INTERVAL-VALUED INTUITIONISTIC FUZZY SETS AND SIMILARITY MEASURE

B. PĘKALA AND K. BALICKI

Abstract. In this paper, the problem of measuring the degree of inclusion and similarity measure for two interval-valued intuitionistic fuzzy sets is considered. We propose inclusion and similarity measure by using order on interval-valued intuitionistic fuzzy sets connected with lexicographical order. Moreover, some properties of inclusion and similarity measure and some correlation, between them and aggregations are examined. Finally, we have included example of ranking problem in car showrooms.

1. Introduction

Many new approaches and theories treating imprecision and uncertainty have been proposed since fuzzy sets were introduced by Zadeh. As extensions of classical fuzzy set theory, intuitionistic fuzzy sets [1] and interval-valued fuzzy sets are very useful in dealing with imprecision and uncertainty. Especially, interval-valued intuitionistic fuzzy set introduced by Atanassov [2], as a combining concept of intuitionistic fuzzy set and interval-valued fuzzy set, greatly furnishes the additional capability to deal with vague information and model non-statistical uncertainty by providing a membership interval and a nonmembership interval. Therefore, interval-valued intuitionistic fuzzy sets have played a significant role in the uncertain system and received much attention from researchers. Recently, many scholars have investigated interval-valued intuitionistic fuzzy sets and obtained some meaningful results in the fields of multicriteria decision making [18] and group decision making with interval-valued intuitionistic fuzzy sets [22]. Many researchers have examined different types of transitivity and have proposed some distance measures, similarity measures ([4], [5], [11], [12], [13], [14], [16], [17], [21] or [23]) and correlation measures of interval-valued intuitionistic fuzzy sets ([22], [25]). Moreover there were presented applications of such considerations to real-life problems involving pattern recognition, medical diagnosis and decision-making. Additionally, some inclusion measures of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets have been proposed in [19], which also play important roles in the application areas such as approximate reasoning, statistical inference and decision making. Thus the motivation of the present paper is to propose a more natural tools for estimating the degree of inclusion between interval-valued intuitionistic

Received: April 2016; Revised: October 2016; Accepted: January 2017

Key words and phrases: Interval-valued intuitionistic fuzzy sets, Inclusion measure, Similarity measure.

fuzzy sets and exploring their properties. The paper is an extension of the results presented at the conference AGOP' 2015 and is organized as follows. In Section 2 we recall basic information on interval-valued intuitionistic fuzzy sets. We also show there crisp definition of inclusion. In Section 3 we present an inclusion measure for interval-valued intuitionistic fuzzy sets by using order connected with lexicographical order. Then, some properties of inclusion measure for interval-valued intuitionistic fuzzy sets are examined. Section 4 is concerning similarity measure and properties connected with transitivity and bisymmetry properties. Finally, we present example of applications considering similarity measure to ranking cars in car showrooms.

2. **Interval-valued Intuitionistic Fuzzy Sets**

Throughout this paper the discourse set is denoted as $X = \{x_1, x_2, \ldots, x_n\}$. A fuzzy set ρ in X is defined as the set of ordered pairs

$$\rho = \{< x_i, R(x_i) >: x_i \in X\},$$

where $\rho : X \to [0, 1]$ and $R(x_i)$ is the grade of belongingness of x_i into ρ. A family of all fuzzy sets in X will be denoted by $FS(X)$. According to Zadeh seminal paper [24] introducing fuzzy sets, we define inclusion for two fuzzy sets ρ and σ in X as follows

$$\rho \leq \sigma \Leftrightarrow R(x_i) \leq S(x_i) \tag{1}$$

for each $i \in \{1, ..., n\}, n \in N$, where we denote membership functions of the sets $\rho = \{< x_i, R(x_i) >: x_i \in X\}$ and $\sigma = \{< x_i, S(x_i) >: x_i \in X\}$, respectively. For each fuzzy set ρ the grade of nonbelongingness of x into ρ is automatically equals to $1 - R(x_i)$. However, in real life the linguistic negation is not always identified with logical negation. This situation is very common in natural language processing, computing with words and their applications in many areas. Thus, although fuzzy set theory provides useful tools for dealing with uncertain information. Atanassov [1] suggested a generalization of classical fuzzy set, called an interval-valued intuitionistic fuzzy set.

Definition 2.1. [3] An interval-valued intuitionistic fuzzy set ρ in the finite universe X can be expressed in the form

$$\rho = \{< x_i, R(x_i), \mathbf{R}(x_i) >: x_i \in X\},$$

where $R(x_i) = [\underline{R}(x_i), \overline{R}(x_i)]$ is called the interval membership degree of an element x_i to interval-valued intuitionistic fuzzy set ρ, while

$\mathbf{R}(x_i) = [\underline{\mathbf{R}}(x_i), \overline{\mathbf{R}}(x_i)]$ is the interval non-membership degree of this element to the set, and the condition $0 \leq \overline{R}(x_i) + \overline{\mathbf{R}}(x_i) \leq 1$ must hold for any $x_i \in X$.

For shorter notation of an interval-valued intuitionistic fuzzy set we use a pair $\rho = (R, \mathbf{R})$. The family of all interval-valued intuitionistic fuzzy sets described in the given sets X is denoted by $IVIFS(X)$. For each interval-valued fuzzy sets we have hesitation index $\pi(x_i)$ whether x_i are in the relation $\rho(x_i)$ or not and

$$\pi_\rho = [1 - (\overline{R} + \overline{\mathbf{R}}), 1 - (\underline{R} + \underline{\mathbf{R}})].$$

The boundary elements in $IVIFS(X)$ are $1^{\cdot} = (\mathbf{1}, \mathbf{0})$ and $0^{\cdot} = (\mathbf{0}, \mathbf{1})$, where $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$. Basic operations for $\rho = (R, \mathbf{R})$, $\sigma = (S, \mathbf{S}) \in IVIFS(X)$ are the union, the intersection and the complement, respectively

$$\rho \vee \sigma = ([\underline{R} \vee \underline{S}, \overline{R} \vee \overline{S}], [\underline{\mathbf{R}} \wedge \underline{\mathbf{S}}, \overline{\mathbf{R}} \wedge \overline{\mathbf{S}}]),$$

$$\rho \wedge \sigma = ([\underline{R} \wedge \underline{S}, \overline{R} \wedge \overline{S}], [\underline{\mathbf{R}} \vee \underline{\mathbf{S}}, \overline{\mathbf{R}} \vee \overline{\mathbf{S}}]),$$

$$\rho' = (\mathbf{R}, R).$$

Moreover, the order is defined by

$$\rho \leq_P \sigma \ \Leftrightarrow \ \underline{R}(x_i) \leq \underline{S}(x_i), \overline{R}(x_i) \leq \overline{S}(x_i), \ \underline{\mathbf{S}}(x_i) \leq \underline{\mathbf{R}}(x_i), \overline{\mathbf{S}}(x_i) \leq \overline{\mathbf{R}}(x_i) \tag{2}$$

for $i = 1, ..., n, n \in N$, where $n = card(X)$.

The pair $(IVIFS(X), \leq_P)$ is a partially ordered set. So operation $\leq_P$ is a lack of a complete property, because we can find two interval-valued intuitionistic fuzzy sets not compared by order $\leq_P$. Also in family

$$L_{IVI} = \{(x, y) : \ x, y \in L[0,1], \ \overline{x} + \overline{y} \leq 1\},$$

where $L[0,1] = \{[\underline{x}, \overline{x}] : \underline{x} \leq \overline{x}, \underline{x}, \overline{x} \in [0,1]\}$ we have a lack of completness property. But we often need completness property in many applications. Thus we use relation $\leq_C$ which is linear and refines the order $\leq_P$ in L_{IVI}, i.e., it is a linear order satisfying that for all $x, y \in L_{IVI}$ such that $x \leq_C y$ it holds $x \leq_P y$. Presented order on L_{IVI} is connected with lexicographical order (see also [8] and [15]), i.e.,

$$(x_1, y_1) \leq_C (x_2, y_2) \Leftrightarrow \underline{x}_1 < \underline{x}_2 \ \ or$$

$$\underline{x}_1 = \underline{x}_2 \ \ and \ \ \overline{x}_1 < \overline{x}_2 \ \ or$$

$$\underline{x}_1 = \underline{x}_2 \ \ and \ \ \overline{x}_1 = \overline{x}_2 \ \ and \ \ \pi_1 \leq_{Lex} \pi_2,$$

where $[a_1, b_1] \leq_{Lex} [a_2, b_2] \Leftrightarrow a_1 < a_2 \ \ or \ \ a_1 = a_2 \ \ and \ \ b_1 < b_2$ for $[a_1, b_1], [a_2, b_2] \in L[0,1]$.

So this order means that smaller element of L_{IVI} by $\leq_C$ has smaller hesitance index, so smaller a measure of non-determinacy. Which is interesting conclusion in ranking problem of information present by pairs of intervals in many real situations.

But $(IVIFS(X), \leq_C)$ is the partially ordered set, where for $\rho, \delta \in IVIFS(X)$, $card(X) = n$, $n \in N$

$$\rho \leq_C \delta \Leftrightarrow \rho(x_i) \leq_P \delta(x_i) \ \ or \ \ else \ \ \rho(x_i) \leq_C \delta(x_i)$$

for all $i = 1, ..., n$.

Reflexivity and transitivity are obvious, moreover antisymmetry of $\leq_C$ we obtain because conditions $\rho \leq_C \delta$ and $\delta \leq_C \rho$ occur only if

$$\underline{x}_1 = \underline{x}_2 \ \ and \ \ \overline{x}_1 = \overline{x}_2 \ \ and \ \ \pi_1 \leq_{Lex} \pi_2,$$

and

$$\underline{x}_2 = \underline{x}_1 \ \ and \ \ \overline{x}_2 = \overline{x}_1 \ \ and \ \ \pi_2 \leq_{Lex} \pi_1,$$

what means that $\leq_C = \leq_P$ and $\rho = \delta$.

We also recall the following definition.

Definition 2.2. [9] A function $\mathcal{A} : L^n \to L$ is called an aggregation function on a bounded lattice L if it is increasing and

$$\mathcal{A}(\underbrace{0_L, ..., 0_L}_{n\times}) = 0_L, \quad \mathcal{A}(\underbrace{1_L, ..., 1_L}_{n\times}) = 1_L.$$

Very interesting are some functions mentioned in the later part of the paper, which are special kind of aggregations, i.e.

Definition 2.3. [10] A triangular norm T on a bounded lattice L is an increasing, commutative, associative function $T : L^2 \to L$ with a neutral element 1_L.
A triangular conorm S on L is an increasing, commutative, associative function $S : L^2 \to L$ with a neutral element 0_L.

Now, we recall the definition of an overlap function which generalizes intersection operators such as the minimum. Overlap functions are special kinds of aggregation operators that have been recently proposed for applications involving the overlap problem and/or when the associativity property is not strongly required, as in imaging processing and decision making based on fuzzy preference relations, respectively. Therefore, in those cases, the use of t-norms or t-conorms as the combination/separation operators is not necessary. For example, overlap functions allowed the development of some construction methods for the concepts of indifference and incomparability, as introduced by Bustince et al. in [7]. The notions of overlapping arise from a common problem in many fields: how to assign a given element or object to exactly one class among several available. The notion of overlap function was presented in [6] and [20] to address the former difficulty in the context of image processing. So it will be interest to examine similarity measure created from the overlap functions.

Definition 2.4. A function $G_0 : [0,1]^2 \to [0,1]$ is called an overlap function, if it satisfies the following conditions:
(GO1) $G_O(x,y) = G_O(y,x)$ for all $x; y \in [0,1]$;
(GO2) $G_O(x,y) = 0$ if and only if $x = 0$ or $y = 0$;
(GO3) $G_O(x,y) = 1$ if and only if $x = y = 1$;
(GO4) G_O is increasing;
(GO5) G_O is continuous.

For our further considerations the condition (GO3) which justifies the choice of the overlap function to create a similarity measure will be the most important.

3. Inclusion Measure

The inclusion measures, also known as subsethood measures, have been studied mainly by constructive approaches (Fan et al. 1999, Ma et al. 1999, Young 1996, Xu et al. 2002, Qiu et al. 2003, Zeng and Li 2006, Grzegorzewski 2011 or Yao and Deng 2014) and axiomatic approaches (Kitainik 1987, Sinha and Dougherty 1993, Young 1996, Zhang and Leung 1996, Fan et al. 1999, Cornelis et al. 2003). The inclusion measure has also been introduced successfully into fuzzy concept lattice theory in (Fan et al. 2006). Many researchers tried to relax the rigidity of Zadeh definition (1) of inclusion to get a soft approach which is more compatible with the spirit of fuzzy logic. Zhang and Leung (1996) thought that quantitative methods were the main approaches in uncertainty inference which is a key problem for artificial intelligence, so they presented a generalized definition for the inclusion measure, called including degree, to represent and measure the uncertainty information. Instead of binary discrimination: being or not being a subset, [27] proposed several indicators giving the degree to which an interval-valued intuitionistic fuzzy set is a subset of another interval-valued intuitionistic fuzzy set. More precisely, let us consider a mapping $Inc: IVIFS(X) \times IVIFS(X) \to [0,1]$, called an inclusion measure (or subsethood measure) such that the value $Inc(\rho,\sigma)$ quantifies the degree of inclusion of ρ into σ, where $\rho, \sigma \in IVIFS(X)$ by using partial order connected with lexicographical order.

Definition 3.1 (cf. [27])**.** A function $Inc : IVIFS(X) \times IVIFS(X) \to [0,1]$ is named an inclusion measure, if it satisfies the following conditions:
(IM1) If $\rho = 1^{\cdot}$, $\sigma = 0^{\cdot}$, then $Inc(\rho,\sigma) = 0$;
(IM2) $Inc(\rho,\sigma) = 1 \Leftrightarrow \rho \leq_C \sigma$;
(IM3) If $\rho \leq_C \sigma \leq_C \gamma$, then $Inc(\gamma,\rho) \leq Inc(\sigma,\rho)$ and $Inc(\gamma,\rho) \leq Inc(\gamma,\sigma)$.

Definition 3.2 (cf. [26])**.** A function $Inc^H : IVIFS(X) \times IVIFS(X) \to [0,1]$ is named a hybrid monotonic inclusion measure, if it satisfies the following conditions:
(HIM1) If $\rho = 1^{\cdot}$, $\sigma = 0^{\cdot}$, then $Inc^H(\rho,\sigma) = 0$;
(HIM2) $Inc^H(\rho,\sigma) = 1 \Leftrightarrow \rho \leq_C \sigma$;
(HIM3) If $\rho \leq_C \sigma$, then $Inc^H(\sigma,\gamma) \leq Inc^H(\rho,\gamma)$ and $Inc^H(\gamma,\rho) \leq Inc^H(\gamma,\sigma)$.

Example 3.3. Here are some examples of inclusion measures:

(i)

$$Inc_{\wedge}(\rho,\sigma) = \begin{cases} 1, & \rho = \sigma = 0^{\cdot}, \\ \frac{|\rho\wedge\sigma|}{|\rho|}, & otherwise, \end{cases}$$

where $|\rho| = \sum_{x_i \in X} \frac{\underline{R}+\overline{R}+2-\underline{\mathbf{R}}-\overline{\mathbf{R}}}{4}$.

(ii)

$$Inc_z(\rho,\sigma) = \begin{cases} 1, & \rho \leq_C \sigma, \\ 0, & otherwise. \end{cases}$$

Now, we examine the transitivity property using the overlap functions.

Proposition 3.4. *Inc_z is a quasi-ordered operation (reflexive and transitive) with G_0-transitivity, where G_0 is an overlap function.*

Proof. Reflexivity is obvious. We consider G_0-transitivity, i.e.

$$G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\gamma)) \leq Inc_z(\rho,\gamma).$$

1. If $\rho \leq_C \sigma \leq_C \gamma$, then $G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\gamma)) = G_0(1,1) = 1 \leq 1 = Inc_z(\rho,\gamma)$.
2. If $\rho \leq_C \gamma \leq_C \sigma$, then $G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\gamma)) = G_0(1,0) = 0 \leq 1 = Inc_z(\rho,\gamma)$.
3. If $\sigma \leq_C \rho \leq_C \gamma$, then $G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\gamma)) = G_0(0,1) = 0 \leq 1 =$

$Inc_z(\rho, \gamma)$.
4. If $\sigma \leq_C \gamma \leq_C \rho$, then $G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \gamma)) = G_0(0, 1) = 0 \leq 0 = Inc_z(\rho, \gamma)$.
5. If $\gamma \leq_C \rho \leq_C \sigma$, then $G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \gamma)) = G_0(1, 0) = 0 \leq 0 = Inc_z(\rho, \gamma)$.
6. If $\gamma \leq_C \sigma \leq_C \rho$, then $G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \gamma)) = G_0(0, 1) = 0 \leq 0 = Inc_z(\rho, \gamma)$.
7. If $\gamma || \sigma, \sigma \leq_C \rho, \gamma \leq_C \rho$, then $G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \gamma)) = G_0(0, 1) = 0 \leq 0 = Inc_z(\rho, \gamma)$.
Similarly we can show the remaining cases of incomparability. So Inc_z is a partially ordered operation. □

4. Similarity Measure

The similarity measure is employed to indicate the similarity degrees of two models or two rules in a system. In this section, we first recall definition of the similarity measure.

Definition 4.1 (cf. [27])**.** A real function $Sim : IVIFS(X) \times IVIFS(X) \to [0, 1]$ is named a similarity measure of IVIFSs on universe X, if it satisfies the following properties:
(SM1) $Sim(\rho, \rho') = 0$, if $\rho \in \{0^{\cdot}, 1^{\cdot}\}$;
(SM2) $Sim(\rho, \sigma) = 1 \Leftrightarrow \rho = \sigma$;
(SM3) $Sim(\rho, \sigma) = Sim(\sigma, \rho)$;
(SM4) If $\rho \leq_C \sigma \leq_C \gamma$, then $Sim(\rho, \gamma) \leq Sim(\rho, \sigma)$ and $Sim(\rho, \gamma) \leq Sim(\sigma, \gamma)$.

Example 4.2. Here are some examples of similarity measures:

(i) $Sim_d(\rho, \sigma) = 1 - \frac{1}{n} \sum_{i=1}^{n} \max(|\underline{R}(x_i) - \underline{S}(x_i)|, |\overline{R}(x_i) - \overline{S}(x_i)|, |\underline{\mathbf{R}}(x_i) - \underline{\mathbf{S}}(x_i)|, |\overline{\mathbf{R}}(x_i) - \overline{\mathbf{S}}(x_i)|)$,
(ii) $Sim_{\wedge}(\rho, \sigma) = Inc(\rho, \sigma) \wedge Inc(\sigma, \rho)$.

We observe an interesting connection between the similarity and the inclusion measure and we present the following similarity measures

Proposition 4.3. *Let G_0 be an overlap function. Then*
$$Sim_{G_0}(\rho, \sigma) = G_0(Inc(\rho, \sigma), Inc(\sigma, \rho))$$
is a similarity measure.

Proof. The proof of (SM1) is obvious. By (GO3) we have
$$Sim_{G_0}(\rho, \sigma) = 1 \Leftrightarrow G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \rho)) = 1 \Leftrightarrow Inc_z(\rho, \sigma) = 1$$
and
$$Inc_z(\sigma, \rho) = 1 \Leftrightarrow \rho \leq_C \sigma, \ \sigma \leq_C \rho.$$
Thus $\rho = \sigma$, which proves (SM2). (SM3) holds by (GO1).
Now, we consider (SM4). Let $\rho \leq_C \sigma \leq_C \gamma$. Then by properties of Inc, (SM3) and (GO4) we obtain
$$Sim_{G_0}(\rho, \gamma) = G_0(Inc(\rho, \gamma), Inc(\gamma, \rho)) \leq G_0(1, Inc(\sigma, \rho)) = G_0(Inc(\rho, \sigma), Inc(\sigma, \rho))$$

$$= Sim_{G_0}(\rho, \sigma)$$

and

$$Sim_{G_0}(\rho, \gamma) = Sim_{G_0}(\gamma, \rho) = G_0(Inc(\gamma, \rho), 1) \leq G_0(Inc(\gamma, \sigma), Inc(\sigma, \gamma)) = Sim_{G_0}(\sigma, \gamma).$$

□

Proposition 4.4. *If G_0 is an overlap function with the neutral element 1, then*

$$Sim_{G_0}(\rho, \sigma) = G_0(Inc^H(\rho, \sigma), Inc^H(\sigma, \rho))$$

is a similarity measure.

Proof. (SM1) and (SM2) hold. We consider (SM4). Let $\rho \leq_C \sigma \leq_C \gamma$ and 1 be the neutral elements of G_0. Then

$$Sim_{G_0}(\rho, \gamma) = G_0(Inc^H(\rho, \gamma), Inc^H(\gamma, \rho)) = G_0(1, Inc^H(\gamma, \rho)) = Inc^H(\gamma, \rho).$$

In a similar way we obtain

$$Sim_{G_0}(\rho, \sigma) = Inc^H(\sigma, \rho), Sim_{G_0}(\sigma, \gamma) = Inc^H(\gamma, \sigma).$$

Now, by (HIM3) we have $Inc^H(\gamma, \rho) \leq Inc^H(\sigma, \rho)$ and $Inc^H(\gamma, \rho) \leq Inc^H(\gamma, \sigma)$. So $Sim_{G_0}(\rho, \gamma) \leq Sim_{G_0}(\rho, \sigma)$ and $Sim_{G_0}(\rho, \gamma) \leq Sim_{G_0}(\sigma, \gamma)$.
(SM3) also holds because
$Sim_{G_0}(\gamma, \rho) = G_0(Inc^H(\gamma, \rho), 1) = Inc^H(\gamma, \rho)$. Thus we finished the proof. □

Moreover, we consider, when the similarity measure is a G_0-equivalence relation.

Similarly to [5], [14], [17] or [23] we will examine the transitivity property and equivalence relation, i.e. a relation $\psi \subset X \times X$ is an F-equivalence relation if it fulfils

- reflexivity, $\psi(x, x) = 1$;
- symmetry, $\psi(x, y) = \psi(y, x)$;
- F-transitivity, $F(\psi(x, z), \psi(z, y)) \leq \psi(x, y)$ for $x, y, z \in X$.

Proposition 4.5. *Let G_0 be a bisymmetric overlap function. Then*

$$Sim_{G_0}(\rho, \sigma) = G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \rho))$$

is the G_0-equivalence relation.

Proof. Let G_0 be a bisymmetric overlap function. If $\rho = \sigma$, then

$$Sim_{G_0}(\rho, \sigma) = G_0(Inc_z(\rho, \rho), Inc_z(\rho, \rho)) = G_0(1, 1) = 1.$$

So Sim_{G_0} has the reflexivity property. Symmetry is obvious.
If G_0 has bisymmetry property, then we can prove G_0-transitivity:

$$G_0(Sim_{G_0}(\rho, \sigma), Sim_{G_0}(\sigma, \delta)) \leq Sim_{G_0}(\rho, \delta).$$

By G_0-transitivity of Inc_z we have

$$G_0(Inc_z(\rho, \sigma), Inc_z(\sigma, \delta)) \leq Inc_z(\rho, \delta)$$

and

$$G_0(Inc_z(\delta, \sigma), Inc_z(\sigma, \rho)) \leq Inc_z(\delta, \rho)$$

Then by isotonicity, symmetry and bisymmetry properties of G_0 we calculate
$G_0(G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\delta)), G_0(Inc_z(\delta,\sigma), Inc_z(\sigma,\rho)))$
$\leq G_0(Inc_z(\rho,\delta), Inc_z(\delta,\rho))$

$$\Leftrightarrow$$

$G_0(G_0(Inc_z(\rho,\sigma), Inc_z(\sigma,\rho)), G_0(Inc_z(\sigma,\delta), Inc_z(\delta,\sigma)))$
$\leq G_0(Inc_z(\rho,\delta), Inc_z(\delta,\rho))$,
which finishes the proof. □

Moreover, directly we have for $\rho, \sigma \in IVIFS(X)$ similarity also after aggregation of similarities where ρ and σ have some properties iff $\rho(x_i)$ and $\sigma(x_i)$ have these properties for all i.

Proposition 4.6. *Let $A : [0,1]^n \to [0,1]$ be aggregation function and fulfils $A(x_1, ..., , x_n) = 1 \Leftrightarrow x_i = 1$ for all $i = 1, ..., n$, then*

$$Sim(\rho,\sigma) = A_{i=1}^{n}(Sim(\rho(x_i), \sigma(x_i)))$$

is a similarity measure.

5. Illustrative Example: Decision Making

In a multi-expert decision making problem we have a set of k alternatives $X = \{x_1, ..., x_k\}$, $(k \geq 2)$ described on set of criteria $C = \{c_1, ..., c_m\}$, $m \geq 1$, a set of n experts $E = \{e_1, ..., e_n\}$, $(n > 2)$ and each of the latter provides his/her opinions on the former set of alternatives by given criteria. It is well known that, depending on the context and/or the level of knowledge of the experts, in some decision making problems it may occur that it is difficult to express the preferences using precise numerical values. Moreover, it can also happen, when there are alternatives pairwise compared, that experts are not sure if they prefer an alternative or another. In these cases, they can choose values close to 0.5. For these reasons, some extensions of fuzzy sets are used. An interval-valued intuitionistic fuzzy relation is defined by: $R_t : X \times C \to L_{IVI}$ and contain experts opinions about alternatives on criteria, which represent as pairs of intervals $R_t(x_i, c_j)$, where first interval denotes the degree of fulfillment j-th criterion by i-th alternative, while the second one represent the non-fulfillment c_j by x_i. A multi-expert decision making algorithm, which we will present, takes as input the opinion of multiple experts. Each of such experts $t \in E$ expresses your estimation as an interval-valued intuitionistic fuzzy relation which is denoted R_t:

$$R_t(x,c) = \begin{array}{c|cccc} & c_1 & c_2 & ... & c_m \\ \hline x_1 & r_{11}^t & r_{12}^t & ... & r_{1m}^t \\ . & ... & ... & ... & ... \\ . & ... & ... & ... & ... \\ x_k & ... & ... & ... & r_{km}^t \end{array}.$$

In algorithm we will calculate aggregation of relations R_t and we obtain interval-valued intuitionistic fuzzy relation R. There, we propose a method to choose the alternative $x_i \in X$, that best meets the customer's expectations. Thus, if we ask new costumer which alternative he look for, which criteria are important for him.

We present this information as interval-valued intuitionistic fuzzy set S with degree of validity of the criteria:

$$S = \{c_1, ...c_m\}.$$

If this set is incomplete for the value c_s, then we put there aggregated value:

$$\mathcal{A}_{1\leq q\leq k} R(x_q, c_s).$$

Then, we find similar interval-valued intuitionistic fuzzy set, i.e. row of relation R and we can precisely and effectively offer the customer the alternative, with the greatest value of similarity and which may satisfy the customer (Proposition 4.6). What has strictly connection with profit from the sale of alternative to the company.

Algorithm discloses a method to solve a multi-expert decision making problem where the interval-valued intuitionistic fuzzy relations model optimization process experts evaluations.

Algorithm 5.1. ***Multi-expert decision making algorithm using similarity of interval-valued intiutionistic fuzzy sets.***

Inputs: *A set of alternatives* $X = \{x_1, ..., x_k\}$, $(k \geq 2)$;
A set of criteria $C = \{c_1, ..., c_m\}$, $m \geq 1$;
A set of n *experts* $E = \{e_1, ..., e_n\}$, $(n > 2)$;
Set of relations R_t, $1 \leq t \leq n$;
Interval-valued intiutionistic fuzzy set S*, with preferences customer to criteria of potential alternative;*
An aggregation function $\mathcal{A}$;
A similarity measure Sim

Outputs: *Ranking of alternatives; The best alternative x' for given customer*

for $t \in \{1, ..., n\}$ *do*

- *calculate an interval-valued intiutionistic fuzzy relation:*

$$R := \mathcal{A}_t R_t$$

end

- *if* S *is incomplete for* c_s*, then calculate:*

$$\mathcal{A}_{1\leq q\leq k} R(x_q, c_s)$$

for $i \in \{1, ..., k\}$ *do*

- *calculate set of similarity preferences of criteria of customer with each row* $\{x_i\}$ *of relation* R*:*

$$\{Sim(\{x_i\}, S)\}$$

- *find chain of values of similarity, which gives chain of preferences given customer of alternatives and the greatest value of these, i.e. the best alternative for given customer*

end

$x_k \to x'$*, such that* x_k *gives maximum of* Sim*;*

Using method with similarity measure of interval-valued intuitionistic fuzzy sets we can omit problem with non comparability elements of IVIFSs. The algorithm aroused a lot of interest in car showrooms in Rzeszów (Poland). Support sellers expertise leads to more effective sales through faster and more accurate proposals to customers. Thus increasing profits car showroom.

Next, we show an example, where a set of experts, $E = \{e_1, e_2, ...e_5\}$ evaluating a set of alternatives, i.e. cars. Thus, the set of alternatives available is
$X = \{x_1 : Honda, x_2 : Volvo, x_3 : Citroen, x_4 : Peugeot\}$, on which experts provide their personal opinion with respect to five criteria:
$C = \{c_1 : power,\ c_2 : combustion,\ c_3 : size,\ c_4 : security,\ c_5 : comfort\}$.
We execute presented above Algorithm to obtain the best alternative considering a consensus process between the experts. Supporting decisions of the client in choosing the car with the criteria that are important to him at a car dealership.

We use the arithmetic mean as representable aggregation function $\mathcal{A}$ and we present aggregated relations experts opinions:

$$R(x,c) =$$

	c_1	c_2	c_3	c_4	c_5
x_1	([0.7, 0.8], [0.1, 0.2])	([0.5, 0.8], [0, 0.1])	([0.9, 1], [0, 0])	([0.3, 0.6], [0.1, 0.2])	([0.6, 0.7], [0.1, 0.2])
x_2	([0.3, 0.5], [0.5, 0.5])	([0.6, 0.7], [0.2, 0.3])	([0.4, 0.6], [0, 0.3])	([0.3, 0.6], [0.4, 0.4])	([0.3, 0.5], [0.1, 0.5])
x_3	([0, 0.2], [0.9, 1])	([0.5, 0.6], [0.2, 0.3])	([0, 0.1], [0.6, 0.9])	([0.3, 0.4], [0.4, 0.5])	([0.3, 0.4], [0.4, 0.6])
x_4	([0.5, 0.6], [0.1, 0.2])	([0.1, 0.2], [0.6, 0.8])	([0.5, 0.6], [0.1, 0.2])	([0.1, 0.1], [0.6, 0.9])	([0.5, 0.6], [0.1, 0.2])

Entered data about customer's expectations:

$S = \{c_1 : ([0.4, 0.6], [0, 0.2]), c_2 : ([0.1, 0.4], [0.6, 0.6]), c_3 : ([0.5, 0.6], [0.2, 0.3]),$
$c_4 : ([0.1, 0.4], [0.5, 0.6]), c_5 : ([0.3, 0.6], [0.1, 0.2])\}$.

Next, we calculate for each alternatives:

$$Sim(\{x_1\}, S) = 0.6, \quad Sim(\{x_2\}, S) = 0.72, \quad Sim(\{x_3\}, S) = 0.54,$$

$$Sim(\{x_4\}, S) = 0.82,$$

where we used $Sim = Sim_d$.
Thus, **Peugeot** is the best car for given customer and we can present to the customer cars in the following order:

$$x_4 \succeq x_2 \succeq x_1 \succeq x_3.$$

6. **Conclusions**

In this paper, we comment on the existing axiomatical definitions of similarity measures for IVIFSs. Some general formula of calculating the similarity between IVIFSs have been proposed. The relationships among the similarity measures and the inclusion measures of IVIFSs and G_0-transitivity have been investigated. In future another transitivity properties and their connection with similarity will be examined. Moreover, dependence between similarity and preference property will be investigated.

Acknowledgements. This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge of University of Rzeszów, Poland, the project RPPK.01.03.00-18-001/10.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., **20** (1986), 87–96.

[2] K. T. Atanassov, *Operators over interval valued intuitionistic fuzzy sets*, Fuzzy Sets Syst., **64** (1994), 159–174.

[3] K. T. Atanassov, *Intuitionistic Fuzzy Sets. Theory and Applications*, Physica-Verlag, Heidelberg/New York, 1999.

[4] U. Bodenhofer, B. De Baets and J. Fodor, *A compendium of fuzzy weak orders: Representations and constructions*, Fuzzy Sets Syst., **158** (2007), 811–829.

[5] K. Bosteels and E. E. Kerre, *On a reflexivity-preserving family of cardinality-based fuzzy comparison measures*, Inform. Sci., **179** (2009), 2342–2352.

[6] H. Bustince, J. Fernandéz, R. Mesiar, J. Montero and R. Orduna, *Overlap functions*, Nonlinear Anal.: Theory Methods Appl., **72** (2010), 1488–1499.

[7] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier and F. Herrera, *Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons*, IEEE Trans. Fuzzy Syst., **20(3)** (2012), 405–415.

[8] H. Bustince, J. Fernandéz, A. Kolesárová and R. Mesiar, *Generation of linear orders for intervals by means of aggregation functions*, Fuzzy Sets Syst., **220** (2013), 69-77.

[9] T. Calvo, A. Kolesárová, M. Komorniková and R. Mesiar, *Aggregation operators: properties, classes and construction methods*, In T. Calvo, G. Mayor, and R. Mesiar (Eds.), Physica-Verlag, New York, Aggregation Operators. Studies in Fuzziness and Soft Computing, **97** (2002), 3-104.

[10] B. De Baets and R. Mesiar, *Triangular norms on product lattices*, Fuzzy Sets Syst., **104** (1999), 61–76.

[11] B. De Baets, H. De Meyer and H. Naessens, *A class of rational cardinality-based similarity measures*, J. Comp. Appl. Math., **132** (2001), 51–69.

[12] B. De Baets and H. De Meyer, *Transitivity frameworks for reciprocal relations:cycle-transitivity versus FG-transitivity*, Fuzzy Sets Syst., **152** (2005), 249–270.

[13] B. De Baets, S. Janssens and H. De Meyer, *On the transitivity of a parametric family of cardinality-based similarity measures*, Int. J. Appr. Reason., **50** (2009), 104–116.

[14] M. De Cock and E. E. Kerre, *Why fuzzy T-equivalence relations do not resolve the Poincar'e paradox, and related issues*, Fuzzy Sets Syst., **133** (2003), 181–192.

[15] L. De Miguel, H. Bustince, J. Fernandéz, E. Induráin, A. Kolesárová and R. Mesiar, *Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making*, Information Fusion, **27** (2016), 189-197.

[16] S. Freson, B. De Baets and H. De Meyer, *Closing reciprocal relations w.r.t. stochastic transitivity*, Fuzzy Sets Syst., **241** (2014), 2–26.

[17] B. Jayaram and R. Mesiar, *I-Fuzzy equivalence relations and I-fuzzy partitions*, Inf. Sci., **179** (2009), 1278–1297.

[18] D. F. Li, *Toposis-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets*, IEEE Trans. Fuzzy Syst., **18** (2010), 299–311.

[19] X. D. Liu, S. H. Zheng and F. L. Xiong, *Entropy and subsethood for general interval-valued intuitionistic fuzzy sets*, Lecture Notes Artif. Intell., **3613** (2005), 42–52.

[20] N. Madrid, A. Burusco, H. Bustince, J. Fernandéz and I. Perfilieva, *Upper bounding overlaps by groupings*, Fuzzy Sets Syst., **264** (2015), 76–99.

[21] S. Ovchinnikov, *Numerical representation of transitive fuzzy relations*, Fuzzy Sets Syst., **126** (2002), 225–232.

[22] D. G. Park, Y. C. Kwun, J. H. Park and I. Y. Park, *Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems*, Math. Comput. Modell., **50** (2009), 1279–1293.

[23] Z. Świtalski, *General transitivity conditions for fuzzy reciprocal preference matrices*, Fuzzy Sets Syst., **137** (2003), 85–100.

[24] L. A. Zadeh, *Fuzzy sets*, Information Contr., **8** (1965), 338 – 353.

[25] W. Y. Zeng and P. Guo, *Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship*, Inf. Sci., **178** (2008), 1334–1342.

[26] H. Y. Zhang and W. X. Zhang, *Hybrid monotonic inclusion measure and its use in measuring similarity and distance between fuzzy sets*, Fuzzy Sets Syst., **160** (2009), 107–118.

[27] Q. Zhang, H. Xing, F. Liu and J. Ye, P. Tang, S*ome new entropy measures for interval-valued intuitionistic fuzzy sets based on distances and their relationships with similarity and inclusion measures*, Inf. Sci., **283** (2014), 55–69.

Barbara Pękala*, Interdisciplinary Centre for Computational Modelling, Faculty of Mathematics and Natural Sciences, University of Rzeszów, Pigonia 1, 35-310 Rzeszów, Poland

E-mail address: bpekala@ur.edu.pl

Krzysztof Balicki, Interdisciplinary Centre for Computational Modelling, Faculty of Mathematics and Natural Sciences, University of Rzeszów, Pigonia 1, 35-310 Rzeszów, Poland

E-mail address: kbalicki@ur.edu.pl

*Corresponding Author