ON FUZZY UPPER AND LOWER CONTRA-CONTINUOUS MULTIFUNCTIONS

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Abstract. This paper is devoted to the concepts of fuzzy upper and fuzzy lower contra-continuous multifunctions and also some characterizations of them are considered.

1. Introduction

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory [7, 31, 32] etc. The initiation of fuzzy multifunctions is due to Papageorgiou [27]. He studied upper and lower semi-continuous multifunctions. Mukherjee and Malakar [21] have studied fuzzy multifunctions with q-coincidence. Recently many authors for example Albrycht and Maltoka, Nouh and El-Shafei [1, 24] and Beg [4, 6] have studied fuzzy multifunctions and have characterized, some properties of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied different types of fuzzy continuity for fuzzy multifunctions, for example see [2, 13, 27, 28] and also for more details on fuzzy multifunctions one can see [6]. On the other hand, Dontchev [12] introduced the notion of contra-continuous functions. It is shown in [12] that contra-continuous images of strongly S-closed spaces are compact. Joseph and Kwack [19] introduced another form of contra-continuity called (θ, s)-continuous functions in order to investigate S-closed spaces due to Thompson [33]. In recent years, several authors have studied some new forms of contra-continuity for functions and multifunctions, for example see [10, 15, 16, 17, 22]. In the present paper, we study the notions of fuzzy upper and fuzzy lower contra-continuous multifunctions. Also, some characterizations and properties of such notions are discussed. Since initiation of the theory of fuzzy sets by Zadeh [38], this theory has found wide applications in economics, engineering, medicine, information sciences, programming, optimization, graphs etc (for example see [8, 9, 18, 30, 35, 37, 39]). Also, fuzzy multifunctions arise in many applications, for example, the budget multifunctions accurs in decision theory, noncooperative games, artificial intelligence, economic theory, medicine, information sciences, fixed point theory (see [3] and its references, [7] etc). Also, Beg [4] and Tsiporkova et al. [34] have initiated the systematic study of different

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types of continuity for fuzzy multifunctions. Moreover, fuzzy closed graph theorem for fuzzy multifunctions was provided and as application the structure of the set of fixed points of the fuzzy multifunctions was studied ([5], [6], [20] etc.). On the other hand, Dontchev considered the contra-continuity for preservation theorems and decomposition theorems which have applications and useful variations in functional analysis and closed graph theory ([23], etc). This paper extends the notion of contra-continuity due to Dontchev and [15] to the setting of fuzzy multifunctions and provides their properties. Hence, the class of fuzzy upper and lower contra-continuous multifunctions can be studied and applied in these practical areas.

2. **Preliminaries**

The class of all fuzzy sets on a universe Y will be denoted by I^Y and fuzzy sets on Y will be denoted by μ, η, etc. A family τ of fuzzy sets in Y is called a *fuzzy topology* for Y [11] if

(1) $\emptyset$, $Y \in \tau$,
(2) $\mu \wedge \eta \in \tau$ whenever μ, $\eta \in \tau$,
(3) If $\mu_i \in \tau$ for each $i \in I$, then $\vee \mu_i \in \tau$.

The pair (Y, σ) is called a *fuzzy topological space*. Every member of σ is called a *fuzzy open set*. A fuzzy set in Y is called a *fuzzy point* if it takes the value 0 for all $y \in Y$ except one, say, $x \in Y$. If its value at x is ε $(0 < \varepsilon \leq 1)$, we denote this fuzzy point by x_ε, where the point x is called its *support* [25, 26]. For any fuzzy point x_ε and any fuzzy set μ, $x_\varepsilon \in \mu$ if and only if $\varepsilon \leq \mu(x)$. A fuzzy point x_ε is called *quasi-coincident* with a fuzzy set η, denoted by $x_\varepsilon q \eta$, if $\varepsilon + \eta(x) > 1$. A fuzzy set μ is called *quasi-coincident* with a fuzzy set η, denoted by $\mu q \eta$, if there exists $x \in Y$ such that $\mu(x) + \eta(x) > 1$ [25, 26]. When they are not quasi-coincident, it will be denoted by $\mu \bar{q} \eta$.

Throughout this paper, (X, τ) or simply X will stand for ordinary topological space and (Y, σ) or simply Y will denote a fuzzy topological space.

Let X be a topological space in classical sense and let Y be a fuzzy topological space. $F : X \multimap Y$ is called a *fuzzy multifunction* [27] if for each $x \in X$, $F(x)$ is a fuzzy set in Y. Throughout the paper, by $F : X \multimap Y$ we mean that F is a fuzzy multifunction from a classical topological space X to a fuzzy topological space Y. For a fuzzy multifunction $F : X \multimap Y$, the upper inverse $F^+(\mu)$ and lower inverse $F^-(\mu)$ of a fuzzy set μ in Y are defined as follows: $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$ and $F^-(\mu) = \{x \in X : F(x) q \mu\}$. For any fuzzy set μ in Y, we have $F^-(Y - \mu) = X - F^+(\mu)$ [21]. We denote the interior and the closure of a subset A of a topological space X by $Int(A)$ and $Cl(A)$, respectively.

3. **Fuzzy Upper and Lower Contra-Continuous Multifunctions**

Definition 3.1. A fuzzy multifunction $F : X \multimap Y$ is called fuzzy lower contra-continuous multifunction if for any fuzzy closed set μ in Y with $x \in F^-(\mu)$, there exists an open set B in X containing x such that $B \subset F^-(\mu)$.

Definition 3.2. A fuzzy multifunction $F : X \multimap Y$ is called fuzzy upper contra-continuous multifunction if for each fuzzy closed set μ in Y with $x \in F^+(\mu)$, there exists an open set B in X containing x such that $B \subset F^+(\mu)$.

Theorem 3.3. *The following are equivalent for a fuzzy multifunction $F : X \multimap Y$:*

(1) *F is fuzzy upper contra-continuous,*

(2) *For each fuzzy closed set μ and $x \in X$ such that $F(x) \leq \mu$, there exists an open set B containing x such that if $y \in B$, then $F(y) \leq \mu$,*

(3) *$F^+(\mu)$ is open for any fuzzy closed set μ in Y,*

(4) *$F^-(\rho)$ is closed for any fuzzy open set ρ in Y.*

Proof. (1) $\Leftrightarrow$ (2) : Obvious.

(1) $\Rightarrow$ (3) : Let μ be any fuzzy closed set in Y and $x \in F^+(\mu)$. By (1), there exists an open set A_x containing x such that $A_x \subset F^+(\mu)$. Thus, $x \in Int(F^+(\mu))$ and hence $F^+(\mu)$ is an open set in X.

(3) $\Rightarrow$ (4) : Let ρ be a fuzzy open set in Y. Then $Y \backslash \rho$ is a fuzzy closed set in Y. By (3), $F^+(Y \backslash \rho)$ is open. Since $F^+(1 \backslash \rho) = X \backslash F^-(\rho)$, then $F^-(\rho)$ is closed in X.

(4) $\Rightarrow$ (3) : It is similar to that of (3) $\Rightarrow$ (4).

(3) $\Rightarrow$ (1) : Let ρ be any fuzzy closed set in Y and $x \in F^+(\rho)$. By (3), $F^+(\rho)$ is an open set in X. Take $A = F^+(\rho)$. Then, $A \subset F^+(\rho)$. Thus, F is fuzzy upper contra-continuous. □

Definition 3.4. The set $\wedge\{\rho \in \tau : \mu \leq \rho\}$ is called the fuzzy kernel of a fuzzy set μ in a fuzzy topological space (X, τ) and is denoted by $Ker(\mu)$.

Lemma 3.5. *For a fuzzy set μ in a fuzzy topological space (X, τ), if $\mu \in \tau$, then $\mu = Ker(\mu)$.*

Theorem 3.6. *Let $F : (X, \tau) \multimap (Y, \sigma)$ be a fuzzy multifunction. If $Cl(F^-(\mu)) \leq F^-(Ker(\mu))$ for any fuzzy set μ in Y, then F is fuzzy upper contra-continuous.*

Proof. Suppose that $Cl(F^-(\mu)) \leq F^-(Ker(\mu))$ for every fuzzy set μ in Y. Let $\rho \in \sigma$. By Lemma 3.5, $Cl(F^-(\rho)) \leq F^-(Ker(\rho)) = F^-(\rho)$. This implies that $Cl((F^-(\rho)) = F^-(\rho)$ and hence $F^-(\rho)$ is closed in X. Thus, by Theorem 3.3, F is fuzzy upper contra-continuous. □

Definition 3.7. A fuzzy multifunction $F : X \multimap Y$ is called

(1) fuzzy lower semi-continuous [21] if for any fuzzy open set μ in Y with $x \in F^-(\mu)$, there exists an open subset B of X containing x such that $B \subset F^-(\mu)$.

(2) fuzzy upper semi-continuous [21] if for any fuzzy open set μ in Y with $x \in F^+(\mu)$, there exists an open subset B of X containing x such that $B \subset F^+(\mu)$.

Remark 3.8. The notions of fuzzy upper contra-continuous multifunctions and fuzzy upper semi-continuous multifunctions are independent as shown in the following examples.

Example 3.9. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, where $\mu(y) = 0,5$, $\rho(y) = 0,6$, $\eta(y) = 0,7$ for $y \in Y$. Define a fuzzy multifunction as follows: $F(a) = \mu$, $F(b) = \rho$, $F(c) = \eta$. Then the fuzzy multifunction $F : (X, \tau) \multimap (Y, \sigma)$ is fuzzy upper contra-continuous, but it is not fuzzy upper semi-continuous.

Example 3.10. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b, c\}\}$ and $Y = [0, 1]$, $\sigma = \{Y, \emptyset, \mu, \rho, \eta\}$, where $\mu(y) = 0,3$, $\rho(y) = 0,2$, $\eta(y) = 0,6$, $\gamma(y) = 0,4$, $\zeta(y) = 0,5$ for $y \in Y$. Define a fuzzy multifunction as follows: $F(a) = \gamma$, $F(b) = \zeta$, $F(c) = \eta$. Then the fuzzy multifunction $F : (X, \tau) \multimap (Y, \sigma)$ is fuzzy upper semi-continuous, but it is not fuzzy upper contra-continuous.

Theorem 3.11. *The following are equivalent for a fuzzy multifunction* $F : X \multimap Y$*:*

(1) F *is fuzzy lower contra-continuous,*

(2) *For any fuzzy closed set* μ *and* $x \in X$ *such that* $F(x)q\mu$*, there exists an open set* B *containing* x *such that if* $y \in B$*, then* $F(y)q\mu$*,*

(3) $F^-(\mu)$ *is open for any fuzzy closed set* μ *in* Y*,*

(4) $F^+(\rho)$ *is closed for any fuzzy open set* ρ *in* Y*.*

Proof. It is similar to that of Theorem 3.3. □

Theorem 3.12. *For a fuzzy multifunction* $F : (X, \tau) \multimap (Y, \sigma)$*, if* $Cl(F^+(\rho)) \leq F^+(Ker(\rho))$ *for every fuzzy set* ρ *in* Y*, then* F *is fuzzy lower contra-continuous.*

Proof. Suppose that $Cl(F^+(\rho)) \leq F^+(Ker(\rho))$ for every fuzzy set ρ in Y. Let $\rho \in \sigma$. We have $Cl(F^+(\rho) \leq F^+(Ker(\rho)) = F^+(\rho)$. Thus, $Cl((F^+(\rho)) = F^+(\rho)$ and hence $F^+(\rho)$ is closed in X. By Theorem 3.11, F is fuzzy lower contra-continuous. □

Definition 3.13. [6] Given a family $\{F_i : X \multimap Y : i \in I\}$ of fuzzy multifunctions, we define the union $\vee_{i \in I} F_i$ and the intersection $\wedge_{i \in I} F_i$ as follows: $\vee_{i \in I} F_i : X \multimap Y$, $(\vee_{i \in I} F_i)(x) = \vee_{i \in I} F_i(x)$ and $\wedge_{i \in I} F_i : X \multimap Y$, $(\wedge_{i \in I} F_i)(x) = \wedge_{i \in I} F_i(x)$.

Theorem 3.14. *If* $F_i : X \multimap Y$ *are fuzzy upper contra-continuous multifunctions for* $i = 1, 2, ..., n$*, then* $\vee_{i=1}^n F_i$ *is a fuzzy upper contra-continuous multifunction.*

Proof. Let μ be a fuzzy closed set of Y. We will show that $(\vee_{i=1}^n F_i)^+(\mu) = \{x \in X : \vee_{i=1}^n F_i(x) \leq \mu\}$ is open in X. Let $x \in (\vee_{i=1}^n F_i)^+(\mu)$. Then $F_i(x) \leq \mu$ for $i = 1, 2, ..., n$. Since $F_i : X \multimap Y$ is fuzzy upper contra-continuous multifunction for $i = 1, 2, ..., n$, then there exists an open set U_x containing x such that for all $z \in U_x$, $F_i(z) \leq \mu$. Let $U = \cup_{i=1}^n U_x$. Then $U \subset (\vee_{i=1}^n F_i)^+(\mu)$. Thus, $(\vee_{i=1}^n F_i)^+(\mu)$ is open and hence $\vee_{i=1}^n F_i$ is a fuzzy upper contra-continuous multifunction. □

Lemma 3.15. [11] *Let* $\{\mu_i\}_{i \in I}$ *be a family of fuzzy sets in a fuzzy topological space* X*. Then a fuzzy point* x *is quasi-coincident with* $\vee \mu_i$ *if and only if there exists an* $i_0 \in I$ *such that* $xq\mu_{i_0}$*.*

Theorem 3.16. *If* $F_i : X \multimap Y$ *are fuzzy lower contra-continuous multifunctions for* $i = 1, 2, ..., n$*, then* $\vee_{i=1}^n F_i$ *is a fuzzy lower contra-continuous multifunction.*

Proof. Let μ be a fuzzy closed set of Y. We will show that $(\vee_{i=1}^{n} F_i)^{-}(\mu) = \{x \in X : (\vee_{i=1}^{n} F_i)(x) q \mu\}$ is open in X. Let $x \in (\vee_{i=1}^{n} F_i)^{-}(\mu)$. Then $(\vee_{i=1}^{n} F_i)(x) q\mu$ and hence $F_{i_0}(x) q\mu$ for an i_0. Since $F_{i_0} : X \multimap Y$ is fuzzy lower contra-continuous multifunction, there exists an open set U_x containing x such that for all $z \in U$, $F_{i_0}(z) q\mu$. Then $(\vee_{i=1}^{n} F_i)(z) q\mu$ and hence $U \subset (\vee_{i=1}^{n} F_i)^{-}(\mu)$. Thus, $(\vee_{i=1}^{n} F_i)^{-}(\mu)$ is open and hence $\vee_{i=1}^{n} F_i$ is a fuzzy lower contra-continuous multifunction. □

Theorem 3.17. *Let $F : X \multimap Y$ be a fuzzy multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

(1) *$F_i = F\,|_{U_i}$ is a fuzzy lower contra-continuous multifunction for all $i \in I$,*
(2) *F is fuzzy lower contra-continuous.*

Proof. (1) ⇒ (2) : Let $x \in X$ and μ be a fuzzy closed set in Y with $x \in F^{-}(\mu)$. Since $\{U_i : i \in I\}$ is an open cover for X, then $x \in U_{i_0}$ for an $i_0 \in I$. We have $F(x) = F_{i_0}(x)$ and hence $x \in F_{i_0}^{-}(\mu)$. Since $F\,|_{U_{i_0}}$ is fuzzy lower contra-continuous, there exists an open set $B = G \cap U_{i_0}$ in U_{i_0} such that $x \in B$ and $F^{-}(\mu) \cap U_{i_0} = F\,|_{U_i}^{-}(\mu) \supset B = G \cap U_{i_0}$, where G is open in X. We have $x \in B = G \cap U_{i_0} \subset F\,|_{U_{i_0}}^{-}(\mu) = F^{-}(\mu) \cap U_{i_0} \subset F^{-}(\mu)$. Hence, F is fuzzy lower contra-continuous.

(2) ⇒ (1) : Let $x \in X$ and $x \in U_i$. Let μ be a fuzzy closed set in Y with $F_i(x) q\mu$. Since F is lower contra-continuous and $F(x) = F_i(x)$, there exists an open set U containing x such that $U \subset F^{-}(\mu)$. Take $B = U_i \cap U$. Then B is open in U_i containing x. We have $B \subset F_i^{-}(\mu)$. Thus F_i is a fuzzy lower contra-continuous. □

Theorem 3.18. *Let $F : X \multimap Y$ be a fuzzy multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

(1) *$F_i = F\,|_{U_i}$ is a fuzzy upper contra-continuous multifunction for all $i \in I$,*
(2) *F is fuzzy upper contra-continuous.*

Proof. It is similar to that of Theorem 3.17. □

Recall that for a multifunction $F_1 : X \multimap Y$ and a fuzzy multifunction $F_2 : Y \multimap Z$, the fuzzy multifunction $F_2 \circ F_1 : X \multimap Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for $x \in X$.

Definition 3.19. Let X and Y be topological spaces. A multifunction $F : X \multimap Y$ is called

(1) lower semi-continuous [28] if for any open subset $A \subset Y$ with $x \in F^{-}(A)$, there exists an open set B in X containing x such that $B \subset F^{-}(A)$.

(2) upper semi-continuous [28] if for any open subset $A \subset Y$ with $x \in F^{+}(A)$, there exists an open set B in X containing x such that $B \subset F^{+}(A)$.

Theorem 3.20. *If $F_1 : X \multimap Y$ is an upper semi-continuous multifunction where X and Y are topological spaces and $F_2 : Y \multimap Z$ is a fuzzy upper contra-continuous* *multifunction where Z is a fuzzy topological space, then $F_2 \circ F_1$ is fuzzy upper contra-continuous.*

Proof. Let $x \in X$ and μ be a fuzzy closed set in Z. We have $(F_2 \circ F_1)^+(\mu) = F_1^+(F_2^+(\mu))$. Since F_2 is fuzzy upper contra-continuous, $F_2^+(\mu)$ is open in Y. Since F_1 is upper semi-continuous, $F_1^+(F_2^+(\mu)) = (F_2 \circ F_1)^+(\mu)$ is open in X. Thus, $F_2 \circ F_1$ is fuzzy upper contra-continuous. □

Definition 3.21. A fuzzy set μ in a fuzzy topological space X is called

(1) a fuzzy cl-neighbourhood of a fuzzy point x_α in X if there exists a fuzzy closed set ρ in X such that $x \in \rho \leq \mu$.

(2) a fuzzy cl-neighbourhood of a fuzzy set η in X if there exists a fuzzy closed set ρ in X such that $\eta \leq \rho \leq \mu$.

Theorem 3.22. *If $F : X \multimap Y$ is a fuzzy upper contra-continuous multifunction, then for each point x of X and each fuzzy cl-neighbourhood μ of $F(x)$, $F^+(\mu)$ is a neighbourhood of x.*

Proof. Let $x \in X$ and μ be a fuzzy cl-neighbourhood of $F(x)$. There exists a fuzzy closed set ρ in Y such that $F(x) \leq \rho \leq \mu$. We have $x \in F^+(\rho) \leq F^+(\mu)$. Since $F^+(\rho)$ is an open set, $F^+(\mu)$ is a neighbourhood of x. □

Remark 3.23. [36] A subset A of a topological space (X, τ) can be considered as a fuzzy set with characteristic function defined by

$$A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

Let (Y, σ) be a fuzzy topological space. The fuzzy sets of the form $A \times \rho$ with $A \in \tau$ and $\rho \in \sigma$ form a basis for the product fuzzy topology $\tau \times \sigma$ on $X \times Y$, where for any $(x, y) \in X \times Y$,

$$(A \times \rho)(x, y) = min\{A(x), \rho(y)\}$$

Definition 3.24. [21] For a fuzzy multifunction $F : X \multimap Y$, the fuzzy graph multifunction $G_F : X \multimap X \times Y$ of F is defined by $G_F(x) = x_1 \times F(x)$ for every $x \in X$.

Theorem 3.25. *If the fuzzy graph multifunction G_F of a fuzzy multifunction $F : X \multimap Y$ is fuzzy lower contra-continuous, then F is fuzzy lower contra-continuous.*

Proof. Suppose that G_F is fuzzy lower contra-continuous and $x \in X$. Let μ be a fuzzy closed set in Y such that $F(x) q \mu$. Then there exists $y \in Y$ such that $(F(x))(y) + \mu(y) > 1$. Then $(G_F(x))(x, y) + (X \times \mu)(x, y) = (F(x))(y) + \mu(y) > 1$. Hence, $G_F(x) q (X \times \mu)$. Since G_F is fuzzy lower contra-continuous, there exists an open set B in X such that $x \in B$ and $G_F(b) q (X \times \mu)$ for all $b \in B$.

Let there exists $b_0 \in B$ such that $F(b_0) \overline{q} \mu$. Then for all $y \in Y$, $(F(b_0))(y) + \mu(y) \leq 1$. For any $(a, c) \in X \times Y$, we have $(G_F(b_0))(a, c) \leq (F(b_0))(c)$ and

$(X \times \mu)(a, c) \leq \mu(c)$. Since for all $y \in Y$, $(F(b_0))(y) + \mu(y) \leq 1$, $(G_F(b_0))(a, c) + (X \times \mu)(a, c) \leq 1$. Thus, $G_F(b_0)\overline{q}(X \times \mu)$, where $b_0 \in B$. This is a contradiction since $G_F(b)q(X \times \mu)$ for all $b \in B$.

Hence, F is fuzzy lower contra-continuous. □

Theorem 3.26. *If the fuzzy graph multifunction G_F of a fuzzy multifunction $F : X \multimap Y$ is fuzzy upper contra-continuous, then F is fuzzy upper contra-continuous.*

Proof. Suppose that G_F is fuzzy upper contra-continuous and let $x \in X$. Let μ be fuzzy closed in Y with $F(x) \leq \mu$. Then $G_F(x) \leq X \times \mu$. Since G_F is fuzzy upper contra-continuous, there exists an open set B containing x such that $G_F(B) \leq X \times \mu$. For any $b \in B$ and $y \in Y$, we have $(F(b))(y) = (G_F(b))(b, y) \leq (X \times \mu)(b, y) = \mu(y)$. Then $(F(b))(y) \leq \mu(y)$ for all $y \in Y$. Thus, $F(b) \leq \mu$ for any $b \in B$. Hence, F is fuzzy upper contra-continuous. □

Theorem 3.27. *Let $F : X \multimap Y$ be a fuzzy multifunction. Then the following are equivalent:*

(1) *F is fuzzy lower contra-continuous,*

(2) *For any $x \in X$ and any net $(x_i)_{i \in I}$ converging to x in X and each fuzzy closed set ρ in Y with $x \in F^-(\rho)$, the net $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.*

Proof. $(1) \Rightarrow (2)$: Let (x_i) be a net converging to x in X and ρ be any fuzzy closed set in Y with $x \in F^-(\rho)$. Since F is fuzzy lower contra-continuous, there exists an open set $A \subset X$ containing x such that $A \subset F^-(\rho)$. Since $x_i \to x$, there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. We have $x_i \in A \subset F^-(\rho)$ for all $i \geq i_0$. Hence, $(x_i)_{i \in I}$ is eventually in $F^-(\rho)$.

$(2) \Rightarrow (1)$: Suppose that F is not fuzzy lower contra-continuous. There exists a point x and a fuzzy closed set μ with $x \in F^-(\mu)$ such that $B \not\subseteq F^-(\mu)$ for any open set $B \subset X$ containing x. Let $x_i \in B$ and $x_i \notin F^-(\mu)$ for each open set $B \subset X$ containing x. Then the neighborhood net (x_i) converges to x but $(x_i)_{i \in I}$ is not eventually in $F^-(\mu)$. This is a contradiction. □

Theorem 3.28. *Let $F : X \multimap Y$ be a fuzzy multifunction. Then the following are equivalent:*

(1) *F is fuzzy upper contra-continuous,*

(2) *For any $x \in X$ and any net (x_i) converging to x in X and any fuzzy closed set ρ in Y with $x \in F^+(\rho)$, the net (x_i) is eventually in $F^+(\rho)$.*

Proof. The proof is similar to that of Theorem 3.27. □

Recall that the frontier of a subset A of a topological space X, denoted by $Fr(A)$, is defined by $Fr(A) = Cl(A) \cap Cl(X \backslash A) = Cl(A) \backslash Int(A)$.

Theorem 3.29. *The set of all points of X at which a fuzzy multifunction $F : X \multimap Y$ is not fuzzy upper contra-continuous is identical with the union of the frontier of the upper inverse image of fuzzy closed sets containing $F(x)$.*

Proof. Suppose F is not fuzzy upper contra-continuous at $x \in X$. Then there exists a fuzzy closed set η in Y containing $F(x)$ such that $A \cap (X \backslash F^+(\eta)) \neq \emptyset$ for every open set A containing x. We have $x \in Cl(X \backslash F^+(\eta)) = X \backslash Int(F^+(\eta))$ and $x \in F^+(\eta)$. Thus, $x \in Fr(F^+(\eta))$.

Conversely, let η be a fuzzy closed set in Y containing $F(x)$ with $x \in Fr(F^+(\eta))$. Suppose that F is fuzzy upper contra-continuous at x. There exists an open set A containing x such that $A \subset F^+(\eta)$. We have $x \in Int(F^+(\eta))$. This is a contradiction. Thus, F is not fuzzy upper contra-continuous at x. □

Theorem 3.30. *The set of all points of X at which a fuzzy multifunction $F : X \multimap Y$ is not fuzzy lower contra-continuous is identical with the union of the frontier of the lower inverse image of fuzzy closed sets which are quasi-coincident with $F(x)$.*

Proof. It is similar to that of Theorem 3.29. □

Definition 3.31. A fuzzy topological space X is called fuzzy strongly S-closed [2] if every fuzzy closed cover of X has a finite subcover.

Theorem 3.32. *Let $F : X \multimap Y$ be a fuzzy upper contra-continuous surjective multifunction. Suppose that $F(x)$ is fuzzy strongly S-closed for each $x \in X$. If X is compact, then Y is fuzzy strongly S-closed.*

Proof. Let $\{\mu_k\}_{k \in I}$ be a fuzzy closed cover of Y. Since $F(x)$ is fuzzy strongly S-closed for any $x \in X$, there exists a finite subset I_x of I such that $F(x) \leq \vee_{k \in I_x} \mu_k$. Take $\mu_x = \vee_{k \in I_x} \mu_k$. Since F is fuzzy upper contra-continuous, there exists a fuzzy open set U_x of X containing x such that $F(U_x) \leq \mu_x$. Then $\{U_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exist x_1, x_2, x_3, ...,x_n in X such that $X = \cup_{i=1}^{n} U_{x_i}$. We have $Y = F(X) = F(\overset{n}{\underset{i=1}{\cup}} U_{x_i}) \leq \overset{n}{\underset{i=1}{\vee}} F(U_{x_i}) \leq \overset{n}{\underset{i=1}{\vee}} \mu_{x_i} = \overset{n}{\underset{i=1}{\vee}} \underset{k \in I_{x_i}}{\vee} \mu_k$. Thus, Y is fuzzy strongly S-closed. □

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