

PIECEWISE CUBIC INTERPOLATION OF FUZZY DATA BASED ON B-SPLINE BASIS FUNCTIONS

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ABSTRACT. In this paper fuzzy piecewise cubic interpolation is constructed for fuzzy data based on B-spline basis functions. We add two new additional conditions which guarantee uniqueness of fuzzy B-spline interpolation. Other conditions are imposed on the interpolation data to guarantee that the interpolation function to be a well-defined fuzzy function. Finally some examples are given to illustrate the proposed method.

1. Introduction

It often occurs that the data to interpolate are not a set of crisp values but they are a set of fuzzy numbers. The topic of interpolation of fuzzy data attracts a growing interest and it has well developed in recent papers (see [1, 5, 6, 7, 10]).

Let $x_0 < \dots < x_n$ be $n + 1$ points in \mathbb{R} and $\{u_i\}_{i=0}^n$ be $n + 1$ fuzzy numbers (see section 2). Then a formulation of Zadeh's interpolation problem is as follows (see [7]).

Construct a fuzzy function $p : \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$ with the following properties:

(i) $p(x_i) = u_i, i = 0, \dots, n;$

(ii) p is continuous, (i.e. for the given fuzzy function p and an element c of its domain, p is said to be continuous at the point c if for $\epsilon > 0$, however small, there exists some number $\delta > 0$ such that for all t in the domain of p with $|x - c| < \delta$, the value of $p(x)$ satisfies

$$D(p(x), p(c)) < \epsilon$$

where D is Hausdorff distance introduced in section 2).

(iii) Let $u_i = \chi_{y_i}, i = 0, \dots, n$, where $y_i \in \mathbb{R}$. Let f be the unique polynomial of degree $\leq n$ passing through the points $(x_i, y_i)_{i=0}^n$. Then $p(x) = \chi_{f(x)}$ for all $x \in \mathbb{R}$.

A solution of this problem was given by R. Lowen using the Lagrange interpolation polynomial (see [7]). This polynomial can be written

$$p^r(x) = \{y \in \mathbb{R} | y = p_{d_0 \dots d_n}(x), d_i \in [u_i]^r\}, \quad (1)$$

where $p^r(x)$ is the r -cut of $p(x)$, $P_{d_0 \dots d_n}$ is Lagrange interpolation polynomial corresponding to the data $(x_i, d_i), i = 0, \dots, n$ and $[u_i]^r$ is the r -cut of u_i (see section 2).

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Successively O. Kaleva provided a method to compute the fuzzy Lagrange polynomial and fuzzy spline interpolations using the sign of the basis functions (see [5]). By using extension principle to the real-valued Lagrange and spline interpolation polynomials, Lowen and Kaleva have constructed the fuzzy interpolation losing the smoothness of the interpolation function. This problem has been overcome by Lodwick and Santos in [6]. They construct fuzzy curves from fuzzy data sets that are consistent as well as smooth. The method given here is simpler, consistent and inherits smoothness properties of the crisp interpolation. In fact, the method in [6] requires an optimization problem to be solved. Thus the authors reduce the interpolation problem as an optimization problem. Our method constructs a fuzzy interpolation based on B-spline functions.

This paper is organized as follows. In section 2, we provide some preliminary results about fuzzy numbers and B-spline functions. In section 3, we construct a fuzzy interpolation based on B-spline functions and prove a theorem that shows the constructed interpolation is a fuzzy function. Finally, the error of this interpolation is investigated.

2. Preliminaries

Definition 2.1. [2] A fuzzy number is a function $\mu : \mathbb{R} \rightarrow I = [0, 1]$, satisfying the following properties:

- (i) μ is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $\mu(x_0) = 1$;
- (ii) μ is convex fuzzy set (i.e. $\mu(tx + (1-t)y) \geq \min\{\mu(x), \mu(y)\}, \forall t \in [0, 1], x, y \in \mathbb{R}$);
- (iii) μ is upper semicontinuous on \mathbb{R} ;
- (iv) $\overline{\{x \in \mathbb{R}; \mu(x) > 0\}}$ is compact, where \bar{A} denotes the closure of A .

The set of all fuzzy real numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_x; x \text{ is a usual real number}\}$.

For $0 \leq r \leq 1$, r -cut of fuzzy number μ is defined by

$$[\mu]^r = \begin{cases} \{x \in \mathbb{R}; \mu(x) \geq r\} & 0 < r \leq 1 \\ \overline{\{x \in \mathbb{R}; \mu(x) > 0\}} & r = 0 \end{cases}$$

Then it is easily shown that μ is a fuzzy number if and only if $[\mu]^r$ is a closed and bounded interval for each $r \in [0, 1]$, and $[\mu]^1 \neq \emptyset$ (see [4]).

For $\mu, \nu \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $\mu + \nu$ and the product $\lambda \cdot \mu$ are defined by

$$[\mu + \nu]^r = [\mu]^r + [\nu]^r, \quad [\lambda \cdot \mu]^r = \lambda[\mu]^r, \quad \forall r \in [0, 1].$$

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+ \cup \{0\}$ be the Hausdorff distance between fuzzy numbers, i.e.

$$D(\mu, \nu) = \sup_{r \in [0, 1]} \max\{|\mu_-^r - \nu_-^r|, |\mu_+^r - \nu_+^r|\},$$

where $[\mu]^r = [\mu_-^r, \mu_+^r]$, $[\nu]^r = [\nu_-^r, \nu_+^r]$. Then the following properties are satisfied (see [9]):

- (i) $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space,
- (ii) $D(\mu + \nu, \mu + \omega) = D(\nu, \omega)$,

- (iii) $D(k.\mu, k.\nu) = |k|D(\mu, \nu)$,
 - (ii) $D(\mu + \nu, \omega + e) \leq D(\mu, \omega) + D(\nu, e)$.
- We define $\|\cdot\| = D(\cdot, \tilde{0})$, where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$, $\tilde{0} = \chi_{\{0\}}$.

Theorem 2.2. [4] *If $\mu \in \mathbb{R}_{\mathcal{F}}$, then*

- (i) μ_-^r is a bounded left continuous nondecreasing function with respect to r , on $(0, 1]$;
 - (ii) μ_+^r is a bounded left continuous nonincreasing function with respect to r , on $(0, 1]$;
 - (iii) $\mu_-^1 \leq \mu_+^1$;
 - (iv) μ_-^r and μ_+^r are right continuous at $r = 0$.
- Moreover, if μ_-^r and μ_+^r satisfy conditions (i)-(iv), then there exists a unique $\nu \in \mathbb{R}_{\mathcal{F}}$ such that $\nu_-^r = \mu_-^r$ and $\nu_+^r = \mu_+^r$.

Consider the partition $\pi : a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$. The space $S_3(\pi)$ is the set of all real functions of real variable $s \in C^2[a, b]$ that can be described as cubic polynomials on each subinterval $(t_i, t_{i+1}) \subseteq [a, b]$, $0 \leq i \leq n - 1$. Let $t_i = a + i(\frac{b-a}{n})$ and then introduce six additional knots $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_{n+3} > t_{n+2} > t_{n+1} > t_n$, and the functions $B_i(t)$ defined by

$$B_i(t) = \frac{1}{h^3} \begin{cases} (t - t_{i-2})^3 & t \in [t_{i-2}, t_{i-1}] \\ h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3 & t \in [t_{i-1}, t_i] \\ h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3 & t \in [t_i, t_{i+1}] \\ (t_{i+2} - t)^3 & t \in [t_{i+1}, t_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

$i = -1, \dots, n + 1$.

Theorem 2.3. *Dim $S_3(\pi) = n + 3$ and $\{B_{-1}, B_0, \dots, B_{n+1}\}$ construct a basis for $S_3(\pi)$.*

Proof. See [8]. □

We prove that there exists a function $s \in S_3(\pi)$ satisfying the interpolatory conditions

$$s(t_i) = f(t_i), \quad 0 \leq i \leq n, \tag{2}$$

where $(t_i, f(t_i))$ are the given interpolation data. Such a cubic spline interpolating function s is not uniquely determined by these interpolatory conditions. There are still two degrees of freedom left, calling for suitable additional requirements. The following additional requirements are commonly considered in case of crisp data:

- (i) $s''(a) = s''(b) = 0$,
- (ii) $s^{(k)}(a) = s^{(k)}(b)$ for $k = 0, 1, 2$: s is periodic,
- (iii) $s'(a) = f'(a)$, $s'(b) = f'(b)$.

In this paper we introduce and use two new conditions of the form

$$s(t_{-2}) = f(t_{-2}), \quad s(t_{n+2}) = f(t_{n+2}) \tag{3}$$

to avoid using the above conditions ((i), (ii), (iii)) in the fuzzy sense, since it is difficult to speak about derivatives of fuzzy functions. Note that for $t_{-2} < a$ and

3.1. Construction of Fuzzy Interpolation Function. To construct p , let us consider the space

$$\tilde{S} = \{f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}} \mid f_-^r, f_+^r \in C^2[a, b] \& f_-^r, f_+^r \in P_3[a, b]\}$$

and introduce the function $p \in \tilde{S}$ which solves the interpolation problem as follows

$$p_-^r(t) = (c_-^r)_{-1}B_{-1}(t) + (c_-^r)_0B_0(t) + \dots + (c_-^r)_{n+1}B_{n+1}(t) \quad (4)$$

$$p_+^r(t) = (c_+^r)_{-1}B_{-1}(t) + (c_+^r)_0B_0(t) + \dots + (c_+^r)_{n+1}B_{n+1}(t), \quad (5)$$

for $r \in [0, 1]$. Now we have to prove that $p : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$.

Theorem 3.1. *Let $a \geq 1$ and*

$$\begin{aligned} e_i^- &= y_i - x_i, \\ e_i^+ &= z_i - y_i \end{aligned}$$

for $i = 0, \dots, n$, $i = -2$ and $i = n + 2$,

$$E_- = (e_{-2}^-, e_0^-, \dots, e_n^-, e_{n+2}^-)^T,$$

$$E_+ = (e_{-2}^+, e_0^+, \dots, e_n^+, e_{n+2}^+)^T.$$

If E_- and E_+ are in the positive cone constructed by columns of matrix A , i.e. if $A_1, A_2, \dots, A_{n+2}, A_{n+3}$ are columns of A , then there exist positive scalars $a_1, a_2, \dots, a_{n+2}, a_{n+3} \geq 0$ and $b_1, b_2, \dots, b_{n+2}, b_{n+3} \geq 0$ such that

$$E_- = \sum_{i=0}^{n+3} a_i A_i, \quad E_+ = \sum_{i=0}^{n+3} b_i A_i,$$

then $p(t) \in \mathbb{R}_{\mathcal{F}}$, for all $t \in [a, b]$.

Proof. Suppose that $t \in [a, b]$ is arbitrary. We prove that the conditions of theorem 2.2 are satisfied by $p(t)$ and so it is a well defined fuzzy number. For this purpose, let $r_1, r_2 \in [0, 1]$, $r_1 \leq r_2$. Then we prove

$$p_-^{r_2}(t) \geq p_-^{r_1}(t), \quad (6)$$

$$p_+^{r_1}(t) \geq p_+^{r_2}(t), \quad (7)$$

and

$$p_+^1(t) \geq p_-^1(t).$$

Clearly (4) and (5) imply $p_-^1(t) = p_+^1(t)$.

In the following we prove the inequality (6). A similar argumentation can be used to prove inequality (7). Since the interpolation points are fuzzy numbers, from theorem 2.2 for the interpolation points we have

$$p_-^{r_2}(t_i) \geq p_-^{r_1}(t_i)$$

for $i = -2, i = 0, \dots, n$ and $i = n + 2$. Therefore, using (4) and (5) implies that

$$\begin{aligned} (c_-^{r_2})_{-1} &= p_-^{r_2}(t_{-2}) \geq (c_-^{r_1})_{-1} = p_-^{r_1}(t_{-2}), \\ (c_-^{r_2})_{-1} + 4(c_-^{r_2})_0 + (c_-^{r_2})_1 &= p_-^{r_2}(t_0) \geq (c_-^{r_1})_{-1} + 4(c_-^{r_1})_0 + (c_-^{r_1})_1 = p_-^{r_1}(t_0), \\ (c_-^{r_2})_0 + 4(c_-^{r_2})_1 + (c_-^{r_2})_2 &= p_-^{r_2}(t_1) \geq (c_-^{r_1})_0 + 4(c_-^{r_1})_1 + (c_-^{r_1})_2 = p_-^{r_1}(t_1), \\ &\vdots \\ (c_-^{r_2})_{n-1} + 4(c_-^{r_2})_n + (c_-^{r_2})_{n+1} &= p_-^{r_2}(t_n) \geq (c_-^{r_1})_{n-1} + 4(c_-^{r_1})_n + (c_-^{r_1})_{n+1} = p_-^{r_1}(t_n), \\ (c_-^{r_2})_{n+1} &= p_-^{r_2}(t_{n+2}) \geq (c_-^{r_1})_{n+1} = p_-^{r_1}(t_{n+2}). \end{aligned}$$

Thus

$$\begin{aligned} (c_-^{r_2})_{-1} - (c_-^{r_1})_{-1} &= p_-^{r_2}(t_{-2}) - p_-^{r_1}(t_{-2}) \geq 0 \\ (c_-^{r_2})_{-1} - (c_-^{r_1})_{-1} + 4((c_-^{r_2})_0 - (c_-^{r_1})_0) &+ (c_-^{r_2})_1 - (c_-^{r_1})_1 = p_-^{r_2}(t_0) - p_-^{r_1}(t_0) \geq 0 \\ (c_-^{r_2})_0 - (c_-^{r_1})_0 + 4((c_-^{r_2})_1 - (c_-^{r_1})_1) &+ (c_-^{r_2})_2 - (c_-^{r_1})_2 = p_-^{r_2}(t_1) - p_-^{r_1}(t_1) \geq 0 \\ &\vdots \\ (c_-^{r_2})_{n-1} - (c_-^{r_1})_{n-1} + 4((c_-^{r_2})_n - (c_-^{r_1})_n) &+ (c_-^{r_2})_{n+1} - (c_-^{r_1})_{n+1} = p_-^{r_2}(t_n) - p_-^{r_1}(t_n) \geq 0 \\ (c_-^{r_2})_{n+1} - (c_-^{r_1})_{n+1} &= p_-^{r_2}(t_{n+2}) - p_-^{r_1}(t_{n+2}) \geq 0 \end{aligned}$$

On the other hand

$$p_-^{r_2}(t_i) - p_-^{r_1}(t_i) = (u_i)_{-}^{r_2} - (u_i)_{-}^{r_1} = (y_i - x_i)(r_2 - r_1) = e_i^-(r_2 - r_1)$$

and so $AD_-^r = (r_2 - r_1)E_-$ where

$$\begin{aligned} D_-^r &= ((d_-^r)_{-1}, (d_-^r)_0, \dots, (d_-^r)_{n+1})^T, \\ (d_-^r)_i &= (c_-^{r_2})_i - (c_-^{r_1})_i, \quad i = -1, \dots, n + 1. \end{aligned}$$

Since E_- is in the positive cone constructed by the columns of matrix A , then $(d_-^r)_i \geq 0, i = -1, \dots, n + 1$. That is

$$(c_-^{r_1})_i \leq (c_-^{r_2})_i.$$

Thus

$$\sum_{i=-1}^{n+1} (c_-^{r_1})_i B_i(t) \leq \sum_{i=-1}^{n+1} (c_-^{r_2})_i B_i(t)$$

since $\{B_i\}_{i=-1}^{n+1}$ are positive functions. Hence

$$p_-^{r_1}(t) \leq p_-^{r_2}(t).$$

Since p_-^r and p_+^r are polynomials, the continuity conditions of theorem 2.2 are obviously true for p_-^r and p_+^r , so that for given t , $p(t)$ satisfies the conditions of theorem 2.2 and so it is a fuzzy number. \square

4. Examples

To illustrate the effectiveness of the proposed algorithm, we consider the following examples. Figures 1 and 2 show the results graphically. In these figures the dotted curves represent upper and lower limits of zero-cut and the solid line represents one-cut.

Example 4.1. Suppose that $[a, b] = [20, 45]$, $n = 5$, $h = 5$ and interpolation data set is as follows:

i	-2	0	1	2	3	4	5	7
t_i	10	20	25	30	35	40	45	55
x_i	29.0	19.0	14.0	5.8	39.0	22.3	33.0	2.5
y_i	29.1	20.2	15.8	7.5	41.1	24.0	34.4	2.6
z_i	29.3	21.1	17.3	9.0	43.2	25.8	35.8	2.9

Then the fuzzy-valued B-spline interpolation are determined as follows:

$$\begin{aligned}
 P_+^r(t) &= (29 + 0.1\alpha)B_{-1}(t) + (-3.754 + 0.189\alpha)B_0(t) + (5.015 + 0.343\alpha)B_1(t) \\
 &\quad + (-2.308 + 0.239\alpha)B_2(t) + (10.016 + 0.4\alpha)B_3(t) \\
 &\quad + (1.242 + 0.26\alpha)B_4(t) + (7.314 + 0.26\alpha)B_5(t) + (2.5 + 0.1\alpha)B_6(t), \\
 P_-^r(t) &= (29.3 - 0.2\alpha)B_{-1}(t) + (-3.465 - 0.1\alpha)B_0(t) + (5.658 - 0.3\alpha)B_1(t) \\
 &\quad + (-1.869 - 0.2\alpha)B_2(t) + (10.817 - 0.4\alpha)B_3(t) + (1.802 - 0.3\alpha)B_4(t) \\
 &\quad + (7.774 - 0.2\alpha)B_5(t) + (2.9 - 0.3\alpha)B_6(t).
 \end{aligned}$$

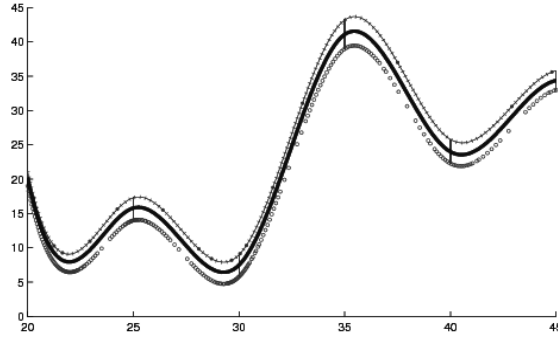


FIGURE 1. The Circles and Dots Show Respectively Lower and Upper Bounds of the Zero-cut and Solid-line Shows One-cut of Interpolation Function of the Example 4.1 and Vertical Solid Lines Show the Interpolated Data

Example 4.2. Suppose that $[a, b] = [3, 8]$, $n = 5$, $h = 1$ and interpolation data set is as follows:

i	-2	0	1	2	3	4	5	7
t_i	1	3	4	5	6	7	8	10
x_i	1.5	-11	-8.1	-1.4	-3.1	-16.5	-3.5	1.5
y_i	2.5	4	3	2	5	1	4	3
z_i	3	13.5	9.2	5.8	14.2	7.3	8.3	5.1

Then the fuzzy-valued B-spline interpolation is determined by the method of section 3 as follows:

$$\begin{aligned}
 P_+^r(t) &= (3 - 0.5\alpha)B_{-1}(t) + (2.2 - 2\alpha)B_0(t) + (1.699 - \alpha)B_1(t) \\
 &\quad + (0.204 - 0.2\alpha)B_2(t) + (3.285 - 2\alpha)B_3(t) + (0.857 - \alpha)B_4(t) \\
 &\quad + (0.586 - 0.3\alpha)B_5(t) + (5.1 - 2.1\alpha)B_6(t), \\
 P_-^r(t) &= (1.5 + \alpha)B_{-1}(t) + (-2.8 + 3\alpha)B_0(t) + (-1.3 + 2\alpha)B_1(t) \\
 &\quad + (-0.96e - 1 + 0.1\alpha)B_2(t) + (0.285 + \alpha)B_3(t) + (-4.143 + 4\alpha)B_4(t) \\
 &\quad + (-0.214 + 0.5\alpha)B_5(t) + (1.5 + 1.5\alpha)B_6(t).
 \end{aligned}$$

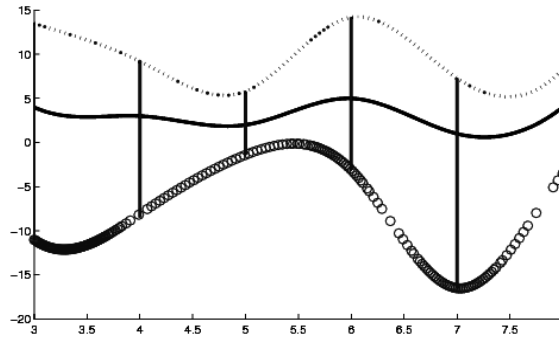


FIGURE 2. The Circles and Dots Show Respectively Lower and Upper Bounds of the Zero-cut and Solid-line Shows One-cut of Interpolation Function of the Example 4.2 and Vertical Solid Lines Show the Interpolated Data

Remark 4.3. Implementation of the method and producing the figures have been done by programming in MatLab 2011 (see the attached MatLab code in Appendix). These figures illustrate that theorem 3.1 is really doing its job.

5. Conclusions

A new cubic piecewise interpolation for fuzzy triangular data by using B-spline functions has been constructed in this paper. Fuzzy B-spline series as an approximation function has been studied in [3], where the interpolation paper, interpolation function as a special case of B-spline series has been introduced briefly. But the α -cuts of the interpolation function lose smoothness in the knots. In this paper, it is proved that the α -cuts of the interpolation function are as smooth as the crisp interpolation function.

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MATLAB CODE

```
function y=bsplt(s,a,b,n,r)
clc
h=(b-a)/n;
t=zeros(n+7);
A=zeros(n+3,n+3);
for i=-3:n+3
t(i+4)=a+i*h;
end
```



```

for i=1:n+3
for j=1:n+3
if i==1 && j==1
    A(i,j)=1;
elseif i==n+3 && j==n+3
    A(i,j)=1;
elseif i>1 && i<n+3 && i==j
    A(i,j)=4;
elseif i>1 && i<n+3 && (i-j)^2==1
    A(i,j)=1;
else
    A(i,j)=0;
end
end
end
m=A\r';
ll=1;
k=4;
while ll>0
    if s>=t(k) && s<=t(k+1)
        y=m(k-3)*bbs(t(k-1),h,s)+m(k-2)*bbs(t(k),h,s)
        +m(k-1)*bbs(t(k+1),h,s)+m(k)*bbs(t(k+2),h,s)
        m(k-2)
        ll=0;
    elseif k==n+4
        ll=0;
    else
        k=k+1;
    end
end
end
hold all
line([3 3],[-11 13.5]);
line([4 4],[-8.5 9.2]);
line([5 5],[-1.4 5.8]);
line([6 6],[-3.1 14.2]);
line([7 7],[-16.5 7.3]);
line([8 8],[-3.5 8.3]);
end
fplot('bsplt(x,3,8,5,
[1.5, -11, -8.1, -1.4, -3.1, -16.5, -3.5, 1.5])',[3 8])
fplot('bsplt(x,3,8,5,[2.5, 4, 3, 2, 5, 1, 4, 3])',[3 8])

```

```
fplot('bsplt(x,3,8,5,
[3, 13.5, 9.2, 5.8, 14.2, 7.3, 8.3, 5.1])',[3 8])
```

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