OPTIMAL CONTROL WITH FUZZY CHANCE CONSTRAINTS

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Abstract. In this paper, a model of an optimal control problem with chance constraints is introduced. The parameters of the constraints are fuzzy, random or fuzzy random variables. To defuzzify the constraints, we consider possibility levels. By chance-constrained programming the chance constraints are converted to crisp constraints which are neither fuzzy nor stochastic and then the resulting classical optimal control problem with crisp constraints is solved by the Pontryagin Minimum Principle and Kuhn-Tucker conditions. The model is illustrated by two numerical examples.

1. Introduction

In recent years, many researchers have extended classical decision making problems using fuzzy, random or fuzzy random parameters [1,4,6,7,8,9,11,12,13]. Chakraborty [1] has dealt with chance-constrained programming in a fuzzy environment where the parameters are random variables, but the probability of the constraints is imprecise. Ramezanzadeh et al. [11] have considered imprecise probability for chance constraints with fuzzy random parameters in the DEA model. Liu [6, 7] has introduced models of decision making with chance constraints where possibility or necessity levels and probability measures are assigned to the constraints and used fuzzy simulation to obtain solutions.

Recently, some optimal control models [8, 9] have been formulated in fuzzy and stochastic environments and some researchers have considered fuzzy and/or stochastic parameters in a single model. However, the constraints were not composed in possibility and probability senses at the same time. In some practical systems, it may happen that the outcomes of a random variable are vague in linguistic terms instead of numeric. A fuzzy random variable describes uncertain parameters in which fuzziness and randomness are fused with each other. For instance, in inventory control systems, we may encounter uncertain and random budgetary capital during a period of so we formulate it by a fuzzy random variable. Note that in such a case, we cannot formulate the parameter only as a fuzzy number or a random variable. So, we propose a model where possibility levels are assigned for chance constraints of the optimal control problem with Following Liu [6, 7], the parameters of the constraints, which are assumed to be fuzzy random variables, are defuzzified using the methods of Dubois and Prade [3]. First the chance constraints

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are converted to optimal control having crisp nonlinear constraints by the chance-constrained programming approach [2] and then the problem is solved using the Pontryagin Minimum Principle and the Kuhn-Tucker conditions. The structure of this paper is as follows: section 2 contains some necessary definitions and concepts. section 3 introduces the model and deals with the process of converting it to crisp model. Section 4 presents the mathematical approach for solving the model. Finally, two numerical examples are given in section 5 to demonstrate the process. Section 6 provides a conclusion.

2. Preliminaries

In this section, we state some necessary definitions and lemmas for fuzzy environments.

Definition 2.1. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, respectively. Then

$$Pos(\tilde{a} \leq \tilde{b}) = \sup \min_{x \leq y} (\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y))$$

where Pos represents possibility([3]).

Lemma 2.2. [3] *Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then*

$$Pos(\tilde{a} \leq \tilde{b}) \geq \beta \quad \textit{iff} \; \frac{b_3 - a_1}{b_3 - b_2 + a_2 - a_1} \geq \beta, \; (a_2 > b_2, b_3 > a_1)$$

Definition 2.3. Let (Ω, A, P) be a probability measure space. The mapping $\tilde{a} : \Omega \longrightarrow F_0(R)$ is called a fuzzy random variable on (Ω, A, P), if for any $\alpha \in (0, 1]$ and $\omega \in \Omega$,

$$\tilde{a}_\alpha(\omega) = \{x | x \in R, \tilde{a}(\omega)(x) \geq \alpha\} = [a_\alpha^-(\omega), a_\alpha^+(\omega)]$$

is a random interval, i.e. $a_\alpha^-(\omega)$ and $a_\alpha^+(\omega)$ are random variables (or finite measurable functions) on (Ω, A, P) ([5]).

3. Optimal Control with Fuzzy Random Constraints

Consider the following optimal control problem:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ x(0) &= x_0, |u(t)| \leq a \end{aligned} \tag{1}$$

such that

$$Pr(\tilde{g}_k(x(t), u(t), t) \leq 0) \geq \alpha, \quad k = 1, \dots, l \tag{2}$$

where $x(t)$ and $u(t)$ are respectively $N \times 1$ and $P \times 1$ state and control vectors and $Pr(\cdot)$ represents the probability and $\alpha \in (0, 1]$. Some parameters in (2) are fuzzy numbers, random variables or fuzzy random variables with known distributions.

The problem is to find the optimal control $u(t)$ with the corresponding trajectory $x(t)$, $0 \leq t \leq T$, satisfying (1) and (2) while minimizing the performance index

$$J = \int_0^T g(x(t), u(t), t)dt \tag{3}$$

where T denotes the finite length of time. The vector functions f and g_k, and scalar function g are generally nonlinear and are assumed to be continuously differentiable with respect to their arguments. For the sake of simplicity, we consider the linear case of problem:

$$MinJ = \int_0^T (\sum_{j=1}^n h_j x_j(t) + \sum_{j=1}^p c_j u_j(t))dt \tag{4}$$

$s.t.$

$$\dot{x}_t(t) = \sum_{j=1}^n a_{ij} x_j(t) + \sum_{j=1}^p b_{ij} u_j(t); \quad i = 1, \cdots, n \tag{5}$$

$$Pr(\sum_{j=1}^n \tilde{h}_{kj}(\omega) x_j(t) + \sum_{j=1}^p \tilde{c}_{kj}(\omega) u_j(t) \leq \tilde{M}_k(\omega)) \geq \alpha_k; \quad k = 1, \cdots l \tag{6}$$

$$x_i(t_0) = x_{i0}, 0 \leq u_i^\ell \leq u_i(t) \leq u_i^u, 0 \leq t \leq T, \alpha_k \in (0, 1] \tag{7}$$

where $\tilde{h}_{kj}(\omega) = (h_{kj}^l(\omega), h_{kj}^m(\omega), h_{kj}^u(\omega)), \tilde{c}_{kj}(\omega) = (c_{kj}^l(\omega), c_{kj}^m(\omega), c_{kj}^u(\omega))$ and $\tilde{M}_k(\omega) = (M_k^l(\omega), M_k^m(\omega), M_k^u(\omega))$ are triangular fuzzy random variables related to random variable ω.

Remark 3.1. It is possible that some parameters of (6) are only fuzzy numbers or crisp random variables. If $\tilde{h}_{kj}(\omega) = h_{kj}(\omega), \tilde{c}_{kj}(\omega) = c_{kj}(\omega)$ and $\tilde{M}_k(\omega) = M_k(\omega)$, then (6) is a stochastic constraint, but not fuzzy. On the other hand, if $\tilde{h}_{kj}(\omega) = \tilde{h}_{kj}, \tilde{c}_{kj} = \tilde{c}_{kj}(\omega)$ and $\tilde{M}_k(\omega) = \tilde{M}_k$, then (6) is only a fuzzy constraint.

According to Liu [6, 7], these constraints can be converted to their nonfuzzy counterparts by considering possibility or necessity where fuzzy numbers are interpreted by a degree of uncertainty. For example (6) becomes

$$Pr(Pos(\sum_{j=1}^n \tilde{h}_{kj}(\omega) x_j(t) + \sum_{j=1}^p \tilde{c}_{kj}(\omega) u_j(t) \leq \tilde{M}_k(\omega)) \geq \beta_k) \geq \alpha_k; k = 1, \cdots, l \tag{8}$$

Where the $\beta_k (k = 1, \cdots, l)$ are predetermined confidence levels.
By Lemma 2.1, the possibility inequality in (8) is converts to the following inequality:

$$\frac{A}{B} \geq \beta_k$$

where

$$A = M_k^u(\omega) - \sum_{j=1}^{n} h_{kj}^l(\omega)x_j(t) - \sum_{j=1}^{p} c_{kj}^l(\omega)u_j(t)$$

$$B = M_k^u(\omega) - M_k^m(\omega) + \sum_{j=1}^{n} h_{kj}^m(\omega)x_j(t) + \sum_{j=1}^{p} c_{kj}^m(\omega)u_j(t) - \sum_{j=1}^{n} h_{kj}^l(\omega)x_j(t)$$

$$- \sum_{j=1}^{p} c_{kj}^l(\omega)u_j(t)$$

Now, we can rewrite (8) as follows:

$$Pr(\sum_{j=1}^{n} A_{kj}(\omega)x_j(t) + \sum_{j=1}^{p} B_{kj}(\omega)u_j(t) \leq b_k(\omega)) \geq \alpha_k; \quad k = 1, \cdots l \tag{9}$$

where

$$A_{kj}(\omega) = (1-\beta_k)h_{kj}^l(\omega) + \beta_k h_{kj}^m(\omega)$$
$$B_{kj}(\omega) = (1-\beta_k)c_{kj}^l(\omega) + \beta_k c_{kj}^m(\omega)$$
$$b_k(\omega) = (1-\beta_k)M_k^u(\omega) + \beta_k M_k^m(\omega),$$

The optimal control problem with constraint (9) is a probability programming problem. Hence it can be converted to a (neither fuzzy nor stochastic) model by the classic technique of chance-constrained programming [2]. Thus we have the following optimal control problem:

$$MinJ = \int_0^T (\sum_{j=1}^{n} h_j x_j(t) + \sum_{j=1}^{p} c_j u_j(t))dt$$

$s.t.$

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{p} b_{ij}u_j(t); \quad i = 1, \ldots, n$$

$$\sum_{j=1}^{n} A_{kj}(\bar{\omega})x_j(t) + \sum_{j=1}^{p} B_{kj}(\bar{\omega})u_j(t) - b_k(\bar{\omega}) - \varphi^{-1}(1-\alpha_i)\sigma_k(x,u)$$
$$\leq 0; \qquad k = 1, \cdots, l \tag{10}$$

$$x_i(t_0) = x_{i0}, 0 \leq u_i^l \leq u_i(t) \leq u_i^u, 0 \leq t \leq T, \alpha \in (0,1]$$

where $\bar{\omega}$ is the mean of ω, φ is the standard normal distribution function, φ^{-1} is inverse of φ and

$$\sigma_k^2(x,u) = \sum_{r=1}^{n}\sum_{s=1}^{n} x_r x_s cov(A_{kr}(\omega), A_{ks}(\omega)) + \sum_{r=1}^{p}\sum_{s=1}^{p} u_r u_s cov(B_{kr}(\omega), B_{ks}(\omega))$$
$$+ 2\sum_{r=1}^{n}\sum_{s=1}^{p} x_r u_s cov(A_{kr}(\omega), B_{ks}(\omega)) - 2\sum_{j=1}^{n} x_j cov(A_{kj}(\omega), b_k(\omega))$$
$$- 2\sum_{j=1}^{n} x_j cov(B_{kj}(\omega), b_k(\omega)) + var(b_k(\omega)).$$

Here, *cov* and *var* denote the covariance and the variance respectively.

Remark 3.2. The nonlinearity in (10) is due to $\sigma_k(x,u)$. If there is only one fuzzy random or random variable in constraint (6), then we have a linear constraint in (10). If we assume that only the right-hand side of (6) is a fuzzy random variable, then (10) will convert to

$$\sum_{j=1}^{n} A_{kj}(\bar{\omega})x_j(t) + \sum_{j=1}^{p} B_{kj}(\bar{\omega})u_j(t) - b_k(\bar{\omega}) - \varphi^{-1}(1-\alpha_i)\sqrt{var(b_k(\bar{\omega}))} \leq 0; \qquad k = 1,\cdots,l \tag{11}$$

4. **Mathematical Technique for Solution**

For the problem (4), (5), (10) and (7) the corresponding Hamiltonian function is

$$H = \sum_{j=1}^{n} h_j x_j(t) + \sum_{j=1}^{p} c_j u_j(t) + p_i(t)(\sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{p} b_{ij}u_j(t))$$

The Lagrangian function for the constraint (10) is

$$L = H + \sum_{k=1}^{l} \lambda_k(\sum_{j=1}^{n} A_{kj}(\bar{\omega})x_j(t) + \sum_{j=1}^{p} B_{kj}(\bar{\omega})u_j(t) - b_k(\bar{\omega}) - \varphi^{-1}(1-\alpha_i)\sigma_k(x,u))$$

where $\lambda_k \geq 0, k = 1,\cdots,l$ are the Lagrange multipliers. Hence, the Kuhn-Tucker conditions are

$$\lambda_k(\sum_{j=1}^{n} A_{kj}(\bar{\omega})x_j(t) + \sum_{j=1}^{p} B_{kj}(\bar{\omega})u_j(t) - b_k(\bar{\omega}) - \varphi^{-1}(1-\alpha_i)\sigma_k(x,u)) = 0,\ k = 1,\cdots,l$$

The corresponding costate $p_i(t)$ is given by the first order differential equation,

$$\dot{p}(t) = -h_i - a_{ii}p_i(t) - \sum_{k=1}^{l} \lambda_k A_{ki}(\bar{\omega})$$
$$p_i(T) = 0$$

By the Pontryagin Minimum Principle we have

$$\frac{\partial L}{\partial u_i(t)} = c_i + b_{ii}p_i(t) + \sum_{k=1}^{l} \lambda_k B_{ki}(\bar{\omega}) \tag{12}$$

$u_j(t)$ is continuous and bounded. Therefore by (12) there are three cases:

$$\begin{cases} \frac{\partial L}{\partial u_j(t)} > 0 & L \text{ is increasing} \\ \frac{\partial L}{\partial u_j(t)} = 0 & L \text{ is independent} \\ \frac{\partial L}{\partial u_j(t)} < 0 & L \text{ is decreasing} \end{cases}$$

So, for $0 \leq u_i^l \leq u_i(t) \leq u_i^u$ we have:

$$u_j(t) = \begin{cases} u_j^u & \text{if} \quad b_{ii}p_i(t) > -\sum_{k=1}^{l} \lambda_k B_{ki}(\bar{\omega}) - c_i, & (13) \\ u_j^l & \text{if} \quad b_{ii}p_i(t) < -\sum_{k=1}^{l} \lambda_k B_{ki}(\bar{\omega}) - c_i & (14) \end{cases}$$

Suppose (13) holds for $0 \leq t \leq t_{il}$ and (14) for $t_{il} \leq t \leq T$, then we can obtain the optimum trajectory in $[0, T]$ and the corresponding J in (3).

5. **Illustrative Examples**

In this section we illustrate our method with two numerical examples.

Example 5.1. Consider the following optimal control problem:

$$MinJ = \frac{1}{2}\int_0^l (x(t) + u(t))dt$$

$$s.t.$$

$$\dot{x}(t) = x(t) + u(t) \tag{15}$$

$$Pr(x(t) + u(t) \leq \tilde{M}(\omega)) \geq \alpha \tag{16}$$

$$x(0) = 0, x(1) = 0, 2 \leq u(t) \leq 4, 0 \leq t \leq 1, \alpha \in (0, 1] \tag{17}$$

where $\tilde{M}(\omega) = (\omega - 1, \omega, \omega + 1)$ and ω has a normal distribution with mean 5 and variance 1. It is assumed that $\beta = 0.9$ and $\alpha = 0.95$. According to (8), the constraint (16) is rewritten as follows:

$$Pr(Pos(x(t) + u(t) \leq \tilde{M}(\omega)) \geq \beta) \geq \alpha$$

We defuzzify the above relation as follows:

$$Pr(x(t) + u(t) \leq \omega - 0.1) \geq 0.95$$

So, the corresponding crisp model is

$$\begin{aligned} &\text{Min}J = \frac{1}{2}\int_0^1 (x(t) + u(t))dt \\ &s.t. \\ &\quad \dot{x}(t) = x(t) + u(t) \\ &\quad x(t) + u(t) \leq 7.23 \\ &\quad x(0) = 0, x(1) = 2, 2 \leq u(t) \leq 4, 0 \leq t \leq 1 \end{aligned}$$

The optimal control function and the corresponding trajectory are obtained as

$$u(t) = \begin{cases} 4, & 0 \leq t \leq 0.09 \\ 2, & 0.09 \leq t \leq 1 \end{cases} \qquad \& \qquad x(t) = \begin{cases} e^{t+ln4} - 4, & 0 \leq t \leq 0.09 \\ e^t - 0.72, & 0.09 \leq t \leq 1 \end{cases}$$

and $J^* = 3.031$.

Example 5.2. We shall discuss the multi-item production-inventory model (Maity [9] with some variations).

Consider a defective n-item production-inventory system with budget and warehouse capacity constraints and time independent demand. The items are produced at a variable rate of $u_j(t)$ in which a fraction δ_j is defective. Defective items are are a natural phenomenon and occur as a the result of an imperfect quality production process. We assume that demand for the items is time independent and the stock level at time t decreases due to defectiveness and consumption. Shortages are allowed but may or may not be backlogged. Moreover, the space and budgetary constraints are satisfied with some predefined possibility and probability, respectively. Assuming the produced quantity-dependent unit production cost and the warehouse to be of finite capacity, minimization of the total cost which consists of holding, shortage and production costs, leads to:

$$\text{Min}J = \sum_{j=1}^{n} \int_{0}^{T} (h_j I_j(t) + h_i' S_i(t) + c_j u_j(t))dt \tag{18}$$

$s.t.$

$$\dot{x}_j(t) = (1 - \delta_j)u_j(t) - d_j,\ i = 1, \cdots, n \tag{19}$$

$$\sum_{j=1}^{n} \tilde{a}_j I_j(t) \leq \tilde{M} \tag{20}$$

$$Pr(\sum_{j=1}^{n} c_j u_j(t) \leq \tilde{Z}(\omega)) \geq \alpha \tag{21}$$

$$x_j(0) = 0, 0 \leq d_j \leq u_j(t) \leq u_j, 0 \leq t \leq T, \alpha \in (0, 1] \tag{22}$$

where,
n:Number of items ($n = 2$),
$\tilde{M} = (85, 100, 115)$: Maximum space available of storage that is fuzzy triangular number,
T: Length of the finite time horizon ($T = 10$),
$\tilde{Z}(\omega) = (\omega - 25, \omega, \omega + 25)$: Available total budgetary capital that is fuzzy random variable, where ω has normal distribution with mean 500 and variance 100,
α: Probability level of second constraint (α =0.95),
σ_j: Fraction of defectives for two items (0.18 and 0.19 respectively),
u_j: Maximum production rate for two items (12 and 13 respectively),
d_j: Demand for two items (8.5 and 9 respectively),
c_j: Production cost per unit item (2 and 2.5 respectively),
h_j: Holding cost per unit item (0.5 and 0.6 respectively),
h_j': Shortage cost per unit item (0.6 and 0.5 respectively),
a_j: Storage area per unit item (fuzzy triangular numbers $\tilde{a_1} = (1.9, 2.0, 2.1)$ and $\tilde{a_2} = (2.4, 2.5, 2.6)$,)
$u_j(t)$: Production rate at time t where $0 \leq d_j \leq u_j(t) \leq u_j$,
$x_j(t)$: Inventory level at time t such that $x_j(t) = I_j(t) - S_j(t)$ where $S_j(t) = max(-x_j(t), 0)$ is the shortage level at time t and $I_j(t) = max(x_j(t), 0)$.

By (8), the constraints (20) and (21) are rewritten as follows:

$$Pos(\sum_{j=1}^{n} \tilde{a_j} I_j(t) \leq \tilde{M}) \geq \beta_1 \tag{23}$$

$$Pr(Pos(\sum_{j=1}^{n} c_j u_j(t) \leq \tilde{Z}(\omega)) \geq \beta_2) \geq \alpha \tag{24}$$

where β_1 and β_2 are possibility levels. Here we assume that $\beta_1 = 0.65$ and $\beta_2 = 0.67$.
Solving process: According to (9), we defuzzify the constraints (23) and (24):

$$20.065 I_l(t) + 2.565 I_2(t) \leq 90.25 \tag{25}$$

$$Pr(2u_1(t) + 2.5u_2(t) \leq \omega - 16.75) \geq 0.95 \tag{26}$$

The constraint (26) is converted to the following crisp constraint:

$$2u_1(t) + 2.5u_2(t) - 506.55 \leq 0 \tag{27}$$

Now, we have the following problem:

$$\text{Min}J = \int_0^{10} (0.5I_1(t) + 0.6I_2(t) + 0.6S_1(t) + 0.5S_2(t) + 2u_1(t) + 2.5u_2(t))dt$$

$$\begin{aligned}
&s.t.\\
&\quad \dot{x_1} = 0.82u_1 - 8.5\\
&\quad \dot{x_2} = 0.81u_2 - 9\\
&\quad 20.065I_1(t) + 2.565I_2(t) \leq 90.25\\
&\quad 2u_1(t) + 2.5u_2(t) - 506.55 \leq 0\\
&\quad x_1(0) = x_1(10) = 0, x_2(0) = x_2(10) = 0\\
&\quad 8.5 \leq u_1 \leq 12, 9 \leq u_2 \leq 13\\
&\quad 0 \leq t \leq 10
\end{aligned}$$

The optimal control function is obtained as

$$u_1(t) = \begin{cases} 12, & 0 \leq t \leq 5.53 \\ 8.5, & 5.53 \leq t \leq 10 \end{cases} \qquad \& \qquad u_2(t) = \begin{cases} 13, & 0 \leq t \leq 5.28 \\ 9, & 5.28 \leq t \leq 10 \end{cases}$$

and we have,

$$I_1(t) = \begin{cases} 1.34t, & 0 \leq t \leq 5.33 \\ -1.53(t - 5.33) + 7.14, & 5.33 \leq t \leq 10 \end{cases}$$

$$\&$$

$$I_2(t) = \begin{cases} 1.53t, & 0 \leq t \leq 5.28 \\ -1.71(t - 5.28) + 8.08, & 5.28 \leq t \leq 10 \end{cases}$$

$S_1(t) = S_2(t) = 0$

The corresponding trajectory is

$$x_j(t) = I_j(t) - S_j(t) = I_j(t)$$

So we obtain $J^* = 473.076$.

6. Conclusion

In this paper we have presented a technique for solving an optimal control problem with chance constraints whose parameters are fuzzy random variables. We assigned possibility levels to the constraints, converted them to a crisp model applying the methods of possibility theory and chance-constrained programming and finally solved the problem using the Pontryagin Minimum Principle and Kuhn-Tucker conditions. This model may be extended to a general model in which coefficients of state variables and control function in the objective function and state equation are also fuzzy random variables. The solution of the crisp model by numerical methods such as wavelets may is also an open problem.

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