S-APPROXIMATION SPACES: A FUZZY APPROACH

A. SHAKIBA, M. R. HOOSHMANDASL, B. DAVVAZ AND
S. A. SHAHZADEH FAZELI

Abstract. In this paper, we study the concept of S-approximation spaces in fuzzy set theory and investigate its properties. Along introducing three pairs of lower and upper approximation operators for fuzzy S-approximation spaces, their properties under different assumptions, e.g. monotonicity and weak complement compatibility are studied. By employing two thresholds for minimum acceptance accuracy and maximum rejection error, these spaces are interpreted in three-way decision systems by defining the corresponding positive, negative and boundary regions.

1. Introduction

Uncertainty is present in many real life situations due to imprecise or incomplete knowledge[34]. Usually, the concept of information system is used to represent knowledge and this representation has many applications in artificial intelligence and data mining[27]. The uncertainty in studying information systems is characterized by a boundary or uncertain region, which for all of its elements if it is non-empty, we cannot certainly decide whether it belongs or does not belong to a concept. This kind of uncertainty is not avoidable in many applications, so the research community proposed several tools to handle them such as Dempster-Shafer theory of evidence [39], theory of fuzzy sets by Zadeh [63, 64, 65], theory of rough sets by Pawlak [30, 29, 27, 28, 31, 32], and S-approximation spaces [17, 40, 41], to name a few.

Rough and fuzzy set theories are two independent[53] and widely used approaches to handle this kind of uncertainty. In rough set theory, a concept is approximated by two sets called the lower and upper approximations with respect to knowledge on the universe of objects whereas in fuzzy set theory, a concept is described by a fuzzy characteristic function [34]. Also, rough set theory and Dempster-Shafer theory of evidence are closely connected [55, 16], e.g. the lower and the upper approximations in rough set theory are very similar to inner and outer reductions of Dempster-Shafer theory [15]. Rough set theory and up to the knowledge of the authors, all of its generalizations [30, 33, 47, 61, 59, 72, 35, 4] are expressible in terms of S-approximation spaces [17, 40].

Among many key differences between these theories, two basic common components can be distinguished, the “knowledge” and the notion of “be a part of” [34].

Received: June 2015; Revised: July 2016; Accepted: October 2016
Key words and phrases: Fuzzy S-approximation Spaces, Fuzzy Sets, Three-way Decisions, Monotonicity, Weak Complement Compatibility, Rough Set Theory, Rough Mereology.

Knowledge in Pawlak's rough set theory is represented by a partition over the set of objects [27], in fuzzy set theory by a fuzzy characteristic function [63], in Dempster-Shafer theory of evidence by a probability distribution over the set of objects [8] and in S-approximation spaces by an arbitrary mapping from a set to a subset of (possibly a different) set of objects [17]. Also, the notion of be a part of in Pawlak's rough set theory and most of its generalizations [30, 33, 47, 61, 59, 72, 35, 4], in fuzzy set theory [63] and in Dempster-Shafer theory of evidence [8] is the inclusion relation or some restrictions of it. However, in S-approximation spaces the notion of "be a part of" is independent of the inclusion relation and can be almost any mapping of the form $S : \mathcal{P}(W) \times \mathcal{P}(W) \to \{0, 1\}$ [17, 40], so it is called a decider.

The Pawlak's rough set theory can be generalized by either relaxing/modifying the equivalence relation on the universe of objects [43, 62, 14, 44, 45, 68, 4], employing precision parameters in the definition of the lower or upper approximation operators [71, 59, 29, 54], combining rough set theory with other uncertainty theories such as fuzzy set theory [9, 20] and neural networks [26], extending the set of object from one set to two sets [33, 42, 19, 66, 46, 48, 47, 4], changing the definition of the lower or upper approximation operators [7, 69, 70], assuming certain constructions on the universe of discourse, e.g. a semigroup or other algebraic structures [6, 50, 5] and any combination of them [49, 42, 19, 46, 51].

The research line in changing the notion of "be a part of" in rough set theories leads to invention of different models of rough sets. For example, in probabilistic rough set models [54, 56], the notion of "be a part of" is quantified using the conditional probability. Also, a set of non-probabilistic models like [67, 36, 10, 11, 12, 13] exists. In [59], the rough set model is extended with quantitative subsethood measures and its properties are studied with respect to certain conditions.

The concept of S-approximation spaces is a new approach for studying approximations, which has been introduced in [17]. This approach is proposed on the basis of the ideas of Dempster's multi-valued mappings [8] and captures, up to the knowledge of authors, all of the extensions of rough set theory like [30, 33, 47, 61, 59, 72, 35, 4]. Hence, it can also be seen as an extension of rough set models. In addition, the concept of S-approximation spaces is a novel tool that can be used for analysis of information systems, which are independent from the inclusion relation. Moreover, S-approximation spaces are over two universes and cast generalized approximation spaces discussed in [33, 22, 4]. The concept of S-approximation spaces is studied in [40] from a three-way decision perspective and also as a unifying theory for rough set models. Moreover, studying S-approximation spaces can lead to a more systematic way of constructing new rough set models or analyzing hybrid models. Also, S-approximation spaces can be viewed in the direction of developing a unified theory for data mining [52] in the following way. For example, using the S-approximation framework, one just needs to select a decider, S measure, and defines the knowledge, T mapping. Note that the exact interpretations of knowledge and decider mappings differ in different applications, since their selection is based on the underlying problem at hand. There exist many similarity measures for both categorical data [1] and numeric data [18]. Moreover, for special types of data like multimedia data, there exist several specialized similarity or difference measures.

These measures can be easily used to create S-approximation spaces. In addition, S-approximation spaces come with a set of operations among them [17] which allow for constructing more complex S-approximation spaces from simpler ones. This feature might be useful in describing uncertainties in distributed environments.

Ordinary S-approximation spaces introduced in [17] and studied in [40] are for non-fuzzy sets. In this paper, we study the approximation of fuzzy concepts using these constructions. We will introduce the notion of fuzzy S-approximation spaces and then investigate three types of lower and upper approximations with respect to different properties of mapping S. We will also define a pair of thresholds for acceptance accuracy and rejection error and use them to refine our approximations to non-fuzzy sets, i.e. defuzzifying. Moreover, the notions of three-way decisions introduced by Yao in [58] are employed in these constructs. Similar to ordinary S-approximation spaces, we will also introduce several fuzzy related operations among fuzzy S-approximation spaces.

The organization of the paper is as follows. In Section 2, we will review some basic concepts of fuzzy set theory and S-approximation spaces. In Section 3, we will introduce the notion of fuzzy S-approximation operators and three pairs of lower and upper approximation operators along introducing a pair of thresholds. Moreover, we will investigate the interpretation of a pair of thresholds and will define five situations regarding these thresholds, that is (a, r)-acceptance, (a, r)-rejection, (a, r)-non-commitment, (a, r)-inconsistent and (a, r)-uncertain. In Section 4, we will employ the pair of thresholds introduced in Section 3 to define three-way regions for a fuzzy S-approximation space and study their properties in details. We will also introduce some measures of inconsistency and generality regarding these regions as well as considering the FG-definability and FG-undefinability measures. In Section 5, we will introduce a new subclass of fuzzy S-approximation spaces and investigate their features in terms of three-way decision regions and lower and upper approximation operators. Weak complement compatible fuzzy S-approximation spaces are introduced in Section 6 and their properties are also studied. For the sake of illustrating an application of fuzzy S-approximation spaces, we will examine a disease diagnosis system along the text. Finally, we conclude the paper.

1.1. **Related Works.** The concept of S-approximation spaces was introduced in [17] as an extension of Dempster's multi-valued mappings [8]. This extension also captures the notion of rough set and up to the knowledge of authors, all of its extensions. Moreover, in [17], a class of decider mappings S called S-min deciders were identified which preserved most of the properties of the lower and upper approximations of concepts in rough set theory. Next, in [40], S-approximation spaces were considered from a three-way decision point of view in order to facilitate them with decision making capabilities. Moreover, two new classes of deciders called partial monotone and complement compatible deciders were introduced which the first class also preserved the properties derived for S-min deciders while it is a more general class and the second class forces the lower approximation to be a subset of the lower approximation.

The notion of rough set theory was proposed in [30] and has many extensions and applications, [27, 61, 4, 69, 33, 35] to mention a few. The basic building blocks in a rough approximation space are a relation and the lower and upper approximation operators. With proper construction of knowledge and decider mappings in S-approximation spaces, one can construct a rough approximation space which its lower and upper approximation operators are identical with that rough approximation space [17]. Moreover, if the decider mapping S satisfies certain properties, the lower and upper operators in S-approximation spaces would have the same set of properties as in rough approximations.

One of the extensions to the rough set theory is the concept of fuzzy rough sets, first introduced by [24] according to [3] and then continued by [9] and more scholars like [37, 25, 23, 60]. Although standard S-approximation spaces can produce the final crisp lower and upper approximation sets in fuzzy rough set theory, it is not applicable to produce fuzzy approximations. Hence, in this paper, these constructions are modified accordingly to provide such capability.

2. Preliminaries

2.1. **Fuzzy Set Theory.** Let U be a non-empty set. A set A is called a fuzzy subset of U if its membership function is a mapping of the form $\mu_A : U \to [0,1]$. For every $x \in U$, $\mu_A(x)$ is called the degree of membership of x in the set A. The family of all fuzzy subsets of a set U is denoted as $\mathcal{F}(U)$. For every fuzzy subset A of U, $\mathrm{CORE}(A)$ is defined as

$$\mathrm{CORE}(A) = \{x \mid \mu_A(x) = 1\}. \tag{1}$$

Let $\alpha \in [0,1]$, the weak α-cut or simply α-cut of fuzzy set A is defined as

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, \tag{2}$$

and is denoted by A_α. Similarly, the strong α-cut of fuzzy set A is defined as

$$A_{\overline{\alpha}} = \{x \mid \mu_A(x) > \alpha\}, \tag{3}$$

and is denoted by $A_{\overline{\alpha}}$.

2.2. **S-approximation Spaces.** In this section we remind some preliminary materials for S-approximation spaces from [40]. An *S-approximation space* is the quadruple $G = (U, W, T, S)$, where U and W are finite non-empty sets, T is a mapping of the form $T : U \to \mathcal{P}(W)$ and S is a mapping of the form $S : \mathcal{P}(W) \times \mathcal{P}(W) \to \{0,1\}$. The lower and upper approximations of $X \subseteq W$ are defined as

$$\underline{G}(X) = \{x \in U \mid S(T(x), X) = 1\}, \tag{4}$$

and

$$\overline{G}(X) = \{x \in U \mid S(T(x), X^c) = 0\}, \tag{5}$$

respectively, where X^c is the complement of X with respect to W.

As it is shown in the following example, rough set theory and up to the knowledge of the authors all of its extensions are expressible in terms of S-approximation spaces.

Example 2.1. [17] Let U and W be two finite non-empty sets, $R \subseteq U \times W$ and $X \subseteq W$. The lower and upper approximations of set X with respect to $G = (U, W, T, S)$ where $T(u) = \{w \in W \mid (u, w) \in R\}$ and

$$S(A,B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases},$$

coincide with the lower and upper approximations of set X with respect to T-rough set (U, W, T). In other words,

$$\begin{aligned} \underline{FG}\,(X) &= \{u \in U \mid S(T(u), X) = 1\} \\ &= \{u \in U \mid T(u) \subseteq X\} = \underline{apr}_T(X), \end{aligned}$$

and

$$\begin{aligned} \overline{FG}\,(X) &= \{u \in U \mid S(T(u), X^c) = 0\} \\ &= \{u \in U \mid T(u) \cap X \neq \emptyset\} = \overline{apr}_T(X). \end{aligned}$$

Note that T-rough sets [4] capture Pawlak's rough sets [30] and rough sets over two universal sets [33] as special cases.

Also, note that in order to express the graded or variable precision rough sets over two universal sets [71, 21], it suffices to plug an appropriate S measure.

Example 2.2. The concept of variable precision rough set was introduced by Ziarko in [71] by introducing a threshold $0.5 < t \leq 1$ as classification error. Note that this condition for threshold was later relaxed in the research line. Given an approximation space (U, R) where $R \subseteq U \times U$ is an equivalence relation and a threshold $0.5 < t \leq 1$, then the t-lower and t-upper approximations of a set $X \subseteq U$ are defined as

$$\begin{aligned} \underline{apr}_R^{\,t}(X) &= \left\{ u \in U \;\middle|\; 1 - \frac{|[u]_R \cap X|}{|[u]_R|} \leq t \right\} \\ &= \left\{ u \in U \;\middle|\; \frac{|[u]_R \cap X|}{|[u]_R|} \geq 1 - t \right\}, \end{aligned}$$

and

$$\begin{aligned} \overline{apr}_R^{\,t}(X) &= \left\{ u \in U \;\middle|\; 1 - \frac{|[u]_R \cap X|}{|[u]_R|} < 1 - t \right\} \\ &= \left\{ u \in U \;\middle|\; \frac{|[u]_R \cap X|}{|[u]_R|} > t \right\} \\ &= \left\{ u \in U \;\middle|\; \frac{|[u]_R \cap X^c|}{|[u]_R|} < 1 - t \right\}, \end{aligned}$$

respectively. Let $G = (U, U, T, S)$ where $T(u) = [u]_R = \{u' \in U \mid (u, u') \in R\}$ and

$$S(A,B) = \begin{cases} 1 & \frac{|A \cap B|}{|A|} \geq 1 - t \\ 0 & \text{otherwise} \end{cases},$$

and $X \subseteq U$. Then, we have

$$\begin{aligned}\underline{G}(X) &= \left\{u \in U \;\middle|\; S(T(u), X) = 1\right\} \\ &= \left\{u \in U \;\middle|\; \frac{|T(u) \cap X|}{|T(u)|} \geq 1 - t\right\} \\ &= \left\{u \in U \;\middle|\; \frac{|[u]_R \cap X|}{|[u]_R|} \geq 1 - t\right\} = \underline{apr}_R^{\,t}(X),\end{aligned}$$

and

$$\begin{aligned}\overline{G}(X) &= \left\{u \in U \;\middle|\; S(T(u), X^c) = 0\right\} \\ &= \left\{u \in U \;\middle|\; \frac{|T(u) \cap X^c|}{|T(u)|} < 1 - t\right\} \\ &= \left\{u \in U \;\middle|\; \frac{|[u]_R \cap X^c|}{|[u]_R|} < 1 - t\right\} = \overline{apr}_R^{\,t}(X).\end{aligned}$$

By investigating different deciders, it can be seen that there are many different deciders than the inclusion relation. Some of them are treated individually in the literature like [71, 12, 13, 36]. In [17], a class of deciders called S-min deciders are investigated. The lower and upper approximations within S-approximation spaces with deciders from this class satisfy certain properties. In [40], an extended class of these deciders are identified with the same set properties. This class of deciders is called partial monotone deciders. Let S be a relation of the form $S : \mathcal{P}(W) \times \mathcal{P}(W) \to \{0, 1\}$ and $A, B \subseteq W$. We say that $A \preccurlyeq_S B$ if $S(A, B) = 1$. A relation S is called *partial monotone* if $X \subseteq Y$ and $A \preccurlyeq_S X$, then $A \preccurlyeq_S Y$. An S-approximation space $G = (U, W, T, S)$ is a partial monotone S-approximation space if S is partial monotone.

Proposition 2.3. [40] *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. For all $X, Y \subseteq W$, the followings hold:*

- $(PS1)$**:** $X \subseteq Y$ *implies* $\overline{G}(X) \subseteq \overline{G}(Y)$,
- $(PS2)$**:** $X \subseteq Y$ *implies* $\underline{G}(X) \subseteq \underline{G}(Y)$,
- $(PS3)$**:** $\overline{G}(X \cup Y) \supseteq \overline{G}(X) \cup \overline{G}(Y)$,
- $(PS4)$**:** $\overline{G}(X \cap Y) \subseteq \overline{G}(X) \cap \overline{G}(Y)$,
- $(PS5)$**:** $\underline{G}(X \cup Y) \supseteq \underline{G}(X) \cup \underline{G}(Y)$,
- $(PS6)$**:** $\underline{G}(X \cap Y) \subseteq \underline{G}(X) \cap \underline{G}(Y)$,
- $(PS7)$**:** $\overline{G}(X) = (\underline{G}(X^c))^c$,
- $(PS8)$**:** $\underline{G}(X) = (\overline{G}(X^c))^c$.

Remark 2.4. The (partial monotone) S mappings can be interpreted as rough inclusion functions as in rough mereology. However, note these S mappings are not restricted to the sets of axioms in rough mereology as in [36], so they are much more general than them.

An S-approximation space $G = (U, W, T, S)$ is called *complement compatible* if for all $x \in U$ and $X \subseteq W$, $S(T(x), X) = 0$ or $S(T(x), X^c) = 0$.

Proposition 2.5. [40] *Let $G = (U, W, T, S)$ be a complement compatible space. Then for any $X \subseteq W$, we have $\underline{G}(X) \subseteq \overline{G}(X)$.*

2.3. **Three-way Decisions.** Let U be a finite non-empty set and X is a finite set of criteria. The problem of three-way decisions with respect to X is to partition the set U based on X into three regions of positive, negative and boundary regions which are denoted by POS, NEG and BND, respectively [58]. The most general case in three-way decisions can be constructed with two posets $(L_a, \preceq_a)$ and $(L_r, \preceq_r)$. Using these posets, a pair of functions $v_a : U \to L_a$ and $v_r : U \to L_r$ called acceptance and rejection evaluations, respectively, can be constructed. For $x \in U$, $v_a(x)$ and $v_r(x)$ are called acceptance and rejection values of x, respectively. With these definitions, the three regions of positive, negative and boundary are defined as follows [58]:

$$\begin{aligned}
\mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r) &= \left\{x \in U \mid v_a(x) \in L_a^+ \wedge v_r(x) \notin L_r^-\right\},\\
\mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r) &= \left\{x \in U \mid v_a(x) \notin L_a^+ \wedge v_r(x) \in L_r^-\right\},\\
\mathrm{BND}_{(L_a^+, L_r^-)}(v_a, v_r) &= U \setminus \left(\mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r) \cup \mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r)\right),
\end{aligned}$$

where $\emptyset \neq L_a^+ \subseteq L_a$ and $\emptyset \neq L_r^- \subseteq L_r$ are designated values for acceptance and rejection, respectively.

3. **Fuzzy S-approximation Spaces**

Embedding fuzzy sets in S-approximation spaces can be done in at least three different ways:

(1) Approximating a fuzzy set $X \in \mathcal{F}(W)$ as two fuzzy sets in $\mathcal{F}(U)$, i.e., lower and upper approximation sets are fuzzy (as in Definition 3.1).
(2) Approximating a fuzzy set $X \in \mathcal{F}(W)$ as two crisp (non-fuzzy) sets in $\mathcal{P}(U)$ by using $T : U \to \mathcal{F}(W)$ and $S : \mathcal{F}(W) \times \mathcal{F}(W) \to \{0, 1\}$.
(3) Approximating a crisp (non-fuzzy) set $X \in \mathcal{P}(W)$ as two fuzzy sets in $\mathcal{F}(U)$ by using $S : \mathcal{P}(W) \times \mathcal{P}(W) \to [0, 1]$.

Among these three different approaches, the first approach captures the second and third as special cases. Hence, we will consider the first approach and at last, we will cover the other two approaches as special cases of the first. We will call the first approach as *fuzzy S-approximation spaces* and the second one as *S-approximation spaces over fuzzy sets.*

Definition 3.1 (Fuzzy S-approximation Spaces)**.** A quadruple $FG = (U, W, T, S)$, where U and W are finite non-empty sets, $T : U \to \mathcal{F}(W)$ and $S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0, 1]$, is called *a fuzzy S-approximation space.* Remind that $\mathcal{F}(W)$ denotes the fuzzy power set of the set W.

Throughout the paper, we will illustrate our ideas using Example 3.2 incrementally. This example states a disease and symptoms expert system.

Example 3.2. Let $U = \{u_1, \ldots, u_5\}$ be a set of five diseases and $W = \{w_1, \ldots, w_7\}$ a set of seven symptoms and the knowledge mapping $T : U \to \mathcal{F}(W)$ be defined as in Table 1. The data in Table 1 shall be interpreted in the context of an S-approximation space. For example, by considering $FG1 = (U, W, T, S_1)$, the column corresponding to the disease u_3 can be interpreted as follows: if the maximum level of symptoms w_2, w_5 and w_7 that can be present in a patient are 0.99, 0.17 and 0.04 and all the other symptoms are not observed, then the patient suffers from disease u_3. This would give an intuition of why mappings T and S are called knowledge and decider mappings, respectively.

Assume that a patient is visited by two specialists, e.g. two doctors, and their observations in clinical tests are given as two fuzzy sets

$$X = \frac{w_1}{0.26} + \frac{w_2}{0.50} + \frac{w_3}{0.00} + \frac{w_4}{0.70} + \frac{w_5}{0.06} + \frac{w_6}{0.40} + \frac{w_7}{0.37},$$

and

$$Y = \frac{w_1}{0.30} + \frac{w_2}{0.20} + \frac{w_3}{0.05} + \frac{w_4}{0.80} + \frac{w_5}{0.00} + \frac{w_6}{0.20} + \frac{w_7}{0.50}.$$

We will also use four different deciders along the text as

$$S_1(A, B) = \begin{cases} 1 & \text{if} \quad A \subseteq B \\ 0 & otherwise \end{cases},$$

$$S_2(A, B) = \max_{x \in U} \left\{ \frac{\mu_A(x) + \mu_B(x)}{2} \right\},$$

$$S_3(A, B) = \max_{x \in U} \{ \mu_A(x) \times \mu_B(x) \},$$

and

$$S_4(A, B) = \frac{1}{2} \left(\frac{|A \cap B|}{|A \cap B| + |A \setminus B|} + \frac{|A \cap B|}{|A \cap B| + |B \setminus A|} \right).$$

Note that $S_4(\cdot, \cdot)$ is taken from [18]. From now on, we use FGi to denote $FGi = (U, W, T, S_i)$ for $i = 1, \ldots, 4$.

Remark 3.3. Note that this example is just for the sake of illustration. The task of disease diagnosis is much complicated than only on the basis of a set of symptoms. A good choose of T mapping can be as

$$T(u_1) = \big(w_1 \geq 0.5 \text{ and } (w_3 < 0.3 \text{ or } w_6 < 0.4) \text{ and } (\text{not } w_8)\big),$$

that is, $T(u)$ be a (fuzzy) boolean formula. However, this is not suitable for our purpose which is just giving an illustration of the results.

At first, we may use the ordinary approximation operators, that is,

$$\underline{FG}\,(X) = \{x \in U \mid S(T(x), X) = 1\}, \tag{6}$$

and

$$\overline{FG}\,(X) = \{x \in U \mid S(T(x), X^c) = 0\}, \tag{7}$$

where $X \in \mathcal{F}(W)$. However, it might be desirable in some applications that approximations of a fuzzy set be also fuzzy. To this end, the following definitions are given.

Symptoms/Disease	u_1	u_2	u_3	u_4	u_5
w_1	0.22	0.53	0.00	0.00	0.31
w_2	0.00	0.11	0.99	0.41	0.43
w_3	0.02	0.00	0.00	0.00	0.52
w_4	0.16	0.00	0.00	1.00	0.82
w_5	0.00	0.25	0.17	0.63	0.55
w_6	0.00	0.00	0.00	0.07	0.95
w_7	0.53	0.00	0.04	0.00	0.73

TABLE 1. Sample Disease-symptoms Table Which is Generated in a Random Fashion. For Every Column, Corresponding to a Disease, There is a Membership Function for Every Symptom

Definition 3.4. Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space and $X \in \mathcal{F}(W)$. Then, the membership functions for the lower and upper approximations of set X are defined as

$$\mu_{\underline{FG}(X)}(x) = S(T(x), X), \tag{8}$$

and

$$\mu_{\overline{FG}(X)}(x) = 1 - S(T(x), X^c). \tag{9}$$

Note that these approximation operators would be the same as the standard approximation operators for crisp (non-fuzzy) sets and mappings, i.e. Equations (4) and (5). Also, these approximation operators absorb several different definitions of fuzzy rough sets like [46, 49, 9].

Example 3.5. Let R be a fuzzy relation from U to W, i.e. $R : U \times W \to [0, 1]$, $\alpha \in [0, 1]$ and

$$R_\alpha(u) = \{v \in W \mid R(u, v) \geq \alpha\},$$

for $u \in U$. Define $FG = (U, W, T, S)$ such that $T(u) = R_\alpha(u)$ for every $u \in U$ and

$$S(A, B) = \begin{cases} 1 & \text{if} \quad A \subseteq B \\ 0 & \text{otherwise} \end{cases},$$

for $A, B \in \mathcal{P}(W)$. It is easy to see that $\underline{FG}(X) = \underline{apr}_{R_\alpha}(X)$ and $\overline{FG}(X) = \overline{apr}_{R_\alpha}(X)$ where

$$\underline{apr}_{R_\alpha}(X) = \{u \in U \mid R_\alpha(u) \subseteq X\},$$

and

$$\overline{apr}_{R_\alpha}(X) = \{u \in U \mid R_\alpha(u) \cap X \neq \emptyset\},$$

as in [46].

Remark 3.6. Note that in [49], there are two pairs of approximation operators, one for approximating a set in W by a set in U and the other for the converse. Example 3.5 can be modified easily in such a way that captures the converse two.

As the following example shows, a fairly general definition of fuzzy rough set as is defined in [38] is also expressible in terms of fuzzy S-approximation spaces.

Example 3.7. Let U be a finite non-empty set, R a fuzzy relation over U, $\mathcal{T}$ and $\mathcal{S}$ a t-norm and a t-conorm respectively and $\mathcal{I}$ a border implicator. Then, for $X \in \mathcal{F}(U)$ and $u \in U$, the lower and upper approximations of X are defined as

$$\mu_{\underline{R}(X)}(u) = \inf_{u' \in U} \mathcal{I}\big(R(u,u'), \mu_X(u')\big),$$

and

$$\mu_{\overline{R}(X)}(u) = \sup_{u' \in U} \mathcal{T}\big(R(u,u'), \mu_X(u')\big),$$

respectively.

Consider a fuzzy S-approximation space $FG = (U, U, T, S)$ where $\mu_{T(u)}(u') = R(u,u')$ and mapping S is defined as

$$S(A,B) = \inf_{u \in U} \mathcal{I}\big(\mu_A(u), \mu_B(u)\big),$$

for $A, B \in \mathcal{F}(U)$. Then, for $X \in \mathcal{F}(U)$ we have

$$\begin{aligned}\mu_{\underline{FG}(X)}(u) =& S(T(u),X) = \inf_{u' \in U} \mathcal{I}\big(\mu_{T(u)}(u'), \mu_X(u')\big)\\ =& \inf_{u' \in U} \mathcal{I}\big(R(u,u'), \mu_X(u')\big) = \mu_{\underline{R}(X)}(u)\end{aligned}$$

and

$$\begin{aligned}\mu_{\overline{FG}(X)}(u) =& 1 - S(T(u), X^c) = 1 - \inf_{u' \in U} \mathcal{I}\big(\mu_{T(u)}(u'), \mu_{X^c}(u')\big)\\ =& 1 - \inf_{u' \in U} \mathcal{I}\big(\mu_{T(u)}(u'), \mu_{X^c}(u')\big)\\ =& 1 - \inf_{u' \in U} \mathcal{S}\Big(\mathcal{N}\big(\mu_{T(u)}(u')\big), \mu_{X^c}(u')\Big)\\ =& 1 - \inf_{u' \in U} 1 - \mathcal{T}\bigg(\mathcal{N}\Big(\mathcal{N}\big(\mu_{T(u)}(u')\big)\Big), \mathcal{N}\big(\mu_{X^c}(u')\big)\bigg)\\ =& 1 - \inf_{u' \in U} 1 - \mathcal{T}\Big(\mu_{T(u)}(u'), \mathcal{N}\big(\mu_{X^c}(u')\big)\Big)\\ =& 1 - \bigg(1 - \sup_{u' \in U} \mathcal{T}\Big(\mu_{T(u)}(u'), \mathcal{N}\big(\mu_{X^c}(u')\big)\Big)\bigg)\\ =& \sup_{u' \in U} \mathcal{T}\Big(\mu_{T(u)}(u'), \mathcal{N}\big(\mu_{X^c}(u')\big)\Big) \qquad (\text{assuming } \mathcal{N}(x) = 1 - x)\\ =& \sup_{u' \in U} \mathcal{T}\big(\mu_{T(u)}(u'), 1 - \mu_{X^c}(u')\big)\\ =& \sup_{u' \in U} \mathcal{T}\big(\mu_{T(u)}(u'), \mu_X(u')\big) = \mu_{\overline{R}(X)}(u).\end{aligned}$$

Example 3.8. Using $FG1$, we obtain the following lower and upper approximations for the set X

$$\begin{aligned}\underline{FG1}(X) =& \emptyset,\\ \overline{FG1}(X) =& \frac{u_4}{1.00} + \frac{u_5}{1.00}.\end{aligned}$$

For $FG2$ to $FG4$, we obtain

$$\begin{aligned}\underline{FG2}\,(X) =&\frac{u_1}{0.45}+\frac{u_2}{0.41}+\frac{u_3}{0.49}+\frac{u_4}{0.55}+\frac{u_5}{0.67},\\ \overline{FG2}\,(X) =&\frac{u_1}{0.42}+\frac{u_2}{0.38}+\frac{u_3}{0.01}+\frac{u_4}{0.05}+\frac{u_5}{0.14},\\ \underline{FG3}\,(X) =&\frac{u_1}{0.20}+\frac{u_2}{0.16}+\frac{u_3}{0.01}+\frac{u_4}{0.10}+\frac{u_5}{0.38},\\ \overline{FG3}\,(X) =&\frac{u_1}{0.67}+\frac{u_2}{0.63}+\frac{u_3}{0.01}+\frac{u_4}{0.10}+\frac{u_5}{0.26},\end{aligned}$$

and

$$\begin{aligned}\underline{FG4}\,(X) =&\frac{u_1}{0.59}+\frac{u_2}{0.31}+\frac{u_3}{0.07}+\frac{u_4}{0.12}+\frac{u_5}{0.70},\\ \overline{FG4}\,(X) =&\frac{u_1}{0.41}+\frac{u_2}{0.41}+\frac{u_3}{0.39}+\frac{u_4}{0.33}+\frac{u_5}{0.19},\end{aligned}$$

respectively. The interpretation of these approximations can be as follows for FGi, $i = 1, \ldots, 4$. The set $\underline{FGi}\,(X)$ represents those elements of U, i.e. diseases, such that their symptoms, i.e. $T(u)$, is similar to the observed symptoms, i.e. set X with respect to decider S_i. Also, the set $\overline{FGi}\,(X)$ represents those elements of U such that their symptoms is not similar to the complement of the observed symptoms with respect to decider S_i. Note that these two approximations are building blocks of three-way decision making in fuzzy S-approximation spaces which are considered in the sequel.

The membership functions of the lower and upper approximations defined in Definition 3.4 satisfy a duality property. This is stated formally in Proposition 3.9.

Proposition 3.9. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space. For all $X, Y \in \mathcal{F}(W)$, the followings hold:*

$(FS7)$**:** $\overline{FG}(X) = (\underline{FG}(X^c))^c$,
$(FS8)$**:** $\underline{FG}(X) = (\overline{FG}(X^c))^c$.

Proof. Since $(FS7)$ and $(FS8)$ are related, so we just show $(FS7)$. For every $x \in U$, we have

$$\begin{aligned}\mu_{(\underline{FG}(X^c))^c}(x) =&1-\mu_{\underline{FG}(X^c)}(x)\\ =&1-S(T(x), X^c)\\ =&\mu_{\overline{FG}(X^c)}(x).\end{aligned}$$

□

Example 3.10. To verify Proposition 3.9, consider $FG3$ and set X. Then we obtain

$$\begin{aligned}\underline{FG3}\,(X) =&\frac{u_1}{0.20}+\frac{u_2}{0.16}+\frac{u_3}{0.01}+\frac{u_4}{0.10}+\frac{u_5}{0.38},\\ \overline{FG3}\,(X) =&\frac{u_1}{0.67}+\frac{u_2}{0.63}+\frac{u_3}{0.01}+\frac{u_4}{0.10}+\frac{u_5}{0.26},\end{aligned}$$

and

$$\underline{FG3}\left(X^{c}\right)=\frac{u_1}{0.33}+\frac{u_2}{0.37}+\frac{u_3}{0.99}+\frac{u_4}{0.90}+\frac{u_5}{0.74},$$
$$\overline{FG3}\left(X^{c}\right)=\frac{u_1}{0.80}+\frac{u_2}{0.84}+\frac{u_3}{0.98}+\frac{u_4}{0.90}+\frac{u_5}{0.62}.$$

It can be easily verified that $\underline{FG3}\left(X\right)=(\overline{FG3}\left(X^{c}\right))^{c}$ and $\overline{FG3}\left(X\right)=(\underline{FG3}\left(X^{c}\right))^{c}$ as in Proposition 3.9[1].

Given a fuzzy S-approximation space $FG=(U,W,T,S)$ and $X\in\mathcal{F}\left(W\right)$. Then, we have

$$\mathrm{CORE}\left(\underline{FG}\left(X\right)\right)=\left\{x\in U\mid \mu_{\underline{FG}(X)}(x)=1\right\}=\{x\in U\mid S(T(x),X)=1\},\tag{10}$$

and

$$\mathrm{CORE}\left(\overline{FG}\left(X\right)\right)=\left\{x\in U\mid \mu_{\overline{FG}(X)}(x)=1\right\}=\{x\in U\mid S(T(x),X^{c})=0\}.\tag{11}$$

Given that $\mathrm{CORE}\left(A\right)=A_1$ for a fuzzy subset A of a set U, we can add acceptance and rejection thresholds into the lower and upper approximations of a set $X\in\mathcal{F}\left(W\right)$ as follows. By acceptance threshold a, we allow for an error in acceptance which is no more than $\%\big((1-a)\times 100\big)$ and for rejection threshold r, we allow for an error in rejection which is no more than $\%(r\times 100)$. These ideas are defined formally in Definition 3.11.

Definition 3.11. Let $FG=(U,W,T,S)$ be a fuzzy S-approximation space and (a,r) be a pair of thresholds, where $0\le r<a\le 1$. Then, the lower and upper approximations of a set $X\in\mathcal{F}\left(W\right)$ are defined as

$$\underline{FG_a}\left(X\right)=\{x\in U\mid S(T(x),X)\ge a\},\tag{12}$$

and

$$\overline{FG_r}\left(X\right)=\{x\in U\mid S(T(x),X^{c})\le r\},\tag{13}$$

respectively with respect to the error threshold pair (a,r).

The intuition behind Definition 3.11 is that we allow those elements $x\in U$ to be in $\underline{FG_a}\left(X\right)$ which can be accepted as X with acceptance accuracy of at least a. Also, those $x\in U$ are allowed to be in $\overline{FG_r}\left(X\right)$ which can be rejected as X^{c} with rejection error of at most r.

Note that from an algebraic point of view, it is clear that $\underline{FG_a}\left(X\right)=\underline{FG}\left(X\right)_a$ and $\overline{FG_r}\left(X\right)=U\setminus\overline{FG}\left(X\right)_{\overline{r}}$.

Regarding the Definition 3.11, one may ask about the effect of changes in acceptance or error thresholds to the lower and upper approximation operators. This question is partially answered in Theorem 3.12 and would be considered more in the sequel.

Theorem 3.12. *Let $FG=(U,W,T,S)$ be a fuzzy S-approximation space, numbers $a,a',r,r'\in[0,1]$ and $X\in\mathcal{F}\left(W\right)$. Then*

(1) *$a<a'$ implies $\underline{FG_{a'}}\left(X\right)\subseteq\underline{FG_a}\left(X\right)$,*

[1]Note that for u_3, $\mu_{\underline{FG3}(X)}(u_3)+\mu_{\overline{FG3}(X^{c})}(u_3)=0.99<1$. This has happened since we have rounded the membership degrees in our examples with 2 significant digits.

(2) $r' < r$ *implies* $\overline{FG_{r'}}(X) \subseteq \overline{FG_r}(X)$.

Proof. The proof easily follows from Definition 3.11. □

Example 3.13. Let $(a, r) = (0.4, 0.6)$, $(a', r') = (0.2, 0.6)$ and $(a'', r'') = (0.4, 0.9)$. Then, for $FG4$ and X, we obtain

$$\begin{aligned}
\underline{FG4_a}(X) &= \{u_1, u_5\}, \\
\overline{FG4_r}(X) &= \{u_1, u_2\}, \\
\underline{FG4_{a'}}(X) &= \{u_1, u_2, u_5\}, \\
\overline{FG4_{r'}}(X) &= \{u_1, u_2\}, \\
\underline{FG4_{a''}}(X) &= \{u_1, u_5\}, \\
\overline{FG4_{r''}}(X) &= U.
\end{aligned}$$

In addition, if the thresholds are swapped, i.e. exchanging a with r, then we have

$$\begin{aligned}
\underline{FG4_a}(X) &= \{u_5\}, \\
\overline{FG4_r}(X) &= \emptyset, \\
\underline{FG4_{a'}}(X) &= \{u_5\}, \\
\overline{FG4_{r'}}(X) &= \emptyset, \\
\underline{FG4_{a''}}(X) &= \emptyset, \\
\overline{FG4_{r''}}(X) &= \emptyset.
\end{aligned}$$

Using these calculations, one can easily verify the results of Theorem 3.12.

Next, we will consider error thresholds for fuzzy S-approximation spaces. It is easy to see that by choosing $a = 1$ and $r = 0$, $\underline{FG_1}(X) = \text{CORE}(\underline{FG}(X))$ and $\overline{FG_0}(X) = \text{CORE}\left(\overline{FG}(X)\right)$. Hence, ordinary S-approximation spaces or S-approximation spaces over fuzzy sets are both expressible in terms of fuzzy S-approximation spaces.

We have previously mentioned the intuition behind defining this pair of error thresholds, now we get back to it in more details. Assuming S as a satisfiability measure, then $\underline{FG_a}(X)$ is consisted of elements $x \in U$ that are satisfied or satisfy X in at least degree of a. Also, for $\overline{FG_r}(X)$, we consider all $x \in U$ whose satisfaction degree to complement of X, i.e. X^c, is less than or equal to degree r. Table 2 explains these ideas in details.

Status	Interpretation
$S(T(x), X) \geq a$	x can be accepted as X with acceptance accuracy at least a.
$S(T(x), X) < a$	x cannot be accepted as X with acceptance accuracy at least a.
$S(T(x), X^c) \leq r$	x can be rejected as X^c with rejection error at most r.
$S(T(x), X^c) > r$	x cannot be rejected as X^c with rejection error at most r.

Table 2. The Intuition of Defining a and r as Acceptance Accuracy and Rejection Error for $x \in U$ and $X \in \mathcal{F}(W)$

Using Table 2, we can characterize five situations, namely (a,r)-positiveness, (a,r)-negativeness, (a,r)-non-commitment, (a,r)-inconsistent and (a,r)-uncertain as X, e.g. it might be the case that x can be accepted as both X and X^c and also can reject it as both X and X^c, i.e. $S(T(x),X) \geq a$, $S(T(x),X) \leq r$, $S(T(x),X^c) \geq a$ and $S(T(x),X^c) \leq r$. Note that we might drop (a,r) from the notation, when it is clear from the context. These five situations are quite natural and are defined in Table 3 and are illustrated in Figures 1 to 3.

As X	$S(T(x),X)$				$S(T(x),X^c)$			
	$\geq a$	$< a$	$\leq r$	$> r$	$\geq a$	$< a$	$\leq r$	$> r$
UC	✓			✓		✓		✓
P	✓			✓		✓	✓	
IC	✓			✓	✓			✓
IC	✓			✓	✓		✓	
IC	✓		✓			✓		✓
IC	✓		✓			✓	✓	
IC	✓		✓		✓			✓
IC	✓		✓		✓		✓	
NC		✓		✓		✓		✓
UC		✓		✓		✓	✓	
UC		✓		✓	✓			✓
IC		✓		✓	✓		✓	
UC		✓	✓			✓		✓
IC		✓	✓			✓	✓	
N		✓	✓		✓			✓
IC		✓	✓		✓		✓	

Table 3. Summary of Five Situations for a Fuzzy S-approximation Space $FG = (U,W,T,S)$ and $X \in \mathcal{F}(W)$ where P: (a,r)-positive, N: (a,r)-negative, NC: (a,r)-Non-commitment, IC: (a,r)-Inconsistent and UC: (a,r)-Uncertainty for arbitrary $a,r \in [0,1]$ discussed in Figures 1 to 3

4. Three-way Decisions in Fuzzy S-approximation Space

In rough set theory, the lower and upper approximations have intuitive interpretations, i.e. the lower approximation is consisted of those elements which belong to a concept with certainty and the elements which are not in the upper approximation do not belong to that concept with certainty. Those elements that fall in the difference of the upper and lower approximations may or may not belong to that concept [27]. To facilitate these kinds of decision makings in S-approximation spaces, the notion of three-way decisions were employed in S-approximation spaces [40]. By a similar approach and using the notions of (a,r)-acceptance or -rejection, -non-commitment and -inconsistencies, we can define the following three-way regions as in [57, 40].

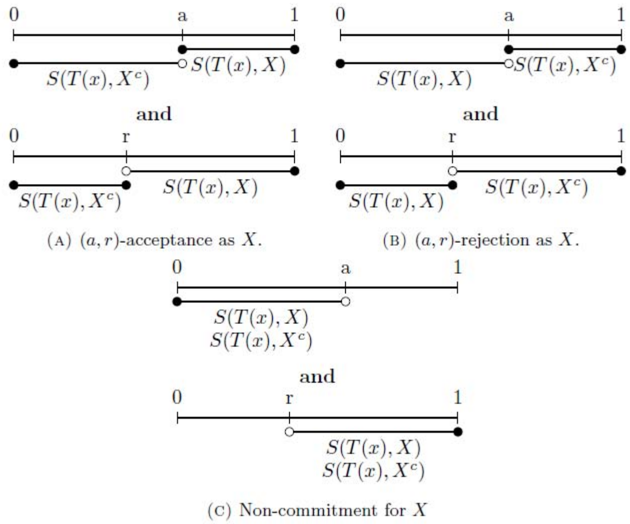

FIGURE 1. Three Situations of (a, r)-acceptance, (a, r)-rejection as X and (a, r)-non-commitment as X for Arbitrary $a, r \in [0, 1]$, $X \in \mathcal{F}(W)$ and Fuzzy S-approximation Space $FG = (U, W, T, S)$

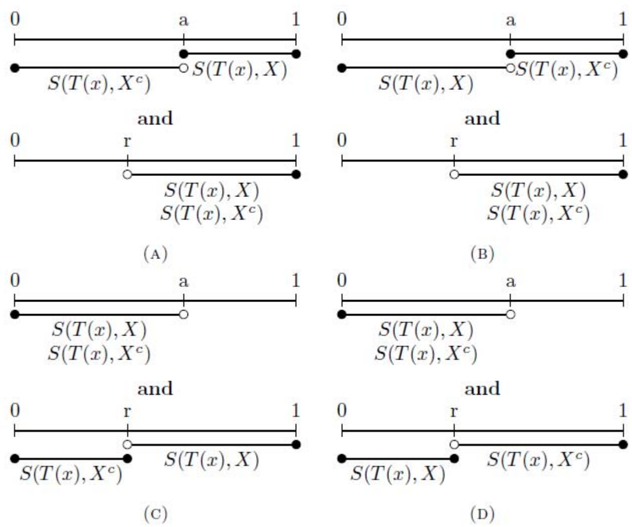

FIGURE 2. Situations of (a, r)-uncertainty as X for Arbitrary $a, r \in [0, 1]$, $X \in \mathcal{F}(W)$ and fuzzy S-approximation space $FG = (U, W, T, S)$

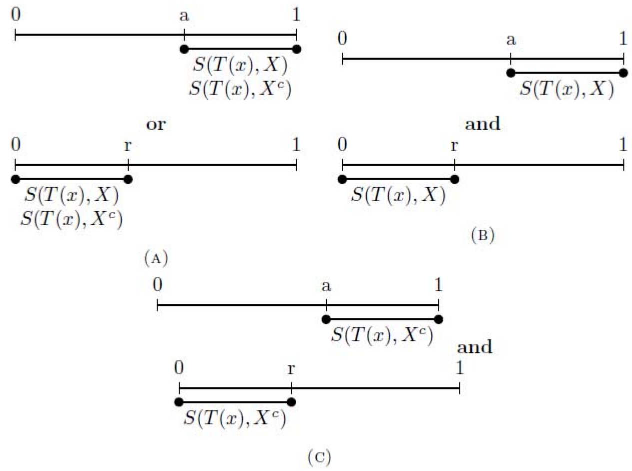

FIGURE 3. Situations of (a, r)-inconsistency as X for Arbitrary $a, r \in [0, 1]$, $X \in \mathcal{F}(W)$ and Fuzzy S-approximation space $FG = (U, W, T, S)$

Definition 4.1. Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then

$$\begin{aligned}\mathrm{POS}_{FG_{(a,r)}}(X) =& \{x \in U \mid S(T(x), X) \geq a,\ S(T(x), X^c) \leq r,\\ & S(T(x), X) > r \text{ and } S(T(x), X^c) < a\},\end{aligned} \tag{14}$$

as *positive region of* X,

$$\begin{aligned}\mathrm{NEG}_{FG_{(a,r)}}(X) =& \{x \in U \mid S(T(x), X) \leq r,\ S(T(x), X^c) \geq a,\\ & S(T(x), X) < a \text{ and } S(T(x), X^c) > r\},\end{aligned} \tag{15}$$

as *negative region of* X and

$$\mathrm{BND}_{FG_{(a,r)}}(X) = U \setminus \big(\mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X)\big), \tag{16}$$

as *boundary region of* X with acceptance accuracy of a and rejection error of r.

Remark 4.2. Note that the concepts of acceptance and rejection discussed in Table 2 are different from the positiveness and negativeness concepts introduced in Definition 4.1, e.g. an element x belongs to the positive region of X with respect to threshold pair (a, r) if it can be accepted as X and cannot be accepted as X^c with accuracy of at least a and at the same time, can be rejected as X^c and cannot be rejected as X with error of at most r.

The sets of non-committable, inconsistent and uncertain elements of X are denoted by $\mathrm{NC}_{FG_{(a,r)}}(X)$, $\mathrm{IC}_{FG_{(a,r)}}(X)$ and $\mathrm{UN}_{FG_{(a,r)}}(X)$, respectively. It is clear that $\mathrm{BND}_{FG_{(a,r)}}(X) = \mathrm{IC}_{FG_{(a,r)}}(X) \cup \mathrm{NC}_{FG_{(a,r)}}(X) \cup \mathrm{UN}_{FG_{(a,r)}}(X)$ and $\mathrm{IC}_{FG_{(a,r)}}(X)$, $\mathrm{NC}_{FG_{(a,r)}}(X)$ and $\mathrm{UN}_{FG_{(a,r)}}(X)$ are pairwise disjoint.

Definition 4.3. Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then, X is called *(a, r)-fuzzy FG-decidable* whenever $\mathrm{BND}_{FG_{(a,r)}}(X) = \emptyset$. If $\mathrm{BND}_{FG_{(a,r)}}(X) = U$, then X is called *(a, r)-fuzzy FG-undecidable*. Finally, if $\emptyset \subsetneq \mathrm{BND}_{FG_{(a,r)}}(X) \subsetneq U$, then X is called *(a, r)-fuzzy partially FG-decidable*

It is easy to see that these three regions are expressible in terms of standard fuzzy approximation operators as

$$\mathrm{POS}_{FG_{(a,r)}}(X) = \underline{FG_a}(X) \cap \overline{FG_r}(X) \cap (\underline{FG_a}(X^c))^c \cap (\overline{FG_r}(X^c))^c, \tag{17}$$

$$\mathrm{NEG}_{FG_{(a,r)}}(X) = \underline{FG_a}(X^c) \cap \overline{FG_r}(X^c) \cap (\underline{FG_a}(X))^c \cap (\overline{FG_r}(X))^c, \tag{18}$$

$$\mathrm{BND}_{FG_{(a,r)}}(X) = U \setminus \Big(\mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X)\Big), \tag{19}$$

for every fuzzy S-approximation space $FG = (U, W, T, S)$, $X \in \mathcal{F}(W)$ and acceptance and rejection errors $a, r \in [0, 1]$. Also, note that

$$\mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X) \cup \mathrm{BND}_{FG_{(a,r)}}(X) = U, \tag{20}$$

for every $X \in \mathcal{F}(W)$ and these three regions are pairwise disjoint.

The following two properties are stated without a proof regarding the relation between the three regions of a set and its complement.

Lemma 4.4. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space and $X \in \mathcal{F}(W)$. Then*

$$\mathrm{POS}_{FG_{(a,r)}}(X) = \mathrm{NEG}_{FG_{(a,r)}}(X^c). \tag{21}$$

Corollary 4.5. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space and $X \in \mathcal{F}(W)$. Then*

$$\mathrm{BND}_{FG_{(a,r)}}(X) = \mathrm{BND}_{FG_{(a,r)}}(X^c). \tag{22}$$

Next, we will discuss the structure of $\mathrm{BND}_{FG_{(a,r)}}(X)$. Each element in set $\mathrm{BND}_{FG_{(a,r)}}(X)$, if $\mathrm{BND}_{FG_{(a,r)}}(X) \neq \emptyset$, can be assigned to one of these three categories uniquely, which are non-commitment, inconsistent or uncertain region.

Now, suppose that $\mathrm{BND}_{FG_{(a,r)}}(X) \neq \emptyset$, i.e. X is either (a, r)-fuzzy partially FG-decidable or (a, r)-fuzzy undecidable. A pair of error thresholds (a, r) is called an *inconsistent threshold for* X if $\mathrm{IC}_{FG_{(a,r)}}(X) \neq \emptyset$ and its degree of inconsistency is computed as

$$d_{IC}(X)_{(a,r)} = \frac{|\mathrm{IC}_{FG}(X)_{(a,r)}|}{|U|}. \tag{23}$$

Also, the generality[2] of (a, r) threshold of a set $X \in \mathcal{F}(W)$ with respect to a fuzzy S-approximation space can be measured by

$$Generality_{G_{(a,r)}}(X) = \frac{|Z|}{|U|} = 1 - \frac{|\mathrm{NC}_{FG_{(a,r)}}(X)|}{|U|}. \tag{24}$$

where $Z = \mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X) \cup \mathrm{UN}_{FG_{(a,r)}}(X) \cup \mathrm{IC}_{FG_{(a,r)}}(X)$.

Note that $Generality(a, r)$ defined here is different from the one defined in [2] as it considers elements from $\mathrm{IC}_{FG_{(a,r)}}(X)$ and $\mathrm{UN}_{FG_{(a,r)}}(X)$, which are subsets of $\mathrm{BND}_{FG_{(a,r)}}(X)$. The intuition behind this difference is that generality of a set X with respect to FG using the threshold (a, r) is the ratio of all elements of U with a decision, either accept, reject, inconsistent or uncertain, to the total number of elements of U.

Up to now, we did not assume any relation between a and r except that $a, r \in [0,1]$. However, from now on we assume that $0 \leq r < a \leq 1$. This assumption makes Table 3 simpler by removing seven rows from it. The simplified version is given in Table 4 for easier reference. Using this assumption, the Figures 1 to 3 can be simplified, too.

As X	$S(T(x),X)$				$S(T(x),X^c)$			
	$\geq a$	$< a$	$\leq r$	$> r$	$\geq a$	$< a$	$\leq r$	$> r$
UC	✓			✓		✓		✓
A	✓			✓		✓	✓	
IC	✓			✓	✓			✓
NC		✓		✓		✓		✓
UC		✓		✓		✓	✓	
UC		✓		✓	✓			✓
UC		✓	✓			✓		✓
IC		✓	✓			✓	✓	
R		✓	✓		✓			✓

Table 4. Definition of Five Situations for Arbitrary $0 \leq r < a \leq 1$, Fuzzy S-approximation Space $FG = (U, W, T, S)$ and $X \in \mathcal{F}(W)$ Where A: (a,r)-accept, R: (a,r)-reject, NC: (a,r)-Non-commitment, IC: (a,r)-Inconsistent and UC: (a,r)-Uncertainty

Using the assumption $r < a$, Definition 4.1 can be stated in a more natural way as

$$\mathrm{POS}_{FG_{(a,r)}}(X) = \{x \in U \mid S(T(x),X) \geq a \text{ and } S(T(x),X^c) \leq r\}, \quad (25)$$

$$\mathrm{NEG}_{FG_{(a,r)}}(X) = \{x \in U \mid S(T(x),X) \leq r \text{ and } S(T(x),X^c) \geq a\}, \quad (26)$$

$$\mathrm{BND}_{FG_{(a,r)}}(X) = U \setminus (\mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X)). \quad (27)$$

These three regions can also be expressed in terms of the lower and upper approximations as

$$\mathrm{POS}_{FG_{(a,r)}}(X) = \underline{FG_a}(X) \cap \overline{FG_r}(X), \quad (28)$$

$$\mathrm{NEG}_{FG_{(a,r)}}(X) = \underline{FG_a}(X^c) \cap \overline{FG_r}(X^c), \quad (29)$$

$$\mathrm{BND}_{FG_{(a,r)}}(X) = U \setminus \left(\mathrm{POS}_{FG_{(a,r)}}(X) \cup \mathrm{NEG}_{FG_{(a,r)}}(X)\right), \quad (30)$$

whenever $0 \leq r < a \leq 1$. The following theorem investigates the effect of changes in thresholds of acceptance and rejection in positive and negative regions.

Theorem 4.6. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $a, a', r, r' \in [0,1]$ such that $r < a$ and $X \in \mathcal{F}(W)$. Then*

(1) *$a < a'$ implies $POS_{FG_{(a',r)}}(X) \subseteq POS_{FG_{(a,r)}}(X)$,*
(2) *$r' < r$ implies $POS_{FG_{(a,r')}}(X) \subseteq POS_{FG_{(a,r)}}(X)$,*
(3) *$a < a'$ implies $NEG_{FG_{(a',r)}}(X) \subseteq NEG_{FG_{(a,r)}}(X)$,*
(4) *$r' < r$ implies $NEG_{FG_{(a,r')}}(X) \subseteq NEG_{FG_{(a,r)}}(X)$.*

Proof. The proof is direct from Equations (28) to (30). □

Corollary 4.7. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $a, r \in [0,1]$ such that $r < a$ and $X \in \mathcal{F}(W)$. Then*

(1) $r < r'$ *implies* $|BND_{FG_{(a,r)}}(X)| \geq |BND_{FG_{(a,r')}}(X)|$,
(2) $a < a'$ *implies* $|BND_{FG_{(a,r)}}(X)| \leq |BND_{FG_{(a',r)}}(X)|$,

The following corollary shows that changing threshold (a,r) to (a',r') such that $0 \leq r' < r < a < a' \leq 1$ can make an (a,r)-fuzzy FG-decidable set $X \in \mathcal{F}(W)$ to a partial FG-decidable or even FG-undecidable with respect to the new threshold pair (a',r').

Corollary 4.8. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $a, a', r, r' \in [0,1]$ such that $r < a$ and $X \in \mathcal{F}(W)$. Then*

(1) $a < a'$ *and* $r' < r$ *implies* $POS_{FG_{(a',r')}}(X) \subseteq POS_{FG_{(a,r)}}(X)$,
(2) $a < a'$ *and* $r' < r$ *implies* $NEG_{FG_{(a',r')}}(X) \subseteq NEG_{FG_{(a,r)}}(X)$,
(3) $a < a'$ *and* $r' < r$ *implies* $BND_{FG_{(a,r)}}(X) \subseteq BND_{FG_{(a',r')}}(X)$.

In [40], two measures of G-definability and G-undefinability are defined for standard or ordinary S-approximation spaces as

$$\alpha_{G_{(a,r)}}(X) = 1 - \frac{|\mathrm{BND}_{G_{(a,r)}}(X)|}{|U|}, \tag{31}$$

and

$$\beta_{G_{(a,r)}}(X) = \begin{cases} \frac{|\mathrm{BND}_{G_{(a,r)}}(X)|}{|\mathrm{POS}_{G_{(a,r)}}(X)|+|\mathrm{NEG}_{G_{(a,r)}}(X)|} & \text{if } \mathrm{POS}_{G_{(a,r)}}(X) \cup \mathrm{NEG}_{G_{(a,r)}}(X) \neq \emptyset \\ \infty & \text{otherwise,} \end{cases} \tag{32}$$

respectively. These measures are extensible to fuzzy S-approximation spaces, too. From now on, by $\alpha_{G_{(a,r)}}(X)$ and $\beta_{G_{(a,r)}}(X)$ we mean the extended ones.

Corollary 4.9. *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, $0 \leq r' < r < a < a' \leq 1$ and $X \in \mathcal{F}(W)$. Then*

$$\alpha_{FG_{(a',r')}}(X) \leq \alpha_{FG_{(a,r)}}(X),$$

and

$$\beta_{FG_{(a,r)}}(X) \leq \beta_{FG_{(a',r')}}(X).$$

Example 4.10. Now assume that a third doctor wants to decide on the possible disease(s) that this patient may suffer. Suppose his/her observations lead to

$$Z = \frac{w_1}{0.50} + \frac{w_2}{0.20} + \frac{w_3}{0.60} + \frac{w_4}{0.14} + \frac{w_5}{0.03} + \frac{w_6}{0.20} + \frac{w_7}{0.40}.$$

Then, he/she will construct the three regions for the set X with respect to $FG = (U, W, T, S)$ where

$$S(A,B) = \frac{|A \cap B|}{|A \cap B| + |A \setminus B| + |B \setminus A| + |A^c \cap B^c|},$$

for every $A, B \in \mathcal{F}(W)$[18], as

$$\begin{aligned}
\mathrm{POS}_{FG_{(a,r)}}(Z) &= \{u_5\},\\
\mathrm{NEG}_{FG_{(a,r)}}(Z) &= \emptyset,\\
\mathrm{BND}_{FG_{(a,r)}}(Z) &= \{u_1, u_2, u_3, u_4\},
\end{aligned}$$

for thresholds $a = 0.40$ and $r = 0.20$. The interpretation of this result is that the patient under consideration suffers from disease u_5, but the decision for other diseases needs further investigations.

Moreover, the FG-definability and FG-undefinability of set Z are obtained as $\alpha_{FG}(Z) = 0.20$ and $\beta_{FG}(Z) = 4$. Surprisingly, all the elements in $\mathrm{BND}_{FG_{(a,r)}}(Z)$ are due to inconsistency of threshold pairs, i.e. the degree of inconsistency of (a, r) for Z with respect to FG is $d_{IC}(Z)_{(a,r)} = 0.80$. Moreover, $Generality_{G(a,r)}(Z) = 1$. Note that if we choose $a = 0.80$ and $r = 0.10$, then the degree of inconsistency would decrease to 0.60.

5. **Monotonicity**

For ordinary S-approximation spaces [40], partial monotonicity was introduced as "$A \subseteq B$ and $S(X, A) = 1$ implies $S(X, B) = 1$". For fuzzy S-approximation spaces, we introduce the following condition as fully monotone or monotone for simplicity, which is a counterpart of partial monotonicity in ordinary S-approximation spaces.

Definition 5.1 (Monotonicity)**.** Let $S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0, 1]$ be a decider. Then, it is called *fully monotone* or *monotone* if

$$A \subseteq B \implies S(X, A) \leq S(X, B), \tag{33}$$

for every $A, B, X \in \mathcal{F}(W)$.

Note that every monotone mapping S is partial monotone, but the converse is not true. In what follows, we will show that monotonicity of fuzzy S-approximation spaces preserves all of the properties of partial monotone S-approximation spaces.

Proposition 5.2. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space. For all $X, Y \in \mathcal{F}(W)$, the followings hold:*

- $(M1)$**:** *$X \subseteq Y$ implies $\overline{FG}(X) \subseteq \overline{FG}(Y)$,*
- $(M2)$**:** *$X \subseteq Y$ implies $\underline{FG}(X) \subseteq \underline{FG}(Y)$,*
- $(M3)$**:** *$\overline{FG}(X \cup Y) \supseteq \overline{FG}(X) \cup \overline{FG}(Y)$,*
- $(M4)$**:** *$\overline{FG}(X \cap Y) \subseteq \overline{FG}(X) \cap \overline{FG}(Y)$,*
- $(M5)$**:** *$\underline{FG}(X \cup Y) \supseteq \underline{FG}(X) \cup \underline{FG}(Y)$,*
- $(M6)$**:** *$\underline{FG}(X \cap Y) \subseteq \underline{FG}(X) \cap \underline{FG}(Y)$.*

Proof. The proof is as follows:

$(M1)$**:** It is clear that by $X \subseteq Y$, we have $Y^c = X^c$. Since S is monotone, it follows that $S(T(x), Y^c) \leq S(T(x), X^c)$ for every $x \in U$ which implies that $1 - S(T(x), X^c) \leq 1 - S(T(x), Y^c)$. Hence, $\mu_{\overline{FG}(X)}(x) \leq \mu_{\overline{FG}(Y)}(x)$ which means that $\overline{FG}(X) \subseteq \overline{FG}(Y)$.

$(M2)$: Since S is monotone, it follows that $S(T(x), X) \leq S(T(x), Y)$ for every $x \in U$. Hence, $\mu_{\underline{FG}(X)}(x) \leq \mu_{\underline{FG}(Y)}(x)$, which means that $\underline{FG}\,(X) \subseteq \underline{FG}\,(Y)$.

$(M3)$: From $X \subseteq X \cup Y$, we have $(X \cup Y)^c \subseteq X^c$. By monotonicity of S, we have $S(T(x), (X \cup Y)^c) \subseteq S(T(x), X^c)$ which implies that $1 - S(T(x), X^c) \leq 1 - S(T(x), (X \cup Y)^c)$ which is by the definition $\overline{FG}(X \cup Y) \supseteq \overline{FG}(X)$. By the same reasoning for Y, we obtain the claim.

$(M4)$: Given that $X \cap Y \subseteq X$, then $X^c \subseteq (X \cap Y)^c$. Hence $S(T(x), X^c) \leq S(T(x), (X \cap Y)^c)$ by monotonicity of S which implies that $1 - S(T(x), (X \cap Y)^c) \leq 1 - S(T(x), X^c)$. This is by the definition $\overline{FG}(X \cap Y) \subseteq \overline{FG}(X)$. By the same reasoning for Y, we obtain $\overline{FG}(X \cap Y) \subseteq \overline{FG}(Y)$. Hence, for every $x \in U$,

$$\mu_{\overline{FG}(X \cap Y)}(x) \leq \min\left\{\mu_{\overline{FG}(X)}(x), \mu_{\overline{FG}(Y)}(x)\right\},$$

and the desired result is obtained.

$(M5)$: From $X \subseteq X \cup Y$, it is clear that $(X \cup Y)^c \subseteq X^c$. Hence by monotonicity of S, we have $S(T(x), (X \cup Y)^c) \leq S(T(x), X^c)$. So, $1 - S(T(x), X^c) \leq 1 - S(T(x), (X \cup Y)^c)$ which implies that $\underline{FG}\,(X) \subseteq \underline{FG}\,(X \cup Y)$. By the same reasoning, we have $\underline{FG}\,(Y) \subseteq \underline{FG}\,(X \cup Y)$. This implies

$$\max\left\{\mu_{\underline{FG}(X)}(x), \mu_{\underline{FG}(Y)}(x)\right\} \leq \mu_{\underline{FG}(X \cup Y)}(x),$$

for every $x \in U$ which concludes the proof.

$(M6)$: Given that $X \cap Y \subseteq X$, then $S(T(x), X \cap Y) \leq S(T(x), X)$ by monotonicity of S. So, $\underline{FG}(X \cap Y) \subseteq \underline{FG}(X)$. With the same reasoning with Y, we can obtain $\underline{FG}(X \cap Y) \subseteq \underline{FG}(Y)$, hence

$$\mu_{\underline{FG}(X \cap Y)}(x) \leq \min\left\{\mu_{\underline{FG}(X)}(x), \mu_{\underline{FG}(Y)}(x)\right\},$$

and the desired result is obtained. □

Corollary 5.3. *Let $FG = (U, W, T, S)$ be a monotone fuzzy approximation space, $X, Y \in \mathcal{F}(W)$ and (a, r) a pair of error thresholds. Then*

$(M1)$: *$X \subseteq Y$ implies $\overline{FG_r}\,(X) \subseteq \overline{FG_r}\,(Y)$,*
$(M2)$: *$X \subseteq Y$ implies $\underline{FG_a}\,(X) \subseteq \underline{FG_a}\,(Y)$,*
$(M3)$: *$\overline{FG_a}(X \cup Y) \supseteq \overline{FG_a}(X) \cup \overline{FG_a}(Y)$,*
$(M4)$: *$\overline{FG_a}(X \cap Y) \subseteq \overline{FG_a}(X) \cap \overline{FG_a}(Y)$,*
$(M5)$: *$\underline{FG_r}(X \cup Y) \supseteq \underline{FG_r}(X) \cup \underline{FG_r}(Y)$,*
$(M6)$: *$\underline{FG_r}(X \cap Y) \subseteq \underline{FG_r}(X) \cap \underline{FG_r}(Y)$.*

Proof. The proof for lower approximations with respect to acceptance threshold a are obtained by application of Proposition 2.3 and the properties of weak and strong α-cuts. □

Theorem 5.4. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $0 \leq r < a \leq 1$. Then*

$(M7)$: *$\overline{FG_r}\,(X) \subseteq (\underline{FG_a}\,(X^c))^c$,*
$(M8)$: *$\underline{FG_a}\,(X) \subseteq (\overline{FG_r}\,(X^c))^c$.*

Lemma 5.5. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then $\underline{FG_a}(W) = U$ if and only if for every $x \in U$, there exists $Y \in \mathcal{F}(W)$ such that $S(T(x), Y) \geq a$.*

Corollary 5.6. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$, $0 \leq r < a \leq 1$ and $\underline{FG_a}(W) = U$. Then, $\overline{FG_r}(\emptyset) = \emptyset$.*

Lemma 5.7. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then $\overline{FG_r}(\emptyset) = \emptyset$ if and only if for every $x \in U$, $S(T(x), W) > r$.*

Lemma 5.8. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then $\underline{FG_a}(\emptyset) = \emptyset$ if and only if for every $x \in U$, $S(T(x), \emptyset) < a$.*

Lemma 5.9. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then $\overline{FG_r}(W) = U$ if and only if for every $x \in U$, $S(T(x), \emptyset) \leq r$.*

Corollary 5.10. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $0 \leq r < a \leq 1$. Then $\underline{FG_a}(\emptyset) = \emptyset$ and $\overline{FG_r}(W) = U$ if and only if for every $x \in U$, $S(T(x), \emptyset) \leq r$.*

Corollary 5.11. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $0 \leq r < a \leq 1$. Then*

- $\underline{FG_a}(\emptyset) = \overline{FG_r}(\emptyset) = \emptyset$,
- $\underline{FG_a}(W) = \overline{FG_r}(W) = \emptyset$,

if and only if

$$\forall x \in U, S(T(x), \emptyset) \leq r \text{ and } \exists Y \subseteq W, S(T(x), Y) \geq a.$$

The Lemmas 5.5 and 5.7 to 5.9 and Corollaries 5.6 and 5.10 are summarized in Table 5.

Condition	$\underline{FG}(\emptyset)_a$	$\overline{FG}(\emptyset)_r$	$\underline{FG}(W)_a$	$\overline{FG}(W)_r$	Description
$\forall x \in U, \exists Y \subseteq W, S(T(x), Y) \geq a$		$\emptyset$	U		5.5 & 5.6
$\forall x \in U, S(T(x), W) > r$		$\emptyset$			5.7
$\forall x \in U, S(T(x), \emptyset) < a$	$\emptyset$				5.8
$\forall x \in U, S(T(x), \emptyset) \leq r$				U	5.9
$\forall x \in U, S(T(x), \emptyset) \leq r < a$	$\emptyset$			U	5.10

TABLE 5. Summary of Lemmas 5.5 and 5.7 to 5.9 and Corollaries 5.6 and 5.10

Theorem 5.12. *Let $FG = (U, W, T, S)$ be a monotone fuzzy S-approximation space, $X, Y \in \mathcal{F}(W)$ and $a, r \in [0, 1]$. Then*

(1) $X \subseteq Y$ *implies* $POS_{FG_{(a,r)}}(X) \subseteq POS_{FG_{(a,r)}}(Y)$,
(2) $X \subseteq Y$ *implies* $NEG_{FG_{(a,r)}}(Y) \subseteq NEG_{FG_{(a,r)}}(X)$,
(3) $POS_{FG_{(a,r)}}(X \cup Y) \supset POS_{FG_{(a,r)}}(X) \cup POS_{FG_{(a,r)}}(Y)$,
(4) $NEG_{FG_{(a,r)}}(X \cup Y) \subseteq NEG_{FG_{(a,r)}}(X) \cup NEG_{FG_{(a,r)}}(Y)$,

(5) $POS_{FG_{(a,r)}}(X \cap Y) \subseteq POS_{FG_{(a,r)}}(X) \cap POS_{FG_{(a,r)}}(Y)$,
(6) $NEG_{FG_{(a,r)}}(X \cap Y) \supseteq NEG_{FG_{(a,r)}}(X) \cap NEG_{FG_{(a,r)}}(Y)$.

6. Weak Complement Compatibility

The notion of complement compatibility was introduced in [40] in order to make the lower approximation of a set included into the upper approximation of that set. For the same reason, we will introduce the notion of weak complement compatibility for fuzzy S-approximation spaces with an slight difference to the complement compatible notion in [40].

Definition 6.1 (Weak Complement Compatible Fuzzy S-approximation Space)**.** Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space. Then, FG is called *weak complement compatible* if

$$S(T(x), X) + S(T(x), X^c) \leq 1, \tag{34}$$

for every $x \in U$ and $X \in \mathcal{F}(W)$.

Proposition 6.2. *Let $FG = (U, W, T, S)$ be a weak complement compatible fuzzy S-approximation space. Then for any $X \in \mathcal{F}(W)$, $\underline{FG}(X) \subseteq \overline{FG}(X)$.*

Proof. By the definition, $\mu_{\underline{FG}(X)}(x) = S(T(x), X)$ for every $X \in \mathcal{F}(W)$ and $x \in U$. Since FG is weak complement compatible, it follows that $S(T(x), X) \leq 1 - S(T(x), X^c)$ which implies that $\mu_{\underline{FG}(X)}(x) \leq \mu_{\overline{FG}(X)}(x)$. Hence, the desired result is obtained. □

Theorem 6.3. *Let $FG = (U, W, T, S)$ be a weak complement compatible fuzzy S-approximation space, $X \in \mathcal{F}(W)$ and $a, b \in [0, 1]$. Then*

(W1): $\overline{FG}(X)_r = \underline{FG}(X)_{1-r}$,
(W2): $\underline{FG}(X)_a = \overline{FG}(X)_{1-a}$.

Example 6.4. Assume that these three doctors has a common supervisor who uses $FG_{final} = (U, W, T, S)$ to make the final decision where

$$S(A, B) = \begin{cases} \frac{|A \cap B|}{|B|} & \text{if} \quad B \neq \emptyset \\ 0 & otherwise \end{cases},$$

for every $A, B \in \mathcal{F}(W)$[40]. It is easy to see that S is monotone.

He/She has access to $X, Y, Z \in \mathcal{F}(W)$ as observations made by his/her three employees, so he/she would obtain the followings using $a = 0.05$ and $r = 0.09$:

$$\begin{array}{ll}
\mathrm{POS}_{FG_{(a,r)}}(X) = \emptyset, & \mathrm{POS}_{FG_{(a,r)}}(Y) = \emptyset, \\
\mathrm{NEG}_{FG_{(a,r)}}(X) = \{u_3, u_4\}, & \mathrm{NEG}_{FG_{(a,r)}}(Y) = \{u_3\}, \\
\mathrm{BND}_{FG_{(a,r)}}(X) = \{u_1, u_2, u_5\}, & \mathrm{BND}_{FG_{(a,r)}}(Y) = \{u_1, u_2, u_4, u_5\},
\end{array}$$

$$\begin{array}{l}
\mathrm{POS}_{FG_{(a,r)}}(Z) = \{u_3, u_4\}, \\
\mathrm{NEG}_{FG_{(a,r)}}(Z) = \emptyset, \\
\mathrm{BND}_{FG_{(a,r)}}(Z) = \{u_1, u_2, u_5\},
\end{array}$$

$$\begin{aligned}
&\mathrm{POS}_{FG_{(a,r)}}(X\cap Y)=\emptyset, && \mathrm{POS}_{FG_{(a,r)}}(X\cup Y)=\emptyset,\\
&\mathrm{NEG}_{FG_{(a,r)}}(X\cap Y)=\{u_3,u_4\}, && \mathrm{NEG}_{FG_{(a,r)}}(X\cup Y)=\{u_3\},\\
&\mathrm{BND}_{FG_{(a,r)}}(X\cap Y)=\{u_1,u_2,u_5\}, && \mathrm{BND}_{FG_{(a,r)}}(X\cup Y)=\{u_1,u_2,u_4,u_5\},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{POS}_{FG_{(a,r)}}(X\cap Z)=\emptyset, && \mathrm{POS}_{FG_{(a,r)}}(X\cup Z)=\{u_3,u_4\},\\
&\mathrm{NEG}_{FG_{(a,r)}}(X\cap Z)=\{u_3,u_4\}, && \mathrm{NEG}_{FG_{(a,r)}}(X\cup Z)=\emptyset,\\
&\mathrm{BND}_{FG_{(a,r)}}(X\cap Z)=\{u_1,u_2,u_5\}, && \mathrm{BND}_{FG_{(a,r)}}(X\cup Z)=\{u_1,u_2,u_5\},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{POS}_{FG_{(a,r)}}(Z\cap Y)=\emptyset, && \mathrm{POS}_{FG_{(a,r)}}(Z\cup Y)=\{u_3,u_4\},\\
&\mathrm{NEG}_{FG_{(a,r)}}(Z\cap Y)=\{u_3\}, && \mathrm{NEG}_{FG_{(a,r)}}(Z\cup Y)=\emptyset,\\
&\mathrm{BND}_{FG_{(a,r)}}(Z\cap Y)=\{u_1,u_2,u_4,u_5\}, && \mathrm{BND}_{FG_{(a,r)}}(Z\cup Y)=\{u_1,u_2,u_5\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{POS}_{FG_{(a,r)}}(X\cap Y\cap Z)=\emptyset,\\
&\mathrm{NEG}_{FG_{(a,r)}}(X\cap Y\cap Z)=\{u_3,u_4\},\\
&\mathrm{BND}_{FG_{(a,r)}}(X\cap Y\cap Z)=\{u_1,u_2,u_5\},\\
&\mathrm{POS}_{FG_{(a,r)}}(X\cup Y\cup Z)=\{u_3,u_4\},\\
&\mathrm{NEG}_{FG_{(a,r)}}(X\cup Y\cup Z)=\emptyset,\\
&\mathrm{BND}_{FG_{(a,r)}}(X\cup Y\cup Z)=\{u_1,u_2,u_5\}.
\end{aligned}$$

7. Conclusion and Future Research Directions

The main contributions of this paper are as follows:

(1) Introducing and studying the notion of fuzzy S-approximation operators and three pairs of lower and upper approximation operators.
(2) Introducing and studying a pair of thresholds and giving its interpretations along defining five situations regarding these thresholds, that is (a, r)-acceptance, (a, r)-rejection, non-commitment, inconsistent and uncertain.
(3) Illustrating the application of fuzzy S-approximation spaces using a sample disease diagnosis system along the paper.
(4) Defining and studying three-way regions for a fuzzy S-approximation space using a pair of thresholds.
(5) Introducing some measures of inconsistency and generality regarding three regions.
(6) Introducing and investigating two new subclasses of fuzzy S-approximation spaces, monotone and weak compatible ones.

Acknowledgements. Authors would like to express their sincere thanks to the anonymous reviewers whose suggestions made this work better.

References

[1] M. Alamuri, B. R. Surampudi and A. Negi, *A survey of distance/similarity measures for categorical data*, In 2014 International Joint Conference on Neural Networks, **IJCNN** (2014), 1907-1914.

[2] N. Azam and J. T. Yao, *Analyzing uncertainties of probabilistic rough set regions with game-theoretic rough sets*, International Journal of Approximate Reasoning, **55(1)** (2014), 142-155, 2014.

[3] C. Cornelis, M. De Cock and A. M. Radzikowska, *Fuzzy rough sets: from theory into practice*, Handbook of Granular computing, (2008), 533-552.

[4] B. Davvaz, *A short note on algebraic T-rough sets*, Information Sciences, **178** (2008), 3247-3252.

[5] B. Davvaz, *Approximations in* n*-ary algebraic systems*, Soft Computing, **12(4)** (2008), 409-418.

[6] B. Davvaz, *Approximations in a semigroup by using a neighbourhood system*, International Journal of Computer Mathematics, **88(4)** (2011), 709-713.

[7] B. Davvaz and M. Mahdavipour, *Rough approximations in a general approximation space and their fundamental properties*, International Journal of General Systems, **37(3)** (2008), 373–386.

[8] A. P. Dempster, *Upper and lower probabilities induced by a multivalued mapping*, The Annals of Mathematical Statistics, **38(2)** (1967), 325–339.

[9] D. Dubois and H. Prade, *Rough fuzzy sets and fuzzy rough sets*, International Journal of General Systems, **17(2-3)** (1990), 191–209.

[10] A. Gomolinska, *Approximation spaces based on relations of similarity and dissimilarity of objects*, Fundamenta Informaticae, **79(3)** (2007), 319–333.

[11] A. Gomolinska, *On certain rough inclusion functions*, Transactions on Rough Sets IX, Springer Berlin Heidelberg, (2008), 35–55.

[12] A. Gomolinska, *Rough approximation based on weak q-RIFs*, Transactions on Rough Sets X, Springer Berlin Heidelberg, (2009), 117–135.

[13] A. Gomolinska and M. Wolski, *Rough inclusion functions and similarity indices*, Fundamenta Informaticae, **133(2)** (2014), 149–163.

[14] S. Greco, B. Matarazzo and R. Slowinski, *Rough approximation of a preference relation by dominance relations*, European Journal of Operational Research, **117(1)** (1999), 63–83.

[15] J. W. Grzymala Busse, *Knowledge acquisition under uncertainty – a rough set approach*, Journal of Intelligent and Robotic Systems, **1(1)** (1988), 3–16.

[16] J. W. Grzymala Busse, *Rough-set and Dempster-Shafer approaches to knowledge acquisition under uncertainty – a comparison*, Technical report, University of Kansas, 1987.

[17] M. R. Hooshmandasl, A. Shakiba, A. K. Goharshady and A. Karimi, *S-approximation: A new approach to algebraic approximation*, Journal of Discrete Mathematics, **2014** (2014), 1–5.

[18] M. J. Lesot, M. Rifqi and H. Benhadda, *Similarity measures for binary and numerical data: a survey*, International Journal of Knowledge Engineering and Soft Data Paradigms, **1(1)** (2009), 63–84.

[19] T. J. Li, *Rough approximation operators on two universes of discourse and their fuzzy extensions*, Fuzzy Sets and Systems, **159(22)** (2008), 3033–3050.

[20] P. Lingras and R. Jensen, *Survey of rough and fuzzy hybridization*, IEEE International Fuzzy Systems Conference, (2007), 1–6.

[21] C. Liu, D. Miao and N. Zhang, *Graded rough set model based on two universes and its properties*, Knowledge-Based Systems, **33(0)** (2012), 65 – 72.

[22] G. Liu, *Rough set theory based on two universal sets and its applications*, Knowledge-Based Systems, **23(2)** (2010), 110–1150.

[23] N. N. Morsi and M. M. Yakout, *Axiomatics for fuzzy rough sets*, Fuzzy Sets and Systems, **100(1)** (1998), 327–342.

[24] A. Nakamura, *Fuzzy rough sets*, Note on Multiple-valued Logic in Japan, **9(8)** (1988), 1–8.

[25] S. Nanda and S. Majumdar, *Fuzzy rough sets*, Fuzzy Sets and Systems, **45(2)** (1992), 157–160.
[26] S. K. Pal, L. Polkowski and A. Skowron, *Rough-Neural Computing: Techniques for Computing with Words*, Artificial intelligence, Springer Berlin Heidelberg, 2004.
[27] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, 1991.
[28] Z. Pawlak and A. Skowron, *Rough membership functions*, Advances in the Dempster-Shafer Theory of Evidence, John Wiley & Sons, (1994), 251–271.
[29] Z. Pawlak, S. K. M. Wong, and W. Ziarko, *Rough sets: Probabilistic versus deterministic approach*, International Journal of Man-Machine Studies, **29** (1988), 81–95.
[30] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, **11** (1982), 341–356.
[31] Z. Pawlak, *Vagueness and uncertainty: A rough set perspective*, Computational Intelligence, **11(2)** (1995), 227–232.
[32] Z. Pawlak, *Rough set theory and its applications to data analysis*, Cybernetics & Systems, **29(7)** (1998), 661–688.
[33] Z. Pei and Z. B. Xu, *Rough set models on two universes*, International Journal of General Systems, **33(5)** (2004), 569–581.
[34] L. Polkowski, *Rough sets: Mathematical foundations*, Physica Verlag, **15** (2002).
[35] L. Polkowski and A. Skowron, *Rough mereology: A new paradigm for approximate reasoning*, International Journal of Approximate Reasoning, **15(4)** (1996), 333–365.
[36] L. Polkowski and A. Skowron, *Rough mereology*, Methodologies for Intelligent Systems, Springer Berlin Heidelberg, (1994), 85–94.
[37] A. M. Radzikowska and E. E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems, **126(2)** (2002), 137–155.
[38] A. M. Radzikowska and E. E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems, **126(2)** (2002), 137–155.
[39] G. Shafer, *A mathematical theory of evidence*, Princeton university press Princeton, **1** (1976).
[40] A. Shakiba and M. R. Hooshmandasl, *S-approximation spaces: A three-way decision approach*, Fundamenta Informaticae, **139(3)** (2015), 307–328.
[41] A. Shakiba and M. R. Hooshmandasl, *Neighborhood system S-approximation spaces and applications*, Knowledge and Information Systems, (2015), 1–46.
[42] Y. Shen and F. Wang, *Variable precision rough set model over two universes and its properties*, Soft Computing, **15(3)** (2011),557–567.
[43] A. Skowron and J. Stepaniuk, *Tolerance approximation spaces*, Fundamenta Informaticae, **27(2)** (1996), 245–253.
[44] R. Slowinski and D. Vanderpooten, *A generalized definition of rough approximations based on similarity*, IEEE Transactions on Knowledge and Data Engineering, **12(2)** (2000), 331–336.
[45] B. Sun, Z. Gong and D. Chen, *Fuzzy rough set theory for the interval-valued fuzzy information systems*, Information Sciences, **178(13)** (2008), 2794–2815.
[46] B. Sun and W. Ma, *Fuzzy rough set model on two different universes and its application*, Applied Mathematical Modelling, **35(4)** (2011), 1798–1809.
[47] S. K. M. Wong, L. S. Wang, and Y. Y. Yao, *Interval structure: A framework for representing uncertain information*, Proceedings of the Eighth International Conference on Uncertainty in Artificial Intelligence, **UAI'92** (1992), 336–343.
[48] W. Z. Wu, J. Sheng Mi and W. Xiu Zhang, *Generalized fuzzy rough sets*, Information Sciences, **151** (2003), 263–282.
[49] W. Xu, W. Sun, Y. Liu and W. Zhang, *Fuzzy rough set models over two universes*, International Journal of Machine Learning and Cybernetics, **4(6)** (2013), 631–645.
[50] S. Yamak, O. Kazanci and B. Davvaz, *Generalized lower and upper approximations in a ring*, Information Sciences, **180(9)** (2010), 1759–1768.
[51] R. Yan, J. Zheng, J. Liu and C. Qin, *Rough set over dual-universes in fuzzy approximation space*, Iranian Journal of Fuzzy Systems, **9(3)** (2012), 79–91.

[52] Q. Yang and X. Wu, *10 challenging problems in data mining research*, International Journal of Information Technology & Decision Making, **5(4)** (2006), 597–604.
[53] Y. Y. Yao, *A comparative study of fuzzy sets and rough sets*, Information Sciences, **109(14)** (1998), 227–242.
[54] Y. Y. Yao, *Probabilistic approaches to rough sets*, Expert Systems, **20(5)** (2003), 287–297.
[55] Y. Y. Yao and P. J. Lingras, *Interpretations of belief functions in the theory of rough sets*, Information Sciences, **104(1-2)** (1998), 81–106.
[56] Y. Y. Yao, *Probabilistic rough set approximations*, International Journal of Approximate Reasoning, **49(2)** (2008), 255–271.
[57] Y. Y. Yao, *Three-way decision: an interpretation of rules in rough set theory*, International Conference on Rough Sets and Knowledge Technology, Springer Berlin Heidelberg, (2009), 642–649.
[58] Y. Y. Yao, *An outline of a theory of three-way decisions*, Rough Sets and Current Trends in Computing, Springer Berlin Heidelberg, (2012), 1–17.
[59] Y. Y. Yao and X. Deng, *Quantitative rough sets based on subsethood measures*, Information Sciences, **267** (2014), 306–322.
[60] Y. Y. Yao, *Combination of rough and fuzzy sets based on α-level sets*, Rough sets and Data Mining, Springer, (1997), 301–321.
[61] Y. Y. Yao, *Generalized rough set models*, Rough Sets in Knowledge Discovery 1: Methodology and Approximations, Studies in Fuzziness and Soft Computing, Physica Verlag Heidelberg, **1** (1998), 286–318.
[62] Y. Y. Yao and T. Y. Lin, *Generalization of rough sets using modal logic*, Intelligent Automation and Soft Computing, **2(2)** (1996), 103–120.
[63] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8(3)** (1965), 338–353.
[64] L. A. Zadeh, *The role of fuzzy logic in the management of uncertainty in expert systems*, Fuzzy Sets and Systems, **11(1)** (1983), 197–198.
[65] L. A. Zadeh, *A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination*, AI magazine, **7(2)** (1986), 85–90.
[66] H. Y. Zhang, W. X. Zhang and W. Z. Wu, *On characterization of generalized interval-valued fuzzy rough sets on two universes of discourse*, International Journal of Approximate Reasoning, **51(1)** (2009), 56–70.
[67] W. X. Zhang and Y. Leung, *Theory of including degrees and its applications to uncertainty inferences*, Fuzzy Systems Symposium, Soft Computing in Intelligent Systems and Information Processing., Proceedings of the 1996 Asian, (1996), 496–501.
[68] W. Zhu and F. Y. Wang, *A new type of covering rough set*, 3rd International IEEE Conference on Intelligent Systems, (2006), 444–449.
[69] W. Zhu and F. Y. Wang, *On three types of covering-based rough sets*, IEEE Transactions on Knowledge and Data Engineering, **19(8)** (2007), 1131–1144.
[70] W. Zhu and F. Y. Wang, *The fourth type of covering-based rough sets*, Information Sciences, **201** (2012), 80–92.
[71] W. Ziarko, *Variable precision rough set model*, Journal of Computer and System Sciences, **46(1)** (1993), 39–59.
[72] W. Ziarko, *Probabilistic decision tables in the variable precision rough set model*, Computational Intelligence, **17(3)** (2001), 593–603.

Ali Shakiba, Department of Computer Science, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

E-mail address: ali.shakiba@vru.ac.ir

MohammadReza Hooshmandasl*, Department of Computer Science, Yazd University, Yazd, Iran

E-mail address: hooshmandasl@yazd.ac.ir

Bijan Davvaz, Department of Mathematics, Yazd University, Yazd, Iran
E-mail address: davvaz@yazd.ac.ir

Seyed Abolfazl Shahzadeh Fazeli, Department of Computer Science, Yazd University, Yazd, Iran
E-mail address: fazeli@yazd.ac.ir

∗Corresponding author