A QUADRATIC PROGRAMMING METHOD FOR RANKING ALTERNATIVES BASED ON MULTIPLICATIVE AND FUZZY PREFERENCE RELATIONS

Y. J. XU, Q. Q. WANG AND H. M WANG

Abstract. This paper proposes a quadratic programming method (QPM) for ranking alternatives based on multiplicative preference relations (MPRs) and fuzzy preference relations (FPRs). The proposed QPM can be used for deriving a ranking from either a MPR or a FPR, or a group of MPRs, or a group of FPRs, or their mixtures. The proposed approach is tested and examined with two numerical examples, and comparative analyses with the existing methods are provided to show the effectiveness and advantages of the QPM.

1. Introduction

Group decision making (GDM) is a prominent area of modern decision science. The decision makers (DMs) often need to select the most desirable alternatives or rank the alternatives from a set of given alternatives. There are often two processes in the process, namely: (1) the preference process; and (2) the priority process. In the former process of decision-making, the DMs generally need to provide their preferences over a set of n alternatives. In other words, the DMs need to compare these alternatives with respect to a single criterion and construct preference relations. In the latter process, the DMs then derive the priority vector by some techniques based on the given preference relations. Pairwise comparison is the most common technique to construct a preference relation. Up to now, there are two common kinds of preference relations, one of the preference relations takes the form of MPR, which was introduced by Saaty [20] firstly, and since then, the analytic hierarchy process (AHP) has been widely studied [2,4–6,9,12,13,16,17,25] and has been applied extensively in many fields, such as economic analysis, technology transfer, and population forecast [24]. The other preference relation takes the form of FPR [3,6–8,10,15,18,19,21–23,40,43,45]. Many methods have been proposed for assessing the priority vector of a MPR, such as the eigenvector method (EM) [20], normalizing rank aggregation method [20], synthetic hierarchy method [16], least square method [13], gradient eigenvector method [4], logarithmic least square method [5], and generalized chi square method [42]. For a FPR, Fernandez and Leyva [10] proposed a multi-objective optimization method for deriving a ranking. Xu and Da [45] transformed a FPR into a multiplicative one

and then derived priorities by using a least-deviation method (LDM). Wang and Parkan [29] brought forward an eigenvector for deriving priorities from a FPR. Xu and Wang [41] extended the EM to priority for an incomplete FPR. Xu, et al. [33] proposed the normalizing rank aggregation method to derive the priority vector from a FPR. Xu, et al. [34] also used the normalizing rank aggregation method to priority for the incomplete FPRs. Xu, et al. [38] presented a logarithmic least squares method (LLSM) to priority for GDM with incomplete FPRs.

Quite lots of researches have been done on how to obtain a priority vector from a MPR or FPR, their mixtures also have got more and more attention. It is an important problem to study how to derive priorities from MPRs and FPRs and their mixtures. Chiclana, et al. [3] studied a fuzzy GDM problem, where the information can be represented by means of preference orderings, utility functions and FPRs. Chiclana, et al. [2] studied the integration of MPRs as a preference representation structure in fuzzy multipurpose decision-making problems. Wang and Fan [27] applied the logarithmic and geometric least squares methods (LLSM and GLSM) to deal with group decision analysis problems with FPRs. Wang and Fan [26] presented two optimization aggregation approaches to determine the relative weights of individual FPRs so that they can be aggregated into a collective FPR in an additively optimal manner. Usually, these methods consist of three steps: (1) Uniform the preference information given by DMs through a transformation function; (2) Aggregate the uniformed preference information into a collective one by means of the aggregation operators, and (3) Rank alternatives or select the most desirable alternative(s) by the selection method. However, the computational procedures of these methods are very complicated. Particularly, in the process of unifying the preference information, preference information may be lost or distorted. Sometimes, it can be difficult to transform preference information from one format into a uniformed format. In view of this argument, Wang, et al. [28] proposed a chi-square method (CSM) for obtaining a priority vector from MPRs and FPRs. Xu, et al. [39] developed the mean deviation method to determine the relative weights of individual different preference relations objectively and then select the most desirable alternative(s) according to the aggregate net flow scores. Fan, et al. [8] constructed a two-objective optimization method, which integrates MPRs and FPRs without the need of preference transformation, to compute the ranking values of alternatives. The two-objective optimization method is referred as TOM in this paper. Fan, et al. [6] also proposed a goal programming method (GPM) to solve the GDM problems with MPRs and FPRs. One of the prominent characteristics of these methods is that they do not need to uniform the different types of preference relations, which would not distort or lost the DMs' original information.

Based on the above idea, in this paper, we propose a new priority method, called quadratic programming method (referred as QPM), for obtaining a priority vector from MPRs and FPRs. The QPM is multifunctional. It can be used to derive priorities from either a single MPR, or a single FPR, or a group of MPRs, or a group of FPRs, or the mixed MPRs and FPRs. The QPM has many advantages: (1) The QPM does not need to uniform the preference relations, this would not need to transform different kinds of preference relations into one kind preference

relation, and thus avoiding loss or distort the DMs' original information, because different transformation functions would lead to different results; (2) Compared with the TOM and the GPM, numerical examples show that the proposed QPM has better fitting performances regarding some criteria; (3) The QPM provides a new methodology to integrate different formats of preference relations in GDM, and thus enriches the decision theory.

The rest of the paper is structured in the following way. In Section 2, we propose the QPM for deriving priorities from the mixed MRPs and FPRs. In Section 3, two numerical examples are used to illustrate the proposed method, and some criteria are proposed to illustrate the better performance of the QPM than the TOM and the GPM. Finally, Section 4 summarizes the work of this paper.

2. The Quadratic Programming Method

This section describes GDM problem with MRPs and FPRs.

Let $X = \{x_1, x_2, ..., x_n\}$ ($n \geq 2$) be a finite set of alternatives and $D = \{d_1, d_2, ..., d_m\}$ ($m \geq 2$) be a finite set of DMs. Let $c_1, c_2, ..., c_m$ be the relative importance weights of the m DMs with $\sum_{k=1}^{m} c_k = 1$ and $c_k \geq 0$ for $k = 1, 2, ..., m$. In multiple attribute decision making problems, the alternatives $x_1, x_2, ..., x_n$ need to be ranked from best to worst according to the DMs' preference information. In this paper, DM's preference information on alternative set X is assumed to be represented in two formats, i.e., DM $d_k \in D^A$ ($k = 1, ..., m_f$) uses MPR and DM $d_k \in D^F$ ($k = m_{f+1}, ..., m$) uses FPR, where $D^A = \{d_1, d_2, ..., d_{m_f}\}$, $D^F = \{d_{m_{f+1}}, d_{m_{f+2}}, ..., d_m\}$ and $D^A \bigcup D^F = D$. A brief description of MPR and FPR is given below.

(1)*Multiplicative preference relation* [20]. The preference relation of $d_k \in D^A$ on X is described by a positive reciprocal matrix $A_k \subset X \times X$, $A_k = (a_{ij}^k)_{n \times n}$, a_{ij}^k denotes a ratio of preference intensity for alternative x_i over x_j. Saaty [20] suggested measuring a_{ij}^k using a ratio scale, $a_{ij}^k \in \{1/9, 1/8, ..., 1/2, 1, 2, ..., 8, 9\}$: (1) $a_{ij}^k = 1$ indicates indifference between x_i and x_j for d_k; (2) $a_{ij}^k \in \{2, 3, ..., 9\}$ indicates that x_i is strictly preferred to x_j ($x_i \succ x_j$) for d_k. Especially, $a_{ij}^k = 9$ indicates that x_i is definitely preferred to x_j; (3) $a_{ij}^k \in \{1/9, 1/8, ..., 1/2\}$ indicates that x_j is strictly preferred to x_i for DM d_k. Especially, $a_{ij}^k = 1/9$ indicates that x_j is definitely preferred to x_i for d_k. It is multiplicative reciprocal, i.e., $a_{ij}^k a_{ji}^k = 1$ and $a_{ii}^k = 1$ for all $i, j = 1, ..., n$. Saaty [20] also defined the consistence on the MPR, i.e. $a_{ij}^k = a_{il}^k a_{lj}^k$ for all $i, j, l = 1, ..., n$.

(2)*Fuzzy preference relation* [14, 19, 21]. The preference information of $d_k \in D^F$ on a set of alternatives is a fuzzy set on the product set $X \times X$, characterized by a membership function

$$\mu_{R_k} : X \times X \to [0, 1]$$

and the preference relation can be represented by an $n \times n$ matrix $R_k = (r_{ij}^k)_{n \times n}$, where $r_{ij}^k = \mu_{R_k}(x_i, x_j)$. r_{ij}^k denotes the preference degree of the alternative x_i over x_j: (1)$r_{ij}^k = 0.5$ denotes indifference between x_i and x_j for d_k; (2) $0.5 < r_{ij}^k < 1$ denotes that x_i is strictly preferred to x_j ($x_i \succ x_j$) for d_k. Especially, $r_{ij}^k = 0$

denotes that x_j is definitely preferred to x_i for d_k; (3) $0 < r_{ij}^k < 0.5$ denotes that x_j is strictly preferred to x_i for d_k. We assume that $R_k = (r_{ij}^k)_{n \times n}$ is a reciprocal FPR, i.e., $r_{ij}^k + r_{ji}^k = 1$ for all $i, j = 1, ..., n$, and in particular $r_{ii}^k = 0.5$. Tanino [21] proposed the definition of multiplicative consistency on FPRs, i.e., $r_{il}^k r_{lj}^k r_{ji}^k = r_{li}^k r_{jl}^k r_{ij}^k$, for all $i, j, l = 1, ..., n$.

Suppose the ranking value of alternative x_i is w_i $(i = 1, 2, ..., n)$ and that w_i is an unknown variable that satisfies $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0$ for $i = 1, ..., n$. Considering the properties of matrices A_k and R_k [8, 20], it is desirable to determine the weight w_i such that

$$a_{ij}^k = w_i / w_j, \quad i, j = 1, ..., n; \ k = 1, ..., m_f, \tag{1}$$

$$r_{ij}^k = w_i / (w_i + w_j), \quad i, j = 1, ..., n; \ k = m_{f+1}, ..., m. \tag{2}$$

Based on eqs.(1) and (2), the deviation degree between a_{ij}^k and w_i / w_j, and the deviation degree between r_{ij}^k and $w_i / (w_i + w_j)$ are given by eqs.(3) and (4), respectively:

$$q_{ij}^k(w) = w_i - a_{ij}^k w_j, \quad i, j = 1, ..., n; \ k = 1, ..., m_f, \tag{3}$$

$$h_{ij}^k(w) = w_i - r_{ij}^k(w_i + w_j), \quad i, j = 1, ..., n; \ k = m_{f+1}, ..., m. \tag{4}$$

Apparently, $q_{ij}^k(w)$ and $h_{ij}^k(w)$ are the explicit functions of w_i $(i = 1, ..., n)$.

Considering the importance degree of each DM, the deviation degrees, $q_{ij}^k(w)$ and $h_{ij}^k(w)$, denoted by eqs.(2) and (3) are combined to form the following optimization problem:

$$\min F(W) = \sum_{k=1}^{m_f} c_k \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(w_i - w_j a_{ij}^k \right)^2 + \sum_{k=m_f+1}^{m} c_k \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[w_i - (w_i + w_j) r_{ij}^k \right]^2 \tag{5}$$

$$\text{s.t.} \sum_{i=1}^{n} w_i = 1 \tag{6}$$

$$w_i \geq 0, \quad i = 1, 2, ..., n. \tag{7}$$

The above model (5)-(7) is equivalent to the following matrix form

$$\min \quad F(w) = w^T G w + w^T H w \tag{8}$$

$$\text{s.t.} \quad e^T w = 1 \tag{9}$$

$$w \geq 0 \tag{10}$$

where $w = (w_1, w_2, ..., w_n)^T$, $e = (1, 1, ..., 1)^T$, $G = (g_{ij})_{n \times n}$, $H = (h_{ij})_{n \times n}$. The elements in G are:

$$g_{ii} = \sum_{k=1}^{m_f} c_k \left[n - 2 + \sum_{h=1}^{n} (a_{hi}^k)^2 \right], \quad i = 1, 2, ..., n, \tag{11}$$

$$g_{ij} = -\sum_{k=1}^{m_f} c_k (a_{ij}^k + a_{ji}^k), \quad i, j = 1, 2, ..., n; \ i \neq j. \tag{12}$$

The elements in matrix H are:

$$h_{ii} = 2 \sum_{k=m_f+1}^{m} c_k \sum_{\substack{h=1 \\ h \neq i}}^{n} (r_{hi}^k)^2, \quad i = 1, 2, ..., n, \tag{13}$$

$$h_{ij} = 2 \sum_{k=m_f+1}^{m} c_k \left((r_{ij}^k)^2 - r_{ij}^k \right), \quad i, j = 1, 2, ..., n; \ i \neq j. \tag{14}$$

The above model can be transformed into the following single objective optimization model

$$\min \quad F(w) = w^T Q w \tag{15}$$

$$\text{s.t.} \quad e^T w = 1 \tag{16}$$

$$w \geq 0 \tag{17}$$

where the elements of matrix Q are given by

$$q_{ii} = \sum_{k=1}^{m_f} c_k \left[n - 2 + \sum_{h=1}^{n} \left(a_{hi}^k \right)^2 \right] + 2 \sum_{k=m_f+1}^{m} c_k \sum_{\substack{h=1 \\ h \neq i}}^{n} \left(r_{hi}^k \right)^2, \quad i = 1, 2, ..., n, \tag{18}$$

$$q_{ij} = -\sum_{k=1}^{m_f} c_k (a_{ij}^k + a_{ji}^k) + 2 \sum_{k=m_f+1}^{m} c_k \left(\left(r_{ij}^k \right)^2 - r_{ij}^k \right), \quad i, j = 1, 2, ..., n; \ i \neq j. \tag{19}$$

The optimization problem determined by eqs.(15)-(17) not only plays the role for integrating two formats of preference relations, but also can be used to obtain the ranking values of alternatives (the collective result).

Model (15)-(17) is a nonlinear programming problem, where both the objective function $F(w)$ and the constraints are convex. In order to solve the optimization model (15)-(17) in the following, we give the Kuhn-Tucher condition.

Lemma 2.1. *(Kuhn-Tucher condition). [30] Consider the following general non-linear programming problem:*

$$\min \quad f(X) \tag{20}$$

$$\text{s.t.} \quad h_i(X) = 0, \quad i = 1, 2, ..., m \tag{21}$$

$$g_j(X) \geq 0, \quad j = 1, 2, ..., l \tag{22}$$

where X is a vector of n variables. Here $f(X)$ is non-linear and both constraints can be linear or non-linear. Let X^* be the Minimal point of non-linear programming problem (20)-(22), the gradient $\nabla h_i(X^*)(i = 1, 2, ..., m)$ and $\nabla g_j(X^*)(j = 1, 2, ..., l)$ are linear-independent, and the ∇ refers to the first-order partial derivatives with respect to X, then there exists the vector $\Lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_m^*)^T$ and $\Gamma^* = (\gamma_1^*, ..., \gamma_l^*)^T$, such that

$$\nabla f(X^*) - \sum_{i=1}^{m} \lambda_i^* \nabla h_i(X^*) - \sum_{j=1}^{l} \gamma_j^* \nabla g_j(X^*) = 0, \tag{23}$$

$$\gamma_j^* g_j(X^*) = 0, \quad j = 1, 2, ..., l, \tag{24}$$

$$\gamma_j^* \geq 0, \quad j = 1, 2, ..., l, \tag{25}$$

where $\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*$ and $\gamma_1^*, \gamma_2^*, \cdots, \gamma_l^*$ are generalized Lagrange multipliers.

As eqs. (15)-(17) is a convex programming problem, by Lemma 2.1, we can obtain the following result.

Theorem 2.2. *The necessary and sufficient conditions of the optimal solutions of the quadratic programming problem (15)-(17) are:*

$$Qw - \lambda e - \gamma = 0, \tag{26}$$

$$e^T w = 1, \tag{27}$$

$$\gamma_j w_j = 0, \quad j = 1, 2, ..., n, \tag{28}$$

$$\gamma \geq 0, \quad w \geq 0, \quad \lambda \; unrestricted \; in \; sign. \tag{29}$$

where Q is given by in (18) and (19), λ is the Lagrange multiplier corresponding to constrain (16), $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^T$ are the Lagrange multiplier corresponding to the constraints (17).

In the Theorem 2.2, λ is unrestricted in sign, so we can replace it by two non-negatives $\lambda^{'}, \lambda^{''}$, where $\lambda = \lambda^{'} - \lambda^{''}$, and $\lambda^{'}, \lambda^{''} \geq 0$, and we also induce the variable v (v is an artificial variable, only v equals to zero, then we can get the solution of the problem), then the above Kuhn-Tucher necessary and optimality conditions can be transformed into the following programming problem:

$$\min \quad z = v \tag{30}$$

$$s.t. \quad Qw - \lambda^{'} e + \lambda^{''} e - \gamma = 0, \tag{31}$$

$$e^T w + v = 1 \tag{32}$$

$$\gamma_j w_j = 0, \quad j = 1, 2, ..., n, \tag{33}$$

$$w_j \geq 0, \quad \gamma_j \geq 0, \quad j = 1, 2, ..., n, \tag{34}$$

$$\lambda^{'}, \lambda^{''} \geq 0, \quad v \geq 0 \tag{35}$$

Solving the above model using optimization software packages such as Microsoft EXCELL, MATLAB and LINGO Solver, we can obtain the priority vector w. The above model also could be solved by Wolf algorithm. And thus we can rank the alternatives (or group decision result).

For convenience, we refer to the above method as the quadratic-programming method (QPM), which can be used to derive a priority vector from MPRs and FPRs. Now, we discuss several special cases of the QPM in detail. If $c_k > 0$ for $k = 1, ..., m_f$ and $c_{m_f+1} = ... = c_m = 0$, then all of the DMs provide their preferences over the given alternatives by means of MPRs. Especially, if only $c_1 = 1$ and $c_k = 0$ for $k = 2, ..., m$, then only one DM provides his/her preferences by means of MPR. If $c_1 = ... = c_{m_f} = 0$ and $c_k > 0$ for $k = m_f+1, ..., m$, then all the DMs provide their preferences over the given alternatives by means of FPRs. Especially, if $c_k = 0$ for $k = 1, ..., m-1$, and $c_m = 1$, then only one DM provides his/her preferences over the given alternatives by means of FPR. Thus, the proposed QPM provides a flexible way to obtain a priority vector from MPRs and FPRs.

3. Numerical Examples

Example 3.1. Consider a GDM problem, an investment company wishes to invest a sum of money in the best option. There are four possible alternatives for the company to invest: x_1 is a car company, x_2 is a food company, x_3 is a computer company and x_4 is an arms company. The investment company has a group of four consultancy departments, and each department is directed by a DM. These DMs e_1, e_2, e_3 and e_4 provide their preferences on the alternative set $X = \{x_1, x_2, x_3, x_4\}$ as follows (adapted from Fan et al. [6,8]):

$$A_1 = \begin{bmatrix} 1 & 1/7 & 1/3 & 1/5 \\ 7 & 1 & 3 & 2 \\ 3 & 1/3 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 & 1/4 & 5 \\ 1/3 & 1 & 2 & 1/3 \\ 4 & 1/2 & 1 & 2 \\ 1/5 & 3 & 1/2 & 1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0.5 & 0.5 & 0.7 & 1 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.8 \\ 0 & 0.4 & 0.2 & 0.5 \end{bmatrix}.$$

To integrate the two formats of preference relations and to obtain the collective ranking values of alternatives, suppose that $c_1 = c_2 = c_3 = c_4 = 1/4$, according to eqs.(18) and (19), we can obtain matrix Q as follows:

$$Q = \begin{bmatrix} 26.9878 & -2.7890 & -2.1208 & -2.7050 \\ -2.7890 & 6.5879 & -1.6183 & -1.6983 \\ -2.1208 & -1.6183 & 6.9459 & -1.3750 \\ -2.7050 & -1.6983 & -1.3750 & 11.5803 \end{bmatrix}$$

By eqs.(30)-(35), we can set up the following model

$$\min \quad z = v$$

$$\text{s.t.} \quad 26.9878w_1 - 2.789w_2 - 2.1208w_3 - 2.7050w_4 - \lambda' + \lambda'' - \gamma_1 = 0,$$

$$- 2.789w_1 + 6.5879w_2 - 1.6183w_3 - 1.6983w_4 - \lambda' + \lambda'' - \gamma_2 = 0,$$

$$- 2.1208w_1 - 1.6183w_2 + 6.9459w_3 - 1.375w_4 - \lambda' + \lambda'' - \gamma_3 = 0,$$

$$- 2.705w_1 - 1.6983w_2 - 1.375w_3 + 11.5803w_4 - \lambda' + \lambda'' - \gamma_4 = 0,$$

$$w_1 + w_2 + w_3 + w_4 + v = 1,$$

$$\gamma_1 w_1 = 0,$$

$$\gamma_2 w_2 = 0,$$

$$\gamma_3 w_3 = 0,$$

$$\gamma_4 w_4 = 0,$$

$$w_1, w_2, w_3, w_4 \geq 0,$$

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0,$$

$$\lambda', \lambda'' \geq 0 \text{ and } v \geq 0.$$

By solving the above linear goal programming problem, we have:

$$v = 0, \ w_1 = 0.1225, \ w_2 = 0.3492, \ w_3 = 0.317, \ w_4 = 0.2113,$$

$$\lambda^{'} = 1.0869, \ \lambda^{''} = 0, \ \gamma_1 = 0, \ \gamma_2 = 0, \ \gamma_3 = 0, \ \gamma_4 = 0.$$

Therefore, the final ranking result of the four alternatives is $x_2 \succ x_3 \succ x_4 \succ x_1$. In [6] and [8], the final weight vector of four alternatives are $w = (0.1280, 0.4301, 0.2515, 0.1903)^T$ and $w = (0.1169, 0.3688, 0.3039, 0.2105)^T$ by the GPM and the TOM, respectively. The weights of the four DMs were also set to be equal, i.e. $c_1 = ... = c_4 = 1/4$. Both GPM and TOM derive the same ranking as the QPM proposed in this paper, which shows the ranking is robust and credible.

In order to compare further the performances of the three methods, we propose the following performance evaluation criterion:

• Deviation Index

$$DI = \frac{1}{n(n-1)m_f} \sum_{k=1}^{m_f} \sum_{1 \le i < j \le n} \left(a_{ij}^k \frac{w_j}{w_i} + a_{ji}^k \frac{w_i}{w_j} - 2 \right) +$$

$$\frac{1}{n(n-1)(m-m_f)} \sum_{k=m_f+1}^{m} \sum_{1 \le i < j \le n} \left| r_{ij}^k - \frac{w_i}{w_i + w_j} \right|$$

The Deviation Index (DI) is the sum of the deviation, the former part is the sum of the deviation for MPRs, and the right part is the sum of the FPRs.

Wang, et al. [28] further proposed the following two performance evaluation criteria:

• Maximum deviation (MD) for MPRs

$$MD = \max_{i,j,k} \left\{ a_{ij}^k \frac{w_j}{w_i} + a_{ji}^k \frac{w_i}{w_j} - 2 \, | i, j = 1, ..., n; k = 1, ..., m_f \right\}$$

The maximum deviation is the largest value for the MPR, which is a changed expression of eq. (1).

• Maximum absolute deviation (MAD) for FPRs

$$MAD = \max_{i,j,k} \left\{ \left| r_{ij}^k - \frac{w_i}{w_i + w_j} \right| | i, j = 1, ..., n; k = m_f + 1, ..., m \right\}$$

The maximum absolute deviation (MAD) is the largest values between the fuzzy preference elements and the weight vector expressed in eq. (2).

The smaller the values DI, MD, MAD, the better the weight will be. If $DI = MD = MAD = 0$, all the preference relations are perfectly consistent.

Criteria	QPM	GPM	TOM
DI	1.0503	1.0989	1.0859
MD	6.7404	8.1797	7.5702
MAD	0.6330	0.5979	0.6429

TABLE 1. Performance Evaluation for Example 3.1

Table 1 shows the performances of the three methods for the above four preference relations. As we can see, QPM is better than TOM with respect to the above three criteria, and DI and MD values are also smaller than GPM, only MAD

value is slightly larger than GPM in the above example. This partly shows the advantages of the QPM.

Example 3.2. Consider the MPR A_1 and the FPR R_1, which are shown as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.5 & 2/3 \\ 1/3 & 0.5 \end{bmatrix}.$$

The TOM uses the following equation to derive the priority from the FPR and MPR:

$$w^* = Q^{-1}e/e^T Q^{-1}e \tag{36}$$

where the elements in matrix Q are given by

$$q_{ii} = \beta \sum_{k=1}^{m_f} c_k \left[n - 2 + \sum_{h=1}^{n} (a_{hi}^k)^2 \right] + 2\alpha \sum_{k=m_f+1}^{m} c_k \sum_{\substack{h=1 \\ h \neq i}}^{n} (r_{hi}^k)^2, \ i = 1, 2, ..., n, \tag{37}$$

$$q_{ij} = -\beta \sum_{k=1}^{m_f} c_k (a_{ij}^k + a_{ji}^k) + 2\alpha \sum_{k=m_f+1}^{m} c_k \left((r_{ij}^k)^2 - r_{ij}^k \right), \ i, j = 1, 2, ..., n; i \neq j. \tag{38}$$

If we use the TOM to integrate the two formats of preference relations and to obtain the collective ranking values of alternatives, suppose that $c_1 = c_2 = 1/2$, $\alpha = 0.5$, $\beta = 0.5$ we can obtain matrix Q as follows:

$$Q = \begin{bmatrix} \frac{53}{144} & -\frac{53}{72} \\ -\frac{53}{72} & \frac{53}{36} \end{bmatrix}$$

For Q is a singular matrix, it has no inverse matrix. Thus, eq.(36) cannot be used to derive the priority for the above preference relations, it shows the limitation of the TOM. If we use the QPM to compute, suppose that $c_1 = c_2 = 1/2$, we can easily get: $w_1 = 0.667$, $w_2 = 0.333$, which shows that the QPM has a broader application scope.

4. Conclusions

This paper presents a new approach to solve the GDM problems with two different formats of preference information, i.e. MPRs and FPRs. Based on the QPM, the proposed approach integrates the two formats of preference relations and computes the ranking values of alternatives. When setting different kinds of weight vectors for the DMs, the QPM can be used to obtain a priority vector from different combination of preference relations. It can be used for either a single MPR or a single FPR or their mixtures. Three criteria are introduced to show the performances of the methods. Two examples are illustrated to show the proposed method and its effectiveness. Compared with the existing method, the proposed QPM has the following features:

(1) By setting different types of c_k, the QPM can be conveniently applied to derive a priority vector from MPRs, FPRs, MPR or FPR and thus, the proposed QPM provides a flexible way to obtain a priority vector from MPRs and FPRs.

(2) As illustrated in Example 3.2, TOM may be not used to derive the weight vector as Q^{-1} does not exist in some situations, and it could be solved by QPM, this shows QPM has broader applications.

(3) As the QPM does not need to uniform all the preference relations into one format, it may not distort the DMs' original preference information.

Current research establishes QPM as a viable and effective tool to handle decision problems with complete MPRs and FPRs. The numerical experiments demonstrate that the QPM often outperforms the other methods such as GPM and TOM in terms of *DI, MD* and *MAD*. However, does it always have better performance than the existing methods? It may be an interesting problem in our future research. Furthermore, in reality, DMs may provide their preference relations in different formats. As a worthy future research topic, it would be interesting to explore how the QPM framework can be extended to deal with other types of decision inputs such as incomplete MPRs [11], incomplete FPRs [31, 32, 35, 36, 43], intuitionistic fuzzy preference relations [44], linguistic preference relations [1, 37], et al.

References

[1] S. Alonso, F. Cabrerizo, F. Chiclana, F. Herrera and E. Herrera-Viedma, *Group decision making with incomplete fuzzy linguistic preference relations*, International Journal of Intelligent Systems, **24** (2009), 201–222.

[2] F. Chiclana, F. Herrera and E. Herrera-Viedma, *Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations*, Fuzzy Sets and Systems, **122** (2001), 277–291.

[3] F. Chiclana, F. Herrera and E. Herrera-Viedma, *Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations*, Fuzzy Sets and Systems, **97** (1998), 33–48.

[4] K. O. Cogger and P. L. Yu, *Eigenweight vectors and least-distance approximation for revealed preference in pairwise weight ratios*, Journal of Optimization Theory and Applications, **46** (1985), 483–491.

[5] G. Crawford and C. Williams, *A note on the analysis of subjective judgement matrices*, Journal of Mathematical Psychology, **29** (1985), 387–405.

[6] Z. P. Fan, J. Ma, Y. P. Jiang, Y. H. Sun and L. Ma, *A goal programming approach to group decision making based on multiplicative preference relations and fuzzy preference relations*, European Journal of Operational Research, **174** (2006), 311–321.

[7] Z. P. Fan, J. Ma and Q. Zhang, *An approach to multiple attribute decision making based on fuzzy preference information on alternatives*, Fuzzy Sets and Systems, **131** (2002), 101–106.

[8] Z. P. Fan, S. H. Xiao and G. F. Hu, *An optimization method for integrating two kinds of preference information in group decision-making*, Computers & Industrial Engineering, **46** (2004), 329–335.

[9] Z. P. Fan and Y. Zhang, *A goal programming approach to group decision-making with three formats of incomplete preference relations*, Soft Computing, **14** (2010), 1083–1090.

[10] E. Fernandez and J. C. Leyva, *A method based on multiobjective optimization for deriving a ranking from a fuzzy preference relation*, European Journal of Operational Research, **154** (2004), 110–124.

[11] P. T. Harker, *Alternative modes of questioning in the analytic hierarchy process*, Mathematical Modelling, **9** (1987), 353–360.

[12] F. Herrera, E. Herrera-Viedma and F. Chiclana, *Multiperson decision-making based on multiplicative preference relations*, European Journal of Operational Research,**129**(2001),372–385.

[13] R. E. Jensen, *An alternative scaling method for priorities in hierarchy structures*, Journal of Mathematical Psychology, **28** (1984), 317–332.

[14] J. Kacprzyk, *Group decision making with a fuzzy linguistic majority*, Fuzzy Sets and Systems, **18** (1986), 105–118.

[15] J. Kacprzyk and M. Roubens, *Non-Conventional Preference Relations in Decision-Making*, Springer, Berlin, 1988.

[16] S. Lipovetsky, *The synthetic hierarchy method: An optimizing approach to obtaining priorities in the AHP*, European Journal of Operational Research, **93** (1996), 550–564.

[17] S. Lipovetsky and A. Tishler, *Interval estimation of priorities in the AHP*, European Journal of Operational Research, **114** (1999), 153–164.

[18] H. Nurmi, *Approaches to collective decision making with fuzzy preference relations*, Fuzzy Sets and Systems, **6** (1981), 249–259.

[19] S. A. Orlovsky, *Decision-making with a fuzzy preference relation*, Fuzzy Sets and Systems, **1** (1978), 155–167.

[20] T. L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.

[21] T. Tanino, *Fuzzy preference orderings in group decision-making*, Fuzzy Sets and Systems, **12** (1984), 117–131.

[22] T. Tanino, *Fuzzy preference relation in group decision making, in: J. Kacprzyk, M. Roubens (Eds.) Non-Conventional Preference Relation in Decision Making*, Springer, Berlin, (1988), 54–71.

[23] T. Tanino, *On group decision making under fuzzy preferences, in: J. Kacprzyk, M. Fedrizzi (Eds.) Multiperson Decision Making Using Fuzzy Sets and Possibility Theory*, Kluwer, Netherlands, 1990, 172–185.

[24] L. G. Vargas, *An overview of the analytic process and its applications*, European Journal of Operational Research, **48** (1990), 2–8.

[25] L. F. Wang and S. B. Xu, *The Introduction to the Analytic Hierarchy Process*, Chinese People University Press, Beijing, 1990.

[26] Y. M. Wang and Z. P. Fan, *Fuzzy preference relations: Aggregation and weight determination*, Computers & Industrial Engineering, **53** (2007), 163–172.

[27] Y. M. Wang and Z. P. Fan, *Group decision analysis based on fuzzy preference relations: Logarithmic and geometric least squares methods*, Applied Mathematics and Computation, **194** (2007), 108–119.

[28] Y. M. Wang, Z. P. Fan and Z. S. Hua, *A chi-square method for obtaining a priority vector from multiplicative and fuzzy preference relations*, European Journal of Operational Research, **182** (2007), 356–366.

[29] Y. M. Wang and C. Parkan, *Multiple attribute decision making based on fuzzy preference information on alternatives: Ranking and weighting*, Fuzzy Sets and Systems, **153** (2005), 331–346.

[30] D. A. Wismer, *Introduction to Nonlinear Optimization: A Problem Solving Approach*, North-Holland Company, New York, 1978.

[31] Y. J. Xu, L. Chen, K. W. Li and H. M. Wang, *A chi-square method for priority derivation in group decision making with incomplete reciprocal preference relations*, Information Sciences, **36** (2015), 166–179.

[32] Y. J. Xu, L. Chen and H. M. Wang, *A least deviation method for priority derivation in group decision making with incomplete reciprocal preference relations*, International Journal of Approximate Reasoning, **66** (2015), 91–102.

[33] Y. J. Xu, Q. L. Da and L. H. Liu, *Normalizing rank aggregation method for priority of a fuzzy preference relation and its effectiveness*, International Journal of Approximate Reasoning, **50** (2009), 1287–1297.

[34] Y. J. Xu, Q. L. Da and H. M. Wang, *A note on group decision-making procedure based on incomplete reciprocal relations*, Soft Computing, **15** (2011), 1289–1300.

[35] Y. J. Xu, J. N. D. Gupta and H. M. Wang, *The ordinal consistency of an incomplete reciprocal preference relation*, Fuzzy Sets and Systems, **246**(2014), 62–77.

[36] Y.J. Xu, K.W. Li and H.M. Wang, *Incomplete interval fuzzy preference relations and their applications*, Computers & Industrial Engineering, **67** (2014), 93–103.

[37] Y. J. Xu, F. Ma, F. F. Tao and H. M. Wang, *Some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making*, Knowledge-Based Systems, **56** (2014), 179–190.

[38] Y.J. Xu, R. Patnayakuni and H. M. Wang, *Logarithmic least squares method to priority for group decision making with incomplete fuzzy preference relations*, Applied Mathematical Modelling, **37** (2013), 2139–2152.

[39] Y. J. Xu, R. Patnayakuni and H. M. Wang, *A method based on mean deviation for weight determination from fuzzy preference relations and multiplicative preference relations*, International Journal of Information Technology & Decision Making, **11** (2012), 627–641.

[40] Y. J. Xu, R. Patnayakuni and H. M. Wang, *The ordinal consistency of a fuzzy preference relation*, Information Sciences, **224** (2013), 152–164.

[41] Y. J. Xu and H. M. Wang, *Eigenvector method, consistency test and inconsistency repairing for an incomplete fuzzy preference relation*, Applied Mathematical Modelling, **37** (2013), 5171–5183.

[42] Z. S. Xu, *Generalized chi square method for the estimation of weights*, Journal of Optimization Theory and Applications, **107** (2000), 183–192.

[43] Z. S. Xu, *Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation*, International Journal of Approximate Reasoning, **36** (2004), 261–270.

[44] Z. S. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Transactions on Fuzzy Systems, **15** (2007), 1179–1187.

[45] Z. S. Xu and Q. L. Da, *A least deviation method to obtain a priority vector of a fuzzy preference relation*, European Journal of Operational Research, **164** (2005), 206–216.

YEJUN XU*, STATE KEY LABORATORY OF HYDROLOGY-WATER RESOURCES AND HYDRAULIC ENGINEERING, HOHAI UNIVERSITY, NO.1 XIKANG ROAD, NANJING, 210098, JIANGSU,CHINA AND BUSINESS SCHOOL, HOHAI UNIVERSITY, JIANGNING CAMPUS, NO.8 FOCHENG WEST ROAD, JIANGNING, NANJING, 211100, JIANGSU, CHINA
E-mail address: xuyejohn@163.com

QIANQIAN WANG, STATE KEY LABORATORY OF HYDROLOGY-WATER RESOURCES AND HYDRAULIC ENGINEERING, HOHAI UNIVERSITY, NO.1 XIKANG ROAD, NANJING, 210098, JIANGSU, CHINA AND BUSINESS SCHOOL, HOHAI UNIVERSITY, JIANGNING CAMPUS, NO.8 FOCHENG WEST ROAD, JIANGNING, NANJING, 211100, JIANGSU, CHINA
E-mail address: 1196297264@qq.com

HUIMIN WANG, STATE KEY LABORATORY OF HYDROLOGY-WATER RESOURCES AND HYDRAULIC ENGINEERING, HOHAI UNIVERSITY, NO.1 XIKANG ROAD, NANJING, 210098, JIANGSU, CHINA AND BUSINESS SCHOOL, HOHAI UNIVERSITY, JIANGNING CAMPUS, NO.8 FOCHENG WEST ROAD, JIANGNING, NANJING, 211100, JIANGSU, CHINA
E-mail address: hmwang@hhu.edu.cn

*CORRESPONDING AUTHOR