FUZZY PREORDERED SET, FUZZY TOPOLOGY AND FUZZY AUTOMATON BASED ON GENERALIZED RESIDUATED LATTICE

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Abstract. This work is towards the study of the relationship between fuzzy preordered sets and Alexandrov (left/right) fuzzy topologies based on generalized residuated lattices here the fuzzy sets are equipped with generalized residuated lattice in which the commutative property doesn't hold. Further, the obtained results are used in the study of fuzzy automata theory.

1. Introduction

The study of fuzzy automata was initiated by Wee [27] and Santos [21] in 1960's after the introduction of fuzzy set theory by Zadeh [29]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced by Malik, Mordeson and Sen [14, 15]. Somewhat different notions were introduced subsequently in [6, 7, 8, 9, 10, 19]. In these studies, the membership values in the closed interval $[0, 1]$ were considered. During the recent years, the researchers began to work on fuzzy automata with membership values in complete residuated lattices, lattice-ordered monoids and some other kind of lattices [3, 12, 16, 17, 18, 20, 28].

Recently, in [4], the concept of subsystem of a fuzzy transition system (fuzzy finite automata over residuated lattice, in the sense of [15]) was introduced and studied. Such concept of subsystems of a fuzzy transition system was proposed in [5] as a natural generalization of the same studied in [15]. Simultaneously, the topological study of subsystems was carried out in [1, 24].

In view of above, in this paper, we introduce Alexandrov (left/right) fuzzy topologies by using the concepts of (left/right) upper sets and (left/right) lower sets of a fuzzy preordered set as well as derive several results. Further, we use these concepts for the study of fuzzy automata (a generalization of the concept of fuzzy transition system introduced in [4]). The study made here differs from the existing studies done in [1, 15, 23, 24], in the sense that here the fuzzy sets are equipped with generalized residuated lattice in which the commutative property doesn't hold.

This paper is organized as follows: In section 2, we recall some notions related to generalized residuated lattices, (left/right) upper set and (left/right) lower set of a

Received: August 2015; Revised: June 2016; Accepted: November 2016

Key words and phrases: Generalized residuated lattice, (left/right) Subsystem, Fuzzy automata, Alexandrov (left/right) fuzzy topology.

fuzzy preordered set. In section 3, we introduce and study the concepts of Alexandrov (left/right) fuzzy topologies by using (left/right) upper sets and (left/right) lower sets of fuzzy preordered set; while in section 4, we study the concepts of fuzzy topologies introduced in previous sections for the study of fuzzy automata.

2. **Preliminaries**

In this section, we introduce some basic concepts related to generalized residuated lattice. We recall the following concepts of generalized residuated lattice from [2].

Definition 2.1. A **generalized residuated lattice** is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top \rangle$ such that the following condition hold.

(i) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice with the least element 0 and the greatest element 1.
(ii) $\langle L, \otimes, \top \rangle$ is a monoid.
(iii) For all $x, y, z \in L$, we have the equivalence:
$$x \otimes y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z.$$

$\rightarrow$ and $\rightsquigarrow$ are called the left and right implication of $\otimes$, respectively.

Definition 2.2. A generalized residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top \rangle$ is said to be

(i) **commutative** (resp., **non-commutative**) if $\otimes$ is commutative (resp., non-commutative),
(ii) **integral** if $\top = 1$,
(iii) **complete** if the underlying lattice $\langle L, \wedge, \vee, 0, 1 \rangle$ is complete.

Clearly, if $\otimes$ is commutative, then $\rightarrow = \rightsquigarrow$ holds.

Example 2.3. Consider the bounded lattice $L = \{0, a, b, 1\}$ with the partial order $\leq$ defined by: $0 < a < b < 1$, whose non-commutative $\otimes$ is defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

It is easy to get.

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	0	b	1
1	0	a	b	1

and

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	0	a	1	1
b	0	a	1	1
1	0	a	b	1

As we have $(b \rightarrow a) \otimes b = 0 \neq b \otimes (b \rightsquigarrow a) = a$. This shows that the $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top \rangle$ is a generalized residuated lattice.

Some of the basic properties of generalized residuated lattices, which we use, are as follows:

Proposition 2.4. [26] *Let $\langle L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top\rangle$ be a generalized residuated lattices. Then the following properties hold for all $a, b, c \in L$:*

(i) $b \rightarrow (a \rightarrow c) = (b \otimes a) \rightarrow c$ *and* $b \rightsquigarrow (a \rightsquigarrow c) = (a \otimes b) \rightsquigarrow c$*;*
(ii) $\top \rightarrow a = a$ *and* $\top \rightsquigarrow a = a$*;*
(iii) $(b \rightarrow c) \otimes (a \rightarrow b) \leq a \rightarrow c$ *and* $(a \rightsquigarrow b) \otimes (b \rightsquigarrow c) \leq a \rightsquigarrow c$*;*
(iv) $a \otimes (\vee_{i \in J} b_i) = \vee_{i \in J}(a \otimes b_i)$ *and* $(\vee_{i \in J} b_i) \otimes a = \vee_{i \in J}(b_i \otimes a)$*;*
(v) $a \rightarrow 1 = 1$ *and* $a \rightsquigarrow 1 = 1$*;*
(vi) $(b \rightarrow a) \otimes b \leq a$ *and* $b \otimes (b \rightsquigarrow a) \leq a$*.*
(vii) *if* $b \leq c$*, then* $a \rightarrow b \leq a \rightarrow c$ *and* $a \rightsquigarrow b \leq a \rightsquigarrow c$*.*

The concepts of fuzzy sets, fuzzy relations, fuzzy topologies and fuzzy automata, we study in this paper, have the membership values in a generalized residuate lattice. For example, a *fuzzy subset* of a nonempty set X is a function from X to L and a *fuzzy relation* on X is a function from $X \times X$ to L. Throughout, L^X denotes the family of all fuzzy subsets of X.

3. Fuzzy Preordered Set and Fuzzy Topology

The concept of fuzzy topology induces by upper sets of a fuzzy preordered sets with the membership values in $[0, 1]$ has been introduced and studied in [11]. In this section, we study the concepts associated with (left/right) upper sets and (left/right) lower sets of a fuzzy preordered set with the membership values in a complete generalized residuated lattice. We show that the collection of (left/right) upper (lower) sets of a fuzzy preordered set form a fuzzy topology. We begin with the following concept of a fuzzy preordered set.

Definition 3.1. [22] Let R be a fuzzy relation on X. Then R is called

(i) **reflexive** if $\forall x \in X, R(x, x) = 1$, and
(ii) **left transitive** if $R(x, y) \otimes R(y, z) \leq R(x, z)$ and
right transitive if $R(y, z) \otimes R(x, y) \leq R(x, z), \forall x, y, z \in X$.

A reflexive and (left/right) transitive fuzzy relation is called a **(left/right) fuzzy preorder**. If R is a (left/right) fuzzy preorder on X, then the pair (X, R) is called a **(left/right) fuzzy preordered set**.

Given a left fuzzy preordered set (X, R), let $R^{op} \in L^{X \times X}$ such that $R^{op}(x, y) = R(y, x)$. Then (X, R^{op}) is a right fuzzy preordered set.

Remark 3.2. If R is a left transitive, then R^{op} is a right transitive, i.e., If R is a left transitive, then $\forall x, y, z \in X, R(x, y) \otimes R(y, z) \leq R(x, z) \rightarrow R^{op}(y, x) \otimes R^{op}(z, y) \leq R^{op}(z, x)$. Thus R^{op} is right transitive.

Definition 3.3. Let (X, R) be a left fuzzy preordered set and $A \in L^X$. Then

(i) A is said to **left upper set** if $A(x) \otimes R(x, y) \leq A(y), \forall x, y \in X$.
(ii) A is said to **right upper set** if $R(x, y) \otimes A(x) \leq A(y), \forall x, y \in X$.
(iii) A is said to **left lower set** if $A(y) \otimes R(x, y) \leq A(x), \forall x, y \in X$.
(iv) A is said to **right lower set** if $R(x, y) \otimes A(y) \leq A(x), \forall x, y \in X$.

Proposition 3.4. *If A is a left upper set in a left fuzzy preordered set (X, R), then A is a left lower set in a right fuzzy preordered set (X, R^{op}). Also if A is a right upper set in a left fuzzy preordered set (X, R), then A is a right lower set in a right fuzzy preordered set (X, R^{op}).*

Proof. Let A be a left upper set in (X, R). Then $\forall x, y \in X, A(x) \otimes R(x, y) \leq A(y) \rightarrow A(x) \otimes R^{op}(y, x) \leq A(y)$. Thus A is a left lower set in (X, R^{op}). Similarly, if A is a right upper set in (X, R). Then $\forall x, y \in X, R(x, y) \otimes A(x) \leq A(y) \rightarrow R^{op}(y, x) \otimes A(x) \leq A(y)$. Thus A is a right lower set in (X, R^{op}). □

Proposition 3.5. *Let (X, R) be a (left/right) fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *λ is a left upper set of a left fuzzy preordered set (X, R) if and only if λ is a left lower set of a right fuzzy preordered set (X, R^{op}).*
(ii) *λ is a right upper set of a left fuzzy preordered set (X, R) if and only if λ is a right lower set of a right fuzzy preordered set (X, R^{op}).*

Proof. We only prove here for left upper set. The other can be proved in a similar way.

(i) Let λ be a left upper set of a left fuzzy preordered set (X, R). Then $\lambda(x) \otimes R(x, y) \leq \lambda(y), \forall x, y \in X$. Hence, we have $\lambda(y) \otimes R^{op}(y, x) \leq \lambda(x), \forall x, y \in X$, or that $\lambda(y) \otimes R(x, y) \leq \lambda(x), \forall x, y \in X$. Thus λ is a left lower set of a right fuzzy preordered set (X, R^{op}). Similarly, it can be prove that if λ is a left lower set of a right fuzzy preordered set (X, R^{op}), then it is also a left upper set of a left fuzzy preordered set (X, R). □

In this paper we are discussing only for left fuzzy preordered set which are similar to fuzzy preordered set (X, R) and the right fuzzy preordered set (X, R^{op}) follows analogously.

Proposition 3.6. *Let (X, R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *λ is a left upper set of (X, R) if and only if $\lambda : (X, R) \longrightarrow (L, \rightsquigarrow)$ is an order preserving map.*
(ii) *λ is a right upper set of (X, R) if and only if $\lambda : (X, R) \longrightarrow (L, \rightarrow)$ is an order preserving map.*

Proof. We only prove here for left upper set. The others can be proved in a similar way. Let $\lambda \in L^X$ be a left upper set of a fuzzy preordered set (X, R). Then $\lambda(x) \otimes R(x, y) \leq \lambda(y), \forall x, y \in X$, or that $R(x, y) \leq \lambda(x) \rightsquigarrow \lambda(y), \forall x, y \in X$. Thus $\lambda : (X, R) \longrightarrow (L, \rightsquigarrow)$ preserves order. Converse follows similarly. □

Proposition 3.7. *Let (X, R) be a fuzzy preordered set and $x, z \in X$. Then*

(i) *$[z]^R \in L^X$ such that $[z]^R(x) = R(z, x)$ is a left upper set of (X, R),* and
(ii) *$[z]_R \in L^X$ such that $[z]_R(x) = R(x, z)$ is a right lower set of (X, R).*

Proof. (i) Let $[z]^R(x) = R(z, x), x, z \in X$. Then $[z]^R(x) \otimes R(x, y) = R(z, x) \otimes R(x, y) \leq R(z, y)$, as R is left transitive. Thus $[z]^R(x) \otimes R(x, y) \leq R(z, y) = [z]^R(y)$. Hence $[z]^R$ is a left upper set of (X, R).

(ii) Let $[z]_R(y) = R(y,z), y,z \in X$. Then $R(x,y) \otimes [z]_R(y) = R(x,y) \otimes R(y,z) \leq R(x,z)$, as R is transitive. Thus $R(x,y) \otimes [z]_R(y) \leq R(x,z) = [z]_R(x)$. Hence $[z]_R$ is a right lower set of (X,R). □

Proposition 3.8. *Let (X,R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *if λ is a left upper set of (X,R), then for each $a \in L, \lambda \rightsquigarrow a$ is a right lower set of (X,R).*

(ii) *if λ is a left lower set of (X,R), then for each $a \in L, \lambda \rightsquigarrow a$ is a right upper set of (X,R).*

Proof. (i) Let λ be a left upper set of a fuzzy preordered set (X,R). Then $\lambda(x) \otimes R(x,y) \leq \lambda(y), \forall x,y \in X$. To show that $\lambda \rightsquigarrow a$ is a right lower set, it is enough to show that $R(x,y) \otimes (\lambda(y) \rightsquigarrow a) \leq \lambda(x) \rightsquigarrow a, \forall x,y \in X$, or that $\lambda(x) \otimes R(x,y) \otimes (\lambda(y) \rightsquigarrow a) \leq a, \forall x,y \in X$. Now, $\lambda(x) \otimes R(x,y) \otimes (\lambda(y) \rightsquigarrow a) \leq \lambda(y) \otimes (\lambda(y) \rightsquigarrow a) \leq a$ (cf., Proposition 2.4). Thus $\lambda \rightsquigarrow a$ is a right lower set.

(ii) Similarly, it can be prove that if λ is a left lower set of a fuzzy preordered set (X,R), then for each $a \in L, \lambda \rightsquigarrow a$ is a right upper set of (X,R). □

Proposition 3.9. *Let (X,R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *if λ is a right upper set of (X,R), then for each $a \in L, \lambda \to a$ is a left lower set of (X,R).*

(ii) *if λ is a right lower set of (X,R), then for each $a \in L, \lambda \to a$ is a left upper set of (X,R).*

Proof. (i) Let λ be a right upper set in a fuzzy preordered set (X,R). Then $R(x,y) \otimes \lambda(x) \leq \lambda(y), \forall x,y \in X$. To show that $\lambda \to a$ is a left lower set, it is enough to show that $(\lambda(y) \to a) \otimes R(x,y) \leq \lambda(x) \to a, \forall x,y \in X$, or that $(\lambda(y) \to a) \otimes R(x,y) \otimes \lambda(x) \leq a, \forall x,y \in X$. Now, $(\lambda(y) \to a) \otimes R(x,y) \otimes \lambda(x) \leq (\lambda(y) \to a) \otimes \lambda(y) \leq a$ (cf., Proposition 2.4). Thus $\lambda \to a$ is a left lower set.

(ii) Similarly, it can be prove that if λ is a right lower set of a fuzzy preordered set (X,R), then for each $a \in L, \lambda \to a$ is a left upper set of (X,R). □

Proposition 3.10. *Let (X,R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *if λ is a right upper set of (X,R), then for each $a \in L, \lambda \otimes a$ is a right upper set of (X,R).*

(ii) *if λ is a right lower set of (X,R), then for each $a \in L, \lambda \otimes a$ is a right lower set of (X,R).*

Proof. (i) Let λ be a right upper set of a fuzzy preordered set (X,R) and $a \in L$. Then $R(x,y) \otimes \lambda(x) \leq \lambda(y), \forall x,y \in X$, which implying that $R(x,y) \otimes \lambda(x) \otimes a \leq \lambda(y) \otimes a, \forall x,y \in X$. Thus $\lambda \otimes a$ is a right upper set in (X,R).

(ii) The proof for the case of right lower set follows similarly. □

Proposition 3.11. *Let (X,R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *if λ is a left upper set of (X,R), then for each $a \in L, a \otimes \lambda$ is a left upper set of (X,R).*

(ii) *if λ is a left lower set of (X,R), then for each $a \in L, a \otimes \lambda$ is a left lower set of (X,R).*

Proof. Follows from [25]. □

The following is the concept of fuzzy topology in the sense of Lowen [13].

Definition 3.12. A **fuzzy topology** τ on X is a subset of L^X, which is closed under arbitrary suprema and finite infima and which contains all constant fuzzy sets. The fuzzy sets in τ are called **fuzzy τ-open**.

Proposition 3.13. *If (X,R) is a fuzzy preordered set then the family τ of its all left (resp. right) upper sets satisfies the following conditions, $\forall \lambda_i \in \tau$ and $\forall \alpha \in L$:*

(i) $\underline{\alpha} \in \tau$,
(ii) $\vee_{i\in J}\lambda_i \in \tau$ and $\wedge_{i\in J}\lambda_i \in \tau$,
(iii) $\alpha \otimes \lambda \in \tau$ (resp.$\lambda \otimes \alpha \in \tau$),
(iv) $\alpha \rightsquigarrow \lambda \in \tau$ (resp.$\alpha \rightarrow \lambda \in \tau$).

Proof. We only prove here for left upper set. The others can be proved in a similar way.
(i) is obvious.
(ii) Let $\lambda_i \in \tau, i \in J$. Then $\lambda_i(x) \otimes R(x,y) \le \lambda_i(y), \forall x, y \in X$ and $\forall i \in J$. Now, $(\vee_{i\in J}\lambda_i(x)) \otimes R(x,y) = \vee_{i\in J}(\lambda_i(x) \otimes R(x,y)) \le \vee_{i\in J}\lambda_i(y)$. Thus $\vee_{i\in J}\lambda_i \in \tau$. Also, $(\wedge_{i\in J}\lambda_i(x)) \otimes R(x,y) \le \lambda_i(x) \otimes R(x,y) \le \lambda_i(y), \forall i \in J$. Thus $(\wedge_{i\in J}\lambda_i(x)) \otimes R(x,y) \le \wedge_{i\in J}\lambda_i(y)$. Hence $\wedge_{i\in J}\lambda_i \in \tau$.
(iii) If λ is a left upper set, then for any $x, y \in X, \lambda(x) \otimes R(x,y) \le \lambda(y)$. Since $\otimes$ is order-preserving in each place, we get $\alpha \otimes \lambda(x) \otimes R(x,y) \le \alpha \otimes \lambda(y)$. Thus $\alpha \otimes \lambda \in \tau$.
(iv) If λ is a left upper set, then for any $x, y \in X, \lambda(x) \otimes R(x,y) \le \lambda(y)$. Then as $(\alpha \rightsquigarrow \lambda(x)) \otimes (\lambda(x) \rightsquigarrow \lambda(y)) \le (\alpha \rightsquigarrow \lambda(y))$ (cf., Proposition 2.4). Thus $(\alpha \rightsquigarrow \lambda(x)) \rightsquigarrow (\alpha \rightsquigarrow \lambda(y)) \ge \lambda(x) \rightsquigarrow \lambda(y) \ge R(x,y)$. Hence $\alpha \rightsquigarrow \lambda$ is a left upper set. □

Thus the family of (left/right) upper sets of a fuzzy preordered set (X,R) forms an Alexandrov-(left/right) fuzzy topology on X, which we shall denote by τ_{lR} and τ_{rR} respectively. The family of (left/right) lower sets of a fuzzy preordered set also form an Alexandrov-(left/right) fuzzy topology, which is given below.

Proposition 3.14. *If (X,R) is a fuzzy preordered set then the family τ of its all left (resp. right) lower sets satisfies the following conditions, $\forall \lambda_i \in \tau$ and $\forall \alpha \in L$:*

(i) $\underline{\alpha} \in \tau$,
(ii) $\vee_{i\in J}\lambda_i \in \tau$ and $\wedge_{i\in J}\lambda_i \in \tau$,
(iii) $\alpha \otimes \lambda \in \tau$ (resp.$\lambda \otimes \alpha \in \tau$),
(iv) $\alpha \rightsquigarrow \lambda \in \tau$ (resp.$\alpha \rightarrow \lambda \in \tau$).

The following gives the characterization of fuzzy relation of a fuzzy preordered set through it's (left/right) upper sets.

Proposition 3.15. *For given a fuzzy preordered set (X,R). We have*

(i) *let* $\mathbf{F}$ *be the family of all left upper sets. Then* $R(x,y) = \wedge\{\lambda(x) \rightsquigarrow \lambda(y) : \lambda \in \mathbf{F}\}, \forall x, y \in X$.
(ii) *let* $\mathbf{F}$ *be the family of all right upper sets. Then* $R(x,y) = \wedge\{\lambda(x) \rightarrow \lambda(y) : \lambda \in \mathbf{F}\}, \forall x, y \in X$.

(iii) *let* $\mathbf{F}'$ *be the family of all left upper sets. Then* $R(x,y) = \wedge\{\lambda(y) \rightsquigarrow \lambda(x) : \lambda \in \mathbf{F}'\}, \forall x, y \in X$.
(iv) *let* $\mathbf{F}'$ *be the family of all right upper sets. Then* $R(x,y) = \wedge\{\lambda(y) \rightarrow \lambda(x) : \lambda \in \mathbf{F}'\}, \forall x, y \in X$.

Proof. We only prove here for left upper set. The others can be proved in a similar way. Let λ be a left upper set of a fuzzy preordered set (X,R). Then for all $x, y \in X, \lambda(x) \otimes R(x,y) \le \lambda(y)$, or that $R(x,y) \le \lambda(x) \rightsquigarrow \lambda(y)$, i.e., $R(x,y) \le \wedge\{\lambda(x) \rightsquigarrow \lambda(y) : \lambda \in \mathbf{F}\}$. Also for $x, z \in X$, as $z^R(x)$ is a left upper set in $(X,R), \wedge\{z^R(x) \rightsquigarrow z^R(y) : z \in X\} \le R(x,x) \rightsquigarrow R(x,y) = 1 \rightsquigarrow R(x,y) = R(x,y)$ (cf., Proposition 2.4). Thus $\wedge\{\lambda(x) \rightsquigarrow \lambda(y) : \lambda \in \mathbf{F}\} \le R(x,y)$. Hence $R(x,y) = \wedge\{\lambda(x) \rightsquigarrow \lambda(y) : \lambda \in \mathbf{F}\}, \forall x, y \in X$. □

Proposition 3.16. *Let* (X,R) *be a fuzzy preordered set and* $\lambda \in L^X$. *Then*

(i) *if* λ *is a left upper set. Then for each* $a \in L, a \rightsquigarrow \lambda$ *is a left upper set of* (X,R).
(ii) *if* λ *is a right upper set. Then for each* $a \in L, a \rightarrow \lambda$ *is a right upper set of* (X,R).

Proof. We only prove here for left upper set. The others can be proved in a similar way. Let $x, y \in X$. Then $(a \rightsquigarrow \lambda(x)) \otimes (\lambda(x) \rightsquigarrow \lambda(y)) \le (a \rightarrow \lambda(y))$, or that $(\lambda(x) \rightsquigarrow \lambda(y)) \le (a \rightsquigarrow \lambda(x)) \rightsquigarrow (a \rightsquigarrow \lambda(y))$. Thus $R(x,y) \le (a \rightsquigarrow \lambda(x)) \rightsquigarrow (a \rightsquigarrow \lambda(y))$ (cf., Proposition 3.15), whereby $(a \rightsquigarrow \lambda(x)) \otimes R(x,y) \le a \rightsquigarrow \lambda(y)$. Hence $a \rightsquigarrow \lambda$ is a left upper set in (X,R). □

Before stating next, recall from [4] that for fuzzy relations $R, S \in L^{X \times X}$, their *composition* $R \circ S$ is a fuzzy relation on X given by $(R \circ S)(x,y) = \vee\{R(x,z) \otimes S(z,y) : z \in X\}$.

Proposition 3.17. *Given a fuzzy preordered set* (X,R), $\lambda \in L^X$ *is a*

(i) *left upper set if and only if* λ *is a solution to a fuzzy relational equation* $\chi \circ R_l = \chi$,
(iv) *right upper set if and only if* λ *is a solution to a fuzzy relational equation* $R_r \circ \chi = \chi$,

where $\chi \in L^X$ *is an unknown.*

Proof. We only prove here for left upper set. The others can be proved in a similar way. Let λ be a left upper set of a fuzzy preordered set (X,R). Then $\lambda \circ R_l \le \lambda$. Also, from the reflexivity of R_l, $\lambda \le \lambda \circ R_l$. Thus $\lambda \circ R_l = \lambda$, or that λ is a solution of fuzzy relational equation $\chi \circ R_l = \chi$. Converse is trivial. □

Proposition 3.18. *Given a fuzzy preordered set* (X,R), $\lambda \in L^X$ *is a*

(i) *left lower set if and only if* λ *is a solution to a fuzzy relational equation* $\chi \circ R_l = \chi$,
(iv) *right lower set if and only if* λ *is a solution to a fuzzy relational equation* $R_r \circ \chi = \chi$,

where $\chi \in L^X$ *is an unknown.*

Proof. Similar to that of Proposition 3.17. □

Proposition 3.19. *Let (X,R) be a fuzzy preordered set and $\lambda \in L^X$. Then*

(i) *if λ is a left upper set, $\neg_l\lambda$ is a right lower set.*
(ii) *if λ is a right upper set, $\neg_r\lambda$ is a left lower set.*
(iii) *if λ is a left lower set, $\neg_l\lambda$ is a right upper set.*
(iv) *if λ is a right lower set, $\neg_r\lambda$ is a left upper set.*

Proof. We only prove here for left upper set and left lower set. The others can be proved in a similar way.
(i) Let λ be a left upper set in a fuzzy preordered set (X,R). Then for all $x,y \in X, \lambda(x) \otimes R(x,y) \leq \lambda(y)$, or that $\neg_l(\lambda(x) \otimes R(x,y)) \geq \neg_l\lambda(y)$. Now, $\neg_l(\lambda(x) \otimes R(x,y)) \geq \neg_l\lambda(y) \Rightarrow (\lambda(x) \otimes R(x,y)) \rightsquigarrow 0 \geq \neg_l\lambda(y) \Rightarrow R(x,y) \rightsquigarrow (\lambda(x) \rightsquigarrow 0) \geq \neg_l\lambda(y)$ (cf., Proposition 2.4) $\Rightarrow R(x,y) \rightsquigarrow \neg_l\lambda(x) \geq \neg_l\lambda(y) \Rightarrow R(x,y) \otimes \neg_l\lambda(y) \leq \neg_l\lambda(x)$. Thus $\neg_l\lambda$ is a right lower set.
(iii) Let λ be a left lower set in a fuzzy preordered set (X,R). Then for all $x,y \in X, \lambda(y) \otimes R(x,y) \leq \lambda(x)$, or that $\neg_l(\lambda(y) \otimes R(x,y)) \geq \neg_l\lambda(x)$. Now, $\neg_l(\lambda(y) \otimes R(x,y)) \geq \neg_l\lambda(x) \Rightarrow (\lambda(y) \otimes R(x,y)) \rightsquigarrow 0 \geq \neg_l\lambda(x) \Rightarrow R(x,y) \rightsquigarrow (\lambda(y) \rightsquigarrow 0) \geq \neg_l\lambda(x)$ (cf., Proposition 2.4) $\Rightarrow R(x,y) \rightsquigarrow \neg_l\lambda(y) \geq \neg_l\lambda(x) \Rightarrow R(x,y) \otimes \neg_l\lambda(x) \leq \neg_l\lambda(y)$. Thus $\neg_l\lambda$ is a right upper set. □

Remark 3.20. Let $\neg_l$ (or, $\neg_r$) be involutive. Then

(i) $\lambda \in L^X$ is a left upper set if and only if $\neg_l\lambda$ is a right lower set.
(ii) $\lambda \in L^X$ is a right upper set if and only if $\neg_r\lambda$ is a left lower set.

Definition 3.21. A map $f : (X,R) \longrightarrow (Y,S)$ between fuzzy preordered sets is called **order preserving** if $R(x,y) \leq S(f(x),f(y)), \forall x,y \in X$.

Proposition 3.22. *Let the map $f : (X,R) \longrightarrow (Y,S)$ between fuzzy preordered sets be order preserving. Then*

(i) *inverse image of a left upper set of (Y,S) is a left upper set of (X,R).*
(ii) *inverse image of a right upper set of (Y,S) is a right upper set of (X,R).*
(iii) *inverse image of a left lower set of (Y,S) is a left lower set of (X,R).*
(iv) *inverse image of a right lower set of (Y,S) is a right lower set of (X,R).*

Proof. We only prove here for left upper set and right upper set. The others can be proved in a similar way. (i) Follows from [25].
(ii) Let $x,y \in X$ and $\lambda \in L^Y$ be a right upper set of (Y,R). Then $R(x,y) \otimes f^{-1}(\lambda)(x) \leq R(x,y) \otimes \lambda(f(x)) \leq S(f(x),f(y)) \otimes \lambda(f(x)) \leq \lambda(f(y)) \leq f^{-1}(\lambda)(y)$. Thus $f^{-1}(\lambda)$ is a right upper set of (X,R). □

4. **Fuzzy Topology and Fuzzy Automaton**

In this section, we show that the results for fuzzy automata introduced in [4] are easy consequences of the results shown in the previous section for fuzzy preordered sets and fuzzy topologies. We begin with the following concept of a fuzzy automaton introduced in [4].

Definition 4.1. A **fuzzy automaton** is a triple $M = (Q, X, \delta)$, where Q is a nonempty set (of states of M), X is a monoid (the input monoid of M) whose identity shall be denoted as e, and δ is an L-valued subset of $Q \times X \times Q$, i.e., a map $\delta : Q \times X \times Q \to L$ such that $\forall p, q \in Q$,

$$\delta(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

and, $\delta(q, xy, p) = \vee\{\delta(q, x, r) \otimes \delta(r, y, p) : r \in Q\}, \forall x, y \in X$.

We now introduce the following concept of homomorphism between fuzzy automata.

Definition 4.2. A **homomorphism** from a fuzzy automaton (Q, X, δ) to a fuzzy automaton (R, Y, λ) is a pair (f, g) of maps, where $f : Q \to R$ and $g : X \to Y$ are functions such that

$$\forall (q, x, p) \in Q \times X \times Q, \quad \lambda(f(q), g(x), f(p)) \geq \delta(q, x, p).$$

The pair (f, g) is called a **strong homomorphism** if

$$\forall (q, x, p) \in Q \times X \times Q, \quad \lambda(f(q), g(x), f(p)) = \vee\{\delta(q, x, t) : t \in Q, f(t) = f(p)\}.$$

It can be seen that the class of all fuzzy automata and their homomorphisms forms a category **FTS** (under obvious composition of maps).

Definition 4.3. A **reverse fuzzy automaton** of a fuzzy automaton $M = (Q, X, \delta)$ is a fuzzy automaton $\bar{M} = (Q, X, \bar{\delta})$, where $\bar{\delta} : Q \times X \times Q \to L$ is a map such that $\bar{\delta}(p, x, q) = \delta(q, x, p), \forall p, q \in Q$ and $\forall x \in X$.

Definition 4.4. Let $M = (Q, X, \delta)$ be a fuzzy automaton. Then $\lambda \in L^Q$ is called

(i) a **left subsystem** of M if $\lambda(p) \otimes \delta(p, x, q) \leq \lambda(q), \forall p, q \in Q$ and $\forall x \in X$;

(ii) a **right subsystem** of M if $\delta(p, x, q) \otimes \lambda(p) \leq \lambda(q), \forall p, q \in Q$ and $\forall x \in X$;

(iii) a **left reverse subsystem** of M if $\lambda(q) \otimes \delta(p, x, q) \leq \lambda(p), \forall p, q \in Q$ and $\forall x \in X$.

(iv) a **right reverse subsystem** of M if $\delta(p, x, q) \otimes \lambda(q) \leq \lambda(p), \forall p, q \in Q$ and $\forall x \in X$.

(v) a **left double subsystem** of M if it is both left subsystem and left reverse subsystem of M.

(vi) a **right double subsystem** of M if it is both right subsystem and right reverse subsystem of M.

Let $M = (Q, X, \delta)$ be a fuzzy automaton. Define $R_\delta(p, q) = \vee\{\delta(p, x, q) : x \in X\}, p, q \in Q$. Then R_δ is a fuzzy preorder on Q. Now, we have the following.

Proposition 4.5. *Let $M = (Q, X, \delta)$ be a fuzzy automaton and $\lambda \in L^Q$. Then the following are equivalent:*

(i) *λ is a left (resp. right) subsystem of M.*

(ii) *λ is $\tau_{R_{l\delta}}$-open (resp. $\tau_{R_{r\delta}}$-open).*

(iii) *λ is a solution to a fuzzy relational equation $\chi \circ R_{l\delta} = \chi$ (resp. $R_{r\delta} \circ \chi = \chi,$), where $\chi \in L^Q$ is an unknown.*

Proof. For left subsystem, the result follows from Propositions 3.13 and 3.17. □

Proposition 4.6. *Let $M = (Q, X, \delta)$ be a fuzzy automaton and $\lambda \in L^Q$. Then the following are equivalent:*

(i) *λ is a left (resp. right) reverse subsystem of M.*
(ii) *λ is $\tau_{R_{l\bar{\delta}}}$-open (resp. $\tau_{R_{r\bar{\delta}}}$-open).*
(iii) *λ is a solution to a fuzzy relational equation $\chi \circ R_{l\bar{\delta}} = \chi$ (resp. $R_{l\bar{\delta}} \circ \chi = \chi$), where $\chi \in L^Q$ is an unknown.*

Proof. For left subsystem, it follows from Propositions 3.14 and 3.18. □

Proposition 4.7. *Let $M = (Q, X, \delta)$ be a fuzzy automaton and $\lambda \in L^Q$. Then*

(i) *if λ is a left reverse subsystem, then $\neg_l \lambda$ is a right subsystem.*
(ii) *if λ is a right reverse subsystem, then $\neg_r \lambda$ is a left subsystem.*
(iii) *if λ is a left subsystem, then $\neg_l \lambda$ is a left reverse subsystem.*
(iv) *if λ is a right subsystem, then $\neg_r \lambda$ is a right reverse subsystem.*

Proof. Follows from Proposition 3.19. □

Proposition 4.8. *Let $M = (Q, X, \delta)$ be a fuzzy automaton and $\lambda \in L^Q$. Then λ is a left (resp. right) double subsystem of M if and only if λ is both $\tau_{R_{l\delta}}$ and $\tau_{R_{l\bar{\delta}}}$-open (resp. both $\tau_{R_{r\delta}}$ and $\tau_{R_{r\bar{\delta}}}$-open).*

Proof. For left subsystem, it follows from Propositions 3.13 and 3.14. □

Definition 4.9. Let $M = (Q, X, \delta)$ and $N = (R, Y, \lambda)$ be fuzzy automata. Let $(f \otimes g) : M \to N$ be a fuzzy homomorphism and μ be a fuzzy subset of Q. Define the fuzzy subset $f(\mu)$ of R by $\forall q' \in R$,

$$f(\mu)(q') = \begin{cases} \vee\{\mu(q) : q \in Q, f(q) = q'\} & \text{if } f^{-1}(q') \neq \phi \\ 0 & \text{if } f^{-1}(q') = \phi \end{cases}$$

Proposition 4.10. *Let $M = (Q, X, \delta)$ and $N = (R, Y, \lambda)$ be fuzzy automata. Let $f : M \to N$ be an onto strong fuzzy homomorphism. Then if μ is a left (resp. right) subsystem of Q, then $f(\mu)$ is a left (resp. right) subsystem of R.*

Proof. We only prove for left subsystem. For which, let $q', p' \in R, x \in X$. Then $f(\mu)(p') \otimes \lambda(p', x, q') = (\vee\{\mu(p) : p \in Q, f(p) = p'\}) \otimes \lambda(p', x, q') = \vee\{\mu(p) \otimes \lambda(p', x, q') : p \in Q, f(p) = p'\}$. Let $p, q \in Q$ be such that $f(p) = p'$ and $f(q) = q'$. Then $\mu(p) \otimes \lambda(p', x, q') = \mu(p) \otimes \lambda(f(p), x, f(q)) = \mu(p) \otimes (\vee\{\delta(p, x, r) : r \in Q, f(r) = f(q) = q'\}) = \vee\{\mu(p) \otimes \delta(p, x, r) : r \in Q, f(r) = f(q) = q'\} \leq \vee\{\mu(r) : r \in Q, f(r) = q'\} = f(\mu)(q')$. Hence $f(\mu)(p') \otimes \lambda(p', x, q') \leq \vee\{f(\mu)(q') : p \in Q, f(p) = p'\} = f(\mu)(q')$. Thus $f(\mu)$ is a left subsystem of R. □

It can be seen that the above proposition is valid only for left (resp. right) subsystem but not for left (resp. right) reverse subsystem.

Remark 4.11. The following counter-example shows that the above result need not be true if f is not onto.

Example 4.12. For the generalized residuated lattice L, consider the integral monoid $(L, \otimes, e)$, where $L = [0, 1], a \otimes b = max(0, a + b - 1), \forall a, b \in L$ and $e = 1$.

Consider a fuzzy automaton (Q, X, δ), where $Q = \{p, q\} = R, X = \{x\}$ and $\delta : Q \times X \times Q \to L$ is given by

$$\delta(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$\delta(p, x, q) = \delta(q, x, p) = 1$ for fixed $p, q \in Q$ and for fixed $x \in X (x \neq 0)$, for other $p, q \in Q$ and $x \in X, \delta(p, x, q) = 0$. Then $M = (Q, X, \delta)$ is a fuzzy automaton. Let $f : Q \to Q$ be a mapping such that $f(p) = f(q) = p$. Then f is not onto. Clearly f is a strong fuzzy homomorphism. Let μ be a fuzzy subset of Q such that $\mu(p) = \mu(q) = 0.5$ for fixed $p, q \in Q$ and for other $p \in Q, \mu(p) = 0$. Then $\mu(q) = 0.5 = \mu(p) \otimes \delta(p, x, q), \forall p, q \in Q$. Thus μ is a left (resp. right) subsystem of M. But $f(\mu)(p) \otimes \delta(p, x, q) = \vee\{\mu(p) \otimes \delta(p, x, q) : p \in Q, f(p) = p'\} = \{0.5 \otimes 1\} = 0.5 > 0 = f(\mu)(q)$. Thus $f(\mu)$ is not a left (resp. right) subsystem of Q.

Some other results for (left/right) subsystems and (left/right) reverse subsystems of fuzzy automata can be derived from Propositions 3.6, 3.8, 3.9, 3.10 and 3.11.

5. **Conclusion**

In this paper, we tried to study the concept of (left/right) upper set and (left/right) lower set of a fuzzy preordered set. We showed that their collection form an Alexandrov (left/right) fuzzy topologies. Finally, we have used such concepts for the study of fuzzy automata.

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