SOME HYPER K-ALGEBRAIC STRUCTURES INDUCED BY MAX-MIN GENERAL FUZZY AUTOMATA

K. ABOLPOUR, M. M. ZAHEDI AND M. GOLMOHAMADIAN

Abstract. We present some connections between the max-min general fuzzy automaton theory and the hyper structure theory. First, we introduce a hyper BCK-algebra induced by a max-min general fuzzy automaton. Then, we study the properties of this hyper BCK-algebra. Particularly, some theorems and results for hyper BCK-algebra are proved. For example, it is shown that this structure consists of different types of (positive implicative) commutative hyper K-ideals. As a generalization, we extend the definition of this hyper BCK-algebra to a bounded hyper K-algebra and obtain relative results.

1. Introduction

Automata are the prime examples of general computational systems over discrete spaces. The incorporation of fuzzy logic into automata theory resulted in fuzzy automata which can handle continuous spaces. In particular, the concept of membership assignment, output mapping, multi-membership resolution, and the concept of acceptance for fuzzy automata were developed in general fuzzy automata.

The concept of BCK-algebra has originated on two different ways. One motivation is based on set theory. In set theory, there are three most elementary and fundamental operations introduced by L. Kantorovic and E. Livenson. These fundamental operations are union, intersection and the set difference. Another motivation is taken from classical and non-classical propositional calculi. There are some systems which contain the only implication functor among the logical functors. These examples are systems of positive implicational calculus, weak positive implicational calculus by A. Church and BCK-systems by C.As. Meredith.

Corsini and Leoreanu [7] presented some connections between a deterministic finite automaton and the hyper algebraic structure theory. Now, here we intend to find some relationships between max-min general fuzzy automata and the hyper K-algebraic structures. So, first we introduce a hyper BCK-algebra induced by max-min general fuzzy automata. Then, we discuss the properties of this BCK-algebra and obtain some useful results. A fuzzy finite-state automaton (FFA) is a six-tuple denoted by $\tilde{F} = (Q, \sum, R, Z, \delta, \omega)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \sum \times Q \longrightarrow [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input

Received: July 2009; Revised: November 2009; Accepted: February 2010

Key words and phrases: (Positive implicative) Commutative hyper K-ideal, (Bounded) Hyper BCK-algebra, Hyper BCK-ideal, Max-min general fuzzy automata.

symbol, attributing a value in the interval $[0,1]$ and $\omega : Q \longrightarrow Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in $[0,1]$. We call this membership value the weight of the transition.

The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refer to the weight of the transition. Otherwise, it specifies the transition itself. Also, the set of all transitions of $\tilde{F}$ is denoted by Δ. The above definition is generally accepted as a formal definition of FFA [13, 18, 20, 21].

There is an important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how to assign a membership value to a next state upon the completion of a transition. Secondly, how to deal with cases where a state is forced to take several membership values simultaneously via overlapping transition.

2. Preliminaries

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [8].

Definition 2.1. [8] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, \sum, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where (i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$, (ii) $\sum$ is a finite set of input symbols, $\sum = \{a_1, a_2, \ldots, a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$, (v) $\omega : Q \longrightarrow Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0,1]) \times \sum \times Q \longrightarrow [0,1]$ is the augmented transition function, and (vii) $F_1 : [0,1] \times [0,1] \longrightarrow [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented by:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \tilde{\delta}(q_i, a_k, q_j)).$$

This means that membership value (mv) of the state q_i at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

There are many choices for the function $F_1(\mu, \delta)$. It can be for example $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\dfrac{\mu + \delta}{2}$, or any other applicable mathematical function.

(Viii) $F_2 : [0,1]^* \longrightarrow [0,1]$: is called multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

We let $Q_{act}(t_i)$ be the set of all active state at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \sum,\ \delta(q', a, q) \in \Delta\}$, $\forall i \geq 1$.

Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in$Domain$(Q_{act}(t_i))$ and $T \subseteq$Domain$(Q_{act}(t_i))$; hereafter, we simply denote them by:

$$q \in Q_{act}(t_i) \quad \text{and} \quad T \subseteq Q_{act}(t_i).$$

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 2.2. [8] (Multi-membership resolution)

If there are several simultaneous transitions to the active state q_j at time $t+1$, then following algorithm assigns a unified membership values.

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this to be ν_i,

$$\nu_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous mv of the active state q_j.

$$\mu^{t+1}(q_j) = \{F_2\}_{i=1}^{n}[\nu_i] = \{F_2\}_{i=1}^{n}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))],$$

where,

n: is the number of simultaneous transitions to the active state q_j at time $t+1$.

$\delta(q_i, a_k, q_j)$: is the weight of a transition from q_i to q_j upon input a_k.

$\mu^t(q_i)$: is the membership value of q_i at time t.

$\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Definition 2.3. [32] Let $\tilde{F} = (Q, \sum, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata of the form $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that $\tilde{\delta}^* : Q_{act} \times \sum^* \times Q \longrightarrow [0, 1]$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), \dots\}$ and for every $i, i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & if \quad q = p \\ 0, & otherwise \end{cases}$$

and for every i, $i \geq 1$, $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$, and recursively,

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$$
$$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \sum$, $\forall 1 \leq i \leq n$, and assuming that the entered input at time t_i is u_i, for $1 \leq i \leq n-1$.

Definition 2.4. [32] Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $p \in Q_{act}(t_j)$,

$q \in Q_{act}(t_i)$, $t_i \leq t_j$ and $0 \leq c < 1$. Then, $(p, \mu^{t_j}(p))$ is called a successor of $(q, \mu^{t_i}(q))$ with threshold c if there exists $x \in \sum^*$ such that $\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) > c$.

Also, we say that there exists a path between $(q, \mu^{t_i}(q))$ and $(p, \mu^{t_j}(p))$ with its length being $|x|$.

Definition 2.5. [16] H is a hyper BCK-algebra if it contains a constant "*zero*" satisfying the following axioms:

(HK1) $(xoz)o(yoz) \ll xoy$,
(HK2) $(xoy)oz = (xoz)oy$,
(HK3) $xoH \ll \{x\}$,
(HK4) $x \ll y, y \ll x \Rightarrow x = y$.

For all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in xoy$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ Such that $a \ll b$.

Definition 2.6. [4] We say that H is a hyper K-algebra if it contains a constant "*zero*" satisfying the following axioms:

(HK1) $(xoz)o(yoz) < xoy$,
(HK2) $(xoy)oz = (xoz)oy$,
(HK3) $x < x$,
(HK4) $x < y, y < x \Rightarrow x = y$,
(HK5) $0 < x$.

For all $x, y, z \in H$, where $x < y$ is defined by $0 \in xoy$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by AoB we mean the subset $U_{a \in A, b \in B}\ aob$ of H.

Theorem 2.7. [22] In a hyper BCK-algebra $(H, o, 0)$, the condition (HK3) is equivalent to the condition:

$$xoy \ll \{x\}, \quad \text{for all} \quad x, y \in H.$$

Theorem 2.8. [22] Every hyper BCK-algebra is a hyper K-algebra.

Definition 2.9. [16] Let I be a non-empty subset of a hyper BCK-algebra H and $0 \in I$. Then,

(1) I is called a weak hyper BCK-ideal of H if $xoy \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in I$.
(2) I is called a hyper BCK-ideal of H if $xoy \ll I$ and $y \in I$ imply that $x \in I$, for all $x, y \in I$.
(3) I is called a strong hyper BCK-ideal of H if $(xoy) \cap I \neq \emptyset$ and $y \in I$ imply that $x \in I$, for all $x, y \in I$.

Theorem 2.10. [16] Any strong hyper BCK-ideal of a hyper BCK-algebra H is a hyper BCK-ideal and a weak hyper BCK-ideal. Also, any hyper BCK-ideal of a hyper BCK-algebra H is a weak hyper BCK-ideal.

Definition 2.11. [22] A hyper BCK-algebra $(H, o, 0)$ is called simple if for all distinct elements $a, b \in H - \{0\}$, $a \not\leq b$ and $b \not\leq a$.

Definition 2.12. [19] Let $(H, o, 0)$ be a hyper K-algebra and I be a subset of H and $\emptyset \neq S \subseteq H$. Then, we say that I is an S-absorbing set, whenever $x \in I$ and $y \in S$ imply that $xoy \subseteq I$.

Definition 2.13. [24] Let I be a non-empty subset of a hyper K-algebra H and $0 \in I$. Then, I is called a commutative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,
(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \emptyset$,
(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $z \in I$ imply that $(xo(yo(yox))) < I$,
(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,
(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \emptyset$,
(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) < I$,
(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,
(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \emptyset$,
(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) < I$.

Theorem 2.14. [22] Let I be a non-empty subset of a hyper K-algebra H and $0 \in I$. Then, the following statements hold:

(i) If I is a commutative hyper K-ideal of type 4, then I is a commutative hyper K-ideal of types 5 and 6.
(ii) If I is a commutative hyper K-ideal of type 6, then I is a commutative hyper K-ideal of type 9.
(iii) If I is a commutative hyper K-ideal of type 5, then I is a commutative hyper K-ideal of type 6.
(iv) If I is a commutative hyper K-ideal of type 9, then I is a commutative hyper K-ideal of type 3.

Definition 2.15. [24] Let I be a non-empty subset of H such that $0 \in I$. Then, I is called a positive implicative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,
(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \emptyset$,
(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,
(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \subseteq I$,
(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \cap I \neq \emptyset$,
(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) < I$,
(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) < I$,
(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \emptyset$,
(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,
(x) type 10, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \emptyset$,
(xi) type 11, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,
(xii) type 12, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,
(xiii) type 13, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \subseteq I$,
(xiv) type 14, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \cap I \neq \emptyset$,
(xv) type 15, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) < I$,
(xvi) type 16, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) < I$ imply that $(xoz) < I$,
(xvii) type 17, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \emptyset$,
(xviii) type 18, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,
(xix) type 19, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) < I$,
(xx) type 20, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \subseteq I$,
(xxi) type 21, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \emptyset$ imply that $(xoz) \cap I \neq \emptyset$,
(xxii) type 22, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,
(xxiii) type 23, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,
(xxiv) type 24, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \emptyset$,
(xxv) type 25, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that

$(xoz) < I$,
(xxvi) type 26, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \emptyset$,
(xxvii) type 27, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$.

Definition 2.16. [3] Let H be a hyper K-algebra. If $0ox = \{0\}$, for all $x \in H$, then we say that H satisfies the zero condition.

Definition 2.17. [3] If there is an element $1 \in X$ of a hyper K-algebra X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called the unit of X. A hyper K-algebra with a unit is a bounded hyper K-algebra.

Theorem 2.18. [3] Let $(H, o, 0)$ be a hyper K-algebra, which satisfies the zero condition. Then, $(H, o, 0)$ can be extended to a bounded hyper K-algebra.

Let H to be a bounded hyper K-algebra with unit 1. So, Nx means *lox*.

Definition 2.19. [25] A non-empty subset D of H is called a dual positive implicative hyper K-ideal of type 3 (DPIHKI- T3) if it satisfies:

(i) $1 \in D$,
(ii) $N((NxoNy)oNz) < D$ and $N(NyoNz) < D$ imply that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.20. [25] Let $1 \in D \subseteq H$. Then, D is a DPIHKI-T3 if and only if $N(NxoNz) \subseteq D$, for all $x, z \in H$.

Definition 2.21. [25] A non-empty subset D of H is called a dual positive implicative hyper K-ideal of type 4 (DPIHKI-T4) if it satisfies:

(i) $1 \in D$,
(ii) $N((NxoNy)oNz) \subseteq D$ and $N(NyoNz) < D$ imply that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.22. [25] D is a DPIHKI-T4 if and only if $N((NxoNy)oNz) \subseteq D$ implies that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.23. [25] If D is a DPIHKI-T3 of H, then D is a DPIHKI-T4.

Definition 2.24. [25] A non-empty subset D of H is called a dual positive implicative hyper K-ideal of type 2 (DPIHKI-T2) if it satisfies:

(i) $1 \in D$,
(ii) $N((NxoNy)oNz) < D$ and $N(NyoNz) \subseteq D$ imply that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.25. [25] D is a DPIHKI-T2 if and only if $N(NyoNz) \subseteq D$ implies that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.26. [25] If D is a DPIHKI-T3, then D is a DPIHKI-T2.

Definition 2.27. [25] A non-empty subset D of H is called a dual positive implicative hyper K-ideal of type 1 (DPIHKI-T1) if it satisfies:

(i) $1 \in D$,
(ii) $N((NxoNy)oNz) \subseteq D$ and $N(NyoNz) \subseteq D$ imply that $N(NxoNz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.28. [25] If D is a DPIHKI-T2, T3 or T4, then D is a DPIHKI-T1.

Definition 2.29. [25] Let D be a non-empty subset of H and $1 \in D$. Then, D is called a dual commutative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $N((NxoNy)oNz) \subseteq D$ and $z \in D$ imply that $N(Nxo(Nyo(NyoNx))) \subseteq D$,
(ii) type 2, if for all $x, y, z \in H$, $N((NxoNy)oNz) < D$ and $z \in D$ imply that $N(Nxo(Nyo(NyoNx))) \subseteq D$.

Note that for simplicity of notation we write DCHKI-T1 (T2) instead of dual commutative hyper K-ideal of types 1 (2).

Theorem 2.30. [25] Every dual commutative hyper K-ideal of type 2 is a dual commutative hyper K-ideal of type 1.

Theorem 2.31. [25] A non-empty subset D of H is a DCHKI-T2 if and only if $N(Nxo(Nyo(NyoNx))) \subseteq D$, for all $x, y \in H$.

Definition 2.32. [26] Let H be a hyper BCK-algebra and A be a non-empty subset of H. Then, the sets $_lA = \{x \in H | a \in aox, \forall a \in A\}$ and $A_r = \{x \in H | x \in xoa, \forall a \in A\}$ are called left hyper BCK-stabilizer of A and right hyper BCK-stabilizer of A, respectively.

Definition 2.33. [26] A hyper BCK-algebra H is called

(i) weak normal, if a_r is a weak hyper BCK-ideal of H for any element $a \in H$.
(ii) normal, if a_r is a hyper BCK-ideal of H for any element $a \in H$.
(iii) strong normal, if a_r is a strong hyper BCK-ideal of H for any element $a \in H$.

3. Hyper BCK-algebra Induced by Max-Min General Fuzzy Automata

Definition 3.1. Let $(p, \mu^{t_j}(p)) \in S_c(q, \mu^{t_i}(q))$, where,
$S_c(q, \mu^{t_i}(q)) = \{(p, \mu^{t_j}(p)) | \exists x \in \sum^* \quad s.t. \quad \tilde{\delta}^*((p, \mu^{t_j}(p)), x, q) > c, \forall t_j \leq t_i\}$. Then, the length between p and q is defined by,

$$L(p, \mu^{t_j}(p)) = \min\{|x| \ |\tilde{\delta}^*((p, \mu^{t_j}(p)), x, q) > c\}.$$

In other words, $L(p, \mu^{t_j}(p))$ is equal to the min of the length of the involved words x, in the path between p and q.

Definition 3.2. Let $S_c(q, \mu^{t_i}(q)) = \{(q_1, \mu^{t_{i_1}}(q_1)), (q_2, \mu^{t_{i_2}}(q_2)), \ldots, (q_n, \mu^{t_{i_n}}(q_n))\}$. Then, the order of $(q, \mu^{t_i}(q))$ is:

$$\max\{L(q_1, \mu^{t_{i_1}}(q_1)), L(q_2, \mu^{t_{i_2}}(q_2)), \ldots, L(q_n, \mu^{t_{i_n}}(q_n))\}$$

and is denoted by $ord(q, \mu^{t_i}(q))$. We define the equivalence relation $\sim$ as follows:

$$(p, \mu^{t_j}(p)) \sim (q, \mu^{t_i}(q)) \Longleftrightarrow ord(p, \mu^{t_j}(p)) = ord(q, \mu^{t_i}(q)).$$

Now, we denote the equivalence class of $(q, \mu^{t_i}(q))$ by $\overline{(q, \mu^{t_i}(q))}$.

Note that the order of $(q_0, \mu^{t_0}(q_0))$ is equal to zero.

Theorem 3.3. Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton. We define the following hyper operation on $\overline{Q}$:

$$\forall \overline{q_1}, \overline{q_2} \in \overline{Q}^2, \overline{q_1} o \overline{q_2} = \begin{cases} \overline{q_1}, & if \quad \overline{q_1} \neq \overline{q_2}, \overline{q_2} \neq \overline{q_0} \neq \overline{q_1} \\ \{\overline{q_0}, \overline{q_1}\}, & if \quad \overline{q_1} = \overline{q_2} \\ \overline{q_0}, & if \quad \overline{q_1} = \overline{q_0}, \overline{q_2} \neq \overline{q_0} \\ \overline{q_1}, & if \quad \overline{q_1} \neq \overline{q_0}, \overline{q_2} = \overline{q_0}. \end{cases}$$

It is easy to check that the hyper operation "o" is well-defined as a function from $\overline{Q} \times \overline{Q}$ to $P^*(\overline{Q}) = P(\overline{Q}) - \{\emptyset\}$, and $\overline{q_0}$ is the zero element of $\overline{Q}$. Then, $(\overline{Q}, o, \overline{q_0})$ is a hyper BCK-algebra.

Proof. First, we need to consider the following situations to show that $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1) and (HK2).

(i) Let $\overline{q_1}, \overline{q_2}, \overline{q_3} \neq \overline{q_0}$ and $\overline{q_3} \neq \overline{q_2} \neq \overline{q_1} \neq \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_2}$. Since $\overline{q} o \overline{q} = \{\overline{q_0}, \overline{q}\}$, we obtain that $\overline{q} \ll \overline{q}$ for any $\overline{q} \in \overline{Q}$. So, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_3} = \overline{q_1}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_2} = \overline{q_1}$. Thus, in this case, (HK2) holds.

(ii) Let $\overline{q_1}, \overline{q_2}, \overline{q_3} \neq \overline{q_0}$ and $\overline{q_1} = \overline{q_2} \neq \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_2}$. So, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \{\overline{q_0}, \overline{q_1}\} o \overline{q_3} = \{\overline{q_0}, \overline{q_1}\}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$. Therefore, in this case (HK2) holds.

(iii) Let $\overline{q_1}, \overline{q_2}, \overline{q_3} \neq \overline{q_0}$ and $\overline{q_1} = \overline{q_3} \neq \overline{q_2}$. Then, $(\overline{q_1} o \overline{q_2}) o (\overline{q_2} o \overline{q_3}) = \{\overline{q_0}, \overline{q_1}\} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1}$. Since $\overline{q_0} o \overline{q_1} = \overline{q_0}$, we obtain that $\overline{q_0} \ll \overline{q_1}$. Also, we know that $\overline{q_1} \ll \overline{q_1}$. Hence, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_3} = \{\overline{q_0}, \overline{q_1}\}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$. So, in this case (HK2) holds.

(iv) Let $\overline{q_1}, \overline{q_2}, \overline{q_3} \neq \overline{q_0}$ and $\overline{q_2} = \overline{q_3} \neq \overline{q_1}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \{\overline{q_0}, \overline{q_2}\} = \overline{q_1}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1}$. Thus, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_3} = \overline{q_1}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_2} = \overline{q_1}$. Therefore, in this case (HK2) holds.

(v) Let $\overline{q_1} = \overline{q_2} = \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \{\overline{q_0}, \overline{q_1}\} o \{\overline{q_0}, \overline{q_1}\} = \{\overline{q_0}, \overline{q_1}\}$ and $\overline{q_1} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$. So, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1).
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = (\overline{q_1} o \overline{q_1}) o \overline{q_1} = (\overline{q_1} o \overline{q_3}) o \overline{q_2}$. Hence, in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK2).

(vi) Let $\overline{q_2}, \overline{q_3} \neq \overline{q_0}$, $\overline{q_1} = \overline{q_0}$ and $\overline{q_2} \neq \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_0} o \overline{q_2} = \overline{q_0}$ and $\overline{q_1} o \overline{q_2} = \overline{q_0} o \overline{q_2} = \overline{q_0}$. Thus, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_0} o \overline{q_3} = \overline{q_0}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_0} o \overline{q_2} = \overline{q_0}$. So, in this case (HK2) holds.

(vii) Let $\overline{q_2}, \overline{q_3} \neq \overline{q_0}$, $\overline{q_1} = \overline{q_0}$ and $\overline{q_2} = \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_0} o \{\overline{q_0}, \overline{q_2}\} = \overline{q_0}$ and $\overline{q_1} o \overline{q_2} = \overline{q_0} o \overline{q_2} = \overline{q_0}$. Therefore, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1).
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_0} o \overline{q_3} = \overline{q_0}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_0} o \overline{q_2} = \overline{q_0}$. So, in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK2).

(viii) Let $\overline{q_1}, \overline{q_3} \neq \overline{q_0}$, $\overline{q_2} = \overline{q_0}$ and $\overline{q_1} \neq \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_0} = \overline{q_1}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1} o \overline{q_0} = \overline{q_1}$. Hence, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_3} = \overline{q_1}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_0} = \overline{q_1}$. Thus, in this case (HK2) holds.

(ix) Let $\overline{q_1}, \overline{q_3} \neq \overline{q_0}$, $\overline{q_2} = \overline{q_0}$ and $\overline{q_1} = \overline{q_3}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \{\overline{q_0}, \overline{q_1}\} o \overline{q_0} = \{\overline{q_0}, \overline{q_1}\}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1} o \overline{q_0} = \overline{q_1}$. Since $\overline{q_0} \ll \overline{q_1}$ and $\overline{q_1} \ll \overline{q_1}$, we obtain that $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1).
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_3} = \{\overline{q_0}, \overline{q_1}\}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\} o \overline{q_0} = \{\overline{q_0}, \overline{q_1}\}$. Hence, in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK2).

(x) Let $\overline{q_1}, \overline{q_2} \neq \overline{q_0}$, $\overline{q_3} = \overline{q_0}$ and $\overline{q_1} \neq \overline{q_2}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_2} = \overline{q_1}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1}$. Therefore, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_0} = \overline{q_1}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_2} = \overline{q_1}$. So, in this case (HK2) holds.

(xi) Let $\overline{q_1}, \overline{q_2} \neq \overline{q_0}$, $\overline{q_3} = \overline{q_0}$ and $\overline{q_1} = \overline{q_2}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$ and $\overline{q_1} o \overline{q_2} = \{\overline{q_1}, \overline{q_0}\}$. Since $\overline{q_0} \ll \overline{q_0}$ and $\overline{q_1} \ll \overline{q_1}$, we get that $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1).
Also, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \{\overline{q_0}, \overline{q_1}\} o \overline{q_0} = \{\overline{q_0}, \overline{q_1}\}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_2} = \{\overline{q_0}, \overline{q_1}\}$. Thus, in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK2).

(xii) Let $\overline{q_1} = \overline{q_2} = \overline{q_0}$ and $\overline{q_3} \neq \overline{q_0}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_0} o \overline{q_0} = \overline{q_0}$ and $\overline{q_1} o \overline{q_2} = \overline{q_0}$. Therefore, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_0} o \overline{q_3} = \overline{q_0}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_0} o \overline{q_0} = \overline{q_0}$. Hence, in this case (HK2) holds.

(xiii) Let $\overline{q_1} = \overline{q_3} = \overline{q_0}$ and $\overline{q_2} \neq \overline{q_0}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_0} o \overline{q_2} = \overline{q_0}$ and $\overline{q_1} o \overline{q_2} = \overline{q_0}$. So, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1).
On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_0} o \overline{q_0} = \overline{q_0}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_0} o \overline{q_2} = \overline{q_0}$. Thus, this case $(\overline{Q}, o, \overline{q_0})$ satisfies (HK2).

(xiv) Let $\overline{q_2} = \overline{q_3} = \overline{q_0}$ and $\overline{q_1} \neq \overline{q_0}$. Then, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) = \overline{q_1} o \overline{q_0} = \overline{q_1}$ and $\overline{q_1} o \overline{q_2} = \overline{q_1}$. Therefore, $(\overline{q_1} o \overline{q_3}) o (\overline{q_2} o \overline{q_3}) \ll \overline{q_1} o \overline{q_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{q_1} o \overline{q_2}) o \overline{q_3} = \overline{q_1} o \overline{q_0} = \overline{q_1}$ and $(\overline{q_1} o \overline{q_3}) o \overline{q_2} = \overline{q_1} o \overline{q_0} = \overline{q_1}$. Hence, in this case (HK2) holds.

So, we showed that $(\overline{Q}, o, \overline{q_0})$ satisfies (HK1) and (HK2).

Now, we should prove that $(\overline{Q}, o, \overline{q_0})$ satisfies (HK3).

By definition of the hyper operation "o" we know that $\overline{q_1} o \overline{q_2} = \overline{q_1}$ or $\{\overline{q_0}, \overline{q_1}\}$ or $\overline{q_0}$, for any $\overline{q_1}, \overline{q_2} \in \overline{Q}$. Also, we know that $\overline{q_1} \ll \overline{q_1}$ and $\overline{q_0} \ll \overline{q_1}$. So, $(\overline{Q}, o, \overline{q_0})$ satisfies (HK3).
To prove (HK4), Let $\overline{q_1} \ll \overline{q_2}$ and $\overline{q_2} \ll \overline{q_1}$. If $\overline{q_1} = \overline{q_2}$, then we are done. Otherwise, since $\overline{q_1} \ll \overline{q_2}$, there exists just one case:

(i) Let $\overline{q_1} = \overline{q_0}$, $\overline{q_2} \neq \overline{q_0}$. Then, $\overline{q_2} o \overline{q_1} = \overline{q_2} o \overline{q_0} = \overline{q_2}$. Thus, $\overline{q_2} \not\ll \overline{q_1}$, and this is a contradiction.

So, we showed that $(\overline{Q}, o, \overline{q_0})$ is a hyper BCK-algebra. □

Theorem 3.4. Let $(\overline{Q}, o, \overline{q_0})$ be the hyper BCK-algebra, as defined in Theorem 3.3. Then, $(\overline{Q}, o, \overline{q_0})$ is

(1) a strong normal hyper BCK-algebra, and (2) a simple hyper BCK-algebra.

Proof. (1) By definition of hyper operation "o", we obtain that $\overline{a} \in \overline{a} o \overline{t}$, for any $\overline{a}$ and $\overline{t}$ in $\overline{Q}$. So, we have:

$$_l\overline{a} = \{\overline{t} \in \overline{Q} | \overline{a} \in \overline{a} o \overline{t}\} = \overline{Q} \quad , \quad \overline{a}_r = \{\overline{t} \in \overline{Q} | \overline{t} \in \overline{t} o \overline{a}\} = \overline{Q}, \ \forall \overline{a} \in \overline{Q}.$$

It is clear that $\overline{Q}$ is a strong hyper BCK-ideal. So, $(\overline{Q}, o, \overline{q_0})$ is a strong normal hyper BCK-algebra.
(2) Let $\overline{q_1} \neq \overline{q_2}$ and $\overline{q_1}, \overline{q_2} \neq \overline{q_0}$. Then, $\overline{q_1} o \overline{q_2} = \overline{q_1}$ and $\overline{q_2} o \overline{q_1} = \overline{q_2}$. Hence, $\overline{q_1} \not\ll \overline{q_2}$ and $\overline{q_2} \not\ll \overline{q_1}$. So, we get that $(\overline{Q}, o, \overline{q_0})$ is a simple hyper BCK-algebra. □

Example 3.5. Consider the max-min GFA in Figure 1 with several transition overlaps. It is specified as:

$\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$, where $Q = \{q_0, q_1, q_2, q_3, q_4\}$ is the set of states, $\sum = \{a, b\}$ is the set of input symbols, $\tilde{R} = \{(q_0, 1)\}$, $Z = \emptyset$ and ω is not applicable, $Q_{act}(t_0) = \{q_0\}$, $Q_{act}(t_1) = \{q_1, q_4\}$, $Q_{act}(t_2) = \{q_1, q_2, q_4\}$, $Q_{act}(t_3) = \{q_2, q_3\}, \ldots$, and

$$^1F_1(\mu,\delta) = \delta, \quad F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^2F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^3F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^4F_1(\mu,\delta) = Max(\mu,\delta), \quad F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^5F_1(\mu,\delta) = Max(\mu,\delta), \quad F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$$^6F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2() = \mu^{t+1}(q_m) = \sum_{i=1}^{n}F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))/n,$$

$$^7F_1(\mu,\delta) = \frac{\mu+\delta}{2}, \quad F_2() = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

where n is the number of simultaneous transitions to the active state q_m at time $t+1$.

If we choose $^1F_1(\mu,\delta) = \delta, \quad F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$, then we have:

$$\begin{aligned}
\mu^{t_0}(q_0) &= 1 \\
\mu^{t_1}(q_1) &= F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4, \\
\mu^{t_1}(q_4) &= F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_4)) = F_1(1, 0.5) = 0.5, \\
\mu^{t_2}(q_1) &= F_1(\mu^{t_1}(q_4), \delta(q_4, a, q_1)) = F_1(0.5, 0.4) = 0.4, \\
\mu^{t_2}(q_2) &= F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.8) = 0.8, \\
\mu^{t_2}(q_4) &= F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_4)) = F_1(0.4, 0.35) = 0.35, \\
\mu^{t_3}(q_2) &= F_1(\mu^{t_2}(q_4), \delta(q_4, b, q_2)) \wedge F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_2)) \\
&= F_1(0.4, 0.1) \wedge F_1(0.8, 0.6) = 0.1 \wedge 0.6 = 0.1, \\
\mu^{t_3}(q_3) &= F_1(\mu^{t_2}(q_1), \delta(q_1, b, q_3)) \wedge F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_3)) \wedge F_1(\mu^{t_2}(q_4), \delta(q_4, b, q_3)) \\
&= F_1(0.4, 0.3) \wedge F_1(0.8, 0.45) \wedge F_1(0.35, 0.7) = 0.3 \wedge 0.45 \wedge 0.7 = 0.3.
\end{aligned}$$

There are two simultaneous transitions to the active state q_2 at time t_3 and there are three simultaneous transitions to the active state q_3 at time t_3. Table 1 shows these findings:

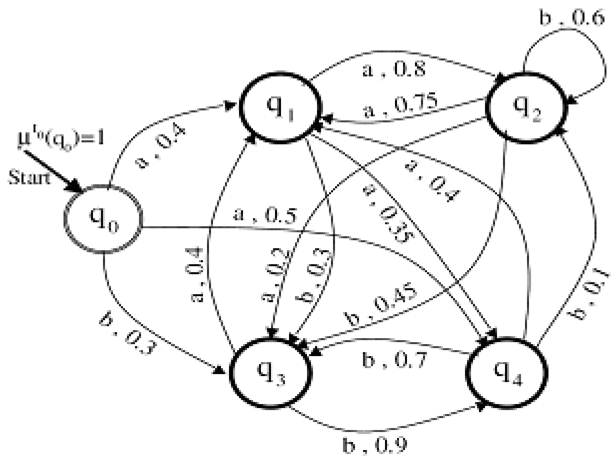

FIGURE 1. The GFA of Example 3.5

time	t_0	t_1		t_2			t_3	
input	Λ	a		a			b	
$Q_{act}(t_i)$	q_0	q_1	q_4	q_1	q_2	q_4	q_2	q_3
mv^1	1.0	0.4	0.5	0.4	0.8	0.35	0.1	0.3
mv^2	1.0	0.4	0.5	0.4	0.4	0.35	0.1	0.3
mv^3	1.0	0.4	0.5	0.4	0.4	0.35	0.4	0.4
mv^4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^6	1.0	0.4	0.5	0.4	0.4	0.35	0.25	0.35
mv^7	1.0	0.7	0.75	0.575	0.75	0.525	0.763	0.613

TABLE 1. Active States and Their Membership Values (mvs) at Different Times in Example 3.5

If $^{5}F_1(\mu,\delta)=Max(\mu,\delta),\quad F_2()=\mu^{t+1}(q_m)=\bigvee_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_i,a_k,q_m)))$, then we have:

$$\begin{aligned}
\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_1) &= \bigvee_{q'\in Q_{act}(t_1)}[\tilde{\delta}(q_0,\mu^{t_0}(q_0),a,q')\wedge\tilde{\delta}((q',\mu^{t_1}(q'),a,q_1)]\\
&= [\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)\wedge\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_1)]\\
&\quad\vee[\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_4)\wedge\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_1)]\\
&= [F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))\wedge F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_1))]\\
&\quad\vee[F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_4))\wedge F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))]\\
&= [F_1(1,0.4)\wedge F_1(1,0)]\vee[F_1(1,0.5)\wedge F_1(1,0.4)]\\
&= [1\wedge 1]\vee[1\wedge 1]=1\vee 1=1.
\end{aligned}$$

If we choose $^2F_1(\mu,\delta)=Min(\mu,\delta), \quad F_2()=\mu^{t+1}(q_m)=\bigwedge_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_i,a_k,q_m))),$
then we have:

$$S_{0.1}(q_2,\mu^{t_2}(q_2))=\{(q_1,\mu^{t_1}(q_1)),(q_4,\mu^{t_1}(q_4)),(q_0,\mu^{t_0}(q_0))\},$$

since,

$$\begin{aligned}
\tilde{\delta}^*((q_1,\mu^{t_1}(q_1)),a,q_2) &= \tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_2)=F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_2))\\
&= Min(0.4,0.8)=0.4>c,\\
\tilde{\delta}^*((q_4,\mu^{t_1}(q_4)),a^2,q_2) &= \bigvee_{q'\in Q_{act}(t_2)}[\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q')\wedge\tilde{\delta}((q',\mu^{t_2}(q')),a,q_2)]\\
&= [\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_1)\wedge\tilde{\delta}((q_1,\mu^{t_2}(q_1)),a,q_2)]\\
&\quad\vee[\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_2)\wedge\tilde{\delta}((q_2,\mu^{t_2}(q_2),a,q_2)]\\
&\quad\vee[\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_4)\wedge\tilde{\delta}((q_4,\mu^{t_2}(q_4),a,q_2)]\\
&= [F_1(0.5,0.4)\wedge F_1(0.4,0.8)]\vee[F_1(0.5,0)\wedge F_1(0.4,0)]\\
&\quad\vee[F_1(0.5,0)\wedge F_1(0.35,0)]\\
&= [0.4]\vee[0]\vee[0]=0.4>c,\\
\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_2) &= \bigvee_{q'\in Q_{act}(t_1)}[\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q')\wedge\tilde{\delta}((q',\mu^{t_1}(q'),a,q_2)]\\
&= [\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)\wedge\tilde{\delta}((q_1,\mu^{t_1}(q_1),a,q_2)]\\
&\quad\vee[\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_4)\wedge\tilde{\delta}((q_4,\mu^{t_1}(q_4),a,q_2)]\\
&= [F_1(1,0.4)\wedge F_1(0.4,0.8)]\vee[F_1(1,0.5)\wedge F_1(0.5,0)]\\
&= [0.4]\vee[0]=0.4>c.\\
\Longrightarrow &\quad ord(q_2,\mu^{t_2}(q_2))=2.\ (3.1)
\end{aligned}$$

Also,

$$S_{0.1}(q_1,\mu^{t_2}(q_1))=\{(q_4,\mu^{t_1}(q_4)),(q_1,\mu^{t_1}(q_1)),(q_0,\mu^{t_0}(q_0))\},$$

since,

$$\begin{aligned}
\tilde{\delta}^*((q_4,\mu^{t_1}(q_4)),a,q_1) &= \tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_1)=F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))\\
&= Min(0.5,0.4)=0.4>c,\\
\tilde{\delta}^*((q_1,\mu^{t_1}(q_1)),a^2,q_1) &= \bigvee_{q'\in Q_{act}(t_2)}[\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q')\wedge\tilde{\delta}((q',\mu^{t_2}(q')),a,q_1)]\\
&= [\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_1)\wedge\tilde{\delta}((q_1,\mu^{t_2}(q_1)),a,q_1)]\\
&\quad\vee[\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_2)\wedge\tilde{\delta}((q_2,\mu^{t_2}(q_2),a,q_1)]\\
&\quad\vee[\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_4)\wedge\tilde{\delta}((q_4,\mu^{t_2}(q_4),a,q_1)]\\
&= [F_1(0.4,0)\wedge F_1(0.4,0)]\vee[F_1(0.4,0.8)\wedge F_1(0.4,0.75)]\\
&\quad\vee[F_1(0.4,0.35)\wedge F_1(0.35,0.4)]\\
&= [0]\vee[0.4]\vee[0.35]=0.4>c,\\
\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a,q_1) &= \tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)=F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))\\
&= Min(1,0.4)=0.4>c.\\
\Longrightarrow &\quad ord(q_1,\mu^{t_2}(q_1))=2.\ (3.2)
\end{aligned}$$

By (3.1) and (3.2) we have $(q_1,\mu^{t_2}(q_1))\sim(q_2,\mu^{t_2}(q_2))$, then $\overline{(q_1,\mu^{t_2}(q_1))}=\overline{(q_2,\mu^{t_2}(q_2))}$. Similarly, we conclude that $\overline{(q_4,\mu^{t_2}(q_4))}=\overline{(q_2,\mu^{t_2}(q_2))}$. Then,

$$\overline{(q_1,\mu^{t_2}(q_1))}=\overline{(q_2,\mu^{t_2}(q_2))}=\overline{(q_4,\mu^{t_2}(q_4))}.$$

Also, we have:

$$\overline{(q_1,\mu^{t_1}(q_1))}=\overline{(q_4,\mu^{t_1}(q_4))},\overline{(q_2,\mu^{t_3}(q_2))}=\overline{(q_3,\mu^{t_3}(q_3))}.$$

O	$\overline{q_0}$	$\overline{q_1}$	$\overline{q_2}$	$\overline{q_3}$
$\overline{q_0}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$
$\overline{q_1}$	$\{\overline{q_1}\}$	$\{\overline{q_0}, \overline{q_1}\}$	$\{\overline{q_1}\}$	$\{\overline{q_1}\}$
$\overline{q_2}$	$\{\overline{q_2}\}$	$\{\overline{q_2}\}$	$\{\overline{q_0}, \overline{q_2}\}$	$\{\overline{q_2}\}$
$\overline{q_3}$	$\{\overline{q_3}\}$	$\{\overline{q_3}\}$	$\{\overline{q_3}\}$	$\{\overline{q_0}, \overline{q_3}\}$

TABLE 2. The Hyper Operation "o" on the Equivalence

Classes of States in Example 3.5

According to the definition of hyper operation "o" as defined in Theorem 3.3, we have Table 2.

Theorem 3.6. Let $(\overline{Q}, o, \overline{q_0})$ be a hyper BCK-algebra as defined in Theorem 3.3. Then, every subset of $\overline{Q}$ which contains $\overline{q_0}$ is a strong hyper BCK-ideal of $\overline{Q}$.

Proof. Let $I = \{\overline{q_0}, \overline{q_1}, \ldots, \overline{q_n}\}$ be a subset of $\overline{Q}$ which contains $\overline{q_0}$ and $n \geq 0$. Now, we show that I is a strong hyper BCK-ideal of H. Let $(xoy) \cap I \neq \emptyset$ and $y \in I$. Then, we consider two cases:

(1) $y = \overline{q_0}$.
Since $xoy = xo\overline{q_0} = x$ and $xoy \cap I \neq \emptyset$, we obtain that $x \in I$.
(2) $y = \overline{q_r}$ and $r \in \{1, \ldots, n\}$.

In this case, if $x \notin I$, then $x \neq \overline{q_0}, x \neq \overline{q_1}, \ldots, x \neq \overline{q_n}$. So, $xoy = xo\overline{q}_r = x$. Since $xoy \cap I \neq \emptyset$, we get that $x \in I$, which is a contradiction. $\square$

Theorem 3.7. Let I be the subset of $\overline{Q}$ which is defined in Theorem 3.3. Then, I is a positive implicative hyper K-ideal of type 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25 and 26 (see [24]).

Proof. First, we show that I is a positive implicative hyper K-ideal of type 1. Let $x, y, z \in \overline{Q}$ and $(xoy)oz \subseteq I$ and $(yoz) \subseteq I$. Then, we should consider two cases:

(1) $x = y$.
Since $yoz = xoz$ and $yoz \subseteq I$, we obtain that $xoz \subseteq I$.
(2) $x \neq y$. By definition of the hyper operation "o", it is easy to see that $x \in xoy$ and $xoy \neq \emptyset$, for all $x, y \in \overline{Q}$. So, $x \in (xoy)oz$. Since $(xoy)oz \subseteq I$, we obtain that $x \in I$. Also, $xoz = x$ or $xoz = \{x, \overline{q_0}\}$. Thus, by definition of I, we get that $(xoz) \subseteq I$. Hence, I is a positive implicative hyper K-ideal of type 1.

Now, we show that I is a positive implicative hyper K-ideal of type 10. Let $x, y, z \in \overline{Q}$, $((xoy)oz) \cap I \neq \emptyset$ and $(yoz) \subseteq I$ Then, we should consider two cases:

(1) $x = y$.
Since $yoz = xoz, yoz \subseteq I$ and $yoz \neq \emptyset$, we obtain that $xoz \subseteq I$ and $(xoz) \cap I \neq \emptyset$.
(2) $x \neq y$. Since $xoy = x$ and $((xoy)oz) \cap I \neq \emptyset$, we get that $(xoz) \cap I \neq \emptyset$. So, we conclude that I is a positive implicative hyper K-ideal of type 10.

The proofs for other cases are similarly developed using suitable modifications. □

Remark 3.8. Note that Theorem 3.7 does not hold for types 11, 13, 18, 20, 22 and 27. To this end, we first consider type 11.

Let $I = \{\overline{q_0}, \overline{q_1}\}$, $x, y, z \neq \overline{q_0}$, $x = z = \overline{q_2}$ and $x \neq y = \overline{q_1}$. Then, $(xoy)oz = xoz = \{\overline{q_0}, \overline{q_2}\}$ and $(yoz) = y = \overline{q_1}$. So $(xoy)oz \cap I \neq \emptyset$ and $(yoz) \subseteq I$. But $xoz = \{\overline{q_0}, \overline{q_2}\} \nsubseteq I$. Hence, I is not a positive implicative hyper K-ideal of type 11.

Then, we show type 27. Since $\overline{q_0} < \overline{q_0}$ and $\overline{q_1} < \overline{q_1}$, we get that $(xoy)oz < I$ and $(yoz) < I$. But, $xoz = \{\overline{q_0}, \overline{q_2}\} \nsubseteq I$. So, I is not a positive implicative hyper K-ideal of types 27. Similarly, we can prove that I is not a positive implicative hyper K-ideal of types 13, 18, 20 and 22.

Theorem 3.9. Let $(\overline{Q}, o, \overline{q_0})$ be the hyper BCK-algebra as defined in Theorem 3.3. Then, every non-empty subset of $\overline{Q}$ is an $\{\overline{q_0}\}$-absorbing set.

Proof. By definition of the hyper operation "o" we know that $\overline{q_1} o \overline{q_0} = \overline{q_1}$, for all $\overline{q_1} \in \overline{Q}$. So, it is easy to see that for all non-empty subset of $\overline{Q}$, we have:

$$x \in I \text{ and } y \in \{\overline{q_0}\} \Longrightarrow xoy \subseteq I.$$

□

Theorem 3.10. Let $(\overline{Q}, o, \overline{q_0})$ be the hyper BCK-algebra as defined in Theorem 3.3. Then, every subset of $\overline{Q}$ which contains $\overline{q_0}$ is a commutative hyper K-ideal of types 2, 3, 4, 5, 6, 8 and 9 (see [24]).

Proof. First, we show that I is a commutative hyper K-ideal of type 4. We know that I is a strong hyper K-ideal. So, I is a weak hyper K-ideal and a hyper K-ideal too. Let $(xoy)oz \subseteq I$ and $z \in I$, for all $x, y, z \in \overline{Q}$. Since $x \in xoy$, for all $x, y \in \overline{Q}$ and I is a weak hyper K-ideal, we obtain that $(xoy) \subseteq I$. Hence, $x \in I$. Now, we consider two cases:

(1) $x \neq y$.
Since $yox = y$, we get that $yo(yox) = yoy = \{\overline{q_0}, y\}$. So, $xo(yo(yox)) = xo\{\overline{q_0}, y\} = x$. Thus, $(xo(yo(yox))) \subseteq I$
(2) $x = y$.

Since $yox = yoy = \{\overline{q_0}, y\}$, we obtain that $yo(yox) = yo\{\overline{q_0}, y\} = \{\overline{q_0}, y\}$. So, $xo(yo(yox)) = xo\{\overline{q_0}, y\} = \{\overline{q_0}, y\} = \{\overline{q_0}, x\}$. Therefore, $xo(yo(yox)) \subseteq I$ and I is a commutative hyper K-ideal of type 4.

Now, we show that I is a commutative hyper K-ideal of type 8. Let $(xoy)oz < I$ and $z \in I$, for all $x, y, z \in \overline{Q}$. Then, we have two cases:

(1) $x = y$.
Since $yox = yoy = \{\overline{q_0}, y\}$, we get that $yo(yox) = yo\{\overline{q_0}, y\} = \{\overline{q_0}, y\}$. Hence, $xo(yo(yox)) = xo\{\overline{q_0}, y\} = \{\overline{q_0}, y\} = \{\overline{q_0}, y\}$. Also, since $\overline{q_0} \in \{\overline{q_0}, y\}$ and $\overline{q_0} \in I$,

we conclude that $xo(yo(yox)) \cap I \neq \emptyset$.
(2) $x \neq y$.

Since $(xoy)oz < I$, $z \in I$ and I is a hyper K-ideal, we obtain that $x \in I$. Also, $yo(yox) = yoy = \{\overline{q_0}, y\}$. So, $xo(yo(yox)) = xo\{\overline{q_0}, y\} = x$ and $xo(yo(yox)) \cap I \neq \emptyset$. Thus, I is a commutative hyper K-ideal of type 8.

By Theorem 2.13, we can prove that I is a commutative hyper K-ideal of types 3, 4, 5, 6, 8 and 9.

Now, it is enough to prove that I is a commutative hyper K-ideal of type 2. Let $((xoy)oz) \cap I \neq \emptyset$ and $z \in I$, for all $x, y, z \in \overline{Q}$. Then, we have two cases:

(1) $x = y$.
Since $\overline{q_0} \in xo(yo(yox)) = \{\overline{q_0}, x\}$, we conclude that $xo(yo(yox)) \cap I \neq \emptyset$.
(2) $x \neq y$.

Since $(xoy)oz \cap I = (xoz) \cap I \neq \emptyset$, $z \in I$ and I is a strong hyper K-ideal, we obtain that $x \in I$. Also, we know that $x \in xo(yo(yox))$. So, $xo(yo(yox)) \cap I \neq \emptyset$ and I is a commutative hyper K-ideal of type 2. □

Remark 3.11. Note that Theorem 3.10 does not hold for types 1 and 7.
To this end, first consider type 1. Let $I = \{\overline{q_0}, \overline{q_1}\}$, $x, y, z \neq \overline{q_0}$, $x = y = \overline{q_2}$ and $x \neq z = \overline{q_1}$.

Then, $(xoy)oz = \{\overline{q_0}, \overline{q_2}\}oz = \{\overline{q_0}, \overline{q_2}\}$. So, $(xoy)oz \cap I \neq \emptyset$ and $z \in I$.
But, $xo(yo(yox)) = \{\overline{q_0}, \overline{q_2}\} \not\subset I$. Hence, I is not a commutative hyper K-ideal of type 1.
Then, we show type 7. Since $\overline{q_0} < \overline{q_0}$, we get that $(xoy)oz < I$.
But, $xo(yo(yox)) = \{\overline{q_0}, \overline{q_1}\} \not\subset I$. So, I is not a commutative hyper K-ideal of type 7.

4. Dual Positive Implicative and Commutative Hyper K-ideal

By Theorem 2.8, let H be a hyper K-algebra as in Theorem 3.3; in other words, $H = (\overline{Q}, o, \overline{q_0})$.

Then, by Definition 2.16, H satisfies the zero condition and hence by Theorem 2.18 can be extended to a bounded hyper K-algebra with unit as follows:
Let $e \notin H$ and $H_1 = H \cup \{e\}$. Define the hyper operation "o_1" on H_1 as follows:

$$\overline{q_1} o_1 \overline{q_2} = \begin{cases} \{e\}, & if \ \ \overline{q_1} = e, \overline{q_2} \in H \\ \{\overline{q_0}\}, & if \ \ \overline{q_1} = \overline{q_2} = e \\ \{\overline{q_0}, \overline{q_1}\}, & if \ \ \overline{q_1} \in H, \overline{q_2} = e \\ \overline{q_1} o \overline{q_2}, & if \ \ \overline{q_1}, \overline{q_2} \in H. \end{cases}$$

In the remainder of this section, let D be a non-empty subset of a bounded hyper K-algebra H_1 containing e.
Example 4.1. Consider the max-min general fuzzy automata $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ in Example 3.5. According to the definition of hyper operation "o_1", we have Table 3.

o_1	e	$\overline{q_0}$	$\overline{q_1}$	$\overline{q_2}$	$\overline{q_3}$
e	$\{\overline{q_0}\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$
$\overline{q_0}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$	$\{\overline{q_0}\}$
$\overline{q_1}$	$\{\overline{q_0},\overline{q_1}\}$	$\{\overline{q_1}\}$	$\{\overline{q_0},\overline{q_1}\}$	$\{\overline{q_1}\}$	$\{\overline{q_1}\}$
$\overline{q_2}$	$\{\overline{q_0},\overline{q_2}\}$	$\{\overline{q_2}\}$	$\{\overline{q_2}\}$	$\{\overline{q_0},\overline{q_2}\}$	$\{\overline{q_2}\}$
$\overline{q_3}$	$\{\overline{q_0},\overline{q_3}\}$	$\{\overline{q_3}\}$	$\{\overline{q_3}\}$	$\{\overline{q_3}\}$	$\{\overline{q_0},\overline{q_3}\}$

TABLE 3. The Hyper Operation ”0 Index 1”on the Equivalence Classes of States in Example 3.5

Theorem 4.2. Let D be a subset of H_1. Then, D is a DPIHKI-T3 if and only if $\overline{q_0} \in D$.

Proof. Let $\overline{q_0} \in D$. By definition of H_1, we have:

$$eo_1((eo_1x)o_1(eo_1y)) = eo_1(eo_1e) = eo_1\overline{q_0} = \{e\} \subseteq D, \forall x, y \in H.$$

Also, $eo_1((eo_1x)o_1(eo_1e)) = eo_1(eo_1\overline{q_0}) = eo_1e = \{\overline{q_0}\}$, $\forall x \in H$.

Furthermore,

$$eo_1((eo_1e)o_1(eo_1x)) = eo_1(\overline{q_0}o_1e) \subseteq \{\overline{q_0}, e\}, \forall x \in H$$

and

$$eo_1((eo_1e)o_1(eo_1e)) = eo_1(\overline{q_0}o_1\overline{q_0}) = eo_1\overline{q_0} = \{e\}.$$

Since e and $\overline{q_0} \in D$, then $eo_1((eo_1x)o_1(eo_1y)) \subseteq D$, $\forall x, y \in H_1$. Thus, D is a DPIHKI-T3.

Conversely, Let D be a DPIHKI-T3. We prove that $\overline{q_0} \in D$. On the contrary, let $\overline{q_0} \notin D$. Then, $eo_1((eo_1e)o_1(eo_1e)) = eo_1\overline{q_0} = \{e\} \subseteq D$. Also, $((eo_1\overline{q_0})o_1(eo_1e)) = \{\overline{q_0}\}$. Since $\overline{q_0} \notin D$, thus D is not a DPIHKI-T3, which is a contradiction. Therefore, $\overline{q_0} \in D$. □

Remark 4.3. If $D = \{e\}$, then D is not a DPIHKI-T4.

By (HK2) and definition of H_1, we have, $\overline{q_0}o_1\overline{q_0} = (eo_1e)o_1\overline{q_0} = (eo_1\overline{q_0})o_1e = eo_1e = \overline{q_0}$. So, we get that

$$e = eo_1\overline{q_0} = eo_1(\overline{q_0}o_1\overline{q_0}) = eo_1((eo_1e)o_1(eo_1e)) = eo_1(((eo_1\overline{q_0})o_1(eo_1\overline{q_0}))o_1(eo_1e)) \subseteq D.$$

Also, $\overline{q_0} = eo_1e = eo_1(eo_1\overline{q_0}) = eo_1((eo_1\overline{q_0})o_1(eo_1e))$. Now, since $\overline{q_0} \notin D$, we have, $N(N\overline{q_0}o_1Ne) \subseteq D$. Thus, D is not a DPIHKI-T4.

Theorem 4.4. Let D be a subset of $H1$. Then, D is a DPIHKI-T4 if and only if $\overline{q_0} \in D$.

Proof. Let $\overline{q_0} \in D$. Then, by Theorems 4.2 and 2.23 we conclude that D is a DPIHKI-T4.

The converse is proved similarly using proof of Theorem 4.2. □

Remark 4.5. If $D = \{e\}$, then D is not a DPIHK-T2.
By definition of H_1, we get that $eo_1\overline{q_0} = \{e\}$ and $\overline{q_0}o_1\overline{q_0} = \{\overline{q_0}\}$. Also, we have,

$$eo_1((eo_1e)o_1(eo_1e)) = eo_1(\overline{q_0}o_1\overline{q_0}) = \{e\} \subseteq D,$$

and

$$eo_1((eo_1\overline{q_0})o_1(eo_1e)) = eo_1(eo_1\overline{q_0}) = eo_1e = \{\overline{q_0}\} \subseteq D.$$

Since $\overline{q_0} \notin D$, thus $N(N\overline{q_0}o_1Ne) \not\subset D$. Therefore, D is not a DPIHKI-T2.

Theorem 4.6. Let D be a subset of H_1. Then, D is a DPIHKI-T2 if and only if $\overline{q_0} \in D$.

Proof. Let $\overline{q_0} \in D$. Then, by Theorems 4.2 and 2.26 we conclude that D is a DPIHKI-T2.
The converse is proved similarly using proof of Theorem 3.2. □

Theorem 4.7. Let D be a subset of H_1. Then, D is a DPIHKI-T1.

Proof. We consider two cases : (i) $\overline{q_0} \in D$ and (ii) $\overline{q_0} \notin D$.

(i) If $\overline{q_0} \in D$, then by Theorems 4.2 and 2.28 we conclude that D is a DPIHKI-T1.
(ii) Let $\overline{q_0} \notin D$ and on the contrary let D not be a DPIHKI-T1.

Then, there are $x, y, z \in H$ such that

$$eo_1((eo_1x)o_1(eo_1y))o_1(eo_1z) \subseteq D, \tag{1}$$

and

$$eo_1((eo_1y)o_1(eo_1z)) \subseteq D, \tag{2}$$

while

$$eo_1((eo_1x)o_1(eo_1z)) \not\subset D. \tag{3}$$

If x and $z \in H$ or $x = z = e$, then by some manipulations we conclude that (3) dose not hold, and this is a contradiction. If $x \in H$ and $z = e$, then for $y = e$, the inclusion (1) dose not hold, and for $y \in H$, (2) dose not hold, and so this case is impossible.
If $x = e$ and $z \in H$. Then, we consider two cases:

(a) $e \in \overline{q_0}o_1e$, and (b) $e \notin \overline{q_0}o_1e$.
(a) If $e \in \overline{q_0}o_1e$, then (1) dose not hold, which is a contradiction.
(b) If $e \notin \overline{q_0}o_1e$, then (2) dose not hold, which is not true.

Therefore, also in this case D is a DPIHKI-T1. □

Theorem 4.8. Let $e \in D \subseteq H_1$ and $D \neq \{e\}$. Then, the following statements are equivalent:

(i) $\overline{q_0} \in D$,
(ii) D is a DCHKI-T1,
(iii) D is a DPIHKI-T2.

Proof. (i)$\longrightarrow$(ii) Let $\overline{q_0} \in D$. Then, for any $x, y \in H_1$, by definition of H_1 we can obtain that $N(Nxo_1(Nyo_1(Nyo_1Nx))) \subseteq \{\overline{q_0}, e\} \subseteq D$. Therefore, D is a DCHKI-T1.
(ii)$\longrightarrow$(i) Let D be a DCHKI-T1. We prove that $\overline{q_0} \in D$. On the contrary, let $\overline{q_0} \notin D$.

Consider $x \in D - \{e\}$. So by definition of H_1 we get that $N((Nxo_1Ne)o_1Nx) = \{e\} \subseteq D$ and also $\overline{q_0} \in N(Nxo_1(Neo_1(Neo_1Nx)))$. Hence,

$$N((Nxo_1(Neo_1(Neo_1Nx))) \not\subseteq D.$$

Therefore, D is not a DCHKI-T1, which is a contradiction.
(i)$\longrightarrow$(iii) This follows from Theorem 4.6. $\square$

Theorem 4.9. Let $D = \{e\}$. Then, D is a DCHKI-T1.

Proof. The proof is similar to the proof of Theorem 4.8. $\square$

Remark 4.10. Let $\overline{q_0} \notin D$. Then, D is not a DCHKI-T2 (DPIHKI-T3).

Since $\overline{q_0} \in (\overline{q_0}o_1e) \cap (eo_1e)$, then we get that $\overline{q_0} \in N(N\overline{q_0}o_1(Neo_1(Neo_1N\overline{q_0})))$, and so $N((N\overline{q_0}o_1(Neo_1(Neo_1N\overline{q_0}))) \not\subseteq D$. Therefore, by Theorem 2.30, D is not a DCHKI-T2. Similarly, we can show that D is not a DPIHKI-T3.

Theorem 4.11. Let $\{\overline{q_0}, e\} \subseteq D \subseteq H_1$. Then, D is a DCHKI-T2 and DPIHKI-T3.

Proof. By definition of H_1, we obtain that $N((Nxo_1(Nyo_1(Nyo_1Nx))) \subseteq \{\overline{q_0}, e\} \subseteq D$ and $N(Nxo_1Ny) \subseteq \{\overline{q_0}, e\} \subseteq D$, for all $x, y \in H_1$. Therefore, D is a DCHKI-T2 and DPIHKI-T3. $\square$

5. Conclusion

The connections of max-min general fuzzy automaton theory with another field, known as hyper structure theory were presented. We pointed out that the concept of automaton leads to important results, both in mathematics and in theoretical computer science. Here, two BCK-algebras were found induced by a max-min general fuzzy automaton. Two important questions arose:
i) Is it possible to find a max-min general fuzzy automaton which is induced by a BCK-algebra?
ii) Is there any relationship between the concept of connectedness (semi-connectedness) between two states p and q with respect to a fuzzy subset of the set of states of a max-min general fuzzy automaton and the concepts of (positive implicative) commutative hyper K-ideals of the induced hyper BCK-algebra?

References

[1] M. A. Arbib, *From automata theory to brain theory*, Int. J. Man-Machine Stud., **7(3)** (1975), 279-295.
[2] W. R. Ashby, *Design for a brain, Chapman and Hall*, London, 1954.
[3] R. A. Borzooei, *Hyper BCK and K-algebras*, Ph.D. Thesis, Department of Mathematics, Shahid Bahonar University of Kerman, Iran, 2000.
[4] R. A. Borzooei, A. Hasankhani, M. M. Zahedi and Y. B. Jun, *On hyper K-algebras*, Scientiae Mathematicae Japonica, **52(1)** (2000), 113-121.
[5] A. W. Burks, *Logic, biology and automata-some historical reflections*, Int. J. Man-Machine Stud., **7(3)** (1975), 297-312.
[6] P. Corsini, *Prolegomena of hyper group theory*, Avian Editor, Italy, 1993.
[7] P. Corsini and V. Leoreanu, *Applications of hyper structure theory*, Advances in Mathematics, Kluwer Academic Publishers, **5** (2003).
[8] M. Doostfatemeh and S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175-214.
[9] J. E. Hopcroft, R. Motwani and J. D. Ullman, *Introduction to automata theory, languages and computation*, seconded, Addison-Wesley, Reading, MA, 2001.
[10] M. Horry and M. M. Zahedi, *Uniform and semi-uniform topology on general fuzzy automata*, Iranian Journal of Fuzzy Systems, **6(2)** (2009), 19-29.
[11] M. Horry and M. M. Zahedi, *Hypergroups and general fuzzy automata*, Iranian Journal of Fuzzy Systems, **6(2)** (2009), 61-74.
[12] Y. Imai and K. Iseki, *On axiom systems of propositional calculi*, XIV Proc. Japan Academy, **42** (1966), 19-22.
[13] D. S. Malik and J. N. Mordeson, *Fuzzy discrete structures*, Physica-Verlag, New York, 2000.
[14] F. Marty, *Sur une generalization de la notion de groups*, 8th Congress Math. Scand naves, Stockholm, (1934), 45-49.
[15] W. S. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bull. Math. Biophysics., **5** (1943), 115-133.
[16] J. Meng and Y. B. Jun, *BCK-algebra*, Kyung Moons, Co., Seoul, 1994.
[17] M. L. Minsky, *Computation: finite and infinite machines*, Prentice-Hall, Englewood Cliffs, NJ, Chapter 3, (1967), 32-66.
[18] J. N. Mordeson and D. S. Malik, *Fuzzy automata and languages, theory and applications*, Chapman and Hall/CRC, London/Boca Raton, FL, 2002.
[19] M. A. Nasr Azadani and M. M. Zahedi, *S-absorbing set and (P)-decomposition in hyper K-algebras*, Italian Journal of Pure and Applied Mathematics, to appear.
[20] W. Omlin, K. K. Giles and K. K. Thornber, *Equivalence in knowledge representation: automata, runs, and dynamic fuzzy systems*, Proc. IEEE, **87(9)** (1999), 1623-1640.
[21] W. Omlin, K. K. Thornber and K. K. Giles, *Fuzzy finite-state automata can be deterministically encoded into recurrent neural networks*, IEEE Trans. Fuzzy Systems, **5(1)** (1998), 76-89.
[22] T. Roodbari, *Positive implicative and commutative hyper K-ideals*, Ph.D. Thesis, Department of Mathematics, Shahid Bahonar University of Kerman, Iran, 2008.
[23] T. Roodbari, L. Torkzadeh and M. M. Zahedi, *Normal hyper K-algebras*, Sciatica Mathematical Japonica, **68(2)** (2008), 265-278.
[24] T. Roodbari and M. M. Zahedi, *Positive implicative hyper K-ideals II*, Scientiae Mathematicae Japonica, **66(3)** (2007), 391-404.
[25] L. Torkzadeh, *Dual positive implicative and commutative hyper K-ideals*, Ph.D. Thesis, Department of Mathematics, Shahid Bahonar University of Kerman, Iran, 2005.
[26] L. Torkzadeh, T. Roodbari and M. M. Zahedi, *Hyper stabilizers and normal hyper BCK-algebras*, Set-Valued Mathematics and Applications, to appear.
[27] A. Turing, *On computable numbers, with an application to the entscheidungs problem*, Proc. London Math. Soc., **42** (1936-37), 220-265.

[28] J. Von Neumann, *Theory of self-reproducing automata*, University of Illinois Press, Urbana, 1966.

[29] W. G. Wee, *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification*, Ph.D. Thesis, Purdue University, Lafayette, IN, 1967.

[30] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

[31] M. M. Zahedi, R. A. Borzooei and H. Rezaei, *Some classifications of hyper K-algebra of order 3*, Math. Japon., (2001), 133-142.

[32] M. M. Zahedi, M. Horry and K. Abolpor, *Bifuzzy (general) topology on max-min general fuzzy automata* , Advances in Fuzzy Mathematics, **3(1)** (2008), 51-68.

khadijeh Abolpour*, Department of Mathematics, Islamic Azad University, Kerman Branch, Kerman, Iran

E-mail address: `abolpor_kh@yahoo.com`

Mohammad Mehdi Zahedi, Department of Mathematics, Tarbiat Modares University, Tehran, Iran

E-mail address: `zahedi_mm@modares.ac.ir`

Masoome Golmohamadian, Department of Mathematics, Tarbiat Modares University, Tehran, Iran

*Corresponding author