FUZZIFYING CLOSURE SYSTEMS AND CLOSURE OPERATORS

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Abstract. In this paper, we propose the concepts of fuzzifying closure systems and Birkhoff fuzzifying closure operators. In the framework of fuzzifying mathematics, we find that there still exists a one to one correspondence between fuzzifying closure systems and Birkhoff fuzzifying closure operators as in the case of classical mathematics. In the aspect of category theory, we prove that the category of fuzzifying closure system spaces is isomorphic to the category of Birkhoff fuzzifying closure spaces. In addition, we obtain an important result that the category of fuzzifying closure spaces and that of fuzzifying closure system spaces can be both embedded in the category of Birkhoff I-closure spaces. Finally, using fuzzifying closure systems of the paper, we introduce a set of separation axioms in fuzzifying closure system spaces, which offer a try how to research the properties of spaces by fuzzifying closure systems.

1. Introduction

Closure operators and closure systems play a significant role in classical mathematics, involving the realm of topology, algebra and analysis etc. In the framework of fuzzy set theory, fuzzy closure operators and fuzzy closure systems themselves (i.e., operators which map fuzzy sets to fuzzy sets and the corresponding systems of closed fuzzy sets) have been studied by many authors in the sense of Čech or in that of Birkhoff. For examples, in the sense of Čech, the concept of a fuzzy closure operator has been introduced and studied by Mashhour and Ghanim [9]. In the sense of Birkhoff, the notion of a fuzzy closure operator has been proposed by Srivastava et al. [12], and also there are the following works to introduce a Birkhoff fuzzy closure space and its separation axioms (see [12, 13]) including T_0- and T_1-fuzzy closure spaces. Note that all the works mentioned above are in the fixed-basis setting $[0, 1]$, the usual unit interval. In [16], Zhou introduced the concept of L-closure operators in the sense of Birkhoff when the fixed-basis L is completely distributive lattice with an ordered-reversing involution "$\prime$", which is a generalization of the concept of fuzzy closure operators studied earlier. In 2001, considering structures of truth values on the fixed-basis L, Bělohlávek [2] outlined a general theory of fuzzy closure operators and fuzzy closure systems.

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From all the works above, we see all kinds of fuzzy closure systems are the role of a classical family of closed fuzzy sets corresponding to fuzzy closure operators. Hence, we point out that all kinds of fuzzy closure systems above may not be closure systems in fuzzy setting.

In this paper we are concerned with a basic question in fuzzy set theory: what is (or what should be) a fuzzifying closure system on a universe set X. For this question, we find that there are some hints in Ying's paper [14]. In 1991, Ying [14] introduced the concept of fuzzifying topological spaces. Briefly speaking, a fuzzifying topology on a set X assigns to every crisp subset of X a certain degree of being open, other than being definitely open or not. Hence, it means that a fuzzifying topology on a set X is a predicate from the classical power set $2^X(\cong \mathscr{P}(X))$ to an appreciate lattice with a suitable structure of truth values. Following the idea of fuzzifying topology, in viewpoint of the paper, a fuzzifying closure system on a set X should be a predicate from the classical power set $2^X(\cong \mathscr{P}(X))$ to an appreciate lattice with logic structure of truth values, which assigns to every crisp subset of X a certain degree of being closed, other than being definitely closed or not.

The contents are arranged as follows. In Section 2, we introduce the necessary concepts and results. In Section 3, the notions of fuzzifying closure systems and fuzzifying closure operators themselves are defined and investigated. Moreover, in categorical aspect, the category **Fss** of fuzzifying closure system spaces and the category **Fcs** of fuzzifying closure spaces are introduced, and it is proved that **Fss** is isomorphic to **Fcs**. In Section 4, the relationship between fuzzifying closure operators and $[0,1]$-closure operators introduced in [16] are discussed. As shall be proved, both **Fss** and **Fcs** could be embedded in the category $[0,1]$**-CLOSURE** of $[0,1]$-closure spaces. In section 5, to introduce T_0, T_1, T_2, R_0 and R_1 separation axioms in fuzzifying closure system spaces, we establish a kind of fuzzifying neighborhood structures of a fuzzifying closure system space, called a fuzzifying remote neighborhood system, and then several separation axioms claimed are established. It should be noted that the topic of separation axioms has been investigated in different areas. To name a few, fuzzy topology [6, 11], fuzzifying topology [7, 10, 15], fuzzy closure spaces [13], and others. However, as the reader shall see, separation axioms occurred in the paper are very different from the previous researches. In addition, Lowen and Xu [8] introduced topological fuzzifying closure operators. There is an example in the paper to show a fuzzifying closure operator of the paper needn't to be a topological fuzzifying closure operator.

2. Preliminaries

Here I will denote the interval $[0,1]$. For a set X, we will make no distinction between a subset of X and its characteristic function. Hence we identify the power set $\mathscr{P}(X)$ with the set 2^X of all characteristic functions on X, and in the paper we use 2^X in place of $\mathscr{P}(X)$. I^X denotes all fuzzy subsets on X.

Now, we recall the definition of a fuzzifying topology on a set X.

Definition 2.1. [14] A fuzzifying topology on a set X is a map $\tau : 2^X \to I$ such that

(1) $\tau(X) = \tau(\emptyset) = 1$;
(2) $\forall A, B \subseteq X$, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) $\forall A_j \subseteq X, j \in J$, $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$.
The pair (X, τ) is called a fuzzifying topological space.

Following Ying's work, Lowen and Xu [8] proposed a topological fuzzifying closure operator, we introduce it as follows.

Definition 2.2. [8] A topological fuzzifying closure operator on a set X is a family of mappings $P = \{p_x : 2^X \to I | x \in X\}$ with the following conditions: for all $x \in X$, $U, V \subseteq X$,

$(N1)$ $p_x(\emptyset) = 0$;
$(N2)$ $p_x(U) < 1 \Rightarrow x \notin U$;
$(N3)$ $p_x(U \cup V) = p_x(U) \vee p_x(V)$;
$(N4)$ $p_x(U) = \bigwedge_{W \supseteq U} \Big(p_x(W) \vee \bigvee_{y \notin W} p_y(W) \Big)$.

In the sense of Birkhoff, Zhou [16] introduced L-closure operators when L is a completely distributive lattice. For the use of the paper, let L be the unit interval I. Then we have the following definition.

Definition 2.3. [16] A map $cl : I^X \to I^X$ satisfying the following conditions:

$(cl1)$ $cl(0_X) = 0_X$;
$(cl2)$ $\forall \lambda \in I^X$, $\lambda \leq cl(\lambda)$;
$(cl3)$ $\forall \lambda,\ \nu \in I^X$, $\lambda \leq \nu \Rightarrow cl(\lambda) \leq cl(\nu)$;
$(cl4)$ $\forall \lambda \in I^X$, $cl(cl(\lambda)) = cl(\lambda)$,
is called an I-closure operator and an ordered pair (X, cl) is called an I-closure space. A continuous map between I-closure spaces (X, cl^X) and (Y, cl^Y) is a map $f : X \to Y$ such that for all $\lambda \in I^Y$, $cl^X(f^{\leftarrow}(\lambda)) \leq f^{\leftarrow}(cl^Y(\lambda))$, where for all $\lambda \in I^X$, $f^{\leftarrow}(\lambda) = \lambda \circ f$. Let I-**CLOSURE** denote the category of I-closure spaces and continuous maps.

In Section 5, when we introduce several separation axioms, the Łukasiewicz's continuous t-norm $*$ on I and its corresponding residuum $\to$ are needed for simplifying formulas therein. Precisely, $\forall a, b \in [0,1]$, $a * b = \max\{a + b - 1, 0\}$, $a \to b = \min\{1 - a + b, 1\}$. Moreover, the t-norm $*$ and its corresponding residuum $\to$ have the following basic properties (see Ćirić et al. [3] and Fang [4]): for all $x, y, z \in I$ and $\{y_i\}_{i\in I} \subseteq I$,

$$x * y \leq z \Leftrightarrow x \leq y \to z, \tag{2.1}$$

$$x \leq y \quad \text{implies} \quad z \to x \leq z \to y \quad \text{and} \quad x \to z \geq y \to z, \tag{2.2}$$

$$x \to 1 = 1; \quad 1 \to x = x, \tag{2.3}$$

$$x \to \bigwedge_{i\in I} y_i = \bigwedge_{i\in I} (x \to y_i), \tag{2.4}$$

$$\bigvee_{i\in I} y_i \to y = \bigwedge_{i\in I} y_i \to y. \tag{2.5}$$

Finally, we refer to [1] for category theory, to [5] for lattice theory.

3. Fuzzifying Closure Systems and Fuzzifying Closure Operators

In classical cases, a closure system is a collection of subsets of a specific universe of discourse, which is closed under arbitrary intersections. In view point of fuzzifying mathematics, we can give the concept of a fuzzifying closure system naturally.

Definition 3.1. A map $\mathcal{S} : 2^X \to [0,1]$ satisfying the following conditions:
($\mathcal{S}1$) $\mathcal{S}(\emptyset) = 1$, $\mathcal{S}(X) = 1$;
($\mathcal{S}2$) $\forall A_t \subseteq X$, $t \in T$, $\mathcal{S}(\bigcap_{t\in T} A_t) \geq \bigwedge_{t\in T} \mathcal{S}(A_t)$,
is called a fuzzifying closure system on a set X and an ordered pair $(X, \mathcal{S})$ is called a fuzzifying closure system space. A continuous map between fuzzifying closure system spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ is a map $f : X \to Y$ such that for all $B \subseteq Y$, $\mathcal{S}(f^{\leftarrow}(B)) \geq \mathcal{T}(B)$, where $f^{\leftarrow}(B) =\{x \mid f(x) \in B\}$. Let **Fss** denote the category of fuzzifying closure system spaces and continuous maps.

In fuzzifying setting, we can obtain a Birkhoff fuzzifying closure operator induced by a fuzzifying closure system. In fact, let $\mathscr{C}_{\mathcal{S}} = \{c_x : 2^X \to I | x \in X\}$, where for all $x \in X,\ A \subseteq X$,

$$c_x(A) = \bigwedge_{x\notin U\supseteq A} \big(1 - \mathcal{S}(U)\big),$$

the following results will show that $\mathscr{C}_{\mathcal{S}}$ plays a role of a closure operator in fuzzifying setting. Note that $c_x(A) = 1$ if $x \in A$. At first, we have the following conclusion about $\mathscr{C}_{\mathcal{S}}$.

Lemma 3.2. *If $\mathscr{C}_{\mathcal{S}}$ is defined as above, then for all $B \subseteq X$, $\bigvee_{x\notin B} c_x(B) = 1-\mathcal{S}(B)$.*

Proof. First, since for any $x \notin B$, $c_x(B) \leq 1-\mathcal{S}(B)$, we always obtain $\bigvee_{x\notin B} c_x(B) \leq 1 - \mathcal{S}(B)$.

Second, we prove the converse inequality.

$$\bigvee_{x\notin B} c_x(B) = \bigvee_{x\notin B} \bigwedge_{x\notin U\supseteq B} \big(1 - \mathcal{S}(U)\big) = \bigwedge_{f\in \prod_{x\notin B} \mathscr{B}_x} \bigvee_{x\notin B} \big(1 - \mathcal{S}(f(x))\big)$$

$$= 1 - \bigvee_{f\in \prod_{x\notin B} \mathscr{B}_x} \bigwedge_{x\notin B} \mathcal{S}(f(x)) \geq 1 - \bigvee_{f\in \prod_{x\notin B} \mathscr{B}_x} \mathcal{S}(\bigwedge_{x\notin B} f(x)) = 1 - \mathcal{S}(B),$$

where $\mathscr{B}_x = \{U | x \notin U \supseteq B\}$. Therefore, the conclusion holds. □

We also have the following conclusion about $\mathscr{C}_{\mathcal{S}}$.

Theorem 3.3. *If $\mathcal{S}$ is a fuzzifying closure system on X, then $\mathscr{C}_{\mathcal{S}} = \{c_x : 2^X \to I | x \in X\}$ induced by $\mathcal{S}$ satisfies the following conditions:*
($\mathscr{C}1$) $\forall x \in X,\ c_x(\emptyset) = 0$;
($\mathscr{C}2$) $c_x(A) < 1 \Rightarrow x \notin A$;
($\mathscr{C}3$) $A \subseteq B \Rightarrow c_x(A) \leq c_x(B)$, $\forall x \in X$;

($\mathscr{C}4$) $\forall A \subseteq X,\ c_x(A) = \bigwedge_{W \supseteq A} \Big(c_x(W) \vee \bigvee_{y \notin W} c_y(W) \Big).$

Proof. ($\mathscr{C}1$) $-$ ($\mathscr{C}3$) are obvious. The proof of ($\mathscr{C}4$) is as follows: First we have $c_x(A) \leq \bigwedge_{W \supseteq A} \Big(c_x(W) \vee \bigvee_{y \notin W} c_y(W) \Big)$.

Conversely, by Lemma 3.2, we obtain

$$\bigwedge_{W \supseteq A} \Big(c_x(W) \vee \bigvee_{y \notin W} c_y(W) \Big) = \bigwedge_{W \supseteq A} \Big(c_x(W) \vee (1 - \mathcal{S}(W)) \Big) = \bigwedge_{W \supseteq A} \big(1 - \mathcal{S}(W) \big)$$
$$\leq \bigwedge_{x \notin W \supseteq A} \big(1 - \mathcal{S}(W) \big) = c_x(A).$$

□

Considering Theorem 3.3, we propose the following definition.

Definition 3.4. (1) A set of $\mathscr{C} = \{c_x : 2^X \to I | x \in X\}$ satisfying ($\mathscr{C}1$) $-$ ($\mathscr{C}4$), is called a (Birkhoff) fuzzifying closure operator on X and an ordered pair $(X, \mathscr{C})$ is called a fuzzifying closure space.

(2) A continuous map between fuzzifying closure spaces $(X, \mathscr{C})$ and $(Y, \mathscr{D})$ is a map $f : X \to Y$ such that for all $B \subseteq Y$, $c_x(f^{\leftarrow}(B)) \leq d_{f(x)}(B)$, where $\mathscr{C} = \{c_x : 2^X \to I | x \in X\}$ and $\mathscr{D} = \{d_y : 2^Y \to I | y \in Y\}$.

(3) Let **Fcs** denote the category of fuzzifying closure spaces and continuous maps.

The following corollary can be obtained from Theorem 3.3 and Definition 3.4.

Corollary 3.5. *If $\mathcal{S}$ is a fuzzifying closure system on X, then $\mathscr{C}_{\mathcal{S}}$ induced by $\mathcal{S}$ is a fuzzifying closure operator.*

Now we will give an example to show that the Birkhoff fuzzifying closure operator defined in this paper is different from a topological fuzzifying closure operator introduced by Lowen and Xu [8].

Example 3.6. Let $X = \{u,\ v,\ w\}$, $\mathcal{S} : 2^X \to I$ such that if $A = \emptyset, \{u\}, \{v\}, \{u, w\}$, X, then $\mathcal{S}(A) = 1$; $\mathcal{S}(A) = 0$ otherwise. Then $\mathcal{S}$ is a fuzzifying closure system by Definition 3.1.

Let $\mathscr{C}_{\mathcal{S}} = \{c_x : 2^X \to I | x \in X\}$, where $c_x(W) = \bigwedge_{x \notin U \supseteq W} \big(1 - \mathcal{S}(U)\big)$. If $x = w \in X$, then $c_w(\{u\} \cup \{v\}) \neq c_w(\{u\}) \vee c_w(\{v\})$, i.e., c_w doesn't preserve union.

In fact, $c_w(\{u\}) = \bigwedge_{w \notin U \supseteq \{u\}} \big(1 - \mathcal{S}(U)\big) = 1 - \mathcal{S}(\{u\}) = 0$, $c_w(\{v\}) = \bigwedge_{w \notin U \supseteq \{v\}} \big(1 - \mathcal{S}(U)\big) = 1 - \mathcal{S}(\{v\}) = 0$, however, $c_w(\{u\} \cup \{v\}) = \bigwedge_{w \notin U \supseteq \{u\} \cup \{v\}} \big(1 - \mathcal{S}(U)\big) = 1 - \mathcal{S}(\{u\} \cup \{v\}) = 1$.

Hence the Birkhoff fuzzifying closure operator defined in this paper may not be a topological fuzzifying closure operator introduced by Lowen and Xu [8].

On the one hand, a fuzzifying closure operator is induced by a fuzzifying closure system. On the other hand, a fuzzifying closure operator induces a fuzzifying closure system. In fact, we have the following definition.

Definition 3.7. Let $\mathscr{C} = \{c_x : 2^X \to I | x \in X\}$ be a fuzzifying closure operator on X, define $\mathcal{S}_{\mathscr{C}}(F) = \bigwedge_{x \notin F}\big(1 - c_x(F)\big)$, for all $F \subseteq X$. Then $\mathcal{S}_{\mathscr{C}}$ is called a fuzzifying closure system induced by $\mathscr{C}$.

In order to show that $\mathcal{S}_{\mathscr{C}}$ is indeed a fuzzifying closure system, we prove the following theorem.

Theorem 3.8. *If $\mathscr{C}$ is a fuzzifying closure operator on X, then $\mathcal{S}_{\mathscr{C}}$ induced by $\mathscr{C}$ is a fuzzifying closure system.*

Proof. $(\mathcal{S}1)$ $\mathcal{S}_{\mathscr{C}}(\emptyset) = \bigwedge\limits_{x \notin \emptyset}\big(1 - c_x(\emptyset)\big) = 1$, $\mathcal{S}_{\mathscr{C}}(X) = \bigwedge\limits_{x \notin X}\big(1 - c_x(X)\big) = \bigwedge \emptyset = 1$.

$(\mathcal{S}2)$ $\forall \{F_t\}_{t \in T} \subseteq 2^X$,

$$\mathcal{S}_{\mathscr{C}}(\bigcap_{t \in T} F_t) = \bigwedge_{x \notin \bigcap\limits_{t \in T} F_t} \big(1 - c_x(\bigcap_{t \in T} F_t)\big) = \bigwedge_{t \in T} \bigwedge_{x \notin F_t} \big(1 - c_x(\bigcap_{t \in T} F_t)\big)$$
$$\geq \bigwedge_{t \in T} \bigwedge_{x \notin F_t} \big(1 - c_x(F_t)\big) = \bigwedge_{t \in T} \mathcal{S}_{\mathscr{C}}(F_t),$$

So we obtain the conclusion. □

In order to prove that there exist functors between the category **Fss** and the category **Fcs**, we point out that the following conclusions hold.

Theorem 3.9. (1)*If a map between fuzzifying closure system spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is continuous, then $f : (X, \mathscr{C}_{\mathcal{S}}) \to (Y, \mathscr{C}_{\mathcal{T}})$ is continuous.*

(2)*If a map between fuzzifying closure spaces $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$ is continuous, then $f : (X, \mathcal{S}_{\mathscr{C}}) \to (Y, \mathcal{S}_{\mathscr{D}})$ is continuous.*

Applying Corollary 3.5 and Theorems 3.8, 3.9, we find a functor $\mathbb{F}$ from **Fss** to **Fcs**, where $\mathbb{F}$ defined by $\mathbb{F}$: **Fss** $\to$ **Fcs** such that for all $(X, \mathcal{S}) \in|$ **Fss** $|$, $\mathbb{F}((X, \mathcal{S})) = (X, \mathscr{C}_{\mathcal{S}})$, for all $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$, $\mathbb{F}(f) = f$. Meanwhile, there exists a functor $\mathbb{G}$ from **Fcs** to **Fss**, such that for all $(X, \mathscr{C}) \in|$ **Fcs** $|$, $\mathbb{G}((X, \mathscr{C})) = (X, \mathcal{S}_{\mathscr{C}})$, and that for all $f : (X, \mathscr{C}) \to (Y, \mathscr{D})$, $\mathbb{G}(f) = f$.

The following results show that $\mathbb{F}$ and $\mathbb{G}$ are both isomorphic functors. In fact, we can prove the following lemmas easily.

Lemma 3.10. *If $\mathcal{S}$ is a fuzzifying closure system on X, then $\mathcal{S}_{\mathscr{C}_{\mathcal{S}}}(A) = \bigwedge\limits_{x \notin A} \bigvee\limits_{x \notin U \supseteq A} \mathcal{S}(U)$, moreover $\mathcal{S} = \mathcal{S}_{\mathscr{C}_{\mathcal{S}}}$.*

Proof. It is easy to obtain $\mathcal{S}_{\mathscr{C}_{\mathcal{S}}}(A) = \bigwedge\limits_{x \notin A} \bigvee\limits_{x \notin U \supseteq A} \mathcal{S}(U)$ by the above conclusions. We only show that $\mathcal{S} = \mathcal{S}_{\mathscr{C}_{\mathcal{S}}}$, i.e., for all $A \subseteq X$, $\mathcal{S}(A) = \bigwedge\limits_{x \notin A} \bigvee\limits_{x \notin U \supseteq A} \mathcal{S}(U)$.

In fact, firstly, if $A = X$, we have $\mathcal{S}(X) = 1$ and $\bigwedge\limits_{x \notin X} \bigvee\limits_{x \notin U \supseteq X} \mathcal{S}(U) = \bigwedge \emptyset = 1$.

Secondly, if $A \neq X$, on the one hand we have $\mathcal{S}(A) \leq \bigwedge_{x \notin A} \bigvee_{x \notin U \supseteq A} \mathcal{S}(U)$.

On the other hand,

$$\bigwedge_{x \notin A} \bigvee_{x \notin U \supseteq A} \mathcal{S}(U) = \bigvee_{f \in \prod_{x \notin A} \mathscr{B}_x} \bigwedge_{x \notin A} \mathcal{S}(f(x)) \leq \bigvee_{f \in \prod_{x \notin A} \mathscr{B}_x} \mathcal{S}(\bigwedge_{x \notin A} f(x))$$
$$= \bigvee_{f \in \prod_{x \notin A} \mathscr{B}_x} \mathcal{S}(A) = \mathcal{S}(A),$$

where $\mathscr{B}_x = \{U | x \notin U \supseteq A\}$. Hence the equality holds. □

Lemma 3.11. *If $\mathscr{C}$ is a fuzzifying closure operator on X, then $c_x(A) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} c_y(B)$, where $c_x \in \mathscr{C}_{\mathcal{S}_\mathscr{C}}$, moreover $\mathscr{C}_{\mathcal{S}_\mathscr{C}} = \mathscr{C}$.*

Proof. It suffices to prove $\mathscr{C}_{\mathcal{S}_\mathscr{C}} = \mathscr{C}$, since we can get $c_x(A) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} c_y(B)$ easily by the above conclusions, where $c_x \in \mathscr{C}_{\mathcal{S}_\mathscr{C}}$.

Indeed, suppose $\mathscr{C} = \{c_z : 2^X \to I | z \in X\}$, firstly, if $z \in A$, we have $c_z(A) = 1$ and $\bigwedge_{z \notin B \supseteq A} \bigvee_{y \notin B} c_y(B) = \bigwedge \emptyset = 1$, i.e. $c_z(A) = \bigwedge_{z \notin B \supseteq A} \bigvee_{y \notin B} c_y(B)$.

Secondly, if $z \notin A$, by ($\mathscr{C}$4) we get $c_z(A) = \bigwedge_{z \notin B \supseteq A} \bigvee_{y \notin B} c_y(B)$. Hence $c_z(A) = c_x(A)$, the claim is true. □

By Theorem 3.9 and Lemmas 3.10, 3.11, we have $\mathbb{F} \circ \mathbb{G} = id_{\mathbf{Fcs}}$, $\mathbb{G} \circ \mathbb{F} = id_{\mathbf{Fss}}$. To sum up, we get the following theorem.

Theorem 3.12. *The category of fuzzifying closure system spaces* **Fss** *is isomorphic to the category of fuzzifying closure spaces* **Fcs**.

4. Embedding Fcs in I-CLOSURE

In this section we will prove that the category **Fcs** can be embedded in the category **I-CLOSURE** [16]. Suppose $\mathscr{C}$ is a fuzzifying closure operator on X. $\mathscr{C}$ induces an operator $cl_\mathscr{C} : I^X \to I^X$ in the following way: for all $x \in X$, $\lambda \in I^X$,

$$cl_\mathscr{C}(\lambda)(x) = \bigwedge_{W \in T_x} \Big(c_x(W) \vee \bigvee_{y \notin W} \lambda(y) \Big), \quad \text{(i)}$$

where $T_x = \{W | x \notin W\}$.

Theorem 4.1. *The operator $cl_\mathscr{C} : I^X \to I^X$ is an I-closure operator if and only if the operator $\mathscr{C}$ is a fuzzifying closure operator on X.*

Proof. Sufficiency: We show that $cl_\mathscr{C}$ satisfies the above conditions (cl1)−(cl4). We only prove that the operator $cl_\mathscr{C}$ satisfies (cl4), where (cl1)−(cl3) are trivial.

Indeed, (cl4): for all $\lambda \in I^X$, $cl_{\mathscr{C}}(cl_{\mathscr{C}}(\lambda)) \geq cl_{\mathscr{C}}(\lambda)$ is obvious. Conversely, we show that $cl_{\mathscr{C}}(cl_{\mathscr{C}}(\lambda)) \leq cl_{\mathscr{C}}(\lambda)$. First,

$$\begin{aligned} cl_{\mathscr{C}}(cl_{\mathscr{C}}(\lambda))(x) &= \bigwedge_{W\in T_x}\Big(c_x(W)\vee \bigvee_{y\notin W} cl_{\mathscr{C}}(\lambda)(y)\Big) \\ &= \bigwedge_{W\in T_x}\Bigg(c_x(W)\vee \bigvee_{y\notin W}\bigwedge_{V\in T_y}\Big(c_y(V)\vee\bigvee_{z\notin V}\lambda(z)\Big)\Bigg) \\ &= \bigwedge_{W\in T_x}\Bigg(c_x(W)\vee \bigwedge_{f\in\prod_{y\notin W} T_y}\bigvee_{y\notin W}\Big(c_y(f(y))\vee\bigvee_{z\notin f(y)}\lambda(z)\Big)\Bigg) \\ &\leq \bigwedge_{W\in T_x}\Bigg(c_x(W)\vee \bigvee_{y\notin W}\Big(c_y(W)\vee\bigvee_{z\notin W}\lambda(z)\Big)\Bigg) \\ &= \bigwedge_{W\in T_x}\Bigg(\bigvee_{y\notin W}\Big(c_y(W)\vee\bigvee_{z\notin W}\lambda(z)\Big)\Bigg). \end{aligned}$$

If $cl_{\mathscr{C}}(\lambda)(x) < a$ $(a > 0)$, then there exists W such that $c_x(W)\vee\bigvee_{y\notin W}\lambda(y) < a$, so $c_x(W) < a$ and $\bigvee_{y\notin W}\lambda(y) < a$. By Lemma 3.11, $c_x(W) = \bigwedge_{x\notin U\supseteq W}\bigvee_{y\notin U} c_y(U)$, and thus there exists some U_0 such that $x\notin U_0\supseteq W$ and $\bigvee_{y\notin U_0} c_y(U_0) < a$. Therefore,

$$\begin{aligned} cl_{\mathscr{C}}(cl_{\mathscr{C}}(\lambda))(x) &\leq \bigwedge_{W\in T_x}\Big(\bigvee_{y\notin W}\big(c_y(W)\vee\bigvee_{z\notin W}\lambda(z)\big)\Big) \leq \bigvee_{y\notin U_0}\Big(c_y(U_0)\vee\bigvee_{z\notin U_0}\lambda(z)\Big) \\ &\leq \bigvee_{y\notin U_0}\Big(c_y(U_0)\vee\bigvee_{z\notin W}\lambda(z)\Big) < a. \end{aligned}$$

Hence $cl_{\mathscr{C}}(cl_{\mathscr{C}}(\lambda)) \leq cl_{\mathscr{C}}(\lambda)$ from the arbitrariness of a.

Necessity: Suppose the operator $cl_{\mathscr{C}}$ is an I-closure operator on X, we show that the operator $\mathscr{C}$ is a fuzzifying closure operator. By (i) we can have, for all $U\subseteq X$,

$$\begin{aligned} cl_{\mathscr{C}}(U)(x) &= \bigwedge_{W\in T_x}\Big(c_x(W)\vee\bigvee_{y\notin W}U(y)\Big) = \bigwedge_{x\notin W\supseteq U}\Big(c_x(W)\vee\bigvee_{y\notin W}U(y)\Big) \\ &= \bigwedge_{x\notin W\supseteq U} c_x(W)\vee 0 = c_x(U). \end{aligned}$$

Thus by the above equality, we can check that $\mathscr{C}$ satisfies $(\mathscr{C}1)-(\mathscr{C}4)$, and the details are omitted. □

Theorem 4.2. *A map $f:(X,\mathscr{C}^X)\to(Y,\mathscr{C}^Y)$ between fuzzifying closure spaces is continuous if and only if it is continuous with respect to the induced I-closure spaces.*

Proof. Necessity: Assume that f is continuous, i.e., for all $x\in X$, $W\subseteq Y$, $c_x^X(f^{\leftarrow}(W)) \leq c_{f(x)}^Y(W)$. In order to prove that f is continuous with respect to the induced I-closure spaces, it suffices to show that for all $x \in X,\ \lambda \in I^Y$,

$$\begin{aligned} cl_{\mathscr{C}^Y}(\lambda)(f(x)) = & \bigwedge_{f(x)\notin W} \Big(c^Y_{f(x)}(W) \vee \bigvee_{y\notin W} \lambda(y) \Big) \\ & \geq \bigwedge_{x\notin f^{\leftarrow}(W)} \Big(c^X_x(f^{\leftarrow}(W)) \vee \bigvee_{y\notin W} \lambda(y) \Big) \\ & \geq \bigwedge_{x\notin f^{\leftarrow}(W)} \Big(c^X_x(f^{\leftarrow}(W)) \vee \bigvee_{z\notin f^{\leftarrow}(W)} \lambda(f(z)) \Big) \\ & \geq \bigwedge_{x\notin U} \Big(c^X_x(U) \vee \bigvee_{z\notin U} \lambda(f(z)) \Big) = cl_{\mathscr{C}^X}(\lambda\circ f)(x). \end{aligned}$$

So a map $f : (X, cl_{\mathscr{C}^X}) \to (Y, cl_{\mathscr{C}^Y})$ is continuous.

Sufficiency: We need to prove that $f : (X, \mathscr{C}^X) \to (Y, \mathscr{C}^Y)$ is continuous. It can be shown as follows: For all $x \in X,\ W \subseteq Y$,

$$\begin{aligned} c^Y_{f(x)}(W) & = cl_{\mathscr{C}^Y}(W)(f(x)) = f^{\leftarrow}(cl_{\mathscr{C}^Y}(W))(x) \\ & \geq cl_{\mathscr{C}^X}(f^{\leftarrow}(W))(x) = c^X_x(f^{\leftarrow}(W)). \end{aligned}$$

The conclusion holds. □

Remark 4.3. Suppose Υ: **Fcs**→I-**CLOSURE** is a functor such that for all $(X,\mathscr{C}) \in |$ **Fcs** $|$, $\Upsilon((X,\mathscr{C})) = (X, cl_{\mathscr{C}})$, for each continuous morphism $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$, $\Upsilon(f) = f$. By the above theorems, Υ is an embedding functor from the category **Fcs** to the category I-**CLOSURE**. In the following, we show that Υ is an embedding functor, i.e., whenever $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ and $g : (X',\mathscr{C}') \to (Y',\mathscr{D}')$ are continuous morphisms (not necessarily parallel) such that $\Upsilon(f) = \Upsilon(g)$, then $f = g$, where $\Upsilon(f)$ and $\Upsilon(g)$ are the continuous morphisms between I-closure spaces.

Since $\Upsilon(f) = \Upsilon(g)$, we have that $\Upsilon(X,\mathscr{C}) = \Upsilon(X',\mathscr{C}')$ and $\Upsilon(Y,\mathscr{D}) = \Upsilon(Y',\mathscr{D}')$ hold. In fact, $\Upsilon(X,\mathscr{C}) = \Upsilon(X',\mathscr{C}') \Rightarrow X = X'$ and $\Upsilon(\mathscr{C}) = \Upsilon(\mathscr{C}')$. $\Upsilon(Y,\mathscr{D}) = \Upsilon(Y',\mathscr{D}') \Rightarrow Y = Y'$ and $\Upsilon(\mathscr{D}) = \Upsilon(\mathscr{D}')$. In order to prove that $f = g$ holds in **Fcs**, it remains to prove that $\mathscr{C} = \mathscr{C}'$, $\mathscr{D} = \mathscr{D}'$ and $f = g$ (as maps) hold.

In fact, we have that

$$\begin{cases} \Upsilon(\mathscr{C}) = \Upsilon(\mathscr{C}') \Rightarrow \mathscr{C} = \mathscr{C}', & (a) \\ \Upsilon(\mathscr{D}) = \Upsilon(\mathscr{D}') \Rightarrow \mathscr{D} = \mathscr{D}'. & (b) \end{cases}$$

Now we show that (a) and (b) hold.

(a): First, $id_X : (X, \Upsilon(\mathscr{C})) \to (X, \Upsilon(\mathscr{C}'))$ is continuous. By Theorem 4.2, $id_X : (X,\mathscr{C}) \to (X,\mathscr{C}')$ is continuous since $X = X'$ and $\Upsilon(\mathscr{C}) = \Upsilon(\mathscr{C}')$, where $\mathscr{C} = \{c_x : 2^X \to I | x \in X\}$, $\mathscr{C}' = \{d_x : 2^X \to I | x \in X\}$. So by the definition of continuity (see Definition 3.4 (2)), $\forall x \in X, W \subseteq X$,

$$d_x(W) = d_{id_X(x)}(W) \geq c_x(id^{\leftarrow}_X(W)) = c_x(W).$$

Second, we also get that $id_X : (X, \Upsilon(\mathscr{C}')) \to (X, \Upsilon(\mathscr{C}))$ is continuous. By Theorem 4.2, $id_X : (X,\mathscr{C}') \to (X,\mathscr{C})$ is continuous. So $\forall x \in X, W \subseteq X$, it holds that

$$c_x(W) = c_{id_X(x)}(W) \geq d_x(id_X^{\leftarrow}(W)) = d_x(W).$$

Therefore $c_x(W) = d_x(W)$ for all $x \in X, W \subseteq X$, i.e., $\mathscr{C} = \mathscr{C}'$. Similarly, we can get (b), as desired.

Finally $f = g$ holds as maps since $\Upsilon(f) = f, \Upsilon(g) = g$ and $\Upsilon(f) = \Upsilon(g)$.

Applying Theorems 4.2, 3.12 and Remark 4.3, we get the following theorem.

Theorem 4.4. *The category* **Fcs** *of fuzzifying closure spaces can be embedded in the category* I**-CLOSURE** *of* I*-closure spaces; Simultaneously, the category* **Fss** *of fuzzifying closure system spaces also can be embedded in the category* I**-CLOSURE** *of* I*-closure spaces.*

5. Separation Axioms

In this section we introduce T_0, T_1, T_2, R_0 and R_1 separation axioms in fuzzifying closure system spaces and study the relations among them. Firstly, we give some preliminary concepts and properties. Note that we write Σ for the class of all fuzzifying closure system spaces.

Lemma 5.1. *Let* $(X, \mathcal{S})$ *be a fuzzifying closure system space and* $\emptyset \neq Y \subseteq X$*.* $(Y, \mathcal{S}|Y)$ *is called the subspace of* $(X, \mathcal{S})$*, where* $\mathcal{S}|Y : 2^Y \to I$ *is defined by* $\mathcal{S}|Y(U) = \bigvee_{V \cap Y = U} \mathcal{S}(V)$*.*

Proof. It suffices to show that $\mathcal{S}|Y$ is a fuzzifying closure system.

(i)$\mathcal{S}|Y(\emptyset) = \bigvee_{V \cap Y = \emptyset} \mathcal{S}(V) \geq \mathcal{S}(\emptyset) = 1$; $\mathcal{S}|Y(X) = \bigvee_{V \cap Y = X} \mathcal{S}(V) = \mathcal{S}(X) = 1$.

(ii)For any $U_t \subseteq Y (t \in T)$,

$$\bigwedge_{t \in T} \mathcal{S}|Y(U_t) = \bigwedge_{t \in T} \bigvee_{V_t \cap Y = U_t} \mathcal{S}(V_t) = \bigvee_{f \in \Pi_{t \in T} \mathscr{M}_t} \bigwedge_{t \in T} \mathcal{S}(f(t)) \leq \bigvee_{f \in \Pi_{t \in T} \mathscr{M}_t} \mathcal{S}(\bigcap_{t \in T} f(t)),$$

where $\mathscr{M}_t = \{V_t \subseteq Y | U_t = V_t \cap Y\} (t \in T)$. Since for all f, $(\bigcap_{t \in T} f(t)) \cap Y = \bigcap_{t \in T} (f(t) \cap Y) = \bigcap_{t \in T} U_t$, we have $\bigwedge_{t \in T} \mathcal{S}|Y(U_t) \leq \mathcal{S}|Y(\bigcap_{t \in T} U_t)$. □

In order to describe separation axioms in fuzzifying closure system spaces, we give the definition of a fuzzifying remote neighborhood system now.

Definition 5.2. Let $(X, \mathcal{S})$ be a fuzzifying closure system space. For each $x \in X$ we define $\eta_x : 2^X \to I$ for each subset $A \subseteq X$, by $\eta_x(A) = \bigvee_{x \notin B \supseteq A} \mathcal{S}(B)$. The $R = \{\eta_x : 2^X \to I | x \in X\}$ is called a fuzzifying remote neighborhood system of $(X, \mathcal{S})$, and for each x, η_x is called a fuzzifying remote neighborhood system at x.

Theorem 5.3. *If* $R = \{\eta_x : 2^X \to I | x \in X\}$ *is a fuzzifying remote neighborhood system of a fuzzifying closure system space* $(X, \mathcal{S})$*, then for all* $x \in X$*,* η_x *has the following properties:*

(1)$\eta_x(\emptyset) = 1, \ \ \eta_x(X) = 0$;

(2)$\forall A, B \subseteq X, A \supseteq B \Rightarrow \eta_x(B) \geq \eta_x(A)$.

Proof. (1) $\eta_x(\emptyset) = \bigvee_{x\notin B\supseteq\emptyset} \mathcal{S}(B) \geq \mathcal{S}(\emptyset) = 1$; $\eta_x(X) = \bigvee_{x\notin B\supseteq X} \mathcal{S}(B) = \bigvee \emptyset = 0$.

(2)If $A \subseteq B$, $\eta_x(A) = \bigvee_{x\notin D\supseteq A} \mathcal{S}(D) \geq \bigvee_{x\notin C\supseteq B} \mathcal{S}(C) = \eta_x(B)$. □

Theorem 5.4. *If $\mathscr{C}_{\mathcal{S}} = \{c_x : 2^X \to I | x \in X\}$ is a fuzzifying closure operator on X induced by $\mathcal{S}$, then $c_x(A) = 1 - \eta_x(A)$, for all $x \in X, A \in 2^X$.*

Proof. The equality is true since

$$1 - \eta_x(A) = 1 - \bigvee_{x\notin B\supseteq A} \mathcal{S}(B) = \bigwedge_{x\notin B\supseteq A} (1 - \mathcal{S}(B)) = c_x(A).$$

□

By Lemmas 3.10, 3.11, Definition 5.2 and the above theorem we have the following lemma.

Lemma 5.5. *If $R = \{\eta_x : 2^X \to I | x \in X\}$ is the remote neighborhood system induced by a fuzzifying closure system space $(X, \mathcal{S})$, then*
(1)$\mathcal{S}(A) = \bigwedge_{x\notin A} \eta_x(A)$;
(2)$\eta_x(A) = \bigvee_{x\notin B\supseteq A} \bigwedge_{y\notin B} \eta_y(B)$.

Proof. It is easy to prove and the proof is omitted. □

Now we give the concepts of separation axioms in fuzzifying closure system spaces with a fuzzifying remote neighborhood system.

Definition 5.6. Let $(X, \mathcal{S})$ be a fuzzifying closure system space and let $\Delta := \{(x, x)|x \in X\}$.

(1) The map $\mathbf{P}_{T_0} : X \times X - \Delta \to I$ is defined as follows:

$$\mathbf{P}_{T_0}(x, y) = \bigvee_{y\in A} \eta_x(A) \vee \bigvee_{x\in B} \eta_y(B),$$

and we call $\mathbf{P}_{T_0}$ the pointwise T_0-degree function. The value $\mathbf{P}_{T_0}(x, y)$ is interpreted as the degree to which two distinguished crisp points x and y are separated in the sense of T_0 in $(X, \mathcal{S})$.

The map $\mathbf{D}_{T_0} : \Sigma \to I$ is defined as follows:

$$\mathbf{D}_{T_0}(X, \mathcal{S}) = \bigwedge \{\mathbf{P}_{T_0}(x, y)|x, y \in X, x \neq y\}, \ \ \forall (X, \mathcal{S}) \in \Sigma,$$

and $\mathbf{D}_{T_0}$ is called a T_0-degree function of a fuzzifying closure system space. A fuzzifying closure system space $(X, \mathcal{S})$ is called a fuzzifying T_0-closure system space (briefly, T_0-css) if $\mathbf{D}_{T_0}(X, \mathcal{S}) = 1$. In general, the value $\mathbf{D}_{T_0}(X, \mathcal{S})$ is interpreted as the degree to which the fuzzifying closure system space $(X, \mathcal{S})$ is T_0.

(2) The map $\mathbf{P}_{T_1} : X \times X - \Delta \to I$ is defined as follows:

$$\mathbf{P}_{T_1}(x, y) = \bigvee_{y\in A} \eta_x(A) \wedge \bigvee_{x\in B} \eta_y(B),$$

and we call $\mathbf{P}_{T_1}$ the pointwise T_1-degree function. The value $\mathbf{P}_{T_1}(x, y)$ is interpreted as the degree to which two distinguished crisp points x and y are separated in the sense of T_1 in $(X, \mathcal{S})$.

The map $\mathbf{D}_{T_1} : \Sigma \to I$ is defined as follows:

$$\mathbf{D}_{T_1}(X, \mathcal{S}) = \bigwedge\{\mathbf{P}_{T_1}(x, y)|x, y \in X, x \neq y\}, \ \ \forall (X, \mathcal{S}) \in \Sigma,$$

and $\mathbf{D}_{T_1}$ is called a T_1-degree function of a fuzzifying closure system space. A fuzzifying closure system space $(X, \mathcal{S})$ is called a fuzzifying T_1-closure system space (briefly, T_1-css) if $\mathbf{D}_{T_1}(X, \mathcal{S}) = 1$. In general, the value $\mathbf{D}_{T_1}(X, \mathcal{S})$ is interpreted as the degree to which the fuzzifying closure system space $(X, \mathcal{S})$ is T_1.

(3) The map $\mathbf{P}_{T_2} : X \times X - \Delta \to I$ is defined as follows:

$$\mathbf{P}_{T_2}(x, y) = \bigvee_{A \cup B = X} \eta_x(A) \wedge \eta_y(B),$$

and we call $\mathbf{P}_{T_2}$ the pointwise T_2-degree function. The value $\mathbf{P}_{T_2}(x, y)$ is interpreted as the degree to which two distinguished crisp points x and y are separated in the sense of T_2 in $(X, \mathcal{S})$.

The map $\mathbf{D}_{T_2} : \Sigma \to I$ is defined as follows:

$$\mathbf{D}_{T_2}(X, \mathcal{S}) = \bigwedge\{\mathbf{P}_{T_2}(x, y)|x, y \in X, x \neq y\}, \ \ \forall (X, \mathcal{S}) \in \Sigma,$$

and $\mathbf{D}_{T_2}$ is called a T_2-degree function of a fuzzifying closure system space. A fuzzifying closure system space $(X, \mathcal{S})$ is called a fuzzifying T_2-closure system space (briefly, T_2-css) if $\mathbf{D}_{T_2}(X, \mathcal{S}) = 1$. In general, the value $\mathbf{D}_{T_2}(X, \mathcal{S})$ is interpreted as the degree to which the fuzzifying closure system space $(X, \mathcal{S})$ is T_2.

(4) The map $\mathbf{P}_{R_0} : X \times X - \Delta \to I$ is defined as follows:

$$\mathbf{P}_{R_0}(x, y) = \mathbf{P}_{T_0}(x, y) \to \mathbf{P}_{T_1}(x, y),$$

and we call $\mathbf{P}_{R_0}$ the pointwise R_0-degree function. The value $\mathbf{P}_{R_0}(x, y)$ is interpreted as the degree to which two distinguished crisp points x and y are separated in the sense of R_0 in $(X, \mathcal{S})$.

The map $\mathbf{D}_{R_0} : \Sigma \to I$ is defined as follows:

$$\mathbf{D}_{R_0}(X, \mathcal{S}) = \bigwedge\{\mathbf{P}_{R_0}(x, y)|x, y \in X, x \neq y\}, \ \ \forall (X, \mathcal{S}) \in \Sigma,$$

and $\mathbf{D}_{R_0}$ is called a R_0-degree function of a fuzzifying closure system space. A fuzzifying closure system space $(X, \mathcal{S})$ is called a fuzzifying R_0-closure system space (briefly, R_0-css) if $\mathbf{D}_{R_0}(X, \mathcal{S}) = 1$. In general, the value $\mathbf{D}_{R_0}(X, \mathcal{S})$ is interpreted as the degree to which the fuzzifying closure system space $(X, \mathcal{S})$ is R_0.

(5) The map $\mathbf{P}_{R_1} : X \times X - \Delta \to I$ is defined as follows:

$$\mathbf{P}_{R_1}(x, y) = \mathbf{P}_{T_0}(x, y) \to \mathbf{P}_{T_2}(x, y),$$

and we call $\mathbf{P}_{R_1}$ the pointwise R_1-degree function. The value $\mathbf{P}_{R_1}(x, y)$ is interpreted as the degree to which two distinguished crisp points x and y are separated in the sense of R_1 in $(X, \mathcal{S})$.

The map $\mathbf{D}_{R_1} : \Sigma \to I$ is defined as follows:

$$\mathbf{D}_{R_1}(X, \mathcal{S}) = \bigwedge\{\mathbf{P}_{R_1}(x, y)|x, y \in X, x \neq y\}, \ \ \forall (X, \mathcal{S}) \in \Sigma,$$

and $\mathbf{D}_{R_1}$ is called a R_1-degree function of a fuzzifying closure system space. A fuzzifying closure system space $(X, \mathcal{S})$ is called a fuzzifying R_1-closure system space (briefly, R_1-css) if $\mathbf{D}_{R_1}(X, \mathcal{S}) = 1$. In general, the value $\mathbf{D}_{R_1}(X, \mathcal{S})$ is interpreted as the degree to which the fuzzifying closure system space $(X, \mathcal{S})$ is R_1.

Theorem 5.7. *For any fuzzifying closure system space $(X, \mathcal{S})$, it holds that*

$$\mathbf{D}_{T_0}\ (X, \mathcal{S}) \geq \mathbf{D}_{T_1}(X, \mathcal{S}) \geq \mathbf{D}_{T_2}(X, \mathcal{S}).$$

Proof. Since

$$\mathbf{P}_{T_1}(x, y) = \bigvee_{y \in A, x \in B} \eta_x(A) \wedge \eta_y(B) \geq \bigwedge_{A \cup B = X} \eta_x(A) \wedge \eta_y(B) = \mathbf{P}_{T_2}(x, y),$$

whenever $x \neq y$, the second inequality is proved by

$$\begin{aligned} \mathbf{D}_{T_2}(X, \mathcal{S}) &= \bigwedge \{\mathbf{P}_{T_2}(x, y) | x, y \in X, x \neq y\} \\ &\leq \bigwedge \{\mathbf{P}_{T_1}(x, y) | x, y \in X, x \neq y\} = \mathbf{D}_{T_1}(X, \mathcal{S}). \end{aligned}$$

Moreover the first inequality is shown by

$$\begin{aligned} \mathbf{D}_{T_0}(X, \mathcal{S}) &= \bigwedge_{x \neq y} \mathbf{P}_{T_1}(x, y) \geq \bigwedge_{x \neq y} \bigvee_{y \in A} \eta_x(A) \\ &\geq \bigwedge_{x \neq y} \bigvee_{y \in A} \eta_x(A) \wedge \bigvee_{x \in B} \eta_y(B) = \mathbf{D}_{T_1}(X, \mathcal{S}). \end{aligned}$$

□

Remark 5.8. Theorem 5.7 implies that a T_2-css must be a T_1-css and a T_1-css must be a T_0-css.

To obtain properties of T_0-, T_1- and T_2-degree functions of fuzzifying closure system spaces further, the following definitions are needed.

Definition 5.9. Let $\mathcal{S}$ and $\mathcal{T}$ be two fuzzifying closure systems on X. The fuzzifying closure system space $(X, \mathcal{S})$ is said to be coarser than $(X, \mathcal{T})$ (or that $(X, \mathcal{T})$ is finer than $(X, \mathcal{S})$) if $\mathcal{S}(A) \geq \mathcal{T}(A)$ for all $A \subseteq X$.

Definition 5.10. Let $\mathscr{C} = \{c_x : 2^X \to I | x \in X\}$ and $\mathscr{D} = \{d_x : 2^X \to I | x \in X\}$ be two fuzzifying closure operators on a set X. The fuzzifying closure space $(X, \mathscr{C})$ is said to be coarser than $(X, \mathscr{D})$ (or that $(X, \mathscr{D})$ is finer than $(X, \mathscr{C})$) if $d_x(A) \leq c_x(A)$ for all $A \subseteq X$.

Proposition 5.11. *The following statements are equivalent:*
(1)$\mathcal{S}(A) \geq \mathcal{T}(A)$ *for all* $A \subseteq X$*;*
(2) *The fuzzifying closure space* $(X, \mathscr{C}_{\mathcal{S}})$ *is finer than* $(X, \mathscr{C}_{\mathcal{T}})$.

Proof. The proof is immediate. □

Proposition 5.12. *Let $(X, \mathcal{S})$ and $(X, \mathcal{T})$ be two fuzzifying closure system spaces such that $(X, \mathcal{S})$ is coarser than $(X, \mathcal{T})$. Then $\mathbf{D}_{T_i}(X, \mathcal{S}) \geq \mathbf{D}_{T_i}(X, \mathcal{T})$, $i = 0, 1, 2$.*

Proof. We only prove $\mathbf{D}_{T_0}(X,\mathcal{S}) \geq \mathbf{D}_{T_0}(X,\mathcal{T})$ and the others can be obtained in a similar way and are omitted. For any two different crisp points x and y in X. From Definition 5.6, we have

$$\mathbf{D}_{T_0}(X,\mathcal{S}) = \bigwedge\{\bigvee_{y\in A}\eta_x(A) \vee \bigvee_{x\in B}\eta_y(B)|x,y\in X, x\neq y\},$$

and

$$\mathbf{D}_{T_0}(X,\mathcal{T}) = \bigwedge\{\bigvee_{y\in A}\eta_x^{\mathcal{T}}(A) \vee \bigvee_{x\in B}\eta_y^{\mathcal{T}}(B)|x,y\in X, x\neq y\}.$$

In fact, we can get $\mathcal{T}(A) \leq \mathcal{S}(A)$ from Definition 5.9. Applying Definition 5.2 we can get $\eta_x^{\mathcal{T}}(A) \leq \eta_x(A)$ and $\eta_y^{\mathcal{T}}(A) \leq \eta_y(A)$. We obtain the conclusion naturally. □

Now in the following theorem we show that there are close relations between the T_i-degree function of a fuzzifying closure system space and that of its subspaces for $i = 0, 1$ and 2.

Theorem 5.13. *If $(Y, \mathcal{S}|Y)$ is the subspace of $(X,\mathcal{S})$, then*

(1)*For any two distinguished crisp points x and y in Y, we have $\mathbf{P}_{T_0}^Y(x,y) = \mathbf{P}_{T_0}^X(x,y)$. Hence $\mathbf{D}_{T_0}^Y(Y,\mathcal{S}|Y) \geq \mathbf{D}_{T_0}^X(X,\mathcal{S})$.*

(2)*For any two distinguished crisp points x and y in Y, we have $\mathbf{P}_{T_1}^Y(x,y) = \mathbf{P}_{T_1}^X(x,y)$. Hence $\mathbf{D}_{T_1}^Y(Y,\mathcal{S}|Y) \geq \mathbf{D}_{T_1}^X(X,\mathcal{S})$.*

(3)*For any two distinguished crisp points x and y in Y, we have $\mathbf{D}_{T_2}^Y(Y,\mathcal{S}|Y) \geq \mathbf{D}_{T_2}^X(X,\mathcal{S})$.*

Proof. We only prove (1) and the others can be obtained in a similar way . For any two different crisp points x and y in X. From Definition 5.6, we have

$$\mathbf{P}_{T_0}^Y(x,y) = \bigvee_{y\in A}\eta_x^Y(A) \vee \bigvee_{x\in B}\eta_y^Y(B),$$

and

$$\mathbf{P}_{T_0}^X(x,y) = \bigvee_{y\in A}\eta_x(A) \vee \bigvee_{x\in B}\eta_y(B).$$

In order to prove that $\mathbf{P}_{T_0}^Y(x,y) = \mathbf{P}_{T_0}^X(x,y)$, it suffices to show that $\eta_x^Y(A) = \eta_x(A)$ and $\eta_y^Y(B) = \eta_y(B)$. In fact,

$$\eta_x^Y(A) = \bigvee_{x\notin U\supseteq A}\mathcal{S}|Y(U) = \bigvee_{x\notin U\supseteq A}\bigvee_{V\cap Y=U}\mathcal{S}(V) = \bigvee_{x\notin W\supseteq A}\mathcal{S}(W) = \eta_x(A).$$

Similarly, we can get $\eta_y^Y(B) = \eta_y(B)$, as desired. Naturally, the conclusion holds. □

Remark 5.14. For each $(X,\mathcal{S}) \in \Sigma$, Theorem 5.13 means that each subspace of $(X,\mathcal{S})$ is T_i-css whenever $(X,\mathcal{S})$ is T_i-css $(i = 0, 1, 2)$.

The degree to which a fuzzifying closure system space is T_1 also can be expressed in the following theorem.

Theorem 5.15. *Let $(X,\mathcal{S})$ be a fuzzifying closure system space. Then*

$$\mathbf{D}_{T_1}(X,\mathcal{S}) = \bigwedge_{x\in X} \mathcal{S}(\{x\}).$$

Proof. For any two crisp points x,y with $x \neq y$, on the one hand, we show

$$\begin{aligned}\mathbf{D}_{T_1}(X,\mathcal{S}) &= \bigwedge_{x\neq y}\bigvee_{y\in A}\eta_x(A)\wedge\bigvee_{x\in B}\eta_y(B)\\ &\leq \bigwedge_{x\neq y}\bigvee_{y\in A}\eta_x(A) = \bigwedge_{x\neq y}\eta_x(\{y\}) \qquad \text{(by Theorem 5.3(2))}\\ &= \bigwedge_{y\in X}\bigwedge_{x\notin\{y\}}\eta_x(\{y\}) = \bigwedge_{y\in X}\mathcal{S}(\{y\}) \qquad \text{(by Lemma 5.5(1))}.\end{aligned}$$

On the other hand, the converse inequality holds iff

$$\begin{aligned}\bigwedge_{z\in X}\mathcal{S}(\{z\}) &= \bigwedge_{z\in X}(\bigwedge_{w\notin\{z\}}\eta_w(\{z\})) \qquad \text{(by Lemma 5.5(1))}\\ &\leq \bigwedge_{w\notin\{y\}}\eta_w(\{y\}) \leq \eta_x(\{y\}) = \bigvee_{y\in A}\eta_x(A) \qquad \text{(by Theorem 5.3(2))}.\end{aligned}$$

Similarly, we have $\bigwedge_{z\in X}\mathcal{S}(\{z\}) \leq \bigvee_{x\in B}\eta_y(B)$. Then $\bigwedge_{z\in X}\mathcal{S}(\{z\}) \leq \mathbf{P}_{T_1}(x,y)$. Hence $\bigwedge_{z\in X}\mathcal{S}(\{z\}) \leq \mathbf{D}_{T_1}(X,\mathcal{S})$. The conclusion holds. □

Definition 5.16. Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be fuzzifying closure system spaces. A map $f:(X,\mathcal{S})\to(Y,\mathcal{T})$ is called a homeomorphism if and only if f is bijective, f and $f^{\leftarrow}$ are both continuous.

Theorem 5.17. *If $f:(X,\mathcal{S})\to(Y,\mathcal{T})$ is homeomorphism. Then for each $x,y\in X$,*

(1) $\mathbf{P}_{T_0}(x,y) = \mathbf{P}_{T_0}(f(x),f(y))$, *moreover* $\mathbf{D}_{T_0}(X,\mathcal{S}) = \mathbf{D}_{T_0}(Y,\mathcal{T})$;
(2) $\mathbf{P}_{T_1}(x,y) = \mathbf{P}_{T_1}(f(x),f(y))$, *moreover* $\mathbf{D}_{T_1}(X,\mathcal{S}) = \mathbf{D}_{T_1}(Y,\mathcal{T})$;
(3) $\mathbf{P}_{T_2}(x,y) = \mathbf{P}_{T_2}(f(x),f(y))$, *moreover* $\mathbf{D}_{T_2}(X,\mathcal{S}) = \mathbf{D}_{T_2}(Y,\mathcal{T})$;
(4) $\mathbf{P}_{R_0}(x,y) = \mathbf{P}_{R_0}(f(x),f(y))$, *moreover* $\mathbf{D}_{R_0}(X,\mathcal{S}) = \mathbf{D}_{R_0}(Y,\mathcal{T})$;
(5) $\mathbf{P}_{R_1}(x,y) = \mathbf{P}_{R_1}(f(x),f(y))$, *moreover* $\mathbf{D}_{R_1}(X,\mathcal{S}) = \mathbf{D}_{R_1}(Y,\mathcal{T})$.

Proof. Omitted. □

In classical cases, the relation between R_0 and R_1 has been studied in literatures, moreover there are some relations among T_0, T_1, T_2, R_0 and R_1. In fuzzifying setting, we find that there also exist those relations among them. Firstly, let us show the relation between R_0- and R_1-degree functions of a fuzzifying closure system space.

Theorem 5.18. *For any fuzzifying closure system space $(X,\mathcal{S})$, it holds that* $\mathbf{D}_{R_1}(X,\mathcal{S}) \leq \mathbf{D}_{R_0}(X,\mathcal{S})$.

Proof. From Theorem 5.7 and the property (2.2), the inequality is true:

$$\begin{aligned}\mathbf{D}_{R_1}(X,\mathcal{S}) &= \bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_2}(x,y)\}\\ &\leq \bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\} = \mathbf{D}_{R_0}(X,\mathcal{S}).\end{aligned}$$

□

Theorem 5.19. *For each* $(X,\mathcal{S})\in\Sigma$*, it holds that:*
(1) $\mathbf{D}_{T_1}(X,\mathcal{S})\leq\mathbf{D}_{R_0}(X,\mathcal{S})$;
(2) $\mathbf{D}_{T_1}(X,\mathcal{S})\leq\mathbf{D}_{R_0}(X,\mathcal{S})\wedge\mathbf{D}_{T_0}(X,\mathcal{S})$;
(3) *If* $\mathbf{D}_{T_0}(X,\mathcal{S})=1$*, then* $\mathbf{D}_{T_1}(X,\mathcal{S})=\mathbf{D}_{R_0}(X,\mathcal{S})\wedge\mathbf{D}_{T_0}(X,\mathcal{S})$.

Proof. (1)Applying Definition 5.6, the properties (2.2) and (2.3) we have

$$\mathbf{D}_{T_1}(X,\mathcal{S}) = \bigwedge_{x\neq y}\mathbf{P}_{T_1}(x,y)\leq\bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\} = \mathbf{D}_{R_0}(X,\mathcal{S}).$$

(2) It is obtained from (1) and Theorem 5.7.

(3) Since $\mathbf{D}_{T_0}(X,\mathcal{S})=1$, for each $x,y\in X$ such that $x\neq y$ we have $\mathbf{P}_{T_0}(x,y)=1$. Now, by the property (2.3)

$$\begin{aligned}\mathbf{D}_{R_0}(X,\mathcal{S})\wedge\mathbf{D}_{T_0}(X,\mathcal{S}) &= \mathbf{D}_{R_0}(X,\mathcal{S})\\ &= \bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\}\\ &= \bigwedge_{x\neq y}\mathbf{P}_{T_1}(x,y) = \mathbf{D}_{T_1}(X,\mathcal{S}).\end{aligned}$$

□

Theorem 5.20. *For each* $(X,\mathcal{S})\in\Sigma$*, it holds that:*
(1)$\mathbf{D}_{R_0}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{T_1}(X,\mathcal{S})$*, equivalently,* $\mathbf{D}_{R_0}(X,\mathcal{S})\leq\mathbf{D}_{T_0}(X,\mathcal{S})\to\mathbf{D}_{T_1}(X,\mathcal{S})$*,* $\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{R_0}(X,\mathcal{S})\to\mathbf{D}_{T_1}(X,\mathcal{S})$*;*
(2)*If* $\mathbf{D}_{T_0}(X,\mathcal{S})=1$*, then* $\mathbf{D}_{R_0}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})=\mathbf{D}_{T_1}(X,\mathcal{S})$.

Proof. (1) Since

$$\begin{aligned}\mathbf{D}_{R_0}(X,\mathcal{S}) &= \bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\}\\ &\leq \bigwedge_{x\neq y}\{\bigwedge_{x\neq y}\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\} \qquad \text{(by the property (2.5))}\\ &= \bigwedge_{x\neq y}\mathbf{P}_{T_0}(x,y)\to\bigwedge_{x\neq y}\mathbf{P}_{T_1}(x,y) \qquad \text{(by the property (2.4))}\\ &= \mathbf{D}_{T_0}(X,\mathcal{S})\to\mathbf{D}_{T_1}(X,\mathcal{S}),\end{aligned}$$

applying the property (2.1), $\mathbf{D}_{R_0}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{T_1}(X,\mathcal{S})$ follows. And because the operator $*$ is commutative, we can obtain $\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{R_0}(X,\mathcal{S})\to\mathbf{D}_{T_1}(X,\mathcal{S})$ easily.

(2)It follows that $\mathbf{P}_{T_0}(x,y)=1$ for each $x,y\in X$ with the property of $x\neq y$ from $\mathbf{D}_{T_0}(X,\mathcal{S})=1$. Thus, the equality can be proved by

$$\begin{aligned}\mathbf{D}_{R_0}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})=\mathbf{D}_{R_0}(X,\mathcal{S})&=\bigwedge_{x\neq y}\{\mathbf{P}_{T_0}(x,y)\to\mathbf{P}_{T_1}(x,y)\}\\&=\bigwedge_{x\neq y}\mathbf{P}_{T_1}(x,y)\qquad\text{(by the property (2.3))}\\&=\mathbf{D}_{T_1}(X,\mathcal{S}).\end{aligned}$$

□

Secondly, we can obtain the relations among R_1-, T_0- and T_2-degree functions of a fuzzifying closure system space. Since the proof of the following three theorems is similar to Theorems 5.19, 5.20, the proof is omitted.

Theorem 5.21. *For each* $(X,\mathcal{S})\in\Sigma$*, it holds that:*
(1) $\mathbf{D}_{T_2}(X,\mathcal{S})\leq\mathbf{D}_{R_1}(X,\mathcal{S})$;
(2) $\mathbf{D}_{T_2}(X,\mathcal{S})\leq\mathbf{D}_{R_1}(X,\mathcal{S})\wedge\mathbf{D}_{T_0}(X,\mathcal{S})$;
(3) *If* $T_0(X,\mathcal{S})=1$*, then* $\mathbf{D}_{T_2}(X,\mathcal{S})=\mathbf{D}_{R_1}(X,\mathcal{S})\wedge\mathbf{D}_{T_0}(X,\mathcal{S})$.

Theorem 5.22. *For each* $(X,\mathcal{S})\in\Sigma$*, it holds that:*
(1)$\mathbf{D}_{R_1}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{T_2}(X,\mathcal{S})$*, equivalently,* $\mathbf{D}_{R_1}(X,\mathcal{S})\leq\mathbf{D}_{T_0}(X,\mathcal{S})\to\mathbf{D}_{T_2}(X,\mathcal{S})$*,* $\mathbf{D}_{T_0}(X,\mathcal{S})\leq\mathbf{D}_{R_1}(X,\mathcal{S})\to\mathbf{D}_{T_2}(X,\mathcal{S})$*;*
(2)*If* $\mathbf{D}_{T_0}(X,\mathcal{S})=1$*, then* $\mathbf{D}_{R_1}(X,\mathcal{S})*\mathbf{D}_{T_0}(X,\mathcal{S})=\mathbf{D}_{T_2}(X,\mathcal{S})$.

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