COUNTING DISTINCT FUZZY SUBGROUPS OF SOME RANK-3 ABELIAN GROUPS

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Abstract. In this paper we classify fuzzy subgroups of a rank-3 abelian group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for any fixed prime p and any positive integer n, using a natural equivalence relation given in [7]. We present and prove explicit polynomial formulae for the number of (i) subgroups, (ii) maximal chains of subgroups, (iii) distinct fuzzy subgroups, (iv) non-isomorphic maximal chains of subgroups and (v) classes of isomorphic fuzzy subgroups of G. Illustrative examples are provided.

1. Introduction

In classical group theory, groups that have similar algebraic properties are usually put together in a class so that one can then analyse and study a simpler representative of the class. This is usually the case in isomorphic groups which are regarded as essentially the same. Fuzzy subgroups of a fixed group G are more numerous than subgroups, thus it is sensible that we divide fuzzy subgroups of a group into equivalence classes under a suitable equivalence relation. In 2001, Murali and Makamba [7] presented an equivalence relation that identifies fuzzy subgroups that can be regarded as essentially the same from an algebraic perspective. In 2008, Tarnaceaunu [20] presented another equivalence on the set of fuzzy subgroups of a group G. This equivalence is weaker than the Murali equivalence in the sense that it is obtained from the Murali equivalence by removing a property on supports. There are other versions of equivalence that have been studied by other researchers, see for example [1], [3], [5], [6], [22] and [23]. However it is not the intention of this paper to present all such equivalences. In this paper we use the Murali equivalence as we believe the condition on supports is important.
In [13], S Ngcibi et al studied the classification of fuzzy subgroups of rank 2 of the form $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ for any prime p and some positive integers n and m, while in [14], J M Oh perfected Ngcibi's work by computing distinct fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ for any positive integers n and m. Similar work has also been considered by Tarnauceanu and Bentea in [20], albeit using a different equivalence.

This paper extends Ngcibi's and Oh's work to a rank-3 group of the form $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$. This is more difficult than having three distinct primes such as $H =$

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$\mathbb{Z}_{p^n} + \mathbb{Z}_q + \mathbb{Z}_r$ for distinct primes p, q and r because p^n, q and r are relatively prime, implying that H is cyclic of order $p^n qr$. It is slightly easier to work with finite cyclic groups.
In our classification of fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$, we will begin with finding and counting all subgroups of G. Thereafter we construct all maximal chains of subgroups of G and finally we count all distinct fuzzy subgroups. We commence with specific powers of primes so as to identify patterns that will lead to general results.
The last part of the paper will deal with numbers of isomorphic classes of fuzzy subgroups of G.

2. **Preliminaries**

A fuzzy subset $\mu : G \longrightarrow I = [0,1]$ of a group G is a fuzzy subgroup of G if $\mu(xy) \geq min\{\mu(x), \mu(y)\}$, $\forall x, y \in G$ and $\mu(x^{-1}) = \mu(x)$, $\forall x \in G$ [17].
For the identity element $e \in G$, $\mu(x) \leq \mu(e)$ $\forall x \in G$.
Throughout this paper we **assume** $\mu(e) = 1$ for any fuzzy subgroup of a group G.

Definition 2.1. A fuzzy subgroup μ of G is equivalent to a fuzzy subgroup ν of G, written as $\mu \sim \nu$, if (i) $\forall x, y \in G$, $\mu(x) > \mu(y) \iff \nu(x) > \nu(y)$ and (ii) $\mu(x) = 0 \iff \nu(x) = 0$ [7].

Clearly $\sim$ is an equivalence relation on the family of all fuzzy subgroups of a group G. Denote by $[\mu]$ the resulting equivalence class containing a fuzzy subgroup μ of G. Two fuzzy subgroups μ and ν are said to be *distinct* iff $[\mu] \neq [\nu]$.
The support of μ is the subset of G defined as $supp\,\mu = \{x \in G : \mu(x) > 0\}$. It is clear that $supp\,\mu$ is a subgroup of G whenever μ is a fuzzy subgroup of G.
A finite n-chain is a collection of numbers on $[0,1]$ of the form $1 > \lambda_1 > \lambda_2 > \lambda_3 > ... > \lambda_{n-1} > \lambda_n$. We simply write $1\lambda_1\lambda_2...\lambda_{n-1}\lambda_n$ in the descending order for the above n-chain. The length of an n-chain is $(n+1)$. The numbers $1, \lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n$ are called pins. An n -chain is called a keychain if $1 \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n \geq 0$.
A chain $\{e\} \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$ of subgroups of G is a maximal chain if no new subgroups can be inserted in the chain. Such a chain is also called a flag on G [8].
For $t \in [0,1]$, a $t-level$ subgroup of a fuzzy subgroup μ is the subgroup $\mu_t = \{x \in G : \mu(x) \geq t\}$ of G. It is easily observed that $t_1 < t_2$ implies $\mu_{t_1} \supseteq \mu_{t_2}$. Thus level subgroups of G always form a chain.
In this paper, a subgroup that distinguishes a maximal chain from others is referred to as a *distinguishing factor*. For more on maximal chains and distinguishing factors, see [9], [11].
Let μ and ν be two fuzzy subgroups of G and H respectively. We say μ is fuzzy isomorphic to ν denoted $\mu \simeq \nu \iff$ there exist an isomorphism $f : G \to H$ such that $\mu(x) > \mu(y) \iff \nu(f(x)) > \nu(f(y))$ and $\mu(x) = 0 \iff \nu(f(x)) = 0$ [7].
Two or more maximal chains are said to be isomorphic if their lenghts are equal and their corresponding components are isomorphic subgroups.

3. Subgroups of $\mathbb{Z}_{\mathbf{p^n}} + \mathbb{Z}_{\mathbf{p}} + \mathbb{Z}_{\mathbf{p}}$

In [12], Ngcibi showed that the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has (1) $n(p+1)+2$ subgroups and (2) $(n-1)(p-1)+(p+1)+(n-1)$ maximal chains. In [13], Ngcibi et al established that the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $2^{n+1}C(n,1)p + 2^{n+2} - 1$ distinct fuzzy subgroups. Further, we have

Theorem 3.1. *The group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ has n non-isomorphic maximal chains.*

Proof. Let $n = 1$. Then $G = \mathbb{Z}_p + \mathbb{Z}_p$ can only have one maximal chain up to isomorphism. Suppose $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ has k maximal chains, $1 \leq k < n$. Let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$. Then the maximal subgroups of G are $\mathbb{Z}_{p^{k+1}} + 0$ and $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ up to isomorphism. By induction, $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ has k maximal chains while $\mathbb{Z}_{p^{k+1}} + 0$ has only 1 maximal chain up to isomorphism. Thus G has $k+1$ non-isomorphic maximal chains. This completes the proof. □

Theorem 3.2. *The total number of non-isomorphic fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ is equal to $2^{n+2} - 1 + (n-1)2^{n+1} = 2^{n+1}(n+1) - 1$.*

Proof. Each maximal chain (up to isomorphism) of subgroups of G is of length $n+2$ and has a single distinguishing factor (subgroup). Thus the number of non-isomorphic fuzzy subgroups is equal to $2^{n+2} - 1 + (n-1)2^{n+1} = 2^{n+1}(n+1) - 1$ since there are n maximal chains (up to isomorphism) in total by Theorem 3.1 above. This completes the proof. □

Remark 3.3. In the above proof, we have used the following general facts:
Pick a maximal chain, of length $n+1$, of subgroups of a group G. Then (i) the maximal chain has $2^{n+1} - 1$ distinct fuzzy subgroups, (ii) the next chain picked will have $2^n = \frac{1}{2} \times 2^{n+1}$ distinct fuzzy subgroups. If a third chain picked has a distinguishing factor, it also will contribute $2^n = \frac{1}{2} \times 2^{n+1}$ distinct fuzzy subgroups. If a fourth chain picked has no single distinguishing factor, then it will contribute $2^{n-1} = \frac{1}{2^2} \times 2^{n+1}$ distinct fuzzy subgroups. The counting continues in that way. See for example [9], [11] for more details.

Subgroups of $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$

It is well known that the number of maximal subgroups of a p-group G of rank d_G is $1 + p + p^2 + p^3 + \cdots + p^{d_G - 1}$. (The rank of a group G is the minimal number of generators of G). This fact will be used in the derivation of a user-friendly formula for the number of subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$. Further, the number of subgroups of order p^n in an elementary p-group G of order p^m is equal to $\prod_{k=0}^{n-1} \frac{p^{m-k}-1}{p^{n-k}-1}$ [18]. There are many other researchers who have attempted to count subgroups of finite abelian groups, see for example [2], [19] and [21]. However, it is sometimes desirable to use explicit polynomial user-friendly formulae for the number of subgroups so as to make it easy to find formulae for the number of distinct fuzzy subgroups. Hence we begin by deriving formulae for the number of subgroups of the abelian rank-3 group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$.

For the group $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$, we have used specific primes such as 2, 3, 5, 7 and computed the subgroups of G manually. The pattern we observed suggests the following polynomial formula:

Proposition 3.4. *The group* $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ *has* $2p^2 + 2p + 4$ *subgroups.*

Proof. By Ngcibi [12], the group $\mathbb{Z}_p + \mathbb{Z}_p$ has $p + 3$ subgroups. Now any maximal subgroup of G is isomorphic to $H = \mathbb{Z}_p + \mathbb{Z}_p + 0$ of order p^2. The number of such maximal subgroups is $\sigma_2 = 1 + p + p^2 + p^3 + \cdots + p^{d_G - 1}$ with $d_G = 3$. Thus $\sigma_2 = 1 + p + p^2$.
Next, the number of subgroups of order p is $\sigma_1 = \prod_{k=0}^{n-1} \frac{p^{m-k}-1}{p^{n-k}-1}$. Here $m = 3$ and $n = 1$, thus $\sigma_1 = \frac{p^3-1}{p-1} = p^2 + p + 1$. Counting all subgroups, including the two trivial subgroups 0 and G, gives the total number of subgroups of G as $2p^2 + 2p + 4$. □

Subgroups of $\mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$

For $p = 2$, we manually computed the number of subgroups to be 27 = 2(2.2.2 + 2 + 1) + 3 + 2.
For $p = 3$, we manually computed the number of subgroups to be 50 = 3(2.3.2 + 2 + 1) + 3 + 2.
Proceeding sequentially, it seems clear that the number of subgroups of $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to $p(2pn + n + 1) + 3 + n$ for any prime p.

Subgroups of $G = \mathbb{Z}_{p^3} + \mathbb{Z}_p + \mathbb{Z}_p$

For $p = 3$, the group $G = \mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 72 crisp subgroups, as listed below:
$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \{0\}$, $\mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3$, $\{0\} + \mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_{3^3} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_3 + \{0\}$,
$\{0\} + \{0\} + \mathbb{Z}_3$, $<(0,1,1)>$, $<(0,1,2)>$, $<(1,0,1)>$, $<(1,0,2)>$,
$<(1,1,0)>$, $<(1,1,1)>$, $<(1,1,2)>$, $<(1,2,0)>$, $<(1,2,1)>$, $<(1,2,2)>$,
$<(1,1,0),(1,1,1)>$, $<(1,1,0),(1,2,1)>$, $<(1,1,0),(1,2,2)>$,
$<(1,1,1),(1,2,0)>$, $<(1,1,1),(1,2,1)>$, $<(1,1,1),(1,2,2)>$,
$<(1,1,2),(1,2,0)>$, $<(1,1,2),(1,2,1)>$, $<(1,1,2),(1,2,2)>$,
$<(1,2,0),(1,2,1)>$, $<(3,0,0),(0,1,0),(0,0,1)>$,
$<(0,0,3)>$, $<(3,0,1)>$, $<(3,0,2)>$, $<(3,1,0)>$, $<(3,1,1)>$,
$<(3,1,2)>$, $<(3,2,0)>$, $<(3,2,1)>$, $<(3,2,2)>$,
$<(3,0,0),(3,0,1)>$, $<(3,0,0),(3,1,0)>$, $<(3,0,0),(3,1,1)>$,
$<(3,0,0),(3,1,2)>$, $<(3,1,0),(3,1,1)>$, $<(3,1,0),(3,2,1)>$,
$<(3,1,0),(3,2,2)>$, $<(3,1,1),(3,2,0)>$, $<(3,1,1),(3,2,1)>$,
$<(3,1,2),(3,2,0)>$, $<(3,1,2),(3,2,2)>$, $<(3,2,0),(3,2,1)>$,
$<(9,0,0),(0,1,0),(0,0,1)>$,
$<(9,0,0)>$, $<(9,0,1)>$, $<(9,0,2)>$, $<(9,1,0)>$, $<(9,1,1)>$,
$<(9,1,2)>$, $<(9,2,0)>$, $<(9,2,1)>$, $<(9,2,2)>$,
$<(9,0,0),(9,0,1)>$, $<(9,0,0),(9,1,0)>$, $<(9,0,0),(9,1,1)>$,
$<(9,0,0),(9,1,2)>$, $<(9,1,0),(9,1,1)>$, $<(9,1,0),(9,2,1)>$,
$<(9,1,0),(9,2,2)>$, $<(9,1,1),(9,2,0)>$, $<(9,1,1),(9,2,1)>$,
$<(9,1,2),(9,2,0)>$, $<(9,1,2),(9,2,2)>$, $<(9,2,0),(9,2,1)>$,

and $\{(0,0,0)\}$.

For $p = 5$, the group $G = \mathbb{Z}_{5^3} + \mathbb{Z}_5 + \mathbb{Z}_5$ contains 176 crisp subgroups, as listed below:
$\mathbb{Z}_{5^3} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^3} + \mathbb{Z}_5 + \{0\}$, $\mathbb{Z}_{5^3} + \{0\} + \mathbb{Z}_5$, $\{0\} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^3} + \{0\} + \{0\}$,
$\{0\}+\mathbb{Z}_5+\{0\}$, $\{0\}+\{0\}+\mathbb{Z}_5$, $<(0,1,1)>$, $<(0,1,2)>$, $<(0,1,3)>$, $<(0,1,4)>$,
$<(1,0,1)>$, $<(1,0,2)>$, $<(1,0,3)>$, $<(1,0,4)>$, $<(1,1,0)>$,
$<(1,1,1)>$, $<(1,1,2)>$, $<(1,1,3)>$, $<(1,1,4)>$, $<(1,2,0)>$,
$<(1,2,1)>$, $<(1,2,2)>$, $<(1,2,3)>$, $<(1,2,4)>$, $<(1,3,0)>$,
$<(1,3,1)>$, $<(1,3,2)>$, $<(1,3,3)>$, $<(1,3,4)>$, $<(1,4,0)>$,
$<(1,4,1)>$, $<(1,4,2)>$, $<(1,4,3)>$, $<(1,4,4)>$,
$<(1,1,0),(1,1,1)>$, $<(1,1,0),(1,2,1)>$, $<(1,1,0),(1,2,2)>$,
$<(1,1,0),(1,2,3)>$, $<(1,1,0),(1,2,4)>$, $<(1,1,1),(1,2,0)>$,
$<(1,1,1),(1,2,1)>$, $<(1,1,1),(1,2,2)>$, $<(1,1,1),(1,2,3)>$,
$<(1,1,1),(1,2,4)>$, $<(1,1,2),(1,2,0)>$, $<(1,1,2),(1,2,1)>$,
$<(1,1,2),(1,2,2)>$, $<(1,1,2),(1,2,3)>$, $<(1,1,2),(1,2,4)>$,
$<(1,1,3),(1,2,0)>$, $<(1,1,3),(1,2,1)>$, $<(1,1,3),(1,2,2)>$,
$<(1,1,3),(1,2,3)>$, $<(1,1,3),(1,2,4)>$, $<(1,1,4),(1,2,0)>$,
$<(1,1,4),(1,2,1)>$, $<(1,1,4),(1,2,2)>$, $<(1,1,4),(1,2,3)>$,
$<(1,1,4),(1,2,4)>$, $<(1,2,0),(1,2,1)>$, $<(1,3,0),(1,3,1)>$,
$<(1,4,0),(1,4,1)>$, $<(5,0,0),(0,1,0),(0,0,1)>$,
$<(5,0,0)>$, $<(5,0,1)>$, $<(5,0,2)>$, $<(5,0,3)>$, $<(5,0,4)>$,
$<(5,1,0)>$, $<(5,1,1)>$, $<(5,1,2)>$, $<(5,1,3)>$, $<(5,1,4)>$,
$<(5,2,0)>$, $<(5,2,1)>$, $<(5,2,2)>$, $<(5,2,3)>$, $<(5,2,4)>$,
$<(5,3,0)>$, $<(5,3,1)>$, $<(5,3,2)>$, $<(5,3,3)>$, $<(5,3,4)>$,
$<(5,4,0)>$, $<(5,4,1)>$, $<(5,4,2)>$, $<(5,4,3)>$, $<(5,4,4)>$,
$<(5,0,0),(5,0,1)>$, $<(5,0,0),(5,1,0)>$, $<(5,0,0),(5,1,1)>$,
$<(5,0,0),(5,1,2)>$, $<(5,0,0),(5,1,3)>$, $<(5,0,0),(5,1,4)>$,
$<(5,1,0),(5,1,1)>$, $<(5,1,0),(5,2,1)>$, $<(5,1,0),(5,2,2)>$,
$<(5,1,0),(5,2,3)>$, $<(5,1,0),(5,2,4)>$, $<(5,1,1),(5,2,0)>$,
$<(5,1,1),(5,2,1)>$, $<(5,1,1),(5,2,3)>$, $<(5,1,1),(5,2,4)>$,
$<(5,1,2),(5,2,0)>$, $<(5,1,2),(5,2,1)>$, $<(5,1,2),(5,2,2)>$,
$<(5,1,2),(5,2,3)>$, $<(5,1,3),(5,2,0)>$, $<(5,1,3),(5,2,2)>$,
$<(5,1,3),(5,2,3)>$, $<(5,1,3),(5,2,4)>$, $<(5,1,4),(5,2,0)>$,
$<(5,1,4),(5,2,1)>$, $<(5,1,4),(5,2,2)>$, $<(5,1,4),(5,2,4)>$,
$<(5,2,0),(5,2,1)>$, $<(5,3,0),(5,3,1)>$, $<(5,4,0),(5,4,1)>$,
$<(25,0,0),(0,1,0),(0,0,1)>$,
$<(25,0,0)>$, $<(25,0,1)>$, $<(25,0,2)>$, $<(25,0,3)>$, $<(25,0,4)>$,
$<(25,1,0)>$, $<(25,1,1)>$, $<(25,1,2)>$, $<(25,1,3)>$, $<(25,1,4)>$,
$<(25,2,0)>$, $<(25,2,1)>$, $<(25,2,2)>$, $<(25,2,3)>$, $<(25,2,4)>$,
$<(25,3,0)>$, $<(25,3,1)>$, $<(25,3,2)>$, $<(25,3,3)>$, $<(25,3,4)>$,
$<(25,4,0)>$, $<(25,4,1)>$, $<(25,4,2)>$, $<(25,4,3)>$, $<(25,4,4)>$,
$<(25,0,0),(25,0,1)>$, $<(25,0,0),(25,1,0)>$, $<(25,0,0),(25,1,1)>$,
$<(25,0,0),(25,1,2)>$, $<(25,0,0),(25,1,3)>$, $<(25,0,0),(25,1,4)>$,
$<(25,1,0),(25,1,1)>$, $<(25,1,0),(25,2,1)>$, $<(25,1,0),(25,2,2)>$,

$<(25,1,0),(25,2,3)>$, $<(25,1,0),(25,2,4)>$, $<(25,1,1),(25,2,0)>$,
$<(25,1,1),(25,2,1)>$, $<(25,1,1),(25,2,3)>$, $<(25,1,1),(25,2,4)>$,
$<(25,1,2),(25,2,0)>$, $<(25,1,2),(25,2,1)>$, $<(25,1,2),(25,2,2)>$,
$<(25,1,2),(25,2,3)>$, $<(25,1,3),(25,2,0)>$, $<(25,1,3),(25,2,2)>$,
$<(25,1,3),(25,2,3)>$, $<(25,1,3),(25,2,4)>$, $<(25,1,4),(25,2,0)>$,
$<(25,1,4),(25,2,1)>$, $<(25,1,4),(25,2,2)>$, $<(25,1,4),(25,2,4)>$,
$<(25,2,0),(25,2,1)>$, $<(25,3,0),(25,3,1)>$, $<(25,4,0),(25,4,1)>$,
and $\{(0,0,0)\}$.
For $p=7$, the group $G=\mathbb{Z}_{7^3}+\mathbb{Z}_7+\mathbb{Z}_7$ contains 328 crisp subgroups. We do not list these subgroups to avoid bulkiness.

In what follows, we counted the subgroups of the group $G=\mathbb{Z}_{p^4}+\mathbb{Z}_p+\mathbb{Z}_p$ for primes $p=2,3,5,7$.
For $p=2$, the group $G=\mathbb{Z}_{2^4}+\mathbb{Z}_2+\mathbb{Z}_2$ has 49 $=p(2p4+4+1)+3+4$ subgroups.
For $p=3$, the group $G=\mathbb{Z}_{3^4}+\mathbb{Z}_3+\mathbb{Z}_3$ has $94=p(2p4+4+1)+3+4$ subgroups.
For $p=5$, the group $G=\mathbb{Z}_{5^4}+\mathbb{Z}_5+\mathbb{Z}_5$ has $232=p(2p4+4+1)+3+4$ subgroups.
For $p=7$, the group $G=\mathbb{Z}_{7^4}+\mathbb{Z}_7+\mathbb{Z}_7$ has $434=p(2p4+4+1)+3+4$ subgroups.
The above observations lead us to the following theorem,

Theorem 3.5. *The number of subgroups of the group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ for $n\geq 1$ is $p(2pn+n+1)+3+n$ for any prime $p\geq 2$.*

Proof. By induction on n. For $n=1$, there are $1+p+p^2$ maximal subgroups of G since G has rank 3. Since $n=1$, all these subgroups must be isomorphic to $\mathbb{Z}_p+\mathbb{Z}_p+0=<(1,0,0);(0,1,0)>$. Each of the maximal subgroups has only cyclic nontrivial proper subgroups of order p. Thus the total number of subgroups of G is obtained by adding all the subgroups of the orders $1,p,p^2$ and p^3, see [18]. The subgroup $H_1=\mathbb{Z}_p+\mathbb{Z}_p+0$ has $1+p$ proper nontrivial subgroups. Next we consider $H_2=\mathbb{Z}_p+0+\mathbb{Z}_p$. The subgroup $\mathbb{Z}_p+0+0$ of H_2 has been counted in H_1. So H_2 has p proper nontrivial subgroups not counted earlier. In $H_3=0+\mathbb{Z}_p+\mathbb{Z}_p$, the subgroups $0+\mathbb{Z}_p+0$ and $0+0+\mathbb{Z}_p$ have already been counted in H_1 and H_2. Thus H_3 has $p-1$ proper nontrivial subgroups not counted above, viz. $<(0,r,1)>$, $r=1,2,\cdots,p-1$.
Next consider $H_4=<(1,0,0),(0,1,1)>$. The maximal subgroups $<(1,0,0)>$ and $<(0,1,1)>$ have already been counted in H_1 and H_3. So the only new subgroups are $<(1,1,1)>$; $<(1,2,2)>$; $<(1,3,3)>$, ... , $<(1,p-1,p-1)>$, a total of $p-1$. Thus H_4 has $p-1$ proper nontrivial subgroups not counted earlier.
In any of the remaining maximal subgroups of G, the only new subgroups (not counted above) will be of the form $<(1,r,s)>$ for $r,s=1,2,\cdots,p-1$ with $r\neq s$. Thus the number of such subgroups is the permutation of $p-1$ distinct symbols taken 2 at a time, i.e. $\frac{(p-1)!}{(p-3)!}=(p-1)(p-2)$.
Thus adding all the subgroups, 1 of order 1; 1 of order p^3; $1+p+p^2$ of order p^2 and $1+p+p+2(p-1)+(p-1)(p-2)$ of order p, we have $2+1+p+p^2+1+p+p+2(p-1)+(p-1)(p-2)=(3+4p+p^2)+p^2-2p+1=2p^2+2p+4=p(2p.1+1+1)+1+3$ is the total number of subgroups of G. Hence the theorem is true for $n=1$.
Now we assume that the theorem is true for all $1\leq k<n$. So the group $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$ has $p(2pk+k+1)+3+k$ subgroups. The group $G=\mathbb{Z}_{p^{k+1}}+\mathbb{Z}_p+\mathbb{Z}_p$ has $1+p+p^2$

maximal subgroups, and one of them $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $p(2pk + k + 1) + 3 + k$ subgroups.
Subgroups of order p^{k+2} other than $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ are all isomorphic. One of them is $H_1 = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$ which has $1 + p$ maximal subgroups. These subgroups are $< (1, 0, 0) >$; $< (0, 1, 0) >$; $< (1, 1, 0) >$; $< (1, 2, 0) >$; ... ; $< (1, p - 1, 0) >$. One, viz $< (0, 1, 0) >$, has already been counted in $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. Thus H_1 gives p maximal subgroups not counted before.
Now consider $H_2 = \mathbb{Z}_{p^{k+1}} + 0 + \mathbb{Z}_p$. One subgroup viz $< (0, 0, 1) >$, has already been counted in $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. Thus, as above, H_2 yields p maximal subgroups not counted before.
The rest of the maximal subgroups not counted above are $< (1, r, s) >$, $r, s = 1, 2, \cdots, p-1$. So the number of such subgroups is $(p-1)^2$ as in the case $n = 1$ above. Summing the numbers of all the subgroups with the zero subgroup already counted in $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$, we have $2p + (p-1)^2 + (1 + p + p^2 - 1) + [p(2pk + k + 1) + 3 + k] + 1 = 2p + p^2 - 2p + 1 + 1 + p + p^2 - 1 + 2P^2k + pk + p + 3 + k = p[2p(k+1) + (k+1) + 1] + (k+1) + 3$ distinct subgroups. Thus the theorem is true for $n = k + 1$. □

4. Maximal Chains of $\mathbf{G} = \mathbb{Z}_{\mathbf{p^n}} + \mathbb{Z}_{\mathbf{p}} + \mathbb{Z}_{\mathbf{p}}$

Here we construct and count the maximal chains of subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ using the information of the previous sections. We start with specific cases.
Let us construct the maximal chains for the group $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ for $p = 2, 3, 5, 7$ in that order.
For $p = 2$, the group $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ has the following 21 maximal chains:
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq < (1, 1, 0) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq < (1, 0, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (0, 1, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0, 0, 0)\}$

$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 0, 1) > \supseteq < (1, 1, 0) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 0, 1) > \supseteq < (1, 0, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 0, 1) > \supseteq < (0, 1, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 1, 1) > \supseteq < (1, 1, 0) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 1, 1) > \supseteq < (1, 1, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 0), (1, 1, 1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 1), (0, 1, 1) > \supseteq < (1, 1, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 1), (0, 1, 1) > \supseteq < (0, 1, 1) > \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 1), (0, 1, 1) > \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0, 0, 0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1, 1, 1), (1, 0, 1) > \supseteq < (1, 1, 1) > \supseteq \{(0, 0, 0)\}$

$\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq <(1,1,1),(1,0,1)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq <(1,1,1),(1,0,1)> \supseteq \{0\}+\mathbb{Z}_2+\{0\} \supseteq \{(0,0,0)\}$

For $p=3$, the group $G=\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3$ has the following 52 maximal chains:
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\mathbb{Z}_3+\{0\} \supseteq \mathbb{Z}_3+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\mathbb{Z}_3+\{0\} \supseteq <(1,1,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\mathbb{Z}_3+\{0\} \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\mathbb{Z}_3+\{0\} \supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\{0\}+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\{0\}+\mathbb{Z}_3 \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\{0\}+\mathbb{Z}_3 \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_3+\{0\}+\mathbb{Z}_3 \supseteq \{0\}+\{0\}+\mathbb{Z}_3 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\{0\}+\mathbb{Z}_3 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \{0\}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_3 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,1,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,1,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(1,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,1)> \supseteq <(1,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,1)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,1)> \supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq <(1,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq \mathbb{Z}_3+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,1)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,1) > \supseteq \mathbb{Z}_3 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,1) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,2) > \supseteq < (1,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,2) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2), (1,2,2) > \supseteq \{0\} + \mathbb{Z}_3 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0), (1,2,1) > \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0), (1,2,1) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0), (1,2,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

Next we consider the maximal chains of subgroups of $\mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$.
For $p = 3$, the group $\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 136 maximal chains. We do not list the chains here.
For $p = 5$, the group $\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5$ has 516 maximal chains.
For $p = 7$ and $n = 2$ the group $G = \mathbb{Z}_{7^2} + \mathbb{Z}_7 + \mathbb{Z}_7$ has 1296 maximal chains. The above observations lead to

Proposition 4.1. *The group $\mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ has $(p+1) + (3p+2)(p^2+p)$ maximal chains for any given prime number p.*

Proof. The group $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ has $1 + p + p^2$ maximal subgroups. One of them is $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ which has $1+p+p^2$ maximal subgroups. By [12], $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $np+1$ maximal chains, implying $\mathbb{Z}_p + \mathbb{Z}_p$ has $p+1$ maximal chains. Each maximal subgroup of $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p + \mathbb{Z}_p$ which has $p+1$ maximal chains. Thus there are $(1+p)(1+p+p^2)$ maximal chains corresponding to $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. The remaining $p + p^2$ maximal subgroups of G are all isomorphic to the group $\mathbb{Z}_{p^2} + \mathbb{Z}_p$ which has $2p + 1$ maximal chains. This yields a further $(2p+1)(p^2+p)$ maximal chains of subgroups of G. Thus the total number of maximal chains is equal to $(1+p)(1+p+p^2) + (p+p^2)(2p+1) = (p+1) + (3p+2)(p^2+p)$. □

We also considered maximal chains of the group $G = \mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ for $k = 3, 5$ and 7. The observed patterns suggest

Theorem 4.2. *For any positive integer n and any prime p, the number of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ is $(p+1) + (p^2+p)[\frac{n(n+1)}{2}p + n]$.*

Proof. We use induction on n. For $n = 1$, $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. One maximal chain of G is $G \supseteq \mathbb{Z}_p + \mathbb{Z}_p + \{0\} \supseteq \mathbb{Z}_p + \{0\} + \{0\} \supseteq \{(0,0,0)\}$.
$\mathbb{Z}_p + \mathbb{Z}_p + \{0\}$ is a maximal subgroup of G of order p^2 and rank 2. All maximal subgroups of G must be of order p^2 and rank 2. There are $1 + p + \cdots + p^{rank(G)-1}$ such subgroups, see [18]. Thus there are $1 + p + p^2$ maximal subgroups, since rank(G) = 3. Each maximal subgroup has maximal subgroups of order p and rank 1, so there are only $1 + p$ such subgroups for each maximal subgroup. Thus there are $(1+p+p^2)(1+p)$ maximal chains of G.
$(1+p+p^2)(1+p) = 1+p+p^2+p+p^2+p^3 = 1+2p+2p^2+p^3$. In the formula, if $n = 1$,

then we have $p+1+(p^2+p)(p+1) = p+1+p^3+p^2+p^2+p = 1+2p+2p^2+p^3$. Therefore the formula is true for $n = 1$.
Suppose now that the formula is true for $n = k$ for $1 \leq k < n$, i.e. $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$ has $p+1+(p^2+p)[\frac{k(k+1)}{2}p+k]$ maximal chains. We show that if $G = \mathbb{Z}_{p^{k+1}}+\mathbb{Z}_p+\mathbb{Z}_p$, then G has $p+1+(p^2+p)[\frac{(k+1)(k+2)}{2}p+k+1]$ maximal chains. Now maximal subgroups of G are isomorphic to $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$ or $\mathbb{Z}_{p^k+1}+\mathbb{Z}_p+\{0\}$. Thus they are of rank 3 or rank 2. The total number of maximal subgroups is $1+p+p^2+\cdots+p^{rank(G)-1} = 1+p+p^2$. Therefore there are $p+p^2$ of rank 2 and 1 of rank 3 maximal subgroups. By induction, the subgroup $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$ of rank 3 yields $p+1+(p^2+p)[\frac{k(k+1)}{2}p+k]$ maximal chains. The $p+p^2$ maximal subgroups of rank 2 of $G = \mathbb{Z}_{p^k+1}+\mathbb{Z}_p+\mathbb{Z}_p$ yield $(p+p^2)[(k+1)p+1]$ maximal chains, since a group $\mathbb{Z}_{p^n}+\mathbb{Z}_p$ has $np+1$ maximal chains by [13] and [12]. Therefore the total number of maximal chains of $\mathbb{Z}_{p^k+1}+\mathbb{Z}_p+\mathbb{Z}_p$ is equal to $p+1+(p^2+p)[\frac{k(k+1)}{2}p+k]+(p^2+p)[(k+1)p+1] = (p+1)+(p^2+p)[\frac{(k+1)(k+2)}{2}p+k+1]$. Thus the result is true for $n = k+1$ and this completes the proof. □

5. Distinct Fuzzy Subgroups

Proposition 5.1. *The number of distinct fuzzy subgroups of* $G = \mathbb{Z}_p+\mathbb{Z}_p+\mathbb{Z}_p$ *is equal to* $2^4-1+2^3(2p^2+2p)+2^2p^3$

Proof. First we consider maximal subgroups of G. One of them is $H_1 = \mathbb{Z}_p+\mathbb{Z}_p+0$. From the proof of the formula for the number of subgroups of G in Section 3, we know that H_1 has $1+p$ maximal subgroups and the length of each maximal chain is 4. Thus all the maximal chains that contain H_1 yield 2^4-1+2^3p distinct fuzzy subgroups. The maximal subgroup $H_2 = \mathbb{Z}_p+0+\mathbb{Z}_p$ has a maximal subgroup $\mathbb{Z}_p+0+0$ already appearing in H_1. However, each maximal chain involving H_2 has a distinguishing factor (new subgroup) not appearing in other maximal chains since H_2 itself does not appear in the maximal chains of H_1. Thus H_2 yields $2^3(1+p)$ distinct fuzzy subgroups.
The maximal subgroup $H_3 = 0+\mathbb{Z}_p+\mathbb{Z}_p$ has maximal subgroups $0+\mathbb{Z}_p+0$ and $0+0+\mathbb{Z}_p$ already appearing in H_1 and H_2 respectively. Thus there is a maximal chain containing H_3 with no distinguishing factor, but only a pair of distinguishing factors. Therefore H_3 yields 2^3p+2^2 distinct fuzzy subgroups.
The rest of the maximal subgroups contribute only the $(p-1)^2$ subgroups $<(1,r,s)>$, $r,s \neq 0$, not appearing earlier, as in the proof of the theorem on the number of subgroups of $G = \mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$, and this yields $2^3(p-1)^2$ distinct fuzzy subgroups. The number of the remaining distinguishing factors not yet used is now $(1+p+p^2-3)$, yielding a further $2^3(1+p+p^2-3)$ distinct fuzzy subgroups.
We have now exhausted all the distinguishing factors. So the rest of the maximal chains can only have pairs of distinguishing factors. The number of the remaining maximal chains is $[p+1+(p^2+p)(\frac{(1+1)}{2}p+1)-3(p+1)-(p-1)^2-(1+p+p^2-3)]$, where $p+1+(p^2+p)(\frac{(1+1)}{2}p+1)$ is the number of all the maximal chains as given in Section 4. This yields $2^2[p+1+(p^2+p)(\frac{(1+1)}{2}p+1)-3(p+1)-(p-1)^2-(1+p+p^2-3)]$ distinct fuzzy subgroups.

Hence the total number of fuzzy subgroups of G is equal to $2^4 - 1 + 2^3p + 2^3(1 + p) + 2^3p + 2^2 + 2^3(p-1)^2 + 2^3(1 + p + p^2 - 3) + 2^2[1 + p + 1 + (p^2 + p)(\frac{(1+1)}{2}p + 1) - 3(p+1) - (p-1)^2 - (1 + p + p^2 - 3)] = 2^4 - 1 + 2^3[p + 1 + p + p + (p-1)^2 + 1 + p + p^2 - 3] + 2^2[1 + p + 1 + (p^2 + p)(p + 1) - 3p - 3 - (p^2 - 2p + 1)^2 - 1 - p - p^2 + 3] = 2^4 - 1 + 2^3[p + 1 + p + p + p^2 - 2p + 1 + 1 + p + p^2 - 3] + 2^2[p + 2 + p^3 + 2p^2 + p - 3p - 3 - p^2 + 2p - 1 - 1 - p - p^2 + 3] = 2^4 - 1 + 2^3(2p^2 + 2p) + 2^2p^3$. □

Proposition 5.2. *The number of distinct fuzzy subgroups of* $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ *is equal to* $2^5 - 1 + 2^4(4p^2 + 3p) + 2^3(3p^3 + p^2)$.

Proof. First we consider maximal subgroups of G. One of them is $H_1 = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. All the maximal chains of G are of length 5. Thus by the above proposition, H_1 yields $2^5 - 1 + 2^4(2p^2 + 2p) + 2^3p^3$ distinct fuzzy subgroups.
Consider $H_2 = \mathbb{Z}_{p^2} + \mathbb{Z}_p + 0$ which has $1 + p$ maximal subgroups: $\mathbb{Z}_p + \mathbb{Z}_p + 0$; $\mathbb{Z}_{p^2}$; $< (1, r, 0) >$, $r = 1, 2, \cdots, p - 1 >$. The proper subgroups of all these subgroups $< (1, r, 0) >$ of H_2 have all been counted in H_1 or in $\mathbb{Z}_{p^2}+0+0$. So H_2 yields $2^4(p+1)$ distinct fuzzy subgroups, corresponding to the subgroups H_2 and $< (1, r, 0) >$, $r = 0, 1, 2, \cdots, p - 1 >$.
Next consider $H_3 = \mathbb{Z}_{p^2} + 0 + \mathbb{Z}_p$ which has $1 + p$ maximal subgroups, similar to H_2. The proper subgroups of all the subgroups of H_3 have all been counted in H_1 or H_2. Thus $< (1, 0, r) >$, $r = 1, 2, \cdots, p - 1 >$, are the only uncounted proper subgroups of H_3. So H_3 yields $2^4p + 2^3$ distinct fuzzy subgroups.
The rest of the maximal chains yield $(p-1)^2$ and $(1+p+p^2-3)$ new subgroups to be used as distinguishing factors, see the above proposition. This yields a further $2^4[(p-1)^2 + (1 + p + p^2 - 3)]$ distinct fuzzy subgroups.
Now the total number of maximal chains is $p + 1 + (p^2 + p)(3p + 2)$, see Section 4. Thus maximal chains not used above yield a further $2^3[p + 1 + (p^2 + p)(3p + 2) - p - 1 - (p^2 + p)(p + 1) - 2(p + 1) - (p - 1)^2 - (1 + p + p^2 - 3)]$ distinct fuzzy subgroups. Hence the total number of distinct fuzzy subgroups is equal to $2^5 - 1 + 2^4(2p^2 + 2p) + 2^3p^3 + 2^4(p+1) + 2^4p + 2^3 + 2^4[(p-1)^2 + (1+p+p^2-3)] + 2^3[p + 1 + (p^2+p)(3p+2) - p - 1 - (p^2+p)(p+1) - 2(p+1) - (p-1)^2 - (1+p+p^2-3)] = 2^5 - 1 + 2^4(4p^2 + 3p) + 2^3(3p^3 + p^2)$. □

Theorem 5.3. *The number of distinct fuzzy subgroups of* $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ *is equal to*
$2^{n+3} - 1 + 2^{n+2}[p(2pn + n + 1)] + 2^{n+1}[p + 1 + (p^2 + p)(\frac{n(n+1)}{2}p + n) - p(2pn + n + 1) - 1]$.

Proof. By induction on n. The above two propositions prove the cases $n = 1$ and $n = 2$. Assume the theorem is true for $1 \leq k < n$ and let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$. The maximal subgroup $H_1 = \mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ of G satisfies the formula of the theorem with $n = k$ by assumption. Now proceed as in the above proposition to work with maximal subgroups of $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$ and use them and the above propositions to show that the formula of the theorem is true for $n = k + 1$.
□

6. On Isomorphic and Non-isomorphic Fuzzy Subgroups of $\mathbf{G} = \mathbb{Z}_{\mathbf{p^n}} + \mathbb{Z}_{\mathbf{p}} + \mathbb{Z}_{\mathbf{p}}$

Recall: **1**. Two fuzzy subgroups μ and ν of a group G are isomorphic if there is an isomorphism $f : G \to G$ such that for $x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\nu(f(x)) > \nu(f(y))$ and $\mu(x) = \mu(y)$ if and only if $\nu(f(x)) = \nu(f(y))$.
If the two fuzzy subgroups are not isomorphic, then they are non-isomorphic.
2. Two maximal chains of a group G are isomorphic if they have the same length and corresponding subgroups in the chains are isomorphic.
For example if $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$, then all the maximal chains of G are isomorphic. When computing the number of non-isomorphic classes of fuzzy subgroups, it surfices to collapse all isomorphic maximal chains into one chain and then use the techniques of distinct fuzzy subgroups to calculate the number of isomorphic classes of fuzzy subgroups. Thus in $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$, the number of isomorphic classes of fuzzy subgroups is $2^4 - 1 = 15$. For more on isomorphism, see [10], [15], [16].
The Fundamental Theorem of finitely generated abelian groups, see for example Fraleigh [4], is useful in deciding which maximal chains of subgroups are isomorphic. We illustrate by means of an example how to compute isomorphic classes of fuzzy subgroups of a group G.

Example 6.1. For $p = 3$ and $n = 2$, the group $G = \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3$ has the following 136 maximal chains:

(1,2)-BLOCK (Each chain has 1 rank-2 subgroup)
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq \mathbb{Z}_{3^2} + \{0\} + \{0\} \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(1,1,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(1,2,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \{0\} \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq <(1,0,1)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq <(1,0,2)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(3,0,0),(3,1,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(3,0,0),(3,1,0)> \supseteq <(3,1,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(3,0,0),(3,1,0)> \supseteq <(3,2,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\} \supseteq <(3,0,0),(3,1,0)> \supseteq \{0\} + \mathbb{Z}_3 + \{0\} \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,1)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,1,1)> \supseteq <(1,1,2)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,1,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,2,1)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,1)> \supseteq <(1,0,2)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,1,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,2,2)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,0),(1,2,2)> \supseteq <(1,0,1)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(1,1,1)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq <(1,1,1),(1,2,0)> \supseteq <(1,2,0)> \supseteq <(3,0,0)> \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,0,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,1,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,2,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,0,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,1,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,2,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq \mathbb{Z}_{3^2} + \{0\} + \{0\} \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,1,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,2,0) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,0,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,0) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,1) > \supseteq < (1,1,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,1) > \supseteq < (1,2,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,1) > \supseteq \mathbb{Z}_{3^2} + \{0\} + \{0\} \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,2) > \supseteq < (1,1,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,2) > \supseteq < (1,2,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,2),(1,2,2) > \supseteq < (1,0,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

(2,2)-BLOCK (Each chain has 2 rank-2 subgroups)

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,1,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,2,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (0,1,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,2) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,2) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,1,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,2) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,2,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,2) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (0,1,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,0) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,0) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,1,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,0) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (3,2,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,0) > \supseteq < (3,0,0),(3,1,2) > \supseteq < (0,1,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (3,0,0),(3,1,0) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (3,0,0),(3,1,0) > \supseteq < (3,1,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (3,0,0),(3,1,0) > \supseteq < (3,2,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,1),(1,2,1) > \supseteq < (3,0,0),(3,1,0) > \supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,1,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,2,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,1),(1,2,2) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (0,1,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,0,0) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,1,1) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (3,2,2) > \supseteq \{0,0,0\}$

$\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq < (1,1,2),(1,2,0) > \supseteq < (3,0,0),(3,1,1) > \supseteq < (0,1,1) > \supseteq \{0,0,0\}$

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(1,3)-BLOCK (Each chain has 1 rank-3 proper subgroup)

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Looking closely at the three blocks of maximal chains, we observe that these blocks are isomorphic classes of maximal chains. Collapsing each block into a single maximal chain, we may say that there are three maximal chains up to isomorphism. We then use the counting techniques for distinct fuzzy subgroups to find the required isomorphic classes of fuzzy subgroups. Hence in total we have $2^5-1+2^4+2^4=31+16+16=63$ non-isomorphic fuzzy subgroups. This agrees with the theorems that follow.

Theorem 6.2. *The number of non-isomorphic maximal chains of subgroups of the group* $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ *is equal to* $\frac{n(n+1)}{2}$.

Proof. By induction on n. For $n = 1$ all the maximal chains are isomorphic and each chain is of length 4. Thus there is only one maximal chain up to isomorphism. The formula $\frac{n(n+1)}{2}$ with $n = 1$ also gives 1 maximal chain up to isomorphism.
Now assume the result is true for all $1 \leq k < n$. Let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$. Then G has 2 non-isomorphic maximal subgroups viz. $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ and $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$. By induction, $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $\frac{k(k+1)}{2}$ non-isomorphic maximal chains. In Section 3, we showed that $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has n non-isomorphic maximal chains. Thus G has $\frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$ non-isomorphic maximal chains. Hence the theorem is true for $n = k + 1$. □

Theorem 6.3. *The number of non-isomorphic fuzzy subgroups of the group* $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ *is equal to* $2^{n+3} - 1 + 2(n-1)2^{n+2} + \frac{(n-1)(n-2)}{2}2^{n+1}$.

Proof. By induction on n. For $n = 1$ we have only one maximal chain up to isomorphism, giving $2^4 - 1$ non-isomorphic fuzzy subgroups. Thus the theorem is true for $n = 1$. Now assume the theorem is true for all $1 \leq k < n$. Let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$, then G has 2 non-isomorphic maximal subgroups viz. $H_1 = \mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ and $H_2 = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$. By induction, H_1 yields $2^{k+4} - 1 + 2(k-1)2^{k+3} + \frac{(k-1)(k-2)}{2}2^{k+2}$ non-isomorphic fuzzy subgroups and H_2 yields 2 new subgroups up to isomorphism viz. $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$ and $\mathbb{Z}_{p^{k+1}} + 0 + 0$. Hence the total number of non-isomorphic fuzzy subgroups is equal to $2^{k+4} - 1 + 2(k-1)2^{k+3} + \frac{(k-1)(k-2)}{2}2^{k+2} + 2.2^{k+3} + [\frac{(k+1)(k+2)}{2} - 1 - 2k - \frac{(k-1)(k-2)}{2}]2^{k+2} = 2^{k+1+3} - 1 + 2(k+1-1)2^{k+1+2} + \frac{(k+1-1)(k+1-2)}{2}2^{k+1+1}$. Hence the theorem is true for $n = k+1$. □

7. Concluding Remarks

For the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$, we have established the following main results:
(1) The number of subgroups of G is equal to $p(2pn + n + 1) + 3 + n$
(2) The number of maximal chains of subgroups of G is equal to $(p + 1) + (p^2 + p)[\frac{n(n+1)}{2}p + n]$
(3) The number of distinct fuzzy subgroups of G is equal to $2^{n+3} - 1 + 2^{n+2}[p(2pn + n + 1)] + 2^{n+1}[p + 1 + (p^2 + p)(\frac{n(n+1)}{2}p + n) - p(2pn + n + 1) - 1]$.
(4) The number of non-isomorphic maximal chains of subgroups of G is equal to $\frac{n(n+1)}{2}$
(5) The number of non-isomorphic fuzzy subgroups of G is equal to
$2^{n+3} - 1 + 2(n-1)2^{n+2} + \frac{(n-1)(n-2)}{2}2^{n+1}$.

We hope to extend these results to the case when $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{p^m} + \mathbb{Z}_p$ where both n and m are greater than 1. Obviously this is going to be a more challenging task. We may also look at the case when $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_q$ where p and q are distinct primes. More variations of the powers of primes may be considered. All these cases are challenging.

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