SOME RESULTS OF MOMENTS OF UNCERTAIN RANDOM VARIABLES

H. AHMADZADE, Y. SHENG AND F. HASSANTABAR DARZI

ABSTRACT. Chance theory is a mathematical methodology for dealing with indeterminate phenomena including uncertainty and randomness. Consequently, uncertain random variable is developed to describe the phenomena which involve uncertainty and randomness. Thus, uncertain random variable is a fundamental concept in chance theory. This paper provides some practical quantities to describe uncertain random variable. The typical one is the expected value, which is the uncertain version of the center of gravity of a physical body. Mathematically, expectations are integrals with respect to chance distributions or chance measures. In fact, expected values measure the center of gravity of a distribution; they are measures of location. In order to describe a distribution in brief terms there exist additional measures, such as the variance which measures the dispersion or spread, and moments. For calculating the moments of uncertain random variable, some formulas are provided through chance distribution and inverse chance distribution. The main results are explained by using several examples.

1. Introduction

In many cases, human uncertainty and objective randomness simultaneously appear in a system. In order to describe this phenomena, chance theory and consequently uncertain random variable are developed.

Prior to today, probability theory and fuzzy set theory are two common mathematical tools to model indeterminacy phenomena and have been widely applied in information theory, engineering, management science, and so on. We know a fundamental premise of applying probability theory is that the estimated probability is closed enough to the real frequency. However, such as 'strength of bridge', 'about one million tons', 'tall', and 'most', due to lack of observed data and the complexity of environment, when making decisions, people have to consult with domain experts. In this case, information and knowledge cannot be described well by random variables. For fuzzy set theory, it was still challenged by many scholars after it was founded. Liu [9] presented several paradoxes to show that fuzzy variable and fuzzy set are not suitable for modeling uncertain quantities and unsharpen concepts, respectively.

Received: October 2015; Revised: July 2016; Accepted: October 2016

Key words and phrases: Chance Theory, Uncertain Random Variable, Chance Distribution, Moments.

Probability theory (Kolmogorov [7]) is a branch of mathematics for studying the behavior of random phenomena. Although probability theory has been applied widely in science and engineering, there exist a lot of vague phenomena that do not behave with randomness.

In order to deal with non-random phenomena, Uncertainty theory founded by Liu [10] as a branch of axiomatic mathematics based on normality, duality, subadditivity, and product axioms. After that, many researchers widely studied the uncertainty theory and made significative progress. Liu [10] presented the concept of uncertain variable and uncertainty distribution. Then, a sufficient and necessary condition of uncertainty distribution was proved by Peng and Iwamura [19]. In addition, a measure inversion theorem was proposed by Liu [12] from which the uncertain measures of some events can be calculated via the uncertainty distribution. After proposing the concept of independence [11], Liu [12] presented the operational law of uncertain variables. In order to sort uncertain variables, Liu [12] proposed the concept of expected value of uncertain variable. A useful formula was presented by Liu and Ha [16] to calculate the expected values of monotone functions of uncertain variables. Based on the expected value, Liu [10] presented the concepts of variance, moments, and distance of uncertain variables. By invoking inverse uncertainty distribution, Yao [25] proposed a formula to calculate the variance of uncertain variable. The concept of moments is introduced by Liu [10] and calculated via uncertainty distribution. By using inverse uncertainty distribution, Sheng and Kar [21] obtained some results of moment of uncertain variable. In order to characterize the uncertainty of uncertain variables, Liu [11] proposed the concept of entropy. Dai and Chen [4] verified the positive linearity of entropy and presented some formulas for calculating the entropy of monotone function of uncertain variables. Chen and Dai [1] discussed the maximum entropy principle for selecting the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints. In order to make an extension of entropy, Chen et al. [2] proposed a concept of cross-entropy for comparing an uncertainty distribution against a reference uncertainty distribution. Liu [13] introduced a paradox of stochastic finance theory based on uncertainty theory and uncertain differential equation. In addition, an uncertain integral was proposed by Chen and Ralescu [3] presented with respect to the general Liu process.

However, in many cases, randomness and uncertainty exist simultaneously in a complex system. Inspired by Puri and Ralescu [20], Kruse and Meyer [8], and Liu and Liu [17, 18], uncertain random variable was first defined by Liu [14] to describe complex systems in which uncertainty and randomness frequently appear together. Thus, in order to describe such a system, Liu [14] first proposed chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, operational law, expected value, variance and so on. Following that, Liu [15] presented the operational law of uncertain random variable, the formula of expected value and proposed uncertain random programming as a branch of mathematical programming involving uncertain random variables.

As everyone knows, the variance of uncertain random variable will provide a degree of the spread of the distribution around its expected value. Several authors devoted their studies to variance of uncertain random variables. For instance, Guo and Wang [5] obtained a formula for calculating the variance of uncertain random variables via uncertainty distribution. By invoking chance distribution and inverse chance distribution, Sheng and Yao [22] established a formula for calculating the variance.

In this paper, moments and central moments of uncertain random variables are studied and some formulas are provided to calculate the moments and the central moments by using chance distribution and inverse chance distribution. This paper is organized as follows. Some preliminary concepts and theorems about chance theory are presented in Section 2. Section 3 introduces the moments and central moments of an uncertain random variable. Also, some formulas about moments and central moments are derived through chance distribution and inverse chance distribution. Furthermore, some useful relationship between the moments and the central moments are discussed in this section. Some formulas about variance of uncertain random variables are derived in Section 4. Finally, a brief conclusion is given in Section 5.

2. **Preliminaries**

In this section, we first review some concepts of uncertainty theory, including uncertain measure, uncertain variable, uncertainty distribution, operational law and expected value. Then we introduce some useful definitions and properties about chance measure, uncertain random variable, chance distribution, operational law and expected value.

2.1. **Uncertainty Theory.** We review some concepts of uncertainty theory, uncertain measure, uncertainty distribution and operational law.

Definition 2.1. (Liu [10]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom.
Axiom 4: (Product Axiom) (Liu [11]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Roughly speaking, an uncertain variable is a measurable function from an uncertainty space to the set of real numbers.

Definition 2.2. (Liu [10]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, *i.e.*, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2.3. (Liu [10]) The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any $x \in \Re$.

Definition 2.4. (Liu [11]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Definition 2.5. (Liu [10]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.

The distribution of a monotonous function of uncertain variables can be obtained by the following theorem.

Theorem 2.6. *(Liu* [12]*) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Definition 2.7. (Liu [10]) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2.8. *(Liu* [10]*) Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\,dx - \int_{-\infty}^{0} \Phi(x)\,dx.$$

Liu and Ha [16] proposed a generalized formula for expected value by inverse uncertainty distribution.

Theorem 2.9. *(Liu and Ha* [16]*) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha.$$

It is mentioned that the expected value operator has linearity property. On the other hand, let ξ and η be two independent uncertain variables, then we have $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ where a and b are real numbers, for more details see [11].

2.2. **Chance Theory.** In this subsection, we review some concepts of chance theory, including chance measure, uncertain random variable, chance distribution, operational law, and expected value, and variance and so on.

The chance space is refer to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, in which $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Definition 2.10. (Liu [14]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} \mathrm{d}r.$$

Liu [14] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is (i) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$; (ii) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for and event Θ; (iii) $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$. Besides, Hou [6] proved the subadditivity of chance measure, that is,

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}$$

for a sequence of events $\Theta_1, \Theta_2, \cdots$.

Definition 2.11. (Liu [14]) An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event for any Borel set B.

Theorem 2.12. *(Liu* [14]*) Let $f : R^n \to R$ be a measurable function, and $\xi_1, \xi_2, ...,$ ξ_n uncertain random variables on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Now, we review an evidence of uncertain random variable. In real life, producing several devices may be take a long time. Now, suppose that we have one and only one production of this device. For talking about the device, we must to invoke expert decisions. This is an example of uncertain variables. In many situations, we use a system as a mixture of rare production without any sample and another production with several frequencies. This case is an example of uncertain random variable. For instance, reliability analysis of a system based on probability theory has been widely studied and used. Nevertheless, it sometimes meets with one problem that the components of a system may have only few or even no samples, so that we cannot estimate their probability distributions via statistics. Then reliability analysis of a system based on uncertainty theory has been proposed. However, in a general system, some components of the system may have enough samples while some others may have no samples, so the reliability of the system cannot be analyzed simply based on probability theory or uncertainty theory. In order to illustrate the method, Wen and Kang [24] considered some common systems such as series system, parallel system, k-out-of-n system and bridge system.

The definition of chance distribution is presented to calculate the chance measure by Liu [15].

Definition 2.13. (Liu [15]) Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\}$$

for any $x \in \mathcal{R}$.

It should be mentioned that, if an uncertain random variable reduces to random variable, the chance distribution is probability distribution, exactly. Similarly, if an uncertain random variable reduces to uncertain variable, the chance distribution is exactly uncertainty distribution.

Theorem 2.14. *(Liu [15]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

where $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable

$$f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

for any real numbers $y_1, y_2, \cdots, y_m$.

Remark 2.15. (Liu [15]) Suppose that $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$. If $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$ are regular uncertainty distribution functions corresponding to $\tau_1, \tau_2, \cdots, \tau_n$, respectively, then $F(x, y_1, \cdots, y_m$ may also be determined by its inverse uncertainty distribution $F^{-1}(\alpha, y_1, \cdots, y_m)$ that is equal to

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).$$

It is mentioned that throughout the Theorems 10-19, the function f satisfies in the conditions of Remark 2.15.

Definition 2.16. (Liu [15]) Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi]=\int_0^{+\infty}\mathrm{Ch}\{\xi\geq r\}\mathrm{d}r-\int_{-\infty}^{0}\mathrm{Ch}\{\xi\leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Let Φ denote the chance distribution of ξ. Liu [15] proved a formula to calculate the expected value of uncertain random variable with chance distribution, that is,

$$E[\xi]=\int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^{0}\Phi(x)\mathrm{d}x.$$

For a random variable η and an uncertain variable τ, Liu [15] proved that $E[\eta+\tau]=E[\eta]+E[\tau]$ and $E[\eta\times\tau]=E[\eta]\times E[\tau]$. In fact, we have the following theorem about the expected value of uncertain random variables.

Theorem 2.17. *(Liu [15]) Let $\eta_1,\eta_2,\cdots,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$, respectively, and let $\tau_1,\tau_2,\cdots,\tau_n$ be uncertain variables (not necessarily independent), then the uncertain random variable*

$$\xi=f(\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n)$$

has an expected value

$$E[\xi]=\int_{\mathbb{R}^m}E[f(y_1,\cdots,y_m,\tau_1,\cdots,\tau_n)]d\Psi_1(y_1)\cdots d\Psi_m(y_m)$$

where $E[f(y_1,\cdots,y_m,\tau_1,\cdots,\tau_n)]$ is the expected value of the uncertain variable

$$f(y_1,\cdots,y_m,\tau_1,\cdots,\tau_n)$$

for any real numbers $y_1,\cdots,y_m$.

Example 2.18. One of the primary objectives of risk theory is to model the distribution of total claim costs for portfolios of policies, so that business decisions can be made regarding various aspects of the insurance contracts. For more details, see Jozef Teugels [23]. In this example, we consider the total claim cost over a fixed time period. The frequency of claims is described by a random variable N with Poisson distribution, i.e.,

$$p_n=\Pr\{N=n\}=e^{-\lambda}\frac{\lambda^n}{n!},n=0,1,2,\cdots,$$

and the sizes of the individual claims separately are described as *iid* uncertain variables $\tau_1,\tau_2,\cdots,\tau_n,\cdots$ with common distribution $\Upsilon(x)$ and expected value e_τ. Then the total claim can be written as,

$$S=\tau_1+\tau_2+\cdots+\tau_N$$

which is a random sum of uncertain variables.

Theorem 2.17 implies that the expected value of ξ is

$$E[\xi] = \sum_{n=0}^{+\infty} n e_\tau p_n = e_N e_\tau = \lambda e_\tau,$$

where e_N and e_τ are the expected values of N and τ, respectively.

Theorem 2.19. *(Liu [15]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent uncertain random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain random variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If*

$$f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

is a strictly increasing or strictly decreasing function with respect to $\tau_1, \tau_2, \cdots, \tau_n$, then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has an expected value

$$E[\xi] = \int_{\Re^m} \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m).$$

Let η be a random variable with probability distribution Ψ, and let τ be an uncertain variable with uncertainty distribution Υ. We proved that

$$E[\eta \vee \tau] = \int_{\Re} \int_0^1 (y \vee \Upsilon^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi(y)$$

and

$$E[\eta \wedge \tau] = \int_{\Re} \int_0^1 (y \wedge \Upsilon^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi(y).$$

Definition 2.20. (Liu [15]) Let ξ be an uncertain random variable with a finite expected value $E[\xi]$. Then the variance of ξ is

$$V[\xi] = E[(\xi - E[\xi])^2].$$

Since $(\xi - E[\xi])^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \mathrm{Ch}\{(\xi - E[\xi])^2 \geq x\} \mathrm{d}x.$$

One question may be arisen. How to we obtain variance from chance distribution? Since the chance measure is a subadditivity measure, the variance of uncertain random variable cannot be derived simply by the chance distribution. In this case, Guo and Wang [5] suggest to a stipulation as follows:

Stipulation 1: (Guo and Wang [5]) Let ξ be an uncertain random variable with a chance distribution Φ. If ξ has a finite expected value $E[\xi]$, then

$$V[\xi] = \int_0^{+\infty} (1 - \Phi\left(E[\xi] + \sqrt{x}\right) + \Phi\left(E[\xi] - \sqrt{x}\right)) \mathrm{d}x.$$

Based on this stipulation, we will give some formulas to calculate the variance of an uncertain random variable with the chance distribution and the inverse chance distribution.

Theorem 2.21. *(Sheng and Yao [22]) Let ξ be an uncertain random variable with a chance distribution Φ. If ξ has a finite expected value $E[\xi]$, then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x - E[\xi])^2 \, d\Phi(x).$$

Theorem 2.22. *(Sheng and Yao [22]) Let ξ be an uncertain random variable with a regular chance distribution Φ. If ξ has a finite expected value $E[\xi]$, then*

$$V[\xi] = \int_{0}^{1} \left(\Phi^{-1}(r) - E[\xi]\right)^2 dr.$$

3. Moments of Uncertain Random Variables

This section will give the concept of moments for uncertain random variable as well as a stipulation on the formula to calculate the moments.

Definition 3.1. (Guo and Wang [5]) Let ξ be an uncertain random variable and let k be a positive integer. If $E[\xi^k]$ is finite, then $E[\xi^k]$ is called the k-th moment of ξ.

Definition 3.2. Let ξ be an uncertain random variable with finite expected value $E[\xi]$ and k be a positive integer. If $E[(\xi - E[\xi])^k]$ is finite, then $E[(\xi - E[\xi])^k]$ is called the k-th central moment of ξ.

Theorem 3.3. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n)$$

. Then

$$\begin{aligned} E[\xi^k] &= \int_{\mathbb{R}^m} \int_0^{\infty} 1 - F(\sqrt[k]{x}, y_1, ..., y_m) dx d\Psi_1(y_1)...d\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m} \int_{-\infty}^{0} F(\sqrt[k]{x}, y_1, ..., y_m) dx d\Psi_1(y_1)...d\Psi_m(y_m). \end{aligned}$$

Proof. Theorem 2.17 implies that

$$E[\xi^k] = \int_{\mathbb{R}^m} Ef^k(\tau_1, ..., \tau_n, y_1, ..., y_m) \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).$$

But,

$$\begin{aligned} Ef^k(\tau_1, ..., \tau_n, y_1, ..., y_m) &= \int_0^{\infty} \mathcal{M}\{f^k(\tau_1, ..., \tau_n, y_1, ..., y_m) \geq x\} \mathrm{d}x \\ &- \int_{-\infty}^{0} \mathcal{M}\{f^k(\tau_1, ..., \tau_n, y_1, ..., y_m) \leq x\} \mathrm{d}x \\ &= \int_0^{\infty} \mathcal{M}\{f(\tau_1, ..., \tau_n, y_1, ..., y_m) \geq \sqrt[k]{x}\} \mathrm{d}x \\ &- \int_{-\infty}^{0} \mathcal{M}\{f(\tau_1, ..., \tau_n, y_1, ..., y_m) \leq \sqrt[k]{x}\} \mathrm{d}x \\ &= \int_0^{\infty} (1 - F(\sqrt[k]{x}, y_1, ..., y_m)) \mathrm{d}x \\ &- \int_{-\infty}^{0} F(\sqrt[k]{x}, y_1, ..., y_m) \mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} E[\xi^k] &= \int_{\mathbb{R}^m}\int_0^{\infty}(1-F(\sqrt[k]{x},y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0}F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

□

However, when k is an even number, the k-th moment of ξ cannot be uniquely determined by the uncertainty distribution F. In this case, we have

$$\begin{aligned} Ef^k(\tau_1,...,\tau_n,y_1,...,y_m) &= \int_0^{\infty}\mathcal{M}\{f^k(\tau_1,...,\tau_n,y_1,...,y_m)\geq x\}\mathrm{d}x \\ &= \int_0^{\infty}\mathcal{M}\{(f(\tau_1,...,\tau_n,y_1,...,y_m)\geq \sqrt[k]{x}) \\ &\bigcup\ (f(\tau_1,...,\tau_n,y_1,...,y_m)\leq -\sqrt[k]{x})\}\mathrm{d}x \\ &\leq \int_0^{\infty}(\mathcal{M}\{f(\tau_1,...,\tau_n,y_1,...,y_m)\geq \sqrt[k]{x}\} \\ &+\ \mathcal{M}\{f(\tau_1,...,\tau_n,y_1,...,y_m)\leq -\sqrt[k]{x}\})\mathrm{d}x \\ &= \int_0^{\infty}(1-F(\sqrt[k]{x},y_1,...,y_m))\mathrm{d}x \\ &+ \int_{-\infty}^{0}F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x. \end{aligned}$$

Thus for the even number k, we have the following stipulation.

Stipulation 2: Let $\eta_1,\eta_2,...,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,...,\Psi_m$, and let $\tau_1,\tau_2,...,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,...,\Upsilon_n$, respectively. Suppose

$$\xi = f(\eta_1,...,\eta_m,\tau_1,...,\tau_n).$$

Then

$$\begin{aligned} E[\xi^k] &= \int_{\mathbb{R}^m}\int_0^{\infty}1-F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0}F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

Theorem 3.4. *Let $\eta_1,\eta_2,...,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,...,\Psi_m$, and let $\tau_1,\tau_2,...,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,...,\Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1,...,\eta_m,\tau_1,...,\tau_n)$$

. Then

$$E[\xi^k] = \int_{\mathbb{R}^m}\int_{-\infty}^{\infty}u^k dF(u,y_1,...,y_m)d\Psi_1(y_1)...d\Psi_m(y_m).$$

Proof. When k is an odd number, Theorem 3.3 says that the k-th moment is

$$\begin{aligned} E[\xi^k] &= \int_{\mathbb{R}^m}\int_0^{\infty}1-F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0}F(\sqrt[k]{x},y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

But, it remains to calculate

$$\int_0^{\infty} 1 - F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x - \int_{-\infty}^{0} F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x,$$

substituting $\sqrt[k]{x}$ with u and x with u^k, the change of variables and integration by parts produce

$$\begin{aligned}\int_0^{\infty} 1 - F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x &= \int_0^{\infty} 1 - F(u, y_1, ..., y_m)\mathrm{d}u^k \\ &= \int_0^{\infty} u^k \mathrm{d}F(u, y_1, ..., y_m)\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{0} F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x &= \int_{-\infty}^{0} F(u, y_1, ..., y_m)\mathrm{d}u^k \\ &= -\int_{-\infty}^{0} u^k \mathrm{d}F(u, y_1, ..., y_m).\end{aligned}$$

Thus,

$$\begin{aligned}E[\xi^k] &= \int_{\mathbb{R}^m}\int_0^{\infty} 1 - F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0} F(\sqrt[k]{x}, y_1, ..., y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).\end{aligned}$$

By invoking a similar method for even number, Stipulation 2 completes the proof. □

Theorem 3.5. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n).$$

Then

$$E[\xi^k] = \int_{\mathbb{R}^m}\int_0^1 (F^{-1}(\alpha, y_1, ..., y_m))^k d\alpha d\Psi_1(y_1)...d\Psi_m(y_m).$$

Proof. Substituting $F(x, y_1, ..., y_m)$ with α and x with $F^{-1}(\alpha, y_1, ..., y_m)$, it follows from the change of variables of integral and Theorem 3.4 that the k-th moment is

$$E[\xi^k] = \int_{\mathbb{R}^m}\int_0^1 (F^{-1}(\alpha, y_1, ..., y_m))^k \mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).$$

□

Theorem 3.6. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n)$$

. Then

$$\begin{aligned}E[(\xi - E(\xi))^k] &= \int_{\mathbb{R}^m}\int_0^{\infty} 1 - F(\sqrt[k]{x} + E[\xi], y_1 ..., y_m) dx d\Psi_1(y_1)...d\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0} F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m) dx d\Psi_1(y_1)...d\Psi_m(y_m).\end{aligned}$$

Proof. Since

$$E[(\xi - E[\xi])^k] = \int_{\mathbb{R}^m} E(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m),$$

but, it remains to calculate

$$\begin{aligned}
& E[(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k] \\
= & \int_0^{\infty} \mathcal{M}\{(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k \geq x\}\mathrm{d}x \\
- & \int_{-\infty}^{0} \mathcal{M}\{(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k \leq x\}\mathrm{d}x \\
= & \int_0^{\infty} \mathcal{M}\{f(y_1,...,y_m,\tau_1,...,\tau_n) \geq \sqrt[k]{x} + E[\xi]\}\mathrm{d}x \\
- & \int_{-\infty}^{0} \mathcal{M}\{f(y_1,...,y_m,\tau_1,...,\tau_n) \leq \sqrt[k]{x} + E[\xi]\}\mathrm{d}x \\
= & \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1,...,y_m))\mathrm{d}x \\
- & \int_{-\infty}^{0} F(\sqrt[k]{x} + E[\xi], y_1,...,y_m)\mathrm{d}x.
\end{aligned}$$

Thus,

$$\begin{aligned}
E[(\xi - E[\xi])^k] = & \int_{\mathbb{R}^m}\int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
- & \int_{\mathbb{R}^m}\int_{-\infty}^{0} F(\sqrt[k]{x} + E[\xi], y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

□

However, when k is an even number, the k-th central moment of ξ cannot be uniquely determined by the uncertainty distribution F. In this case, we have

$$\begin{aligned}
& E[(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k] \\
= & \int_0^{\infty} \mathcal{M}\{(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi])^k \geq x\}\mathrm{d}x \\
= & \int_0^{\infty} \mathcal{M}\{(f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi] \geq \sqrt[k]{x}) \\
\bigcup & (f(y_1,...,y_m,\tau_1,...,\tau_n) - E[\xi] \leq -\sqrt[k]{x})\}\mathrm{d}x \\
\leq & \int_0^{\infty} (\mathcal{M}\{f(y_1,...,y_m,\tau_1,...,\tau_n \geq \sqrt[k]{x} + E[\xi]\} \\
+ & \mathcal{M}\{f(y_1,...,y_m,\tau_1,...,\tau_n) \leq -\sqrt[k]{x} + E[\xi]\})\mathrm{d}x \\
= & \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1,...,y_m))\mathrm{d}x \\
+ & \int_0^{\infty} F(-\sqrt[k]{x} + E[\xi], y_1,...,y_m)\mathrm{d}x.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned} E[(\xi - E[\xi])^k] &= \int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &+ \int_{\mathbb{R}^m} \int_0^{\infty} F(-\sqrt[k]{x} + E[\xi], y_1, ..., y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

Stipulation 3: Let ξ be an uncertain random variable with uncertainty distribution F, and let k be an even number. Then the k-th central moment of ξ is

$$\begin{aligned} E[(\xi - E[\xi])^k] &= \int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &+ \int_{\mathbb{R}^m} \int_0^{\infty} F(-\sqrt[k]{x} + E[\xi], y_1, ..., y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

Theorem 3.7. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n),$$

then

$$E[(\xi - E[\xi])^k] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} (x - E[\xi])^k dF(x, y_1, ..., y_m) d\Psi_1(y_1)...d\Psi_m(y_m).$$

Proof. When k is an odd number, Theorem 3.6 says that the k-th central moment is

$$\begin{aligned} E[(\xi - E[\xi])^k] &= \int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- \int_{\mathbb{R}^m} \int_{-\infty}^{0} F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m), \end{aligned}$$

substituting $\sqrt[k]{x} + E[\xi]$ with u and x with $(u - E[\xi])^k$, using the change of variables and integration by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(\sqrt[k]{x} + E[\xi], y_1, ..., y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(u, y_1, ..., y_m))\mathrm{d}(u - E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m} \int_0^{\infty} (u - E[\xi])^k\mathrm{d}F(u, y_1, ..., y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m} \int_0^{\infty} (x - E[\xi])^k\mathrm{d}F(x, y_1, ..., y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{-\infty}^{0} F(\sqrt[k]{x}+E[\xi],y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{-\infty}^{0} F(u,y_1,...,y_m)\mathrm{d}(u-E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &-\int_{\mathbb{R}^m}\int_{-\infty}^{0} (u-E[\xi])^k\mathrm{d}F(u,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &-\int_{\mathbb{R}^m}\int_{-\infty}^{0} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

Hence, for an odd number k, we have

$$\begin{aligned}
E[(\xi-E[\xi])^k] &= \int_{\mathbb{R}^m}\int_{0}^{\infty} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
&+ \int_{\mathbb{R}^m}\int_{-\infty}^{0} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
&= \int_{\mathbb{R}^m}\int_{-\infty}^{\infty} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)
\end{aligned}$$

When k is an even number, by Stipulation 3, the k-th central moment is

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{0}^{\infty} (1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
+\ &\int_{\mathbb{R}^m}\int_{0}^{\infty} F(-\sqrt[k]{x}+E[\xi],y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

For the first term substituting $\sqrt[k]{x}+E[\xi]$ with u and x with $(u-E[\xi])^k$, using the change of variables and integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{0}^{\infty} (1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{0}^{\infty} (1-F(u,y_1,...,y_m))\mathrm{d}(u-E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{0}^{\infty} (u-E[\xi])^k\mathrm{d}F(u,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{0}^{\infty} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

Similarly, substituting $-\sqrt[k]{x}+E[\xi]$ with u and x with $(u-E[\xi])^k$ in the second term, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{0}^{\infty} (F(-\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{-\infty}^{0} (F(u,y_1,...,y_m))\mathrm{d}(u-E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{-\infty}^{0} (u-E[\xi])^k\mathrm{d}F(u,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\
=\ &\int_{\mathbb{R}^m}\int_{-\infty}^{0} (x-E[\xi])^k\mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

Hence, for even number k, we have

$$\begin{aligned} E[(\xi-E[\xi])^k] &= \int_{\mathbb{R}^m}\int_0^{\infty}(x-E[\xi])^k \mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &+ \int_{\mathbb{R}^m}\int_{-\infty}^{0}(x-E[\xi])^k \mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &= \int_{\mathbb{R}^m}\int_{-\infty}^{\infty}(x-E[\xi])^k \mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m). \end{aligned}$$

So the k-th central moment of uncertain random variable ξ is

$$E[(\xi-E[\xi])^k] = \int_{\mathbb{R}^m}\int_{-\infty}^{\infty}(x-E[\xi])^k \mathrm{d}F(x,y_1,...,y_m)\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).$$

□

Theorem 3.8. *Let $\eta_1,\eta_2,...,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,...,\Psi_m$, and let $\tau_1,\tau_2,...,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,...,\Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1,...,\eta_m,\tau_1,...,\tau_n).$$

Then

$$E[(\xi-E[\xi])^k] = \int_{\mathbb{R}^m}\int_0^1 (F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k d\alpha d\Psi_1(y_1)...d\Psi_m(y_m).$$

Proof. Firstly, let k be an odd number. By Theorem 3.6, the k-th central moment of ξ is

$$\begin{aligned} E[(\xi-E[\xi])^k] &= \int_{\mathbb{R}^m}\int_0^{\infty}(1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi(y_m) \\ &- \int_{\mathbb{R}^m}\int_{-\infty}^{0}F(\sqrt[k]{x}+E[\xi],y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m), \end{aligned}$$

substituting $F(\sqrt[k]{x}+E[\xi],y_1,...,y_m)$ with α and x with $(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k$ the first term is expressed as

$$\begin{aligned} &\int_{\mathbb{R}^m}\int_0^{\infty}(1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{1}(1-\alpha)\mathrm{d}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m), \end{aligned}$$

and the second term is expressed as

$$\begin{aligned} &\int_{\mathbb{R}^m}\int_{-\infty}^{0}F(\sqrt[k]{x}+E[\xi],y_1,...,y_m)\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &\int_{\mathbb{R}^m}\int_0^{F(E[\xi],y_1,...,y_m)}\alpha\mathrm{d}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ = &-\int_{\mathbb{R}^m}\int_0^{F(E[\xi],y_1,...,y_m)}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m), \end{aligned}$$

and

$$
\begin{aligned}
&E[(\xi - E[\xi])^k] \\
= & \int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
+ & \int_{\mathbb{R}^m}\int_{0}^{F(E[\xi],y_1,...,y_m)}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi(y_m) \\
= & \int_{\mathbb{R}^m}\int_{0}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}
$$

Secondly, let k be an even number, by Stipulation 3, the k-th central moment of ξ is

$$
\begin{aligned}
&E[(\xi - E[\xi])^k] \\
= & \int_{\mathbb{R}^m}\int_{0}^{\infty}(1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
+ & \int_{\mathbb{R}^m}\int_{-\infty}^{0}F(-\sqrt[k]{x}+E[\xi],y_1,...,y_m)\mathrm{d}x \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m),
\end{aligned}
$$

substituting $F(\sqrt[k]{x}+E[\xi],y_1,...,y_m)$ with α and x with $(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k$ the first term is expressed as

$$
\begin{aligned}
& \int_{\mathbb{R}^m}\int_{0}^{\infty}(1-F(\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
= & \int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{1}(1-\alpha)\mathrm{d}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
= & \int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}
$$

Similarly, substituting $F(-\sqrt[k]{x}+E[\xi],y_1,...,y_m)$ with α and x with $(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k$ in the second term, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^m}\int_{0}^{\infty}F(-\sqrt[k]{x}+E[\xi],y_1,...,y_m))\mathrm{d}x \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
= & \int_{\mathbb{R}^m}\int_{F(E[\xi],y_1,...,y_m)}^{0}\alpha \mathrm{d}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\
= & \int_{\mathbb{R}^m}\int_{0}^{F(E[\xi],y_1,...,y_m)}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
\end{aligned}
$$

Hence, for an even number k, we have

$$
E[(\xi - E[\xi])^k] = \int_{\mathbb{R}^m}\int_{0}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
$$

So the k-th central moment of uncertain random variable ξ is

$$
E[(\xi - E[\xi])^k] = \int_{\mathbb{R}^m}\int_{0}^{1}(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).
$$

□

Example 3.9. Let η be a random variable with probability distribution Ψ,and let τ be an uncertain variable with uncertainty distribution Υ, such that

$$\Psi(x)=\begin{cases} 0, & x\le c\\ \dfrac{x-c}{d-c}, & c\le x\le d\\ 1, & otherwise\end{cases}$$

and

$$\Upsilon(x)=\begin{cases} 0, & x\le a\\ \dfrac{x-a}{b-a}, & a\le x\le b\\ 1, & otherwise.\end{cases}$$

Suppose $\xi=\eta+\tau$, then we have

$$E[\xi]=E[\eta]+E[\tau].$$

Also, it is obvious that $\Upsilon^{-1}_{\tau+y}(\alpha)=\Upsilon^{-1}(\alpha)+y$. By invoking Theorem 3.8, we obtain

$$\begin{aligned} E[(\xi-E[\xi])^3] &= \int_{\mathbb{R}^m}\int_0^1(\Upsilon^{-1}_{\tau+y}(\alpha)-E[\xi])^3\mathrm{d}\alpha\mathrm{d}\Psi(y)\\ &= \int_{\mathbb{R}}\int_0^1(\alpha(b-a)+a+y-\frac{a+b}{2}-\frac{c+d}{2})^3\mathrm{d}\alpha\mathrm{d}\Psi(y)\\ &= \int_0^1(\alpha(b-a)+a-\frac{a+b}{2})^3\mathrm{d}\alpha+\int_{\mathbb{R}}(y-\frac{c+d}{2})^3\mathrm{d}\Psi(y)\\ &+ 3\int_0^1(\alpha(b-a)+a-\frac{a+b}{2})\mathrm{d}\alpha\times\int_{\mathbb{R}}(y-\frac{c+d}{2})^2\mathrm{d}\Psi(y)\\ &+ 3\int_0^1(\alpha(b-a)+a-\frac{a+b}{2})^2\mathrm{d}\alpha\times\int_{\mathbb{R}}(y-\frac{c+d}{2})\mathrm{d}\Psi(y)=0.\end{aligned}$$

Following theorem shows relationship between the moments and the central moments of an uncertain random variable.

Theorem 3.10. *Let $\eta_1,\eta_2,...,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,...,\Psi_m$, and let $\tau_1,\tau_2,...,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,...,\Upsilon_n$, respectively. Suppose*

$$\xi=f(\eta_1,...,\eta_m,\tau_1,...,\tau_n).$$

Then $E[(\xi-E[\xi])^k]=\sum_{j=0}^k(-1)^{k-j}C_k^jE[\xi^j]E[\xi]^{k-j}$ where $C_k^j=\frac{k!}{j!(k-j)!}$.

Proof. It follows from Theorems 3.8 and 3.5 that

$$\begin{aligned} &E[(\xi-E[\xi])^k]\\ =&\int_{\mathbb{R}^m}\int_0^1(F^{-1}(\alpha,y_1,...,y_m)-E[\xi])^k\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\ =&\int_{\mathbb{R}^m}\int_0^1\sum_{j=0}^k(-1)^{k-j}C_k^j(F^{-1}(\alpha,y_1,...,y_m))^jE[\xi]^{k-j}\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\ =&\sum_{j=0}^k(-1)^{k-j}C_k^jE[\xi]^{k-j}\int_{\mathbb{R}^m}\int_0^1(F^{-1}(\alpha,y_1,...,y_m))^j\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m)\\ =&\sum_{j=0}^k(-1)^{k-j}C_k^jE[\xi^j]E[\xi]^{k-j}.\end{aligned}$$

□

4. **Variance of Uncertain Random Variable**

In this section, we derive some results about variance of uncertain random variables based on the inverse of uncertainty distribution.

Theorem 4.1. *(Guo and Wang [5]) Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Then*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n)$$

has a variance with formula

$$\int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(e + \sqrt{x}; y_1, ..., y_m) + F(e - \sqrt{x}, y_1, ..., y_m))\,dx\,d\Psi_1(y_1)...d\Psi_m(y_m)$$

where $F(x, y_1, ..., y_m)$ is the uncertainty distribution of the uncertain variable

$$f(y_1, ..., y_m, \tau_1, ..., \tau_n)$$

and is determined by $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$.

Theorem 4.2. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Then*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n)$$

has a variance

$$V[\xi] = \int_{\mathbb{R}^m} \int_0^1 (F^{-1}(\alpha, y_1, ..., y_m) - e)^2\,d\alpha\,d\Psi_1(y_1)...d\Psi_m(y_m)$$

where $F^{-1}(x, y_1, ..., y_m)$ is the inverse uncertainty distribution of the uncertain variable $f(y_1, ..., y_m, \tau_1, ..., \tau_n)$ and is determined by $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$.

Proof. Theorem 4.1 implies that variance of uncertain random variable is equal with

$$\int_{\mathbb{R}^m} \int_0^{\infty} (1 - F(e + \sqrt{x}; y_1, ..., y_m) + F(e - \sqrt{x}, y_1, ..., y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m).$$

But, we want to calculate

$$\int_0^{\infty} (1 - F(e + \sqrt{x}; y_1, ..., y_m) + F(e - \sqrt{x}, y_1, ..., y_m))\mathrm{d}x$$

substituting $e + \sqrt{x}$ with u and x with $(u - e)^2$, the change of variables and integration by parts produce

$$\begin{aligned}\int_0^{\infty} (1 - F(e + \sqrt{x}, y_1, ..., y_m))\mathrm{d}x &= \int_e^{\infty} (1 - F(u, y_1, ..., y_m))\mathrm{d}(u - e)^2 \\ &= \int_e^{\infty} (u - e)^2 \mathrm{d}F(u, y_1, ..., y_m).\end{aligned}$$

Similarly, substituting $e - \sqrt{x}$ with u and x with $(u - e)^2$, we obtain

$$\begin{aligned}\int_0^{\infty} F(e - \sqrt{x}, y_1, ..., y_m) &= \int_e^{\infty} F(u, y_1, ..., y_m)\mathrm{d}(u - e)^2 \\ &= \int_{-\infty}^{e} (u - e)^2 \mathrm{d}F(u, y_1, .., y_m).\end{aligned}$$

It follows that the variance is

$$\begin{aligned}\int_0^{\infty}(1-F(e+\sqrt{x};y_1,...,y_m)+F(e-\sqrt{x},y_1,...,y_m))\mathrm{d}x \\ =\int_e^{\infty}(x-e)^2\mathrm{d}F(x,y_1,...,y_m) \\ +\int_{-\infty}^{e}(x-e)^2\mathrm{d}F(x,y_1,...,y_m) \\ =\int_{-\infty}^{\infty}(x-e)^2\mathrm{d}F(x,y_1,...,y_m).\end{aligned}$$

Now, substituting $F(x,y_1,...,y_m)$ with α and x with $F^{-1}(\alpha,y_1,...,y_m)$, we can write

$$\begin{aligned}&\int_0^{\infty}(1-F(e+\sqrt{x};y_1,...,y_m)+F(e-\sqrt{x},y_1,...,y_m))\mathrm{d}x\\ =&\int_{-\infty}^{\infty}(x-e)^2\mathrm{d}F(x,y_1,...,y_m)\\ =&\int_0^1(F^{-1}(\alpha,y_1,...,y_m)-e)^2\mathrm{d}\alpha.\end{aligned}$$

Thus,

$$V[\xi]=\int_{\mathbb{R}^m}\int_0^1(F^{-1}(\alpha,y_1,...,y_m)-e)^2\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)...\Psi_m(y_m).$$

□

Example 4.3. Let η be a random variable with probability distribution Ψ,and let τ be an uncertain variable with uncertainty distribution Υ, such that

$$\Psi(x)=\begin{cases}0, & x\le c\\ \dfrac{x-c}{d-c}, & c\le x\le d\\ 1, & otherwise\end{cases}$$

and

$$\Upsilon(x)=\begin{cases}0, & x\le a\\ \dfrac{x-a}{b-a}, & a\le x\le b\\ 1, & otherwise.\end{cases}$$

Then variance of uncertain random variable $\xi=\eta+\tau$ is

$$\begin{aligned}V[\xi]\ &=\ \int_{\mathbb{R}}\int_0^1(\Upsilon^{-1}_{\tau+y}(\alpha)-e)^2\mathrm{d}\alpha\mathrm{d}\Psi(y)\\ &=\ \int_{\mathbb{R}}\int_0^1(\alpha(b-a)+a+y-\frac{a+b}{2}-\frac{c+d}{2})^2\mathrm{d}\alpha\mathrm{d}\Psi(y)\\ &=\ \int_0^1(\alpha(b-a)+a-\frac{a+b}{2})^2\mathrm{d}\alpha+\int_{\mathbb{R}}(y-\frac{c+d}{2})^2\mathrm{d}\Psi(y)\\ &=\ V[\tau]+V[\eta].\end{aligned}$$

Theorem 4.4. *Let $\eta_1, \eta_2, ..., \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, ..., \Psi_m$, and let $\tau_1, \tau_2, ..., \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$, respectively. Suppose*

$$\xi = f(\eta_1, ..., \eta_m, \tau_1, ..., \tau_n).$$

Then

$$V[\xi] = E[\xi^2] - E[\xi]^2.$$

Proof. By using Theorem 4.2, we obtain

$$\begin{aligned} V[\xi] &= \int_{\mathbb{R}^m} \int_0^1 (F^{-1}(\alpha, y_1, ..., y_m) - e)^2 \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &= \int_{\mathbb{R}^m} \int_0^1 (F^{-1}(\alpha, y_1, ..., y_m))^2 \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) \\ &- 2e \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y_1, ..., y_m) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)...\mathrm{d}\Psi_m(y_m) + e^2. \end{aligned}$$

Now, Theorem 3.5 implies that

$$V[\xi] = E[\xi^2] - 2eE[\xi] + e^2 = E[\xi^2] - E[\xi]^2.$$

□

5. Conclusions

In this paper, the moments and the central moments of an uncertain random variable were studied. Also, by invoking chance distribution and inverse chance distribution, we obtained several formulas to calculate the moments of an uncertain random variable. Furthermore, the relationships between moments and central moments were investigated. It should be mentioned that, if uncertain random variables reduce to uncertain variables, all results hold.

Acknowledgements. This work was completed during the first author's stay in the Department of Mathematical Sciences at Tsinghua University, China. He is grateful for the hospitality of Prof. B. Liu. Also, the work was supported by National Natural Science Foundation of China Grant (No. 61462086, No. 61563050) and Doctoral Fund of Xinjiang University (No. BS150206).

References

[1] X. Chen and W. Dai, *Maximum entropy principlefor uncertain variables*, International Journal of Fuzzy Systems, **13(3)** (2011), 232–236.
[2] X. Chen, S. Kar and D. Ralescu, *Cross-entropy measure of uncertain variables*, Information Sciences, **201** (2012), 53–60.
[3] X. Chen and D. Ralescu, *Liu process and uncertain calculus*, Journal of Uncertainty Analysis and Applications, **1(3)** (2013), 1– 12.
[4] W. Dai and X. Chen, *Entropy of function of uncertain variables*, Mathematics and Computer Modelling, **55** (2012), 754–760.
[5] H. Y. Guo and X. S. Wang, *Variance of uncertain random variables*, Journal of Uncertainty Analysis and Applications, **2(6)** (2014), 1–7.
[6] Y. C. Hou, *Subadditivity of chance measur*, Journal of Uncertainty Analysis and Applications, **2(14)** (2014), 1–8.

[7] A. N. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung, Julius Springer, Berlin, 1933.
[8] R. Kruse and K. Meyer, Statistics with Vague Data, Reidel Publishing Company, Dordrecht, 1987.
[9] B. Liu, Uncertainty Theory, 5th ed., http://orsc.edu.cn/liu/ut.pdf. 2014.
[10] B. Liu, Uncertainty Theory, 2nd ed., Springer-Verlag, Berlin, 2007.
[11] B. Liu, *Some research problems in uncertainty theory*, Journal of Uncertain Systems, **31** (2009), 3–10.
[12] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer-Verlag, Berlin, 2010.
[13] B. Liu, *Toward uncertain finance theory*, Journal of Uncertainty Analysis and Applications, **1(1)** (2013), 1–15.
[14] Y. H. Liu, *Uncertain random variables: A mixture of uncertainty and randomness*, Soft Computing, **17(4)** (2013), 625–634.
[15] Y. H. Liu, *Uncertain random programming with applications*, Fuzzy Optimization and Decision Making, **12(2)** (2013), 153–169.
[16] Y. H. Liu and M. H. Ha, *Expected value of function of uncertain variables*, Journal of Uncertain Systems, **4(3)** (2010), 181–186.
[17] Y. K. Liu and B. Liu, *Fuzzy random variables: a scalar expected value operator*, Fuzzy Optimization and Decision Making, **2(2)** (2003), 143–160.
[18] Y. K. Liu and B. Liu, *Fuzzy random programming with equilibrium chance constraints*, Informtion Sciences **170** (2005), 363–395.
[19] Z. X. Peng and K. Iwamura, *A sufficient and necessary condition of uncertainty distribution*, Journal of Interdisciplinary Mathematics, **13(3)** (2010), 277–285.
[20] M. Puri and D. Ralescu, *Fuzzy random variables*, Journal of Mathmatical Application, **114** (1986), 409–422.
[21] Y. H. Sheng and S. Kar, *Some results of moments of uncertain variable through inverse uncertainty distribution*, Fuzzy Optimization and Decision Making, **14** (2015), 57–76.
[22] Y. H. Sheng and K. Yao, *Some formulas of variance of uncertain random variable*, Journal of Uncertainty Analysis and Applications, **2(12)** (2014), 1–10.
[23] J. L. Teugels and B. Sundt, *Encyclopedia of actuarial science*, Wiley & Sons, **1** (2004).
[24] M. Wen and R. Kang, *Reliability analysis in uncertain random system*, Fuzzy Optimization and Decision Making, doi:10.1007/s10700-016-9235-y, (2016).
[25] K. Yao, *A formula to calculate the variance of uncertain variable*, Soft Computing, **19(10)** (2015), 2947–2953.

Hamed Ahmadzade, Department of Statistics, University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: ahmadzadeh.h.63@gmail.com

Yuhong Sheng*, College of Mathematical and System Sciences, Xinjiang University, Urumqi 830046, China
E-mail address: shengyuhong1@sina.com

Fatemeh Hassantabar Darzi, Department of Statistics, University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: hasantabar_f@math.usb.ac.ir

*Corresponding author