma \int_0^t f(s, y(s))ds, \widetilde{0}) \\
&\leq d_\infty(\frac{\gamma}{\lambda - 1} \int_0^T f(s, y(s))ds, \widetilde{0}) + d_\infty(\gamma \int_0^t f(s, y(s))ds, \widetilde{0}) \\
&\leq \frac{\gamma}{\lambda - 1} \int_0^T d_\infty(f(s, y(s)), \widetilde{0})ds + \gamma \int_0^t d_\infty(f(s, y(s)), \widetilde{0})ds \\
&\leq \frac{\gamma}{\lambda - 1} \int_0^T d_\infty(f(s, y(s)), \widetilde{0})ds + \gamma \int_0^T d_\infty(f(s, y(s)), \widetilde{0})ds \\
&\leq \gamma H \int_0^T d_\infty(f(s, y(s)), \widetilde{0})ds < H \int_0^T d_\infty(f(s, y(s)), \widetilde{0})ds \\
&\leq HMT = P.
\end{aligned}$$

Thus (8) holds.

The operator $\mathcal{A} : \overline{B_P} \to C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$ is compact by the Arzela-Ascoli theorem, since it is a completely continuous map restricted to a closed ball. Nonlinear Alternativity ensures the existence of at least one solution in B_P for (3) and hence

(2) admits at least one solution. By an elementary compact argument involving a suitable sequence of solutions, this solution is also in $C^1(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$ □

Theorem 3.3. (**Uniqueness**) *Suppose $\lambda > 1$ and $f \in C(I \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$. If there exists a constant $k \geq 0$ such that*

$$d_\infty(f(t,u), f(t,v)) \leq k d_\infty(u,v), \text{ for all } u, v \in \mathbb{R}_{\mathcal{F}} \tag{10}$$

and $\frac{\lambda k T}{\lambda - 1} < 1$, then the boundary value problem (2) has a unique solution in $C(I, \mathbb{R}_{\mathcal{F}})$.

Proof. Let $C(J, \mathbb{R}_{\mathcal{F}})$ denote the set of all continuous functions from J to $\mathbb{R}_{\mathcal{F}}$. Define the metric

$$D(u,v) = \sup_J d_\infty(u(t), v(t))$$

for $u, v \in C(J, \mathbb{R}_{\mathcal{F}})$. Since $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space [22, 23], a standard argument shows that $(C(J, \mathbb{R}_{\mathcal{F}}), D)$ is also complete.

Now suppose there exist two solutions y_1, y_2 for (2). Then

$$\begin{aligned}
d_\infty(y_1, y_2) &= d_\infty(\mathcal{A}y_1, \mathcal{A}y_2) \\
&= d_\infty\Big(\frac{1}{\lambda - 1}\int_0^T f(s, y_1(s))ds + \int_0^t f(s, y_1(s))ds, \\
&\qquad \frac{1}{\lambda - 1}\int_0^T f(s, y_2(s))ds + \int_0^t f(s, y_2(s))ds\Big) \\
&\leq \frac{1}{\lambda - 1} d_\infty\Big(\int_0^T f(s, y_1(s))ds, \int_0^T f(s, y_2(s))ds\Big) \\
&\quad + d_\infty\Big(\int_0^t f(s, y_1(s))ds, \int_0^t f(s, y_2(s))ds\Big) \\
&\leq \frac{1}{\lambda - 1}\left(\int_0^T d_\infty(f(s, y_1(s)), f(s, y_2(s)))ds\right) \\
&\quad + \int_0^t d_\infty(f(s, y_1(s)), f(s, y_2(s)))ds \\
&\leq \frac{1}{\lambda - 1}\left(\int_0^T k d_\infty(y_1(s), y_2(s))ds\right) + \int_0^t k d_\infty(y_1(s), y_2(s))ds \\
&\leq \frac{1}{\lambda - 1}\left(\int_0^T k d_\infty(y_1(s), y_2(s))ds\right) + \int_0^T k d_\infty(y_1(s), y_2(s))ds \\
&\leq \frac{1}{\lambda - 1}(kTD(y_1, y_2)) + kTD(y_1, y_2) = \frac{\lambda k T D(y_1, y_2)}{\lambda - 1};
\end{aligned}$$

and rearranging, we obtain

$$D(y_1, y_2)\left(1 - \frac{\lambda k T}{\lambda - 1}\right) \leq 0.$$

So, we have $D(y_1, y_2) = 0$ for all $t \in [0,T]$ by $\frac{\lambda kT}{\lambda - 1} < 1$, and hence the solution is unique. $\square$

3.2. **Case $\lambda = 1$.** Here, we consider the following fuzzy differential equation

$$y' = f(t,y),\ t \in I = [0,T],\ y(0) = y(T). \tag{11}$$

The equivalent integral expression and the boundary condition imply that

$$y(0) = y(T) = y(0) + \int_0^T f(s, y(s))ds,$$

that is,

$$\widetilde{0} = y(0) \ominus y(0) = \int_0^T f(s, y(s))ds,$$

where

$$\widetilde{0}(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This expression is equivalent to

$$0 = \int_0^T \underline{f}^r(s, y(s))ds \leq \int_0^T \overline{f}^r(s, y(s))ds = 0,\ \text{for every } r \in [0,1].$$

Hence

$$\int_0^T \underline{f}^r(s, y(s)) - \overline{f}^r(s, y(s))ds = 0,\ \text{for every } r \in [0,1],$$

and by continuity

$$\underline{f}^r(s, y(s)) = \overline{f}^r(s, y(s)),\ \text{for every } r \in [0,1], s \in [0,T]. \tag{12}$$

Therefore, (12) and $0 = \int_0^T \underline{f}^r(s, y(s))ds$ are two necessary conditions to obtain (periodic) solutions for problem (11).

On the other hand, for each $r \in [0,1]$,

$$\begin{aligned} &\operatorname{diam}([y(t)]^r) \\ &= \operatorname{diam}\left(\left[\underline{y}^r(0) + \textstyle\int_0^t \underline{f}^r(s, y(s))ds, \overline{y}^r(0) + \int_0^t \overline{f}^r(s, y(s))ds\right]\right) \\ &= \operatorname{diam}([y(0)]^r) + \textstyle\int_0^t \operatorname{diam}([f(s, y(s))]^r)\, ds. \end{aligned}$$

Here, $\operatorname{diam}([u]^r) = \underline{u}^r - \overline{u}^r$.

For the function $y(t)$ to be constant in variable t, for every fixed $r \in [0,1]$, it is necessary that $\operatorname{diam}([f(s, y(s))]^r) = 0$, for all $r \in [0,1]$ and $s \in [0,T]$. In other words, assuming that f is continuous, the solution y has level sets with constant diameter if, for every $r \in [0,1]$, and every $s \in [0,T]$, $\operatorname{diam}([f(s,y(s))]^r) = 0$, that is $f(t,y(t))$ is crisp, for every $t \in [0,T]$.

In particular, if $f(t,x)$ is crisp, for every $t \in [0,T]$ and $x \in \mathbb{R}_{\mathcal{F}}$, then the diameter of each level set for the solutions to the initial value problem associated to equation $y'(t) = f(t,y(t)), t \in [0,T]$, is constant. Note that this does not mean that the solutions are crisp, but $\operatorname{diam}([y(t)]^r) = \operatorname{diam}([y(0)]^r)$, for every $t \in [0,T]$ and $r \in [0,1]$, that is, the diameter of each level set of the solution is diameter of the corresponding level set of the initial condition. Under this assumption, there could be fuzzy periodic solutions, too.

Example 3.4. Take $f(t,x) = -1 + \frac{2}{5}t$, for $t \in I = [0,5]$ and $x \in \mathbb{R}$, which is a continuous crisp function satisfying

$$\int_0^5 f(s,x)ds = \int_0^5 \left(-1 + \frac{2}{5}s\right) ds = \left[-s + \frac{s^2}{5}\right]_0^5 = 0,$$

for every $x \in \mathbb{R}$.

Consider the initial value problem

$$y'(t) = \chi_{\{f(t,y)\}},\ t \in I = [0,5],\ y(0) = \chi_{[0,1]}, \tag{13}$$

which can be easily solved, obtaining that $[y(t)]^r = \left[-t + \frac{t^2}{5}, 1 - t + \frac{t^2}{5}\right]$, for every $t \in I$ and $r \in [0,1]$, that is, $y(t) = \chi_{\{-t+\frac{t^2}{5}\}} + \chi_{[0,1]}, t \in I$. Then, for $t = 5$, we have $y(5) = \chi_{[0,1]}$.

Hence $y(t) = \chi_{\left[-t+\frac{t^2}{5}, 1-t+\frac{t^2}{5}\right]}, t \in I$ is a periodic solution for (13), and $\operatorname{diam}([y(t)]^r) = 1$ for every $t \in I$ and every $r \in [0,1]$. Furthermore, if we take the triangular fuzzy number $y_0 = (0;1,1)$, then

$$[y(t)]^r = \left[-(1-r) - t + \frac{t^2}{5}, (1-r) - t + \frac{t^2}{5}\right], \forall t,\ \forall r,$$

which defines a triangular fuzzy number $y(t) = \chi_{\{-t+\frac{t^2}{5}\}} + (0;1,1)$, for every $t \in I$. Also, $\operatorname{diam}([y(t)]^r) = 2(1-r) = \operatorname{diam}([y(0)]^r)$. Note that for all $r \in [0,1]$, $[y(5)]^r = [y(0)]^r$, then $y(0) = y(5) = (0;1,1)$.

Example 3.5. For the problem

$$y'(t) + \chi_{\{1\}} = \chi_{\{\frac{2}{3}\}} y(t),\ t \in I = [0,3],\ y(0) = y(3), \tag{14}$$

we have at least, the periodic solution $y(t) = \chi_{\{\frac{3}{2}\}}$. If we start at a crisp initial condition, the periodic solutions are crisp, since $\operatorname{diam}([y(t)]^r) = 0$, for every $t \in I$ and $r \in [0,1]$. Moreover, if the initial condition is not crisp, it is necessary for the diameter of the solution to be constant.

Remark 3.6. If $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is such that $f(t, \chi_{\{x\}}) = \chi_{\{g(t,x)\}}$, for every $t \in I$ and $x \in \mathbb{R}$ where $g : I \times \mathbb{R} \to \mathbb{R}$, and the crisp equation $u'(t) = g(t,u(t)), t \in I$, has a real solution y satisfying $\lambda u(0) = u(T)$, then $y(t) = \chi_{\{u(t)\}},\ t \in I$, is a solution of the boundary value problem (2).

3.3. **Case** $0 < \lambda < 1$**.** Since for any solution y of problem (2), $\operatorname{diam}\left([y(t)]^r\right)$ is nondecreasing in the variable t, for each $r \in [0,1]$ (fixed), therefore the boundary conditions

$$\lambda \underline{y}^r(0) = \underline{y}^r(T), \quad \lambda \overline{y}^r(0) = \overline{y}^r(T)$$

imply that

$$\begin{aligned}\operatorname{diam}\left([y(T)]^r\right) &= \overline{y}^r(T) - \underline{y}^r(T) = \lambda\left(\overline{y}^r(0) - \underline{y}^r(0)\right) \\ &= \lambda \operatorname{diam}\left([y(0)]^r\right) < \overline{y}^r(0) - \underline{y}^r(0) = \operatorname{diam}\left([y(0)]^r\right).\end{aligned}$$

If $y(0)$ is not crisp, then for some r,

$$\operatorname{diam}\left([y(T)]^r\right) < \operatorname{diam}\left([y(0)]^r\right),$$

so that we cannot find a solution of (2). For the existence of solution, it is necessary that

$$\lambda y(0) = y(T) = y(0) + \int_0^T f(s, y(s))ds;$$

hence

$$(\lambda - 1)\underline{y}^r(0) = \int_0^T \underline{f}^r(s, y(s))ds$$

and

$$(\lambda - 1)\overline{y}^r(0) = \int_0^T \overline{f}^r(s, y(s))ds.$$

Consequently,

$$(\lambda - 1) \operatorname{diam}\left([y(0)]^r\right) = \int_0^T \operatorname{diam}\left([f(s, y(s)]^r)\right) ds \geq 0,$$

and $\operatorname{diam}\left([y(0)]^r\right) > 0$ leads to contradiction. Therefore the unique possibility is $\operatorname{diam}\left([y(0)]^r\right) = 0$.

If $\lambda \in (0,1)$, and $y(0)$ is crisp, then the solution is crisp.

4. **Summary and Conclusions**

In this work, we studied a class of fuzzy differential equations with boundary value conditions. Indeed, considering fuzzy BVP (2) and using the approach of Hukuhara differentiability and a nonlinear alternative argument, we showed that problem (2) possesses at least one solution, under appropriate condition on nonlinearity f (see Theorem 2.6). We also proved that this solution is unique, if f satisfies a Lipschitz-type condition (see Theorem 3.3).

We would like to note that an improvement of our uniqueness result and also the existence theorem can be done by replacing condition (10) with a weaker one. Our results will be used in further works to verify numerical solutions of (2) which their results will be appeared somewhere in our future studies.

The existence and uniqueness of solutions of (2), using the generalized differentiability approach ([6]), and also the existence and uniqueness of solutions of (2) with the mixed condition $\lambda y(0) = y(T) + \mu$ are some of the interesting issues.

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