BIPOLAR FUZZY HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

Y. B. JUN, M. S. KANG AND H. S. KIM

ABSTRACT. Using the notion of bipolar-valued fuzzy sets, the concepts of bipolar fuzzy (weak, s-weak, strong) hyper BCK-ideals are introduced, and their relations are discussed. Moreover, several related properties are investigated.

1. Introduction

The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [11] at the 8th congress of Scandinavian Mathematiciens. In [8], Jun et al. applied the hyper structures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. They also introduced the notion of a (weak, s-weak, strong) hyper BCK-ideal, and gave relations among them. Harizavi [3] studied prime weak hyper BCK-ideals of lower hyper BCK-semilattices. Jun et al. discussed the notion of hyperatoms and scalar elements of hyper BCK-algebras (see [5]). Fuzzy set theory is established in the paper [13]. Jun et al. also discussed the fuzzy structures of (implicative) hyper BCK-ideals in hyper BCK-algebras (see [4, 6]). Davvaz [1] introduced the notions of fuzzy hyperideal and fuzzy bi-hyperideal in ternary semihyperrings, and considered their characterizations. Torkzadeh et al. [12] considered the notion of intuitionistic fuzzy (weak) dual hyper K-ideals and obtained related results.

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [2, 14]) In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The

Received: January 2009; Revised: May 2010; Accepted: July 2010

Key words and phrases: (Strong, weak, s-weak) hyper BCK-ideal, Bipolar fuzzy hyper BCK-ideal, Bipolar fuzzy strong hyper BCK-ideal, Bipolar fuzzy weak hyper BCK-ideal, Bipolar fuzzy s-weak hyper BCK-ideal, Inf-sup property.

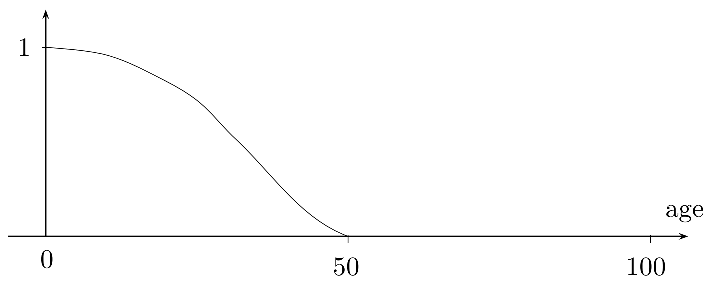

FIGURE 1. A Fuzzy Set "*Young*"

traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see FIGURE 1). Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that the age 95 is more apart from the property rather than the age 50 (see [9]).

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [9] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In this paper, using the notion of bipolar-valued fuzzy set, we establish the bipolar fuzzification of the notion of (strong, weak, s-weak) hyper BCK-ideals in hyper BCK-algebras, and investigate some of their properties. We give characterizations of bipolar fuzzy hyper BCK-ideals. We show that every bipolar fuzzy s-weak hyper BCK-ideal is a bipolar fuzzy weak hyper BCK-ideal, but the converse is not valid in general by example. We provide conditions for a bipolar fuzzy weak hyper BCK-ideal to be a bipolar fuzzy s-weak hyper BCK-ideal.

2. Preliminaries

2.1. **Basic Results on Bipolar Valued Fuzzy Sets.** Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat

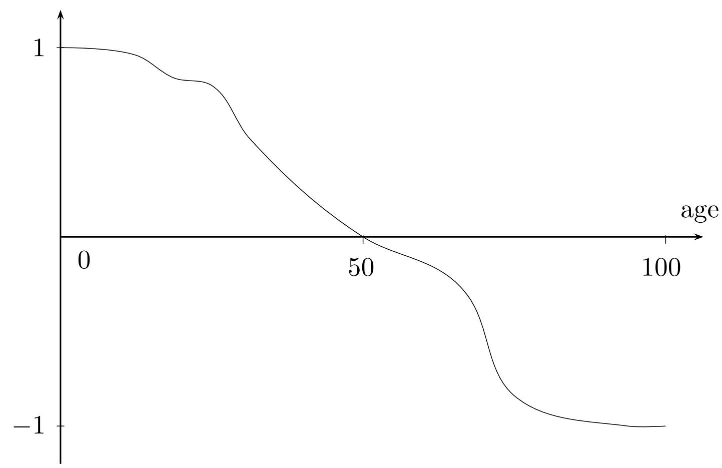

FIGURE 2. A Bipolar Fuzzy Set "*Young*"

satisfy the implicit counter-property (see [9]). FIGURE 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of FIGURE 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of FIGURE 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of FIGURE 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [9]).

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form

$$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$$

where $\mu_\Phi^P : X \to [0, 1]$ and $\mu_\Phi^N : X \to [-1, 0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set Φ, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of Φ. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for Φ. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x

does not satisfy the property of Φ but somewhat satisfies the counter-property of Φ. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [10]). For the sake of simplicity, we shall use the symbol $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

2.2. **Basic Results on Hyper BCK-algebras.** We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [6], [7], and [8].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)(x \circ y \ll x). \tag{1}$$

In any hyper BCK-algebra H, the following hold:

$$(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}. \tag{2}$$

$$0 \circ 0 = \{0\}. \tag{3}$$

$$0 \ll x. \tag{4}$$

$$A \ll A. \tag{5}$$

$$A \subseteq B \ \Rightarrow\ A \ll B. \tag{6}$$

$$0 \circ A = \{0\}. \tag{7}$$

$$A \ll \{0\} \ \Rightarrow\ A = \{0\}. \tag{8}$$

$$x \in x \circ 0. \tag{9}$$

$$x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y. \tag{10}$$

$$y \ll z \ \Rightarrow\ x \circ z \ll x \circ y. \tag{11}$$

$$x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z. \tag{12}$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

A nonempty subset I of a hyper BCK-algebra H is said to be a *hyper BCK-ideal* of H if it satisfies

(I1) $0 \in I$,

(I2) $(\forall x \in H)\ (\forall y \in I)\ (x \circ y \ll I \ \Rightarrow\ x \in I)$.

A nonempty subset I of a hyper BCK-algebra H is called a *strong hyper BCK-ideal* of H if it satisfies (I1) and

(I3) $(\forall x \in H)\ (\forall y \in I)\ ((x \circ y) \cap I \neq \emptyset \ \Rightarrow\ x \in I)$.

Note that every strong hyper BCK-ideal of a hyper BCK-algebra is a hyper BCK-ideal.

A nonempty subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I4) $(\forall x \in H)\ (\forall y \in I)\ (x \circ y \subseteq I \Rightarrow x \in I)$.

We now review some fuzzy logic concepts. A *fuzzy set* in a set H is a function $\mu : H \to [0,1]$. For any $t \in [0,1]$ and a fuzzy set μ in a nonempty set H, the set

$$U(\mu; t) = \{x \in H \mid \mu(x) \geq t\} \ \text{ (resp. } L(\mu; t) = \{x \in H \mid \mu(x) \leq t\})$$

is called an *upper* (resp. *lower*) *level set* of μ.

A fuzzy set μ in a hyper BCK-algebra H is called a *fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \ \Rightarrow\ \mu(y) \leq \mu(x)), \tag{13}$$

$$(\forall x, y \in H)(\mu(x) \geq \min\Big\{\inf_{a \in x \circ y} \mu(a), \mu(y)\Big\}). \tag{14}$$

3. Bipolar Fuzzy Hyper BCK-ideals

In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \ \Rightarrow\ \mu_\Phi^P(x) \geq \mu_\Phi^P(y),\ \mu_\Phi^N(x) \leq \mu_\Phi^N(y)), \tag{15}$$

and

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\Big\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\ \mu_\Phi^P(y)\Big\}, \\ \mu_\Phi^N(x) &\leq \max\Big\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\ \mu_\Phi^N(y)\Big\} \end{aligned} \tag{16}$$

for all $x, y \in H$.

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.7	0.4	0.2
μ_Φ^N	-0.8	-0.6	-0.2

It is easily verified that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H.

Definition 3.3. A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy strong hyper BCK-ideal* of H if it satisfies

$$\begin{aligned} &\inf_{a\in x\circ x} \mu_\Phi^P(a) \geq \mu_\Phi^P(x) \geq \min\{\sup_{b\in x\circ y} \mu_\Phi^P(b),\ \mu_\Phi^P(y)\}, \\ &\sup_{c\in x\circ x} \mu_\Phi^N(c) \leq \mu_\Phi^N(x) \leq \max\{\inf_{d\in x\circ y} \mu_\Phi^N(d),\ \mu_\Phi^N(y)\} \end{aligned} \tag{17}$$

for all $x, y \in H$.

Example 3.4. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.9	0.6	0.3
μ_Φ^N	-0.8	-0.5	-0.2

It is routine to check that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H.

Definition 3.5. A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy s-weak hyper BCK-ideal* of H if it satisfies

$$(\forall x \in H)(\mu_\Phi^P(0) \geq \mu_\Phi^P(x),\ \mu_\Phi^N(0) \leq \mu_\Phi^N(x)) \tag{18}$$

and for every $x, y \in H$ there exist $a, b \in x \circ y$ such that

$$\begin{aligned} &\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(a), \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(b), \mu_\Phi^N(y)\}. \end{aligned} \tag{19}$$

Definition 3.6. A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy weak hyper BCK-ideal* of H if it satisfies

$$\begin{aligned} &\mu_\Phi^P(0) \geq \mu_\Phi^P(x) \geq \min\Big\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\, \mu_\Phi^P(y)\Big\}, \\ &\mu_\Phi^N(0) \leq \mu_\Phi^N(x) \leq \max\Big\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\, \mu_\Phi^N(y)\Big\} \end{aligned} \tag{20}$$

for all $x, y \in H$.

Theorem 3.7. *Every bipolar fuzzy s-weak hyper BCK-ideal is a bipolar fuzzy weak hyper BCK-ideal.*

Proof. Let a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H be a bipolar fuzzy s-weak hyper BCK-ideal of H and let $x, y \in H$. Then there exist $a, b \in x \circ y$ such that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(a), \mu_\Phi^P(y)\} \text{ and } \mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(b), \mu_\Phi^N(y)\}.$$

Since $\mu_\Phi^P(a) \geq \inf_{c \in x \circ y} \mu_\Phi^P(c)$ and $\mu_\Phi^N(b) \leq \sup_{d \in x \circ y} \mu_\Phi^N(d)$, it follows that

$$\mu_\Phi^P(x) \geq \min\{\inf_{c \in x \circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x) \leq \max\{\sup_{d \in x \circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\}.$$

Hence $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H. □

The converse of Theorem 3.7 is not true as it can be seen in the following example.

Example 3.8. Let $H = \mathbb{N} \cup \{0, \alpha\}$, where $\alpha(\neq 0) \notin \mathbb{N}$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0\} & \text{if } x = 0, \\ \{0, x\} & \text{if } (x \leq y,\ x \in \mathbb{N}) \text{ or } (x \in \mathbb{N},\ y = \alpha), \\ \{x\} & \text{if } x > y,\ x \in \mathbb{N}, \\ \{0\} \cup \mathbb{N} & \text{if } x = y = \alpha, \\ \mathbb{N} & \text{if } x = \alpha,\ y \in \mathbb{N}, \\ \{\alpha\} & \text{if } x = \alpha,\ y = 0. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	1	2	3	$\cdots$	α
μ_Φ^P	$4-3$	$4-3.1$	$4-3.14$	$4-3.141$	$\cdots$	$4-\pi$
μ_Φ^N	$-4+3$	$-4+3.1$	$-4+3.14$	$-4+3.141$	$\cdots$	$-4+\pi$

Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H, but it is not a bipolar fuzzy s-weak hyper BCK-ideal of H because if $x = \alpha$ and $y = 1$, then $x \circ y = \mathbb{N}$. For every $a \in \mathbb{N} = x \circ y$, we have

$$\mu_\Phi^P(\alpha) = 4 - \pi < \min\{\mu_\Phi^P(a), \mu_\Phi^P(1)\}$$

and/or

$$\mu_\Phi^N(\alpha) = -4 + \pi > \max\{\mu_\Phi^N(a), \mu_\Phi^N(1)\}.$$

Hence Φ is not a bipolar fuzzy s-week hyper BCK-algebra.

A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is said to satisfy the **inf-sup** *property* if for any nonempty subset T of H there exist $x_0, y_0 \in T$ such that $\mu_\Phi^P(x_0) = \inf_{x\in T} \mu_\Phi^P(x)$ and $\mu_\Phi^N(y_0) = \sup_{y\in T} \mu_\Phi^N(y)$.

Note that, in a finite hyper BCK-algebra, every bipolar fuzzy set satisfies the **inf-sup** property. The following example shows that there exists a bipolar fuzzy set which does not satisfy the **inf-sup** *property*.

Example 3.9. Let $H = \mathbb{N} \cup \{0\} \cup \{\alpha, \beta\}$, where $\alpha(\neq 0) \notin \mathbb{N}$ and $\beta(\neq 0) \notin \mathbb{N}$ with $\alpha \neq \beta$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } (x \leq y,\ x, y \in \mathbb{N} \cup \{0\}) \text{ or } (x \in \mathbb{N} \cup \{0\}, y \in \{\alpha, \beta\}) \\ \{x\} & \text{if } x > y,\ x, y \in \mathbb{N} \cup \{0\}, \\ \{\alpha\} & \text{if } x = \alpha,\ y \neq \alpha, \\ \{\beta\} & \text{if } x = \beta,\ y \neq \beta, \\ \{0\} & \text{if } x = y = \alpha \text{ or } x = y = \beta. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra (see [3]). Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	1	2	3	4	$\cdots$	α	β
μ_Φ^P	$2-1$	$2-1.4$	$2-1.41$	$2-1.414$	$2-1.4142$	$\cdots$	0	0
μ_Φ^N	$-2+1$	$-2+1.4$	$-2+1.41$	$-2+1.414$	$-2+1.4142$	$\cdots$	0	0

Let $T = \mathbb{N} \cup \{0\} \subseteq H$. Then $\inf_{x\in T} \mu_\Phi^P(x) = 2 - \sqrt{2}$ and $\sup_{y\in T} \mu_\Phi^N(y) = -2 + \sqrt{2}$. But there does not exist $x_0, y_0 \in T$ such that $\mu_\Phi^P(x_0) = 2 - \sqrt{2}$ and $\mu_\Phi^N(y_0) = -2 + \sqrt{2}$. Hence $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ does not satisfy the **inf-sup** *property*.

The following example shows that there exists a bipolar fuzzy set which satisfies the **inf-sup** *property*.

Example 3.10. Let $H = \mathbb{N} \cup \{0\}$ and define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{if } x > y. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algerba.

(1) Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H where

$$\mu_\Phi^P(n) := \begin{cases} 0 & \text{if } n \in \{0, 2, 4, \cdots\}, \\ \alpha & \text{if } n \in \{1, 3, 5, \cdots\}, \end{cases}$$

$$\mu_\Phi^N(n) := \begin{cases} 0 & \text{if } n \in \{0, 2, 4, \cdots\}, \\ \beta & \text{if } n \in \{1, 3, 5, \cdots\}. \end{cases}$$

where $\alpha \in (0, 1]$ and $\beta \in [-1, 0)$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the **inf-sup** *property.*

(2) Let $\Psi = (H; \mu_\Psi^P, \mu_\Psi^N)$ be a bipolar fuzzy set in H defined by

	0	1	2	3	4	$\cdots$
μ_Ψ^P	$2-1$	$2-1.7$	$2-1.73$	$2-1.732$	$2-1.7320$	$\cdots$
μ_Ψ^N	$-2+1$	$-2+1.7$	$-2+1.73$	$-2+1.732$	$-2+1.7320$	$\cdots$

Then $\Psi = (H; \mu_\Psi^P, \mu_\Psi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. Let $T = \mathbb{N} \subseteq H$. Then $\inf_{x\in T} \mu_\Psi^P(x) = 2 - \sqrt{3}$ and $\sup_{y\in T} \mu_\Psi^N(y) = -2 + \sqrt{3}$. But there does not exist $x_0, y_0 \in T$ such that $\mu_\Psi^P(x_0) = 2 - \sqrt{3}$ and $\mu_\Psi^N(y_0) = -2 + \sqrt{3}$. Hence $\Psi = (H; \mu_\Psi^P, \mu_\Psi^N)$ does not satisfy the **inf-sup** *property.*

A bipolar fuzzy weak hyper BCK-ideal may not be a bipolar fuzzy s-weak hyper BCK-ideal. But we have the following proposition.

Proposition 3.11. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H satisfying the* **inf-sup** *property, then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy s-weak hyper BCK-ideal of H.*

Proof. Since $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the **inf-sup** property, there exist $a_0, b_0 \in x \circ y$ such that $\mu_\Phi^P(a_0) = \inf_{a\in x\circ y} \mu_\Phi^P(a)$ and $\mu_\Phi^N(b_0) = \sup_{b\in x\circ y} \mu_\Phi^N(b)$. It follows that

$$\mu_\Phi^P(x) \geq \min\{\inf_{a\in x\circ y} \mu_\Phi^P(a),\ \mu_\Phi^P(y)\} = \min\{\mu_\Phi^P(a_0), \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x) \leq \max\{\sup_{b\in x\circ y} \mu_\Phi^N(b),\ \mu_\Phi^N(y)\} = \max\{\mu_\Phi^N(b_0), \mu_\Phi^N(y)\}.$$

This completes the proof. □

Since in a finite hyper BCK-algebra, every fuzzy set satisfies the **inf-sup** property, the concept of bipolar fuzzy weak hyper BCK-ideals and bipolar fuzzy s-weak hyper BCK-ideals coincide in a finite hyper BCK-algebra.

Proposition 3.12. *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy strong hyper BCK-ideal of H and let $x, y \in H$. Then*

(1) *$\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$.*
(2) *$x \ll y$ implies $\mu_\Phi^P(x) \geq \mu_\Phi^P(y)$ and $\mu_\Phi^N(x) \leq \mu_\Phi^N(y)$.*
(3) *$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(a), \mu_\Phi^P(y)\}$ and $\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(b), \mu_\Phi^N(y)\}$ for all $a, b \in x \circ y$.*

Proof. (1) Since $0 \in x \circ x$ for all $x \in H$, we have $\mu_\Phi^P(0) \geq \inf_{a \in x \circ x} \mu_\Phi^P(a) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \sup_{b \in x \circ x} \mu_\Phi^N(b) \leq \mu_\Phi^N(x)$, which proves (1).

(2) Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$ and so $\sup_{c \in x \circ y} \mu_\Phi^P(c) \geq \mu_\Phi^P(0)$ and $\inf_{d \in x \circ y} \mu_\Phi^N(d) \leq \mu_\Phi^N(0)$. It follows from (1) that

$$\mu_\Phi^P(x) \geq \min\{\sup_{c \in x \circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(0),\, \mu_\Phi^P(y)\} = \mu_\Phi^P(y)$$

and

$$\mu_\Phi^N(x) \leq \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\} \leq \max\{\mu_\Phi^N(0),\, \mu_\Phi^N(y)\} = \mu_\Phi^N(y).$$

(3) Let $x, y \in H$. Since

$$\mu_\Phi^P(x) \geq \min\{\sup_{c \in x \circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(a),\, \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x) \leq \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\} \leq \max\{\mu_\Phi^N(b),\, \mu_\Phi^N(y)\}$$

for all $a, b \in x \circ y$, we have the desired result. □

The following corollaries are straightforward.

Corollary 3.13. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H, then*

$$\mu_\Phi^P(x) \geq \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\, \mu_\Phi^P(y)\} \quad \text{and} \quad \mu_\Phi^N(x) \leq \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\, \mu_\Phi^N(y)\}$$

for all $x, y \in H$.

Corollary 3.14. *Every bipolar fuzzy strong hyper BCK-ideal is both a bipolar fuzzy s-weak hyper BCK-ideal (and hence a bipolar fuzzy weak hyper BCK-ideal) and a bipolar fuzzy hyper BCK-ideal.*

Proposition 3.15. *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy hyper BCK-ideal of H and let $x, y \in H$. Then*

(1) $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ *and* $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$.

(2) *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the* **inf-sup** *property, then*

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\{\mu_\Phi^P(a),\, \mu_\Phi^P(y)\}, \\ \mu_\Phi^N(x) &\leq \max\{\mu_\Phi^N(b),\, \mu_\Phi^N(y)\} \end{aligned} \tag{21}$$

for some $a, b \in x \circ y$.

Proof. (1) Since $0 \ll x$ for all $x \in H$, it follows from (15) that $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$.

(2) Since $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the **inf-sup** property, there exist $a_0, b_0 \in x \circ y$ such that $\mu_\Phi^P(a_0) = \inf_{a \in x \circ y} \mu_\Phi^P(a)$ and $\mu_\Phi^N(b_0) = \sup_{b \in x \circ y} \mu_\Phi^N(b)$. Hence

$$\mu_\Phi^P(x) \geq \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\, \mu_\Phi^P(y)\} = \min\{\mu_\Phi^P(a_0),\, \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x) \leq \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\, \mu_\Phi^N(y)\} = \max\{\mu_\Phi^N(b_0),\, \mu_\Phi^N(y)\}.$$

This completes the proof. □

Corollary 3.16. (1) *Every bipolar fuzzy hyper BCK-ideal is a bipolar fuzzy weak hyper BCK-ideal.*

(2) *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H satisfying the* **inf-sup** *property, then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy s-weak hyper BCK-ideal of H.*

Proof. Straightforward. □

In Proposition 3.15, if a bipolar fuzzy hyper BCK-ideal $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ does not satisfy the **inf-sup** property, then (21) is not valid. In fact, in Example 3.8, $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. If $T = \mathbb{N}$, then there does not exist $x_0, y_0 \in T$ such that $\mu_\Phi^P(x_0) = \inf_{x \in T} \mu_\Phi^P(x)$ and $\mu_\Phi^N(y_0) = \sup_{y \in T} \mu_\Phi^N(y)$. Hence $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ does not satisfy the **inf-sup** property. If $x = \alpha$, $y = 1$, then $x \circ y = \mathbb{N}$. Hence

$$\mu_\Phi^P(x) = \mu_\Phi^P(\alpha) = 4 - \pi < \min\{\mu_\Phi^P(a), \mu_\Phi^P(1)\},$$
$$\mu_\Phi^N(y) = \mu_\Phi^N(\alpha) = -4 + \pi > \max\{\mu_\Phi^N(a), \mu_\Phi^N(1)\}$$

for every $a \in \mathbb{N} = x \circ y$. Therefore $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ does not satisfy (21).

The following example shows that the converse of Corollary 3.14 and Corollary 3.16(1) may not be true.

Example 3.17. (1) Consider the hyper BCK-algebra H in Example 3.2. Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.7	0.5	0.3
μ_Φ^N	−0.8	−0.7	−0.5

Then we can see that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H and hence it is also a bipolar fuzzy weak hyper BCK-ideal of H. But it is not a bipolar fuzzy strong hyper BCK-ideal of H since

$$\min\{\sup_{w \in b \circ a} \mu_\Phi^P(w),\, \mu_\Phi^P(a)\} = \min\{\mu_\Phi^P(a),\, \mu_\Phi^P(a)\} = 0.5 \nleqq \mu_\Phi^P(b)$$

also

$$\max\{\inf_{w\in b\circ a}\mu_\Phi^N(w),\ \mu_\Phi^N(a)\} = \max\{\mu_\Phi^N(a),\ \mu_\Phi^N(a)\} = -0.7 \not\geq \mu_\Phi^N(b).$$

(2) Consider the hyper BCK-algebra H in Example 3.2. Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.7	0.3	0.5
μ_Φ^N	-0.8	-0.5	-0.7

Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is both a bipolar fuzzy weak hyper BCK-ideal of H and a bipolar fuzzy s-weak hyper BCK-ideal of H. But it is not a bipolar fuzzy hyper BCK-ideal of H since $a \ll b$ but $\mu_\Phi^P(a) \not\geq \mu_\Phi^P(b)$, also $\mu_\Phi^N(a) \not\leq \mu_\Phi^N(b)$.

For a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in a set H, the *positive level set* and *negative level set* are denoted by $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$, and are defined as follows:

$$P(\mu_\Phi^P; \alpha) := \{x \in H \mid \mu_\Phi^P(x) \geq \alpha\},\ \alpha \in [0,1],$$
$$N(\mu_\Phi^N; \beta) := \{x \in H \mid \mu_\Phi^N(x) \leq \beta\},\ \beta \in [-1,0],$$

respectively.

Theorem 3.18. *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H if and only if for every $(\alpha, \beta) \in [0,1] \times [-1,0]$, the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are weak hyper BCK-ideals of H.*

Proof. Assume that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H and $P(\mu_\Phi^P; \alpha) \neq \emptyset \neq N(\mu_\Phi^N; \beta)$ for $(\alpha, \beta) \in [0,1] \times [-1,0]$. It is clear from (20) that $0 \in P(\mu_\Phi^P; \alpha) \cap N(\mu_\Phi^N; \beta)$. Let $x, y \in H$ be such that $x \circ y \subseteq P(\mu_\Phi^P; \alpha)$ and $y \in P(\mu_\Phi^P; \alpha)$. Then for any $a \in x \circ y$, $a \in P(\mu_\Phi^P; \alpha)$. It follows that $\mu_\Phi^P(a) \geq \alpha$ for all $a \in x \circ y$ so that $\inf_{a\in x\circ y}\mu_\Phi^P(a) \geq \alpha$. Thus

$$\mu_\Phi^P(x) \geq \min\{\inf_{a\in x\circ y}\mu_\Phi^P(a),\ \mu_\Phi^P(y)\} \geq \alpha,$$

and so $x \in P(\mu_\Phi^P; \alpha)$. Therefore $P(\mu_\Phi^P; \alpha)$ is a weak hyper BCK-ideal of H. Now let $x, y \in H$ be such that $x \circ y \subseteq N(\mu_\Phi^N; \beta)$ and $y \in N(\mu_\Phi^N; \beta)$. Then $x \circ y \subseteq N(\mu_\Phi^N; \beta)$ implies that for every $b \in x \circ y$, $b \in N(\mu_\Phi^N; \beta)$. It follows that $\mu_\Phi^N(b) \leq \beta$ for all $b \in x \circ y$ so that $\sup_{b\in x\circ y}\mu_\Phi^N(b) \leq \beta$. Using (20) we have

$$\mu_\Phi^N(x) \leq \max\{\sup_{b\in x\circ y}\mu_\Phi^N(b),\ \mu_\Phi^N(y)\} \leq \beta,$$

which implies that $x \in N(\mu_\Phi^N; \beta)$. Consequently, $N(\mu_\Phi^N; \beta)$ is a weak hyper BCK-ideal of H.

Conversely, suppose that the nonempty positive and negative level sets $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are weak hyper BCK-ideals of H for each $(\alpha,\beta)\in[0,1]\times[-1,0]$. Let $\mu_\Phi^P(x)=\alpha$ and $\mu_\Phi^N(x)=\beta$ for $x\in H$. By (I1), $0\in P(\mu_\Phi^P;\alpha)$ and $0\in N(\mu_\Phi^N;\beta)$. Hence $\mu_\Phi^P(0)\geq\alpha$ and $\mu_\Phi^N(0)\leq\beta$, and so $\mu_\Phi^P(0)\geq\mu_\Phi^P(x)$ and $\mu_\Phi^N(0)\leq\mu_\Phi^N(x)$. Now, let

$$\alpha=\min\{\inf_{c\in x\circ y}\mu_\Phi^P(c),\,\mu_\Phi^P(y)\}$$

and

$$\beta=\max\{\sup_{d\in x\circ y}\mu_\Phi^N(d),\,\mu_\Phi^N(y)\}$$

for $x,y\in H$. Then $y\in P(\mu_\Phi^P;\alpha)\cap N(\mu_\Phi^N;\beta)$, and for each $a,b\in x\circ y$ we have

$$\mu_\Phi^P(a)\geq\inf_{c\in x\circ y}\mu_\Phi^P(c)\geq\min\{\inf_{c\in x\circ y}\mu_\Phi^P(c),\,\mu_\Phi^P(y)\}=\alpha$$

and

$$\mu_\Phi^N(b)\leq\sup_{d\in x\circ y}\mu_\Phi^N(d)\leq\max\{\sup_{d\in x\circ y}\mu_\Phi^N(d),\,\mu_\Phi^N(y)\}=\beta.$$

Hence $a\in P(\mu_\Phi^P;\alpha)$ and $b\in N(\mu_\Phi^N;\beta)$, which imply that $x\circ y\subseteq P(\mu_\Phi^P;\alpha)$ and $x\circ y\subseteq N(\mu_\Phi^N;\beta)$. Combining $y\in P(\mu_\Phi^P;\alpha)\cap N(\mu_\Phi^N;\beta)$ and $P(\mu_\Phi^P;\alpha)\cap N(\mu_\Phi^N;\beta)$ being weak hyper BCK-ideal of H, we conclude that

$$x\in P(\mu_\Phi^P;\alpha)\cap N(\mu_\Phi^N;\beta),$$

and so

$$\mu_\Phi^P(x)\geq\alpha=\min\{\inf_{c\in x\circ y}\mu_\Phi^P(c),\,\mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x)\leq\beta=\max\{\sup_{d\in x\circ y}\mu_\Phi^N(d),\,\mu_\Phi^N(y)\}.$$

This completes the proof. □

Lemma 3.19. [5] *Let A be a subset of H. If I is a hyper BCK-ideal of H such that $A\ll I$, then A is contained in I.*

Theorem 3.20. *Let $\Phi=(H;\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy set in H. Then $\Phi=(H;\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H if and only if for every $(\alpha,\beta)\in[0,1]\times[-1,0]$, the nonempty positive level set $P(\mu_\Phi^P;\alpha)$ and the nonempty negative level set $N(\mu_\Phi^N;\beta)$ are hyper BCK-ideals of H.*

Proof. Assume that $\Phi=(H;\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H and $P(\mu_\Phi^P;\alpha)\neq\emptyset\neq N(\mu_\Phi^N;\beta)$, for $(\alpha,\beta)\in[0,1]\times[-1,0]$. It is clear that $0\in P(\mu_\Phi^P;\alpha)\cap N(\mu_\Phi^N;\beta)$, by Proposition 3.15(1). Let $x,y\in H$ be such that $x\circ y\ll P(\mu_\Phi^P;\alpha)$ and $y\in P(\mu_\Phi^P;\alpha)$. Then for any $a\in x\circ y$, there exists $a_0\in P(\mu_\Phi^P;\alpha)$ such that $a\ll a_0$. It follows from (3.1) that $\mu_\Phi^P(a)\geq\mu_\Phi^P(a_0)\geq\alpha$, for all $a\in x\circ y$ so that $\inf_{a\in x\circ y}\mu_\Phi^P(a)\geq\alpha$. Thus

$$\mu_\Phi^P(x)\geq\min\{\inf_{a\in x\circ y}\mu_\Phi^P(a),\,\mu_\Phi^P(y)\}\geq\alpha,$$

and so $x\in P(\mu_\Phi^P;\alpha)$. Therefore $P(\mu_\Phi^P;\alpha)$ is a hyper BCK-ideal of H. Now let $x,y\in H$ be such that $x\circ y\ll N(\mu_\Phi^N;\beta)$ and $y\in N(\mu_\Phi^N;\beta)$. Then $x\circ y\ll N(\mu_\Phi^N;\beta)$

implies that for every $b \in x \circ y$ there is $b_0 \in N(\mu_\Phi^N;\beta)$ such that $b \ll b_0$, so $\mu_\Phi^N(b) \le \mu_\Phi^N(b_0)$ by (3.1). It follows that $\mu_\Phi^N(b) \le \mu_\Phi^N(b_0) \le \beta$ for all $b \in x \circ y$ so that $\sup\limits_{b\in x\circ y} \mu_\Phi^N(b) \le \beta$. Then

$$\mu_\Phi^N(x) \le \max\{\sup_{b\in x\circ y} \mu_\Phi^N(b),\, \mu_\Phi^N(y)\} \le \beta,$$

which implies that $x \in N(\mu_\Phi^N;\beta)$. Consequently, $N(\mu_\Phi^N;\beta)$ is a hyper BCK-ideal of H.

Conversely, suppose that for each $(\alpha,\beta) \in [0,1]\times[-1,0]$ the nonempty positive and negative level sets $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are hyper BCK-ideals of H. Let $x, y \in H$ be such that $x \ll y$, $\mu_\Phi^P(y) = \alpha$ and $\mu_\Phi^N(y) = \beta$. Then $y \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$, and so $x \ll P(\mu_\Phi^P;\alpha)$ and $x \ll N(\mu_\Phi^N;\beta)$. It follows from Lemma 3.19 that $x \in P(\mu_\Phi^P;\alpha)$ and $x \in N(\mu_\Phi^N;\beta)$ so that $\mu_\Phi^P(x) \ge \alpha = \mu_\Phi^P(y)$ and $\mu_\Phi^N(x) \le \beta = \mu_\Phi^N(y)$. Now, let

$$\alpha = \min\{\inf_{c\in x\circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\}$$

and

$$\beta = \max\{\sup_{d\in x\circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\}$$

for $x, y \in H$. Then $y \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$, and for each $a, b \in x \circ y$ we have

$$\mu_\Phi^P(a) \ge \inf_{c\in x\circ y} \mu_\Phi^P(c) \ge \min\{\inf_{c\in x\circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\} = \alpha$$

and

$$\mu_\Phi^N(b) \le \sup_{d\in x\circ y} \mu_\Phi^N(d) \le \max\{\sup_{d\in x\circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\} = \beta.$$

Hence $a \in P(\mu_\Phi^P;\alpha)$ and $b \in N(\mu_\Phi^N;\beta)$, which imply that $x \circ y \subseteq P(\mu_\Phi^P;\alpha)$ and $x \circ y \subseteq N(\mu_\Phi^N;\beta)$. Using (2.8), we get $x \circ y \ll P(\mu_\Phi^P;\alpha)$ and $x \circ y \ll N(\mu_\Phi^N;\beta)$. Combining $y \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$ and $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ being hyper BCK-ideals of H, we conclude that $x \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$, and so

$$\mu_\Phi^P(x) \ge \alpha = \min\{\inf_{c\in x\circ y} \mu_\Phi^P(c),\, \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(x) \le \beta = \max\{\sup_{d\in x\circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\}.$$

This completes the proof. □

Theorem 3.21. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H, then for every $(\alpha,\beta) \in [0,1]\times[-1,0]$, the nonempty positive level set and the negative level set $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are strong hyper BCK-ideals of H.*

Proof. Assume that $P(\mu_\Phi^P;\alpha) \ne \emptyset \ne N(\mu_\Phi^N;\beta)$, for $(\alpha,\beta) \in [0,1]\times[-1,0]$. Then there exist $a \in P(\mu_\Phi^P;\alpha)$ and $b \in N(\mu_\Phi^N;\beta)$, and so $\mu_\Phi^P(a) \ge \alpha$ and $\mu_\Phi^N(b) \le \beta$. Using Proposition 3.12(1), we get $\mu_\Phi^P(0) \ge \mu_\Phi^P(a) \ge \alpha$ and $\mu_\Phi^N(0) \le \mu_\Phi^N(b) \le \beta$, so $0 \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$. Let $x, y \in H$ be such that $(x \circ y) \cap P(\mu_\Phi^P;\alpha) \ne \emptyset$ and

$y \in P(\mu_\Phi^P;\alpha)$. Then there exists $a_0 \in (x \circ y) \cap P(\mu_\Phi^P;\alpha)$ and hence $\mu_\Phi^P(a_0) \geq \alpha$. It follows that

$$\mu_\Phi^P(x) \geq \min\{\sup_{a \in x \circ y} \mu_\Phi^P(a),\, \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(a_0),\, \mu_\Phi^P(y)\} \geq \alpha$$

so that $x \in P(\mu_\Phi^P;\alpha)$. Now let $u, v \in H$ be such that $(u \circ v) \cap N(\mu_\Phi^N;\beta) \neq \emptyset$ and $v \in N(\mu_\Phi^N;\beta)$. Then we can take $b_0 \in (u \circ v) \cap N(\mu_\Phi^N;\beta)$ and so $\mu_\Phi^N(b_0) \leq \beta$. Hence

$$\mu_\Phi^N(u) \leq \max\{\inf_{b \in u \circ v} \mu_\Phi^N(b),\, \mu_\Phi^N(v)\} \leq \max\{\mu_\Phi^N(b_0),\, \mu_\Phi^N(v)\} \leq \beta,$$

which implies $u \in N(\mu_\Phi^N;\beta)$. Consequently, $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are strong hyper BCK-ideals of H. □

We now consider the converse of Theorem 3.21.

Theorem 3.22. *Let H satisfies $|x \circ y| < \infty$ for all $x, y \in H$. Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H such that the nonempty positive level sets and the negative level sets $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are strong hyper BCK-ideals of H for every $(\alpha, \beta) \in [0,1] \times [-1,0]$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H.*

Proof. Assume that for every $(\alpha, \beta) \in [0,1] \times [-1,0]$, nonempty level sets $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are strong hyper BCK-ideals of H. Then $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are hyper BCK-ideals of H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H by Theorem 3.20. Note that $x \circ x \subseteq x \circ H \ll \{x\}$ for all $x \in H$. It follows that $a \ll x$ and $c \ll x$ for every $a, c \in x \circ x$ so that $\mu_\Phi^P(a) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(c) \leq \mu_\Phi^N(x)$ for all $a, c \in x \circ x$. Hence $\inf_{a \in x \circ x} \mu_\Phi^P(a) \geq \mu_\Phi^P(x)$ and $\sup_{c \in x \circ x} \mu_\Phi^N(c) \leq \mu_\Phi^N(x)$. Let

$$\min\{\sup_{b \in x \circ y} \mu_\Phi^P(b),\, \mu_\Phi^P(y)\} = \alpha$$

and

$$\max\{\inf_{d \in x \circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\} = \beta$$

for $x, y \in H$. Then $\sup_{b \in x \circ y} \mu_\Phi^P(b) \geq \alpha$ and $\mu_\Phi^P(y) \geq \alpha$, $\inf_{d \in x \circ y} \mu_\Phi^N(d) \leq \beta$ and $\mu_\Phi^N(y) \leq \beta$. Since $|x \circ y| < \infty$ for all $x, y \in H$, there exist $b_0, d_0 \in x \circ y$ such that $\mu_\Phi^P(b_0) \geq \alpha$, $\mu_\Phi^P(y) \geq \alpha$, $\mu_\Phi^N(d_0) \leq \beta$, and $\mu_\Phi^N(y) \leq \beta$. Then $(x \circ y) \cap P(\mu_\Phi^P;\alpha) \neq \emptyset$, $y \in P(\mu_\Phi^P;\alpha)$, $(x \circ y) \cap N(\mu_\Phi^N;\beta) \neq \emptyset$, and $y \in N(\mu_\Phi^N;\beta)$. Since $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are strong hyper BCK-ideals, it follows that $x \in P(\mu_\Phi^P;\alpha)$ and $x \in N(\mu_\Phi^N;\beta)$ so that $\mu_\Phi^P(x) \geq \alpha = \min\{\sup_{b \in x \circ y} \mu_\Phi^P(b),\, \mu_\Phi^P(y)\}$ and $\mu_\Phi^N(x) \leq \beta = \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d),\, \mu_\Phi^N(y)\}$. Hence $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H. □

Acknowledgements. Authors would like to express their sincere thanks to the anonymous reviewers for their valuable suggestions.

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Young Bae Jun, Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Korea
E-mail address: skywine@gmail.com

Min Su Kang*, Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: sinchangmyun@hanmail.net

Hee Sik Kim, Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: heekim@hanyang.ac.kr

*Corresponding author