M-FUZZIFYING INTERVAL SPACES

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ABSTRACT. In this paper, we introduce the notion of M-fuzzifying interval spaces, and discuss the relationship between M-fuzzifying interval spaces and M-fuzzifying convex structures. It is proved that the category **MYCSA2** can be embedded in the category **MYIS** as a reflective subcategory, where **MYCSA2** and **MYIS** denote the category of M-fuzzifying convex structures of M-fuzzifying arity ≤ 2 and the category of M-fuzzifying interval spaces, respectively. Under the framework of M-fuzzifying interval spaces, subspaces and product spaces are presented and some of their fundamental properties are obtained.

1. Introduction

Convexity theory has been accepted to be of increasing importance in recent years in the study of extremum problems in many areas of applied mathematics. The concept of convexity which was mainly defined and studied in R^n in the pioneering works of Newton, Minkowski and others as described in [1], now finds a place in several other mathematical structures such as vector spaces, posets, lattices, metric spaces, graphs and median algebras. This development is motivated not only by the need for an abstract theory of convexity generalizing the classical theorems in R^n due to Helly, Caratheodory etc; but also by the necessity to unify geometric aspects of all these mathematical structures. Some more details can be found in [9, 24].

In 1994, Rosa presented the notion of fuzzy convex structures in [15, 16]. In 2009, Maruyama generalized it to M-fuzzy setting in [11]. A fuzzy convex structure is a pair of $(X, \mathcal{C})$ in which $\mathcal{C}$ is a crisp subset of the set of M-fuzzy subsets of a nonempty set X satisfying certain set of axioms. Recently, a new approach to the fuzzification of convex structures is introduced in [21]. It is called an M-fuzzifying convex structure, in which each subset of X can be regarded as a convex set to some degree.

In classical theory of convex structures, interval operators provide a natural and frequent method of describing or constructing convex structures. In [13], fuzzy interval operators were defined to describe fuzzy convex structures. The present paper is a continuation of investigations on M-fuzzifying convex structures, started in [21]. Now we are concerned with the so-called M-fuzzifying interval spaces,

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and discuss the relationship between M-fuzzifying interval spaces and M-fuzzifying convex structures from the category theory point view . We also define the notions of a subspace and a product space of M-fuzzifying interval spaces and study some of their fundamental properties.

2. Preliminaries

Throughout this paper, $(M, \bigvee, \bigwedge, ')$ denotes a completely distributive lattice with an order-reversing involution $'$. The smallest element (or zero element) and the largest element (or unit element) in M are denoted by $\perp$ and $\top$, respectively. $\mathbf{2}^X$, $\mathbf{2}^X_{fin}$, denotes the collection of all subsets, all finite subsets of a nonempty set X respectively.

An element a in M is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [5]. The set of non-zero co-prime elements in M is denoted by $J(M)$.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [2]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [25], denoted by $\beta(b)$. Moreover, for $b \in M$, we define $\alpha(b) = \{a \in M : b \prec^{op} a\}$.

In a completely distributive lattice M with an order-reversing involution $'$, there exist $\alpha(b)$ and $\beta(b)$ for each $b \in M$, $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ and $a \prec b \Leftrightarrow b' \prec^{op} a'$(see[25]).

For $A \in M^X$ and $a \in M$, we use the following notations:
$A_{[a]} = \{x \in X : A(x) \geq a\}$, $A_{(a)} = \{x \in X : a \in \beta(A(x))\}$,
$A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}$, $A^{(a)} = \{x \in X : A(x) \not\leq a\}$.

Some properties of these cut sets can be found in [7, 12, 17, 18, 19, 20].

Let $f : X \longrightarrow Y$ be a mapping. Define $f^{\rightarrow}_M : M^X \longrightarrow M^Y$ and $f^{\leftarrow}_M : M^Y \longrightarrow M^X$ by $f^{\rightarrow}_M(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in M^X$ and $y \in Y$, and $f^{\leftarrow}_M(B) = B \circ f$ for $B \in M^Y$, respectively.

Here we point out that $f^{\rightarrow}_M$ and $f^{\rightarrow}_M$ are the usual Zadeh image and preimage operators, respectively. For more details we refer the reader to [14].

In [8, 6, 22], the concept of M-fuzzy non-negative real numbers were introduced as follows:

An M-fuzzy non-negative real number is an equivalence class $[\lambda]$ of antitone maps $\lambda : \mathbb{R} \longrightarrow M$ satisfying $\lambda(0-) = \bigvee_{t<0} \lambda(t) = \top, \lambda(+\infty) = \bigwedge_{t \in R} \lambda(t) = \perp$, where $[\lambda] = \{\mu : \forall t > 0, \lambda(t-) = \mu(t-)\}$.

We do not distinguish between an M-fuzzy real number $[\lambda]$ and a left continuous fuzzy function λ representing it. The set of all non-negative M-fuzzy real numbers is denoted by $[0, +\infty)(M)$.

Theorem 2.1. [25] *For a subfamily $\{a_i : i \in \Omega\}$ of M, we have*

(1) $\alpha\left(\bigwedge_{i\in\Omega} a_i\right) = \bigcup_{i\in\Omega} \alpha(a_i)$, *i.e., α is a $\bigwedge - \bigcup$ mapping.*
(2) $\beta\left(\bigvee_{i\in\Omega} a_i\right) = \bigcup_{i\in\Omega} \beta(a_i)$, *i.e., β is a $\bigvee - \bigcup$ mapping.*

Theorem 2.2. [17] *For each M-fuzzy set A in M^X and $a \in M$,* $\left(\bigvee_{i\in\Omega} A_i\right)_{(a)} = \bigcup_{i\in\Omega} (A_i)_{(a)}$.

Based on [10] and [22], we can give the following definition:

Definition 2.3. A map $d : X \times X \longrightarrow [0, +\infty)(M)$ is called an M-fuzzifying pseudo-quasi-metric on X if it satisfies the following conditions. For all $x, y, z \in X$ and for all $s, t > 0$,

(Md1) $x = y \Longrightarrow d(x, y)(0+) = \bot$;
(Md2) $d(x, z)(r + s) \leq d(x, y)(r) \vee d(y, z)(s)$;
(Md3) $d(x, y) = d(y, x)$.

Definition 2.4. [24] Let $\mathcal{I} : X \times X \longrightarrow \mathbf{2}^X$ be a mapping satisfying the following conditions: for all $x, y \in X$,

(I1) $x, y \in \mathcal{I}(x, y)$;
(I2) $\mathcal{I}(x, y) = \mathcal{I}(y, x)$.

Then $\mathcal{I}$ is called an interval operator on X, and $\mathcal{I}(x, y)$ is the interval between x and y. The resulting pair $(X, \mathcal{I})$ is called an interval space.

Definition 2.5. [24] A convex structure $(X, \mathcal{C})$ is of *arity* $\leq n$ provided its convex sets are precisely the sets with the property that $co(F) \subseteq C$ for each subset F with $|F| \leq n$.

Theorem 2.6. [24] *A convex structure is induced by an interval operator if and only if it is of arity* ≤ 2.

Theorem 2.7. [24] *Let* $(X, \mathcal{C}_X)$ *and* $(Y, \mathcal{C}_Y)$ *be convex structures and* co_X, co_Y *denote the hull operators, respectively.*

(1) $f : X \longrightarrow Y$ *is a convexity preserving function if and only if* $f(co_X(F)) \subseteq co_Y(f(F))$ *for any* $F \in \mathbf{2}^X_{fin}$. *If* $(X, \mathcal{C}_X)$ *is of arity* $\leq n$, *then it suffices to consider sets* $|F| \leq n$.

(2) $f : X \longrightarrow Y$ *is a convex-to-convex function if and only if* $f(co_X(F)) \supseteq co_Y(f(F))$ *for any* $F \in \mathbf{2}^X_{fin}$. *If* $(X, \mathcal{C}_X)$ *is of arity* $\leq n$, *then it suffices to consider sets* $|F| \leq n$.

Definition 2.8. [4] Let $A \in M^X$, $\emptyset \neq Y \subseteq X$. We define an M-fuzzy set $A|Y \in M^Y$ as follows: $\forall y \in Y$, $(A|Y)(y) = A(y)$. $A|Y$ is called the restriction of A to Y.

Definition 2.9. [13] Let X be a set and let $\mathcal{I} : X \times X \longrightarrow [0, 1]^X$ be a mapping with the following properties: for all $x, y \in X$,

(FI1) $\mathcal{I}(x, y)(x) > 0$, $\mathcal{I}(x, y)(y) > 0$;
(FI2) $\mathcal{I}(x, y) = \mathcal{I}(y, x)$.

Then $\mathcal{I}$ is called a fuzzy interval operator on X. The resulting pair $(X, \mathcal{I})$ is called a fuzzy interval space and $\mathcal{I}(x, y)$ is called the fuzzy interval between x and y for each $x, y \in X$.

Definition 2.10. [21] A mapping $\mathcal{C} : \mathbf{2}^X \longrightarrow M$ is called an M-fuzzifying convexity on X if it satisfies the following conditions:

(MYC1) $\mathcal{C}(\emptyset) = \mathcal{C}(X) = \top$;
(MYC2) if $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ is nonempty, then $\mathcal{C}(\bigcap_{i \in \Omega} A_i) \geq \bigwedge_{i \in \Omega} \mathcal{C}(A_i)$;

(MYC3) if $\{A_i : i \in \Omega\} \subseteq \mathbf{2}^X$ is nonempty and totally ordered by inclusion, then $\mathcal{C}(\bigcup_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{C}(A_i)$.

If $\mathcal{C}$ is an M-fuzzifying convexity on X, then the pair $(X, \mathcal{C})$ is called an M-fuzzifying convex structure.

Theorem 2.11. [21] *Let $\mathcal{C} : \mathbf{2}^X \longrightarrow M$ be a mapping. Then the following are equivalent.*

(1) *$(X, \mathcal{C})$ is an M-fuzzifying convex structure.*

(2) *For each $a \in M\backslash\{\bot\}$, $(X, \mathcal{C}_{[a]})$ is a convex structure.*

(3) *For each $a \in \alpha(\bot)$, $(X, \mathcal{C}^{[a]})$ is a convex structure.*

Definition 2.12. [21] Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures. A function $f : X \longrightarrow Y$ is called an M-fuzzifying convex-to-convex function if $\mathcal{C}_Y(f(A)) \geq \mathcal{C}_X(A)$ for all $A \in \mathbf{2}^X$. $f : X \longrightarrow Y$ is called an M-fuzzifying convexity preserving function if $\mathcal{C}_X\left(f^{-1}(B)\right) \geq \mathcal{C}_Y(B)$ for all $B \in \mathbf{2}^Y$.

Theorem 2.13. [21] *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures. Then a function $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is an M-fuzzifying convexity preserving function if and only if $f : (X, \mathcal{C}_X^{[a]}) \longrightarrow (Y, \mathcal{C}_Y^{[a]})$ is a convexity preserving function for any $a \in \alpha(\bot)$.*

Theorem 2.14. [21] *Let $(X, \mathcal{C})$ be an M-fuzzifying convex structure. Define a mapping $co_{\mathcal{C}} : \mathbf{2}^X \longrightarrow M$ by*

$$\forall x \in X,\ \forall A \subseteq X,\ co_{\mathcal{C}}(A)(x) = \bigwedge_{x\notin B\supseteq A} \mathcal{C}(B)'.$$

Then $co_{\mathcal{C}}$ satisfies the following conditions:

(C0) *$co_{\mathcal{C}}(\emptyset)(x) = \bot$ for every $x \in X$.*

(C1) *$co_{\mathcal{C}}(A)(x) = \top$ for every $x \in A$.*

(C2) *$A \subseteq B \Longrightarrow co_{\mathcal{C}}(A) \leq co_{\mathcal{C}}(B)$.*

(C3) *$co_{\mathcal{C}}(A)(x) = \bigwedge_{x\notin B\supseteq A} \bigvee_{y\notin B} co_{\mathcal{C}}(B)(y)$.*

(MDF) *$co_{\mathcal{C}}(A)(x) = \bigvee\{co_{\mathcal{C}}(F)(x) : F \in \mathbf{2}^A_{fin}\}$.*

Conversely, let a mapping $co : \mathbf{2}^X \longrightarrow M^X$ satisfy $(C0)-(C3)$ and (MDF). Define a mapping $\mathcal{C}_{co} : \mathbf{2}^X \longrightarrow M$ by

$$\forall A \subseteq X,\ \mathcal{C}_{co}(A) = \bigwedge_{x\notin A} (co(A)(x))'.$$

Then $(X, \mathcal{C}_{co})$ is an M-fuzzifying convex structure. Moreover, $co_{\mathcal{C}_{co}} = co$.

Theorem 2.15. [21] *Let $(X, \mathcal{C})$ be an M-fuzzifying convex structure. Then $\mathcal{C}_{co_{\mathcal{C}}} = \mathcal{C}$.*

Remark 2.16. If $(X, \mathcal{C})$ is an M-fuzzifying convex structure, then we call $co_{\mathcal{C}}$ an M-fuzzifying hull operator for $(X, \mathcal{C})$. For each $x, y \in X$, we call $co_{\mathcal{C}}\{x, y\}$ the M-fuzzifying segment joining x and y.

Theorem 2.17. [23] *Let $\mathcal{C}$ be the M-fuzzifying convex structure on X. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in M$, then for any $a \in \alpha(\bot)$ and $A \subseteq X$, $co_{(\mathcal{C}^{[a]})}(A) = co_{\mathcal{C}}(A)_{(a')}$.*

Theorem 2.18. [21] *Let $(X, \mathcal{C})$ be an M-fuzzifying convex structure, $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an M-fuzzifying convex structure on Y, where $(\mathcal{C}|Y)(A) = \bigvee\{\mathcal{C}(B) : B \in \mathbf{2}^X,\ B \bigcap Y = A\}$ for every $A \in \mathbf{2}^Y$. We call $(Y, \mathcal{C}|Y)$ an M-fuzzifying substructure of $(X, \mathcal{C})$.*

Definition 2.19. [21] Let $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex structures. Let X be the product of the sets X_t for $t \in T$, and let $\pi_t : X \longrightarrow X_t$ denote the projection for each $t \in T$. Define a mapping $\varphi : \mathbf{2}^X \longrightarrow M$ by $\varphi(A) = \bigvee_{t\in T} \bigvee_{(\pi_t)^{-1}(B)=A} \mathcal{C}_t(B)$ for each $A \in \mathbf{2}^X$. Then the product convexity $\mathcal{C}$ of X, denoted by $\prod_{t\in T} \mathcal{C}_t$, is the one generated by φ. That is $\prod_{t\in T} \mathcal{C}_t$ is the finest convexity containing φ. The resulting M-fuzzifying convex structure $(X, \mathcal{C})$ is called the product of $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ and is denoted by $\prod_{t\in T}(X_t, \mathcal{C}_t)$.

Theorem 2.20. [21] *Let $(Y, \mathcal{D})$ be an M-fuzzifying convex structure and $f : X \to Y$ a surjective function. Define a mapping $f^{-1}(\mathcal{D}) : \mathbf{2}^X \to M$ by*

$$\forall A \in \mathbf{2}^X,\ f^{-1}(\mathcal{D})(A) = \bigvee \left\{\mathcal{D}(B) : f^{-1}(B) = A\right\}.$$

Then $(X, f^{-1}(\mathcal{D}))$ is an M-fuzzifying convex structure.

Definition 2.21. [21] Let $\{(X, \mathcal{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex structures on X. Define a maping $\varphi : \mathbf{2}^X \longrightarrow M$ by $\varphi(A) = \bigvee_{t\in T} \mathcal{C}_t(A)$ for each $A \in \mathbf{2}^X$. Then the join of $\{\mathcal{C}_t\}_{t\in T}$, denoted by $\bigsqcup_{t\in T} \mathcal{C}_t$, is the one generated by φ. That is $\bigsqcup_{t\in T} \mathcal{C}_t$ is the finest convexity containing φ. The resulting M-fuzzifying convex structure $(X, \bigsqcup_{t\in T} \mathcal{C}_t)$ is called the join of $\{(X, \mathcal{C}_t)\}_{t\in T}$ and is denoted by $\bigsqcup_{t\in T}(X, \mathcal{C}_t)$.

Theorem 2.22. [21] *Let $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex structures. Let X be the product of the sets X_t for $t \in T$. Then $\prod_{t\in T}(X_t, \mathcal{C}_t) = \bigsqcup_{t\in T}(X, (\pi_t)^{-1}(\mathcal{C}_t))$.*

3. M-fuzzifying Interval Spaces

In this section, we generalize the notion of interval spaces to M-fuzzy setting and discuss the relationship between M-fuzzifying interval spaces and M-fuzzifying convex structures.

Definition 3.1. A mapping $\mathcal{I} : X \times X \longrightarrow M^X$ is called an M-fuzzifying interval operator on X if it satisfies the following conditions: for all $x, y \in X$,
(MYI1) $\mathcal{I}(x, y)(x) = \mathcal{I}(x, y)(y) = \top$;
(MYI2) $\mathcal{I}(x, y) = \mathcal{I}(y, x)$.
If $\mathcal{I}$ is an M-fuzzifying interval operator on X, then the pair $(X, \mathcal{I})$ is called an M-fuzzifying interval space. For $x, y \in X$, $\mathcal{I}(x, y)$ is the M-fuzzifying interval between x and y.

Remark 3.2. Definitions 2.9 and 3.1 are both generalizations of interval spaces. However, Definition 2.9 is related to fuzzy convex structures in [15, 16], and an M-fuzzifying interval space is defined for M-fuzzifying convex structures introduced in [21].

Example 3.3. Let d be an M-fuzzifying pseudo-quasi-metric on X. Define an operator $\mathcal{I}_d : X \times X \longrightarrow M^X$ by

$$\forall x, y, z \in X,\ \mathcal{I}_d(x,y)(z) = \bigwedge_{r>0} \bigvee_{p \in \{x,y\}} (d(p,z)(r))'.$$

It is easy to verify that $\mathcal{I}_d$ is an M-fuzzifying interval operator on X.

Theorem 3.4. *Let $\mathcal{I} : X \times X \longrightarrow M^X$ be a mapping. Then the following conditions are equivalent:*
(1) *$(X, \mathcal{I})$ is an M-fuzzifying interval space.*
(2) *For each $a \in M \backslash \{\top\}$, $(X, \mathcal{I}^a)$ is an interval space, where $\mathcal{I}^a(x,y) = \mathcal{I}(x,y)^{(a)}$ for all $x, y \in X$.*
(3) *For each $a \in \alpha(\bot)$, $(X, \mathcal{I}_a)$ is an interval space, where $\mathcal{I}_a(x,y) = \mathcal{I}(x,y)_{(a)}$ for all $x, y \in X$.*

Proof. (2) $\Rightarrow$ (1) (MYI1) For each $a \in M \backslash \{\top\}$, let $x, y \in \mathcal{I}^a$. Then we have $x, y \in \mathcal{I}^a(x,y) = \mathcal{I}(x,y)^{(a)}$, that is, $\mathcal{I}(x,y)(x) \nleq a$ and $\mathcal{I}(x,y)(y) \nleq a$. Hence $\mathcal{I}(x,y)(x) = \mathcal{I}(x,y)(y) = \top$.

(MYI2) For any $x, y, z \in X$ and $a \in M \backslash \{\top\}$,

$$\begin{aligned} \mathcal{I}(x,y)(z) \nleq a &\Leftrightarrow z \in \mathcal{I}(x,y)^{(a)} \\ &\Leftrightarrow z \in \mathcal{I}^a(x,y) \\ &\Leftrightarrow z \in \mathcal{I}^a(y,x) \\ &\Leftrightarrow z \in \mathcal{I}(y,x)^{(a)} \\ &\Leftrightarrow \mathcal{I}(y,x)(z) \nleq a. \end{aligned}$$

This implies $\mathcal{I}(x,y) = \mathcal{I}(y,x)$.

(1) $\Rightarrow$ (2) Suppose that $\mathcal{I} : X \times X \longrightarrow M^X$ is an M-fuzzifying interval operator and $a \in M \backslash \{\top\}$. Now we prove that $(X, \mathcal{I}^a)$ is an interval space.

(I1) For all $x, y \in X$, by $\mathcal{I}(x,y)(x) = \mathcal{I}(x,y)(y) = \top$, we know that

$$x, y \in \mathcal{I}(x,y)^{(a)} = \mathcal{I}^a(x,y).$$

(I2) For all $x, y \in X$,, by $\mathcal{I}(x,y) = \mathcal{I}(y,x)$, we know that

$$\mathcal{I}^a(y,x) = \mathcal{I}(y,x)^{(a)} = \mathcal{I}(x,y)^{(a)} = \mathcal{I}^a(x,y).$$

The proof of (1) $\Leftrightarrow$ (3) is similar to that of (1) $\Leftrightarrow$ (2) and is omitted. □

The next theorem shows that an M-fuzzifying convex structure induces an M-fuzzifying interval space in a natural way.

Theorem 3.5. *Let $(X, \mathcal{C})$ be an M-fuzzifying convex structure. Define a mapping $Seg_{\mathcal{C}} : X \times X \longrightarrow M^X$ by*

$$\forall x, y, z \in X,\ Seg_{\mathcal{C}}(x,y)(z) = co_{\mathcal{C}}\{x,y\}(z) = \bigwedge_{z \notin B \supseteq \{x,y\}} \mathcal{C}(B)'.$$

Then $Seg_{\mathcal{C}}$ is an M-fuzzifying interval operator. We call $Seg_{\mathcal{C}}$ the M-fuzzifying segment operator for $(X, \mathcal{C})$.

Proof. It is obvious that $Seg_{\mathcal{C}}$ satisfies (MYI1) and (MYI2). □

From [24], we know that if $(X,\mathcal{I})$ is an interval space, then the convex structure $(X,\mathcal{C})$, introduced by $(X,\mathcal{I})$, is obtained as follows: $\mathcal{C}=\{A\subseteq X:\forall x,y\in A,\mathcal{I}(x,y)\subseteq A\}$, that is, $A\in\mathcal{C}\Leftrightarrow A\supseteq\bigcup\{\mathcal{I}(x,y):\forall x,y\in A\}\Leftrightarrow\forall z\notin A,\forall x,y\in A,z\notin\mathcal{I}(x,y)$.

Based on the above fact, we can get the following theorem, which shows that an M-fuzzifying interval space induces an M-fuzzifying convex structure.

Theorem 3.6. *Let $(X,\mathcal{I})$ be an M-fuzzifying interval space. Define a mapping $\mathcal{C}_{\mathcal{I}}:\mathbf{2}^X\longrightarrow M$ by*

$$\forall A\subseteq X,\ \mathcal{C}_{\mathcal{I}}(A)=\bigwedge_{z\notin A}\bigwedge_{x,y\in A}(\mathcal{I}(x,y)(z))'.$$

Then $(X,\mathcal{C}_{\mathcal{I}})$ is an M-fuzzifying convex structure. Moreover, $\mathcal{I}\leq Seg_{\mathcal{C}_{\mathcal{I}}}$.

Proof. Since $(X,\mathcal{I})$ is an M-fuzzifying interval space, by Theorem 3.4, for each $a\in M\backslash\{\top\}$, $(X,\mathcal{I}^a)$ is an interval space. Let $(X,\mathcal{C}^a)$ denote the convex strucure induced by $(X,\mathcal{I}^a)$. By Theorem 2.6, $(X,\mathcal{C}^a)$ is of *arity* ≤ 2. Then for each $a\in M\backslash\{\bot\}$, we have

$$\begin{array}{rcl} A\in(\mathcal{C}_{\mathcal{I}})_{[a]} & \Leftrightarrow & \mathcal{C}_{\mathcal{I}}(A)\geq a\\ & \Leftrightarrow & \bigwedge_{z\notin A}\bigwedge_{x,y\in A}(\mathcal{I}(x,y)(z))'\geq a\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ (\mathcal{I}(x,y)(z))'\geq a\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ \mathcal{I}(x,y)(z)\leq a'\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ z\notin\mathcal{I}(x,y)^{(a')}\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ z\notin\mathcal{I}^{a'}(x,y)\\ & \Leftrightarrow & A\in\mathcal{C}^{a'}. \end{array}$$

This implies $(\mathcal{C}_{\mathcal{I}})_{[a]}=\mathcal{C}^{a'}$ for each $a\in M\backslash\{\bot\}$. By Theorem 2.11, $(X,\mathcal{C}_{\mathcal{I}})$ is an M-fuzzifying convex structure.

To show the last inequality, let $p,q\in X$. Then for every $h\in X$,

$$\begin{array}{rcl} Seg_{\mathcal{C}_{\mathcal{I}}}(p,q)(h) & = & \bigwedge_{h\notin B\supseteq\{p,q\}}\mathcal{C}_{\mathcal{I}}(B)'\\ & = & \bigwedge_{h\notin B\supseteq\{p,q\}}(\bigwedge_{z\notin B}\bigwedge_{x,y\in B}(\mathcal{I}(x,y)(z))')'\\ & = & \bigwedge_{h\notin B\supseteq\{p,q\}}(\bigvee_{z\notin B}\bigvee_{x,y\in B}\mathcal{I}(x,y)(z))\\ & \geq & \bigwedge_{h\notin B\supseteq\{p,q\}}\mathcal{I}(p,q)(h)\\ & = & \mathcal{I}(p,q)(h). \end{array}$$

This implies $\mathcal{I}\leq Seg_{\mathcal{C}_{\mathcal{I}}}$. □

Lemma 3.7. *Let $(X,\mathcal{I})$ be an M-fuzzifying interval space and $\mathcal{C}_{\mathcal{I}}$ be the M-fuzzifying convexity induced by $(X,\mathcal{I})$. Then for each $a\in\alpha(\bot)$, $(\mathcal{C}_{\mathcal{I}})^{[a']}=\mathcal{C}_{\mathcal{I}_a}$.*

Proof. For each $A\subseteq X$ and $a\in\alpha(\bot)$,

$$\begin{array}{rcl} A\in(\mathcal{C}_{\mathcal{I}})^{[a']} & \Leftrightarrow & a'\notin\alpha(\mathcal{C}_{\mathcal{I}}(A))\\ & \Leftrightarrow & a'\notin\alpha(\bigwedge_{z\notin A}\bigwedge_{x,y\in A}(\mathcal{I}(x,y)(z))')\\ & \Leftrightarrow & a\notin\bigcup_{z\notin A}\bigcup_{x,y\in A}\alpha((\mathcal{I}(x,y)(z))')\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ a'\notin\alpha((\mathcal{I}(x,y)(z))')\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ a\notin\beta(\mathcal{I}(x,y)(z))\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ z\notin\mathcal{I}(x,y)_{(a)}\\ & \Leftrightarrow & \forall z\notin A,\ \forall x,y\in A,\ z\notin\mathcal{I}_a(x,y)\\ & \Leftrightarrow & A\in\mathcal{C}_{\mathcal{I}_a}. \end{array}$$

This implies for each $a\in\alpha(\bot)$, that $(\mathcal{C}_{\mathcal{I}})^{[a']}=\mathcal{C}_{\mathcal{I}_a}$. □

Next we give the formula of the M-fuzzifying hull operator of an M-fuzzifying convex structure induced by an M-fuzzifying interval space.

Theorem 3.8. *Let $(X,\mathcal{I})$ be an M-fuzzifying interval space and $\mathcal{C}_{\mathcal{I}}$ be the M-fuzzifying convexity induced by $(X,\mathcal{I})$. Then $co_{\mathcal{C}_{\mathcal{I}}}(A)=\bigvee_{k=0}^{\infty}A_k$, where $A_0=A$, $A_{k+1}(z)=\bigvee_{x,y\in X}\mathcal{I}(x,y)(z)\wedge A_k(x)\wedge A_k(y)$.*

Proof. For each $a\in\alpha(\bot)$,

$$\begin{aligned} z\in(A_{k+1})_{(a)} &\Leftrightarrow a\in\beta(A_{k+1}(z))\\ &\Leftrightarrow a\in\beta(\textstyle\bigvee_{x,y\in X}\mathcal{I}(x,y)(z)\wedge A_k(x)\wedge A_k(y))\\ &\Leftrightarrow a\in\textstyle\bigcup_{x,y\in X}\beta(\mathcal{I}(x,y)(z)\wedge A_k(x)\wedge A_k(y))\\ &\Leftrightarrow a\in\textstyle\bigcup_{x,y\in X}[\beta(\mathcal{I}(x,y)(z))\cap\beta(A_k(x))\cap\beta(A_k(y))]\\ &\Leftrightarrow \exists x,y\in X, s.t., a\in\beta(\mathcal{I}(x,y)(z)), a\in\beta(A_k(x)), a\in\beta(A_k(y))\\ &\Leftrightarrow \exists x,y\in X, s.t., z\in\mathcal{I}(x,y)_{(a)}, x,y\in(A_k)_{(a)}\\ &\Leftrightarrow \exists x,y\in X, s.t., z\in\mathcal{I}_a(x,y), x,y\in(A_k)_{(a)}. \end{aligned}$$

This implies $(A_{k+1})_{(a)}=\bigcup\{\mathcal{I}_a(x,y)|x,y\in(A_k)_{(a)}\}$, for each $a\in\alpha(\bot)$. Since $(X,\mathcal{I}_a)$ is an interval space for each $a\in\alpha(\bot)$, by Theorem 3.4, we can obtain $co_{\mathcal{C}_{\mathcal{I}_a}}(A)=\bigcup_{n=0}^{\infty}(A_k)_{(a)}$, where $A_0=A$, $(A_{k+1})_{(a)}=\bigcup\{\mathcal{I}_a(x,y)|x,y\in(A_k)_{(a)}\}$. Furthermore, $co_{\mathcal{C}_{\mathcal{I}_a}}(A)=(\bigvee_{n=0}^{\infty}A_k)_{(a)}$. By Lemma 3.7, for each $a\in\alpha(\bot)$, $(\mathcal{C}_{\mathcal{I}})^{[a']}=\mathcal{C}_{\mathcal{I}_a}$. So we have $co_{(\mathcal{C}_{\mathcal{I}})^{[a']}}=co_{\mathcal{C}_{\mathcal{I}_a}}$ and then $co_{(\mathcal{C}_{\mathcal{I}})^{[a']}}(A)=(\bigvee_{n=0}^{\infty}A_k)_{(a)}$. By Theorem 2.17, for each $a\in\alpha(\bot)$, $co_{(\mathcal{C}_{\mathcal{I}})^{[a]}}(A)=co_{\mathcal{C}_{\mathcal{I}}}(A)_{(a')}$. Hence for each $a\in\alpha(\bot)$, $co_{\mathcal{C}_{\mathcal{I}}}(A)_{(a)}=(\bigvee_{n=0}^{\infty}A_k)_{(a)}$. Therefore $co_{(\mathcal{C}_{\mathcal{I}})}(A)=\bigvee_{n=0}^{\infty}A_k$, where $A_0=A$, $A_{k+1}(z)=\bigvee_{x,y\in X}\mathcal{I}(x,y)(z)\wedge A_k(x)\wedge A_k(y)$. □

From [24], we know that if $(X,\mathcal{C})$ is a convex structure and $co_{\mathcal{C}}$ its hull operator, then the following holds: $A\in\mathcal{C}\Leftrightarrow A\supseteq co_{\mathcal{C}}(A)\Leftrightarrow\forall x\notin A, x\notin co_{\mathcal{C}}(A)=\bigcup\{co_{\mathcal{C}}(F):F\in\mathbf{2}^A_{fin}\}\Leftrightarrow\forall x\notin A,\forall F\in\mathbf{2}^A_{fin}, x\notin co_{\mathcal{C}}(F)$.

Based on the above fact, we can give the following theorem, which gives a relation between an M-fuzzifying convex structure (X,C) and its hull operator $co_{\mathcal{C}}$.

Theorem 3.9. *Let $(X,\mathcal{C})$ be an M-fuzzifying convex structure and $co_{\mathcal{C}}$ be the M-fuzzifying hull operator. Then $\forall A\subseteq X,\ \mathcal{C}(A)=\bigwedge_{x\notin A}\bigwedge_{F\in\mathbf{2}^A_{fin}}(co_{\mathcal{C}}(F)(x))'$.*

Proof. Since $co_{\mathcal{C}}$ satisfies (MDF), for all $A\in\mathbf{2}^X$,

$$\begin{aligned} \mathcal{C}(A) &= \textstyle\bigwedge_{x\notin A}(co_{\mathcal{C}}(A)(x))'\\ &= \textstyle\bigwedge_{z\notin A}(\bigvee\{co_{\mathcal{C}}(F)(x):F\in\mathbf{2}^A_{fin}\})'\\ &= \textstyle\bigwedge_{x\notin A}\bigwedge_{F\in\mathbf{2}^A_{fin}},(co_{\mathcal{C}}(F)(x))'. \end{aligned}$$

□

Based on Definition 2.5 and Theorem 3.9, we can give the following definition.

Definition 3.10. Let $(X,\mathcal{C})$ be an M-fuzzifying convex structure. $(X,\mathcal{C})$ is of M-fuzzifying *arity* $\leq n$ if it satisfies

$$\forall A\subseteq X,\ \mathcal{C}(A)=\bigwedge_{x\notin A}\ \bigwedge_{F\in\mathbf{2}^A_{fin},|F|\leq n}(co_{\mathcal{C}}(F)(x))'.$$

Theorem 3.11. *An M-fuzzifying convex structure is induced by an M-fuzzifying interval operator if and only if it is of M-fuzzifying arity ≤ 2.*

Proof. Necessity. Let $(X, \mathcal{C}_{\mathcal{I}})$ be the M-fuzzifying convex structure induced by an M-fuzzifying interval operator $(X, \mathcal{I})$ and $Seg_{\mathcal{C}_{\mathcal{I}}}$ denote the M-fuzzifying segment operator of $(X, \mathcal{C}_{\mathcal{I}})$. Next we prove $(X, \mathcal{C}_{\mathcal{I}})$ is of M-fuzzifying *arity* ≤ 2, that is, $\mathcal{C}_{\mathcal{I}}(A) = \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(co_{\mathcal{C}_{\mathcal{I}}}\{x,y\}(z))'$. Since $\mathcal{I} \leq Seg_{\mathcal{C}_{\mathcal{I}}}$, we know that $\mathcal{C}_{\mathcal{I}}(A) \geq \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(co_{\mathcal{C}_{\mathcal{I}}}\{x,y\}(z))'$. For each $a \in M\backslash\{\perp\}$, let $\mathcal{C}_{\mathcal{I}}(A) \geq a$ and co_a denote the hull operator of $(X, (\mathcal{C}_{\mathcal{I}})_{[a]})$. Then

$$\begin{array}{rcl}
\mathcal{C}_{\mathcal{I}}(A) \geq a & \Rightarrow & A \in (\mathcal{C}_{\mathcal{I}})_{[a]} \\
& \Rightarrow & A = \bigcup\{co_a\{x,y\} : \{x,y\} \subseteq A\} \\
& \Rightarrow & \forall z \notin A,\ \forall\{x,y\} \subseteq A,\ z \notin co_a(\{x,y\}) \\
& \Rightarrow & z \notin co_a\{x,y\} \supseteq \{x,y\},\ co_a\{x,y\} \in (\mathcal{C}_{\mathcal{I}})_{[a]} \\
& \Rightarrow & z \notin co_a\{x,y\} \supseteq \{x,y\},\ \mathcal{C}_{\mathcal{I}}(co_a\{x,y\}) \geq a \\
& \Rightarrow & \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(\bigvee_{z\notin B\supseteq\{x,y\}} \mathcal{C}_{\mathcal{I}}(B)) \geq a \\
& \Rightarrow & \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(\bigwedge_{z\notin B\supseteq\{x,y\}} \mathcal{C}_{\mathcal{I}}(B)')' \geq a \\
& \Rightarrow & \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(co_{\mathcal{C}_{\mathcal{I}}}\{x,y\}(z))' \geq a.
\end{array}$$

This implies $\mathcal{C}_{\mathcal{I}}(A) \leq \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A}(co_{\mathcal{C}_{\mathcal{I}}}\{x,y\}(z))'$. Therefore the proof of necessity is completed.

Sufficiency. If $(X, \mathcal{C})$ is of M-fuzzifying *arity* ≤ 2, then

$$\mathcal{C}(A) = \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A} (co_{\mathcal{C}}\{x,y\}(z))' = \bigwedge_{z\notin A} \bigwedge_{x,y\in A} (Seg_{\mathcal{C}}(x,y)(z))'.$$

Let $\mathcal{I}_{\mathcal{C}} = Seg_{\mathcal{C}}$. Obviously, $\mathcal{I}_{\mathcal{C}}$ is an M-fuzzifying interval operator on X. So $(X, \mathcal{C})$ can be induced by the M-fuzzifying interval operator $\mathcal{I}_{\mathcal{C}}$. □

Theorem 3.12. *Let $(X, \mathcal{C})$ be an M-fuzzifying convex structure of M-fuzzifying arity ≤ 2. Then $\mathcal{C} = \mathcal{C}_{\mathcal{I}_{\mathcal{C}}}$.*

Proof. By Theorems 3.6 and 3.11, for all $A \in \mathbf{2}^X$,

$$\mathcal{C}_{\mathcal{I}_{\mathcal{C}}}(A) = \bigwedge_{z\notin A} \bigwedge_{x,y\in A} (\mathcal{I}_{\mathcal{C}}(x,y)(z))' = \bigwedge_{z\notin A} \bigwedge_{\{x,y\}\subseteq A} (co_{\mathcal{C}}\{x,y\}(z))' = \mathcal{C}(A).$$

Therefore $\mathcal{C} = \mathcal{C}_{\mathcal{I}_{\mathcal{C}}}$. □

4. MYCSA2 as a Subcategory of MYIS

In this section, we intend to investigate relations between M-fuzzifying interval spaces and M-fuzzifying convex structures of M-fuzzifying arity ≤ 2 from a categorical aspect. The category of M-fuzzifying convex structures of M-fuzzifying arity ≤ 2 and M-fuzzifying convexity preserving functions is denoted by **MYCSA2**.

Definition 4.1. Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be M-fuzzifying interval spaces.

(1) If a function $f : X \longrightarrow Y$ satisfies the following condition: for $x, y \in X$,

$$f_M^{\rightarrow}(\mathcal{I}_X(x,y)) \leq \mathcal{I}_Y(f(x), f(y)),$$

then f is called an M-fuzzifying interval preserving function.

(2) If a function $f : X \longrightarrow Y$ satisfies the following condition: for $x, y \in X$,

$$f_M^{\rightarrow}(\mathcal{I}_X(x,y)) = \mathcal{I}_Y(f(x), f(y)),$$

then f is called an M-fuzzifying interval-to-interval function.

The category of M-fuzzifying interval spaces and M-fuzzifying interval preserving functions is denoted by **MYIS**.

Theorem 4.2. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures and co_X, co_Y denote their M-fuzzifying hull operators, respectively. Then $f : X \longrightarrow Y$ is an M-fuzzifying convexity preserving function if and only if $f_M^{\rightarrow}(co_X(F)) \leq co_Y(f(F))$ for any $F \in \mathbf{2}^X_{fin}$. If $(X, \mathcal{C}_X)$ is of M-fuzzifying arity $\leq n$, then it suffices to consider sets $|F| \leq n$.*

Proof. Necessity. We need to prove $\bigvee_{f(x)=y} co_X(F)(x) \leq co_Y(f(F))(y)$ for any $F \in \mathbf{2}^X_{fin}$ and for any $y \in Y$, that is, for $y \in Y$, $\forall x \in X$, if $f(x) = y$, then $co_X(F)(x) \leq co_Y(f(F))(y)$ for any $F \in \mathbf{2}^X_{fin}$. For each $D \in \mathbf{2}^Y_{fin}$, if $y \notin D \supseteq f(F)$, then $x \notin f^{-1}(D) \supseteq F$. Since $\mathcal{C}_X\left(f^{-1}(D)\right) \geq \mathcal{C}_Y(D)$, we know that

$$\begin{array}{rcl} co_X(F)(x) & = & \bigwedge_{x\notin B\supseteq F} \mathcal{C}_X(B)' \\ & \leq & \bigwedge_{x\notin f^{-1}(D)\supseteq F} \mathcal{C}_X(f^{-1}(D))' \\ & \leq & \bigwedge_{y\notin D\supseteq f(F)} \mathcal{C}_Y(D)' \\ & = & co_Y(f(F))(y). \end{array}$$

This implies $\bigvee_{f(x)=y} co_X(F)(x) \leq co_Y(f(F))(y)$ for any $F \in \mathbf{2}^X_{fin}$ and for any $y \in Y$.

Sufficiency. We need to prove $\mathcal{C}_X\left(f^{-1}(B)\right) \geq \mathcal{C}_Y(B)$ for all $B \in \mathbf{2}^Y$, that is,

$$\bigwedge_{p\notin f^{-1}(B)} \bigwedge_{F\in \mathbf{2}^{f^{-1}(B)}_{fin}} (co_X(F)(p))' \quad \geq \quad \bigwedge_{q\notin B} \bigwedge_{G\in \mathbf{2}^B_{fin}} (co_Y(G)(q))'.$$

$\forall p \notin f^{-1}(B)$ and $F \in \mathbf{2}^{f^{-1}(B)}_{fin}$, let $y = f(p)$. Then $y = f(p) \notin B$ and $f(F) \in \mathbf{2}^B_{fin}$. By $f_M^{\rightarrow}(co_X(F)) \leq co_Y(f(F))$ for any $F \in \mathbf{2}^X_{fin}$, we know that $co_X(F)(p) \leq co_Y(f(F))(y)$ and then $(co_X(F)(p))' \geq (co_Y(f(F))(y))'$. So

$$\begin{array}{rcl} \mathcal{C}_X\left(f^{-1}(B)\right) & = & \bigwedge_{p\notin f^{-1}(B)} \bigwedge_{F\in \mathbf{2}^{f^{-1}(B)}_{fin}} (co_X(F)(p))' \\ & \geq & \bigwedge_{f(p)\notin B} \bigwedge_{f(F)\in \mathbf{2}^B_{fin}} (co_Y(f(F))(f(p)))' \\ & \geq & \bigwedge_{q\notin B} \bigwedge_{G\in \mathbf{2}^B_{fin}} (co_Y(G)(q))' \\ & = & \mathcal{C}_Y(B). \end{array}$$

By Definition 3.10, we can easily obtain that if $(X, \mathcal{C}_X)$ is of M-fuzzifying *arity* $\leq n$, then in the above proof it suffices to consider sets $|F| \leq n$. □

Theorem 4.3. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures and co_X, co_Y denote their M-fuzzifying hull operators, respectively. Then $f : X \longrightarrow Y$ is an M-fuzzifying convex-to-convex function if and only if $f_M^{\rightarrow}(co_X(F)) \geq co_Y(f(F))$ for any $F \in \mathbf{2}^X_{fin}$. If $(Y, \mathcal{C}_Y)$ is of M-fuzzifying arity $\leq n$, then it suffices to consider sets $|F| \leq n$.*

Proof. Necessity. We need to prove $\bigvee_{f(x)=y} co_X(F)(x) \geq co_Y(f(F))(y)$ for any $F \in \mathbf{2}^X_{fin}$ and for any $y \in Y$, that is, $\bigvee_{f(x)=y} \bigwedge_{x\notin B\supseteq F} \mathcal{C}_X(B)' \geq \bigwedge_{y\notin D\supseteq f(F)} \mathcal{C}_Y(D)'$ for any $F \in \mathbf{2}^X_{fin}$ and for any $y \in Y$. For any $a \in \beta(\top)$, let co^a_X and co^a_Y denote hull operators of $(X, (\mathcal{C}_X)^{[a']})$ and $(Y, (\mathcal{C}_Y)^{[a']})$, respactively. By Theorems 2.7 and 2.13, we can obtain $co^a_Y(f(F)) \subseteq f(co^a_X(F))$ for any $F \in \mathbf{2}^X_{fin}$ and for any $a \in \beta(\top)$. Let a be any element in $\beta(\top)$ such that $\bigwedge_{y\notin D\supseteq f(F)} \mathcal{C}_X(D)' \succ a$. Then we have

$$\begin{aligned}
\textstyle\bigwedge_{y\notin D\supseteq f(F)} \mathcal{C}_Y(D)' \succ a \quad &\Rightarrow \quad \forall D \supseteq f(F),\ if\ y \notin D,\ then\ a \prec \mathcal{C}_Y(D)' \\
&\Rightarrow \quad \forall D \supseteq f(F),\ if\ y \notin D,\ then\ \mathcal{C}_Y(D) \prec^{op} a' \\
&\Rightarrow \quad \forall D \supseteq f(F),\ if\ y \notin D,\ then\ a' \in \alpha(\mathcal{C}_Y(D)) \\
&\Rightarrow \quad \forall D \supseteq f(F),\ if\ y \notin D,\ then\ D \notin (\mathcal{C}_Y)^{[a']} \\
&\Rightarrow \quad y \in co^a_Y(f(F)) \\
&\Rightarrow \quad y \in co^a_Y(f(F)) \subseteq f(co^a_X(F)) \\
&\Rightarrow \quad \exists x \in co^a_X(F),\ s.t.,\ f(x) = y \\
&\Rightarrow \quad \forall B \supseteq F,\ if\ x \notin B,\ then\ B \notin (\mathcal{C}_X)^{[a']} \\
&\Rightarrow \quad \forall B \supseteq F,\ if\ x \notin B, then\ a' \in \alpha(\mathcal{C}_X(B)) \\
&\Rightarrow \quad \forall B \supseteq F,\ if\ x \notin B, then\ \mathcal{C}_X(B) \prec^{op} a' \\
&\Rightarrow \quad \forall B \supseteq F,\ if\ x \notin B, then\ a \prec \mathcal{C}_X(B)' \\
&\Rightarrow \quad \textstyle\bigwedge_{x\notin B\supseteq F} \mathcal{C}_X(B)' \geq a \\
&\Rightarrow \quad \textstyle\bigvee_{f(x)=y} \bigwedge_{x\notin B\supseteq F} \mathcal{C}_X(B)' \geq a \\
&\Rightarrow \quad \textstyle\bigvee_{f(x)=y} co_X(F)(x) \geq a.
\end{aligned}$$

This implies $\bigvee_{f(x)=y} co_X(F)(x) \geq co_Y(f(F))(y)$.

Sufficiency. We need to prove $\mathcal{C}_Y(f(A)) \geq \mathcal{C}_X(A)$ for all $A \in \mathbf{2}^X$, that is,

$$\bigwedge_{z\notin f(A)} \bigwedge_{G\in \mathbf{2}^{f(A)}_{fin}} (co_Y(G)(z))' \geq \bigwedge_{x\notin A} \bigwedge_{F\in \mathbf{2}^A_{fin}} (co_X(F)(x))'.$$

Note that for any $F \in \mathbf{2}^X_{fin}$ and for any $y \in Y$, $f^{\rightarrow}_M(co_X(F)) \geq co_Y(f(F))$, that is, $\bigvee_{f(x)=y} co_X(F)(x) \geq co_Y(f(F))(y)$. It follows that $(\bigvee_{f(x)=y} co_X(F)(x))' \leq (co_Y(f(F))(y))'$ and then $\alpha((\bigvee_{f(x)=y} co_X(F)(x))') \supseteq \alpha((co_Y(f(F))(y))')$. Hence we obtain $\bigcup_{f(x)=y} \alpha((co_X(F)(x))') \supseteq \alpha((co_Y(f(F))(y))')$.

Let a be any element in M such that $a \in \alpha(\bigwedge_{z\notin f(A)} \bigwedge_{G\in \mathbf{2}^{f(A)}_{fin}} (co_Y(G)(z))')$. Then

$$\begin{aligned}
& a \in \alpha(\textstyle\bigwedge_{z\notin f(A)} \bigwedge_{G\in \mathbf{2}^{f(A)}_{fin}} (co_Y(G)(z))') \\
\Rightarrow\quad & a \in \textstyle\bigcup_{z\notin f(A)} \alpha(\bigwedge_{G\in \mathbf{2}^{f(A)}_{fin}} (co_Y(G)(z))') \\
\Rightarrow\quad & a \in \textstyle\bigcup_{z\notin f(A)} \bigcup_{G\in \mathbf{2}^{f(A)}_{fin}} \alpha((co_Y(G)(z))') \\
\Rightarrow\quad & \exists z \notin f(A),\ G \in \mathbf{2}^{f(A)}_{fin}, s.t.,\ a \in \alpha((co_Y(G)(z))') \\
\Rightarrow\quad & \exists z \notin f(A),\ G \in \mathbf{2}^{f(A)}_{fin}, s.t.,\ a \in \alpha((\textstyle\bigvee_{f(x)=z} (co_X(H)(x))'), \\
& where\ f(H) = G\ and\ H\ is\ finite \\
\Rightarrow\quad & \exists z \notin f(A),\ G \in \mathbf{2}^{f(A)}_{fin}, s.t.,\ a \in \textstyle\bigcup_{f(x)=z} \alpha((co_X(H)(x))') \\
\Rightarrow\quad & \exists x \in X,\ f(x) = z,\ x \notin A, s.t.,\ a \in \alpha((co_X(H)(x))') \\
\Rightarrow\quad & a \in \alpha(\textstyle\bigwedge_{x\notin A} \bigwedge_{F\in \mathbf{2}^A_{fin}} (co_X(F)(x))'),
\end{aligned}$$

This implies $\mathcal{C}_Y\left(f(A)\right) \geq \mathcal{C}_X(A)$ for any $A \in \mathbf{2}^X$.

By Definition 3.10, we can easily obtain that if $(Y, \mathcal{C}_Y)$ is of M-fuzzifying *arity* $\leq n$, then in the above proof it suffices to consider sets $|F| \leq n$. □

By Theorems 4.2 and 4.3, we can easily obtain the following three theorems.

Theorem 4.4. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures and let $Seg_{\mathcal{C}_X}$, $Seg_{\mathcal{C}_Y}$ be their M-fuzzifying segment operators, respectively.*

(1) If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is M-fuzzifying convexity preserving, then $f : (X, Seg_{\mathcal{C}_X}) \longrightarrow (Y, Seg_{\mathcal{C}_Y})$ is M-fuzzifying interval preserving.

(2) If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is M-fuzzifying convexity preserving and M-fuzzifying convex-to-convex, then $f : (X, Seg_{\mathcal{C}_X}) \longrightarrow (Y, Seg_{\mathcal{C}_Y})$ is M-fuzzifying interval-to-interval.

Theorem 4.5. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex structures of M-fuzzifying arity ≤ 2.*

(1) If $f : (X, Seg_{\mathcal{C}_X}) \longrightarrow (Y, Seg_{\mathcal{C}_Y})$ is M-fuzzifying convexity preserving, then $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is M-fuzzifying convexity preserving.

(2) If $f : (X, Seg_{\mathcal{C}_X}) \longrightarrow (Y, Seg_{\mathcal{C}_Y})$ is M-fuzzifying interval-to-interval, then $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is M-fuzzifying convexity preserving and M-fuzzifying convex-to-convex.

Theorem 4.6. *Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be M-fuzzifying interval spaces. If $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is M-fuzzifying convexity preserving, then $f : (X, \mathcal{C}_{\mathcal{I}_X}) \longrightarrow (Y, \mathcal{C}_{\mathcal{I}_Y})$ is M-fuzzifying convexity preserving.*

Applying Theorems 3.6, 3.11, 4.4, 4.5 and 4.6, we find a functor $\mathbb{F}$ from **MYCSA2** to **MYIS**, where $\mathbb{F}$ is defined by $\mathbb{F}$: **MYCSA2** $\longrightarrow$ **MYIS** such that for all $(X, \mathcal{C}) \in |$**MYCSA2**$|$, $\mathbb{F}((X, \mathcal{C})) = (X, \mathcal{I}_{\mathcal{C}})$, for all $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$, $\mathbb{F}(f) = f$. Meanwhile, there exists a functor $\mathbb{G}$ from **MYIS** to **MYCSA2**, where $\mathbb{G}$ is defined by $\mathbb{G}$: **MYIS** $\longrightarrow$ **MYCSA2** such that for all $(X, \mathcal{I}) \in |$**MYIS**$|$, $\mathbb{F}((X, \mathcal{I})) = (X, \mathcal{C}_{\mathcal{I}})$, for all $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$, $\mathbb{G}(f) = f$. Moreover, we have $\mathbb{G} \circ \mathbb{F}((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{F} \circ \mathbb{G}((X, \mathcal{I})) \geq (X, \mathcal{I})$. To sum up, we get the following theorem.

Theorem 4.7. *The category* **MYCSA2** *can be embedded in the category* **MYIS** *as a reflective subcategory.*

Next we give an example to show **MYCSA2** is not isomorphic to **MYIS** with respect to $\mathbb{F}$ and $\mathbb{G}$.

Example 4.8. Let (X, ρ) be the finite metric space, presented in FIGURE 1 as a weighted graph.
Let $(X, \mathcal{I})$ be an interval space induced by (X, ρ), where $\mathcal{I}(x, y) = \{z | \rho(x, z) + \rho(z, y) = \rho(x, y)\}$ and then $(X, \chi_{\mathcal{I}})$ is an M-fuzzifying interval space, where $M = \{\perp, \top\}$. Let $(X, \mathcal{C}_{\mathcal{I}})$ be a convex structure induced by $(X, \mathcal{I})$ and then $(X, \chi_{\mathcal{C}_{\mathcal{I}}})$ is an M-fuzzifying convex structure. We can see that $\mathcal{I}(x, y) = \{x, y, z, q\}$ and $\mathcal{I}(z, q) = \{x, y, z, p, q\}$. It is obvious that $\mathcal{I}(z, q) \subseteq \mathcal{I}(x, y)$ does not hold. So $\mathcal{I}(x, y)$ is not a convex set in $(X, \mathcal{C}_{\mathcal{I}})$. Let Seg be the segment operator of $(X, \chi_{\mathcal{C}_{\mathcal{I}}})$. Then $\chi_{\mathcal{I}}(x, y) \neq Seg(x, y)$. Therefore $\mathbb{F} \circ \mathbb{G}((X, \chi_{\mathcal{I}})) \neq (X, \chi_{\mathcal{I}})$.

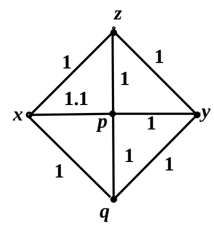

FIGURE 1. Weighted Graph of (X, ρ)

5. **Subspaces and Product Spaces**

Theorem 5.1. *Let $(X, \mathcal{I}_X)$ be an M-fuzzifying interval space, $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{I}_Y)$ is an M-fuzzifying interval space on Y, where $\mathcal{I}_Y = \mathcal{I}_X|Y$. We call $(Y, \mathcal{I}_Y)$ a subspace of $(X, \mathcal{I}_X)$.*

Proof. It is easy to verify that $\mathcal{I}_Y$ satisfies (MYI1) and (MYI2). □

Theorem 5.2. *Let $(X, \mathcal{I}_X)$ be an M-fuzzifying interval space, $\emptyset \neq Y \subseteq X$, and let $i : Y \longrightarrow X$ be the inclusion mapping. Then i is an M-fuzzifying interval preserving function from $(Y, \mathcal{I}_Y)$ to $(X, \mathcal{I}_X)$.*

Proof. The proof is obvious and is omitted. □

Next we can easily obtain the following result, which is the generalization of the theorem in [24]: the relative interval operator induces a convexity which is finer than or equal to the relative convexity. That is if $(X, \mathcal{I}_X)$ is an interval space and $\emptyset \neq Y \subseteq X$, then $\mathcal{C}_{\mathcal{I}_X}|Y \subseteq \mathcal{C}_{\mathcal{I}_Y}$.

Theorem 5.3. *Let $(X, \mathcal{I}_X)$ be an M-fuzzifying interval space, $\emptyset \neq Y \subseteq X$. Then $\mathcal{C}_{\mathcal{I}_X}|Y \leq \mathcal{C}_{\mathcal{I}_Y}$.*

Proof. For $A \subseteq Y$, by Theorems 2.18 and 3.6, we have

$$\begin{aligned} (\mathcal{C}_{\mathcal{I}_X}|Y)(A) &= \bigvee\{\mathcal{C}_{\mathcal{I}_X}(B) : B \in \mathbf{2}^X, B \cap Y = A\} \\ &= \bigvee_{B \in \mathbf{2}^X, B \cap Y = A} \ \bigwedge_{p \notin B} \ \bigwedge_{m,n \in B} (\mathcal{I}_X(m,n)(p))'. \end{aligned}$$

Let a be any element in $M \backslash \{\perp\}$ such that $(\mathcal{C}_{\mathcal{I}_X}|Y)(A) \succ a$. Then

$$\begin{aligned} \mathcal{C}_{\mathcal{I}_X}|Y(A) \succ a \ &\Rightarrow\ \exists B \in \mathbf{2}^X,\ B \cap Y = A, \\ &\qquad \forall p \notin B,\ \forall m,n \in B, (\mathcal{I}_X(m,n)(p))' \geq a \\ &\Rightarrow\ \exists B \in \mathbf{2}^X,\ B \cap Y = A,\ B \in \mathcal{C}_{(\mathcal{I}_X)^{a'}} \\ &\Rightarrow\ A \in \mathcal{C}_{(\mathcal{I}_X)^{a'}}|Y \\ &\Rightarrow\ A \in \mathcal{C}_{(\mathcal{I}_X)^{a'}|Y} \\ &\Rightarrow\ \forall z \notin A,\ \forall x,y \in A, z \notin (\mathcal{I}_X)^{a'}|Y(x,y) \\ &\Rightarrow\ \forall z \notin A,\ \forall x,y \in A, z \notin \mathcal{I}_Y(x,y)^{(a')} \\ &\Rightarrow\ \bigwedge_{z \notin A} \bigwedge_{x,y \in A} (\mathcal{I}_Y(x,y)(z))' \geq a \\ &\Rightarrow\ \mathcal{C}_{(\mathcal{I}_Y)}(A) \geq a. \end{aligned}$$

This implies $\mathcal{C}_{\mathcal{I}_X}|Y \leq \mathcal{C}_{\mathcal{I}_Y}$. □

Theorem 5.4. *Let $\{(X_t, \mathcal{I}_t)\}_{t\in T}$ be a family of M-fuzzifying interval spaces. Let X be the product of the sets X_t for $t \in T$. Define a mapping $\mathcal{I} : X \times X \longrightarrow M^X$ by $\mathcal{I}(x,y)(z) = \bigwedge_{t\in T} \mathcal{I}_t(x_t, y_t)(z_t)$, where $x = (x_t)_{t\in T}$, $y = (y_t)_{t\in T}$, $z = (z_t)_{t\in T}$. Then $(X, \mathcal{I})$ is an M-fuzzifying interval space on X. We call $(X, \mathcal{I})$ the product space of $\{(X_t, \mathcal{I}_t)\}_{t\in T}$ and denote it by $(X, \prod_{t\in T} \mathcal{I}_t)$.*

Proof. The proof is obvious and is omitted. □

Theorem 5.5. *Let $(X, \mathcal{I})$ be the product space of a family of M-fuzzifying interval spaces $\{(X_t, \mathcal{I}_t)\}_{t\in T}$. Let $\pi_t : X \longrightarrow X_t$ denote the projection for each $t \in T$. Then for each $t \in T$, π_t is an M-fuzzifying interval preserving function from $(X, \mathcal{I})$ to $(X_t, \mathcal{I}_t)$.*

Proof. The proof is obvious and is omitted. □

Theorem 5.6. *Let $(X, \mathcal{I})$ be the product space of a family of M-fuzzifying interval spaces $\{(X_t, \mathcal{I}_t)\}_{t\in T}$, where T is finite. Then for each $a \in J(M)$, $\mathcal{I}^{(a')} = \prod_{t\in T} \mathcal{I}_t^{(a')}$.*

Proof. For each $a \in J(M)$ and for all $x, y, z \in X$,

$$\begin{aligned}
\forall z \notin \mathcal{I}^{(a')}(x,y) \quad &\Leftrightarrow \quad \mathcal{I}(x,y)(z) \leq a' \\
&\Leftrightarrow \quad (\mathcal{I}(x,y)(z))' \geq a \\
&\Leftrightarrow \quad (\textstyle\bigwedge_{t\in T} \mathcal{I}_t(x_t, y_t)(z_t))' \geq a \\
&\Leftrightarrow \quad \textstyle\bigvee_{t\in T} (\mathcal{I}_t(x_t, y_t)(z_t))' \geq a \\
&\Leftrightarrow \quad \exists t \in T, (\mathcal{I}_t(x_t, y_t)(z_t))' \geq a \\
&\Leftrightarrow \quad \exists t \in T, \mathcal{I}_t(x_t, y_t)(z_t) \leq a' \\
&\Leftrightarrow \quad \exists t \in T, z_t \notin \mathcal{I}_t(x_t, y_t)^{(a')} \\
&\Leftrightarrow \quad z \notin \textstyle\prod_{t\in T} \mathcal{I}_t^{(a')}(x,y).
\end{aligned}$$

This implies for each $a \in J(M)$, $\mathcal{I}^{(a')} = \prod_{t\in T} \mathcal{I}_t^{(a')}$. □

Theorem 5.7. *Let $\{(X, \mathcal{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex structures, where T is finite and $\mathcal{C} = \bigsqcup_{t\in T} \mathcal{C}_t$. Then $\mathcal{C}(A) = \bigvee_{\bigcap_{t\in T} B_t = A} \bigwedge_{t\in T} \mathcal{C}_t(B_t)$ for all $A \in \mathbf{2}^X$.*

Proof. Let $\widehat{\mathcal{C}}(A) = \bigvee_{\bigcap_{t\in T} B_t = A} \bigwedge_{t\in T} \mathcal{C}_t(B_t)$. Next we prove $\widehat{\mathcal{C}}$ is an M-fuzzifying convexity.

(MYC1) It is easy to verify that $\widehat{\mathcal{C}}(\emptyset) = \widehat{\mathcal{C}}(X) = \top$.

(MYC2) We need to prove that $\widehat{\mathcal{C}}(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \widehat{\mathcal{C}}(A_\lambda)$ for any nonempty $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathbf{2}^X$. Let a be any element in $M\backslash\{\bot\}$ such that

$$a \prec \bigwedge_{\lambda\in\Lambda} \bigvee_{\bigcap_{t\in T} B_{t,\lambda} = A_\lambda} \bigwedge_{t\in T} \mathcal{C}_t(B_{t,\lambda}) = \bigwedge_{\lambda\in\Lambda} \widehat{\mathcal{C}}(A_\lambda).$$

Then for each $\lambda \in \Lambda$, there exists a set $\{B_{t,\lambda} : t \in T\} \subseteq \mathbf{2}^X$ such that $\bigcap\{B_{t,\lambda} : t \in T\} = A_\lambda$ and $\forall t \in T$, $\mathcal{C}_t(B_{t,\lambda}) \geq a$, that is $B_{t,\lambda} \in (\mathcal{C}_t)_{[a]}$. Hence $\bigcap\{B_{t,\lambda} :$

$\lambda \in \Lambda\} \in (\mathcal{C}_t)_{[a]}$ and then $\mathcal{C}_t(\bigcap_{\lambda\in\Lambda} B_{t,\lambda}) \geq a$. By $\bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda}\bigcap_{t\in T} B_{t,\lambda} = \bigcap_{t\in T}\bigcap_{\lambda\in\Lambda} B_{t,\lambda}$, we have

$$\mathcal{C}(\bigcap_{\lambda\in\Lambda} A_\lambda) = \bigvee_{\bigcap_{t\in T}\bigcap_{\lambda\in\Lambda} B_{t,\lambda}=\bigcap_{\lambda\in\Lambda} A_\lambda} \bigwedge_{t\in T} \mathcal{C}_t(\bigcap_{\lambda\in\Lambda} B_{t,\lambda}) \geq a.$$

This implies $\widehat{\mathcal{C}}(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \widehat{\mathcal{C}}(A_\lambda)$.

(MYC3) We need to prove that $\widehat{\mathcal{C}}(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \widehat{\mathcal{C}}(A_\lambda)$ for any nonempty and up-directed $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathbf{2}^X$. Let a be any element in $M\backslash\{\perp\}$ such that $a \prec \bigwedge_{\lambda\in\Lambda}\bigvee_{\bigcap_{t\in T} B_{t,\lambda}=A_\lambda}\bigwedge_{t\in T}\mathcal{C}_t(B_{t,\lambda}) = \bigwedge_{\lambda\in\Lambda}\widehat{\mathcal{C}}(A_\lambda)$. Then for each $\lambda \in \Lambda$, there exists an up-directed set $\{B_{t,\lambda} : t \in T\} \subseteq \mathbf{2}^X$ such that $\bigcap\{B_{t,\lambda} : t \in T\} = A_\lambda$ and $\forall t \in T$, $\mathcal{C}_t(B_{t,\lambda}) \geq a$, that is $B_{t,\lambda} \in (\mathcal{C}_t)_{[a]}$. Let co_t^a denote the hull operator of $(X, (\mathcal{C}_t)_{[a]})$. Then $co_t^a(A_\lambda) \subseteq B_{t,\lambda}$ and $A_\lambda = \bigcap\{co_t^a(A_\lambda) : t \in T\}$. Since $\{A_\lambda : \lambda \in \Lambda\}$ is up-directed, $\{co_t^a(A_\lambda) : \lambda \in \Lambda\}$ is up-directed. Hence $\bigcup\{co_t^a(A_\lambda) : \lambda \in \Lambda\} \in (\mathcal{C}_t)_{[a]}$. Then $\mathcal{C}_t(\bigcup_{\lambda\in\Lambda} co_t^a(A_\lambda)) \geq a$. Let $D_t = \bigcup_{\lambda\in\Lambda} co_t^a(A_\lambda)$. Since $\bigcap_{t\in T} D_t = \bigcap_{t\in T}\bigcup_{\lambda\in\Lambda} co_t^a(A_\lambda) = \bigcup_{\lambda\in\Lambda}\bigcap_{t\in T} co_t^a(A_\lambda) = \bigcup_{\lambda\in\Lambda} A_\lambda$, we obtain $\widehat{\mathcal{C}}(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigvee_{\bigcap_{t\in T} B_t=\bigcup_{\lambda\in\Lambda} A_\lambda}\bigwedge_{t\in T}\mathcal{C}_t(B_t) \geq \bigwedge_{t\in T}\mathcal{C}_t(D_t) \geq a$. Therefore $\widehat{\mathcal{C}}(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda}\widehat{\mathcal{C}}(A_\lambda)$.

Let $\mathcal{D}$ be an M-fuzzifying convexity satisfying $\mathcal{D}(A) \geq \varphi(A) = \bigvee_{t\in T}\mathcal{C}_t(A)$ for each $A \in \mathbf{2}^X$. Then for any set $\{B_t : t \in T\} \subseteq \mathbf{2}^X$ with $\bigcap_{t\in T} B_t = A$,

$$\bigwedge_{t\in T}\mathcal{C}_t(B_t) \leq \bigwedge_{t\in T}\mathcal{D}(B_t) \leq \mathcal{D}(\bigcap_{t\in T} B_t) = \mathcal{D}(A).$$

Hence $\widehat{\mathcal{C}}(A) = \bigvee_{\bigcap_{t\in T} B_t=A}\bigwedge_{t\in T}\mathcal{C}_t(B_t) \leq \mathcal{D}(A)$. We know that φ is a subbase of $\widehat{\mathcal{C}}$ and thus $\widehat{\mathcal{C}}$ is the joint of $\{\mathcal{C}_t : t \in T\}$. Therefore $\widehat{\mathcal{C}} = \mathcal{C}$. □

Theorem 5.8. *Let $\{(X_t, \mathcal{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex structures, where T is finite. Let $(X, \mathcal{C})$ be the product of $\{(X_t, \mathcal{C}_t)\}_{t\in T}$. Then $\mathcal{C}(A) = \bigvee_{\prod_{t\in T} B_t=A}\bigwedge_{t\in T}\mathcal{C}_t(B_t)$ for all $A \in \mathbf{2}^X$.*

Proof. By Theorem 2.20, $\{\pi_t^{-1}(\mathcal{C}_t) | t \in T\}$ is a family of M-fuzzifying convexities on X. By Theorem 5.7, $\mathcal{C}(A) = \bigvee_{\bigcap_{t\in T} B_t=A}\bigwedge_{t\in T}\bigvee_{\pi_t^{-1}(D_t)=B_t}\mathcal{C}_t(D_t)$ for all $A \in \mathbf{2}^X$, where $D_t \subseteq X_t$ and $B_t \subseteq X$. Next we prove that $\mathcal{C}(A) = \bigvee_{\prod_{t\in T} G_t=A}\bigwedge_{t\in T}\mathcal{C}_t(G_t)$.

On one hand, let a be any element in $M\backslash\{\perp\}$ such that $a \prec \mathcal{C}(A)$. Then there exists $\{B_t | t \in T\} \subseteq \mathbf{2}^X$ such that $\bigcap_{t\in T} B_t = A$ and for each $t \in T$, $\pi_t^{-1}(D_t) = B_t$ and $\mathcal{C}_t(D_t) \geq a$. Let $G_t = D_t$. Then we get $\prod_{t\in T} G_t = \prod_{t\in T} D_t = \bigcap_{t\in T}\pi_t^{-1}(D_t) = \bigcap_{t\in T} B_t = A$ and $\mathcal{C}_t(G_t) \geq a$. Hence $\bigvee_{\prod_{t\in T} G_t=A}\bigwedge_{t\in T}\mathcal{C}_t(G_t) \geq a$. This implies $\bigvee_{\prod_{t\in T} G_t=A}\bigwedge_{t\in T}\mathcal{C}_t(G_t) \geq \mathcal{C}(A)$. On the other hand, let a be any element in $M\backslash\{\perp\}$ such that $a \prec \bigvee_{\prod_{t\in T} G_t=A}\bigwedge_{t\in T}\mathcal{C}_t(G_t)$. Then there exists $\{G_t | t \in T\}$ such that $\prod_{t\in T} G_t = A$ and for each $t \in T$, $\mathcal{C}_t(G_t) \geq a$. Let $D_t = G_t$ and $B_t = \pi_t^{-1}(D_t)$. Then we get $\bigcap_{t\in T} B_t = \bigcap_{t\in T}\pi_t^{-1}(D_t) = \prod_{t\in T} D_t = \prod_{t\in T} G_t = A$ and $\bigvee_{\pi_t^{-1}(D_t)=B_t}\mathcal{C}_t(D_t) = \bigvee_{\pi_t^{-1}(D_t)=B_t}\mathcal{C}_t(G_t) \geq a$. Hence

$$\bigvee_{\bigcap_{t\in T} B_t=A}\bigwedge_{t\in T}\bigvee_{\pi_t^{-1}(D_t)=B_t}\mathcal{C}_t(D_t) \geq a.$$

This implies $\mathcal{C}(A) \geq \bigvee_{\prod_{t\in T} G_t = A} \bigwedge_{t\in T} \mathcal{C}_t(G_t)$.

Therefore $\mathcal{C}(A) = \bigvee_{\prod_{t\in T} B_t = A} \bigwedge_{t\in T} \mathcal{C}_t(B_t)$ for all $A \in \mathbf{2}^X$. □

Theorem 5.9. [24] *Suppose that $(X, \mathcal{I})$ is the product of a family of interval spaces $\{(X_t, \mathcal{I}_t)\}_{t\in T}$, where T is finite. Let $\mathcal{C}_{\mathcal{I}}$ be the convexity induced by $\mathcal{I}$ and $\forall t \in T$, $\mathcal{C}_{\mathcal{I}_t}$ be the convexity induced by $\mathcal{I}_t$. Then $\mathcal{C}_{\mathcal{I}} = \prod_{t\in T} \mathcal{C}_{\mathcal{I}_t}$.*

Based on Theorem 5.9, we give the following analysis:

$$\begin{array}{rcl} A \in \mathcal{C}_{\mathcal{I}} & \Leftrightarrow & \forall z \notin A, \forall x, y \in A, z \notin \mathcal{I}(x, y) \\ & \Leftrightarrow & \forall z \notin A, \forall x, y \in A, \exists t \in T, s.t., z_t \notin \mathcal{I}_t(x_t, y_t) \\ & \Leftrightarrow & A \in \prod_{t\in T} \mathcal{C}_{\mathcal{I}_t} \\ & \Leftrightarrow & \forall t \in T, A_t \in \mathcal{C}_{\mathcal{I}_t}, \text{where } \prod_{t\in T} A_t = A \\ & \Leftrightarrow & \forall t \in T, \forall h \notin A_t, \forall p, q \in A_t, h \notin \mathcal{I}_t(p, q). \end{array}$$

Next, we give the generalization of Theorem 5.9.

Theorem 5.10. *Let $(X, \mathcal{I})$ be the product of a family of M-fuzzifying interval spaces $\{(X_t, \mathcal{I}_t)\}_{t\in T}$, where T is finite. Then $\prod_{t\in T} \mathcal{C}_{\mathcal{I}_t} = \mathcal{C}_{\mathcal{I}}$.*

Proof. For all $A \in \mathbf{2}^X$, by Theorems 3.6 and 5.4, we have

$$\begin{array}{rcl} \mathcal{C}_{\mathcal{I}}(A) & = & \bigwedge_{z\notin A} \bigwedge_{x,y\in A} (\mathcal{I}(x, y)(z))' \\ & = & \bigwedge_{z\notin A} \bigwedge_{x,y\in A} (\bigwedge_{t\in T} \mathcal{I}_t(x_t, y_t)(z_t))' \\ & = & \bigwedge_{z\notin A} \bigwedge_{x,y\in A} \bigvee_{t\in T} (\mathcal{I}_t(x_t, y_t)(z_t))'. \end{array}$$

Hence, for each $a \in J(M)$,

$$\begin{array}{rcl} A \in (\mathcal{C}_{\mathcal{I}})_{[a]} & \Leftrightarrow & \mathcal{C}_{\mathcal{I}}(A) \geq a \\ & \Leftrightarrow & \forall z \notin A, \forall x, y \in A, \exists t \in T, \\ & & s.t., (\mathcal{I}_t(x_t, y_t)(z_t))' \geq a \\ & \Leftrightarrow & \forall z \notin A, \forall x, y \in A, \exists t \in T, \\ & & s.t., z_t \notin (\mathcal{I}_t(x_t, y_t))^{(a')} \\ & \Leftrightarrow & A \in \mathcal{C}_{\mathcal{I}^{(a')}}. \end{array}$$

By Theorem 5.8, for each $a \in J(M)$,

$$\begin{array}{rcl} A \in (\prod_{t\in T} \mathcal{C}_{\mathcal{I}_t})_{[a]} & \Leftrightarrow & \prod_{t\in T} \mathcal{C}_{\mathcal{I}_t}(A) \geq a \\ & \Leftrightarrow & \bigvee_{\prod_{t\in T} B_t = A} \bigwedge_{t\in T} \mathcal{C}_{\mathcal{I}_t}(B_t) \geq a \\ & \Leftrightarrow & \exists \{B_t | t \in T\}, s.t., \prod_{t\in T} B_t = A, \text{and } \forall t \in T, \mathcal{C}_{\mathcal{I}_t}(B_t) \geq a \\ & \Leftrightarrow & \forall t \in T, \bigwedge_{h\notin A} \bigwedge_{p,q\in B_t} (\mathcal{I}_t(p, q)(h))' \geq a \\ & \Leftrightarrow & \forall t \in T, \forall h \notin A, \forall p, q \in B_t, h \notin \mathcal{I}_t(p, q)^{(a')} \\ & \Leftrightarrow & \forall t \in T, B_t \in \mathcal{C}_{\mathcal{I}_t^{(a')}} \\ & \Leftrightarrow & A \in \prod_{t\in T} \mathcal{C}_{\mathcal{I}_t^{(a')}}. \end{array}$$

By Theorem 5.6, for each $a \in J(M)$, $\mathcal{C}_{\mathcal{I}^{(a')}} = \prod_{t\in T} \mathcal{C}_{\mathcal{I}_t^{(a')}}$. Hence for each $a \in J(M)$, $(\prod_{t\in T} \mathcal{C}_{\mathcal{I}_t})_{[a]} = (\mathcal{C}_{\mathcal{I}})_{[a]}$. Therefore $\prod_{t\in T} \mathcal{C}_{\mathcal{I}_t} = \mathcal{C}_{\mathcal{I}}$. □

6. **Conclusion**

In this paper, the notion of M-fuzzifying interval spaces is introduced. This approach to the fuzzification of interval spaces preserves many basic properties of crisp interval spaces. It is proved that the category **MYCSA2** can be embedded in the category **MYIS** as a reflective subcategory. Moreover, the formula of the M-fuzzifying hull operator of an M-fuzzifying convex structure induced by an M-fuzzifying interval space is given. Equivalent characterizations of M-fuzzifying convexity preserving functions and M-fuzzifying convexity-to-convexity functions are obtained by means of M-fuzzifying hull operators. Subspaces and product spaces of M-fuzzifying interval spaces are presented and some of their fundamental properties are obtained. These facts will be useful to help further investigations of M-fuzzifying convex structures.

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