# FUZZY INCLUSION LINEAR SYSTEMS

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Abstract. In this manuscript, we introduce a new class of fuzzy problems, namely "fuzzy inclusion linear systems" and propose a fuzzy solution set for it. Then, we present a theoretical discussion about the relationship between the fuzzy solution set of a fuzzy inclusion linear system and the algebraic solution of a fuzzy linear system. New necessary and sufficient conditions are derived for obtaining the unique algebraic solution for a fuzzy linear system. Also, all new concepts are illustrated by numerical examples.

## 1. Introduction

One field of applied mathematics that plays a major role in several applications in various areas of sciences is solving systems of linear equations where some of the system's parameters are proposed as fuzzy numbers. A general model for solving an $n \times n$ fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number-valued vector was first proposed by Friedman et al. [11]. Based on Friedman et al.'s method, Allahviranloo [1, 3] used various numerical methods to solve fuzzy linear systems. Also, Ezzati [10] developed a new method for solving fuzzy linear systems by using embedding method [17] and replaced an $n \times n$ fuzzy linear system by two $n \times n$ crisp linear systems. But in 2011, Allahviranloo et al. [4] showed by an counterexample that the so-called weak solution defined by Friedman et al. [11], is not always a fuzzy number-valued vector. A simple and practical method to obtain fuzzy symmetric solutions of fuzzy linear systems is also proposed in [7]. In 2012, Allahviranloo et al. [6, 8] obtained the fuzzy exact solutions and the nearest fuzzy symmetric solutions of fuzzy linear systems by proposing a new metric (for more information see [2, 13, 14, 16] ). Also, in 2012, Allahviranloo and Ghanbari [5] proposed a new approach for solving fuzzy linear systems by introducing a new concept, namely "interval inclusion linear system". They defined an interval inclusion linear system as following

$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \in [b_1], \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \in [b_2], \\ \vdots \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n \in [b_n], \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})_{n\times n}$ is an $n \times n$ real-valued matrix and $[b_i] = [\underline{b_i}, \overline{b_i}]$, $1 \leqslant i \leqslant n$ are interval numbers. Then, they presented a solution set

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for it and investigated the relationship between the solution set of equation (1) and the exact solution of interval linear system

$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n = [b_1], \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n = [b_2], \\ \qquad\qquad \vdots \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n = [b_n], \end{cases} \tag{2}$$

where the coefficient matrix and the right hand side vector are defined as above. It should be noted that, in 2012, the author et al. [12] presented a simple method for obtaining the exact solution of equation (2).

In this paper, we are going to expand our previous works from interval theory to fuzzy theory. For this end, we define a new class of fuzzy problems, namely "fuzzy inclusion linear systems" where the coefficient matrix is real-valued and the right hand side vector is fuzzy-valued. Then, we present a definition of it's fuzzy solution set and give a detailed discussion about the relationship between the fuzzy solution set of a fuzzy inclusion linear system and the fuzzy exact solution (algebraic solution) of a fuzzy linear system. Also, we obtain a new necessary and sufficient condition that guarantees a fuzzy linear system to have the unique algebraic solution.

The outline of the paper is as follows. In Section 2 we present some basic definitions, remarks and lemmas. In Section 3, we define a fuzzy inclusion linear system and it's fuzzy solution set and investigate the relationship between the fuzzy solution set of a fuzzy inclusion linear system and the algebraic solution of a fuzzy linear system. Conclusion is drawn in Section 4.

## 2. Preliminaries

**Definition 2.1.** A fuzzy subset $\widetilde{x}$ of the real line $\mathbb{R}$, with membership function $\mu_{\widetilde{x}}$, is a fuzzy number if

- **(i):** $\widetilde{x}$ is normal, i.e. $\exists t_0 \in \mathbb{R}$ with $\mu_{\widetilde{x}}(t_0) = 1$,
- **(ii):** $\widetilde{x}$ is a convex fuzzy set, i.e., $\mu_{\widetilde{x}}\,(\lambda s + (1-\lambda)t) \geqslant \min\{\mu_{\widetilde{x}}(s), \mu_{\widetilde{x}}(s)\}$, $\forall s, t \in \mathbb{R}$, $\forall \lambda \in [0,1]$,
- **(iii):** $\mu_{\widetilde{x}}$ is upper semi-continuous on $\mathbb{R}$,
- **(iv):** $\overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of $A$.

The set of all these fuzzy numbers is denoted by $E$. Obviously, $\mathbb{R} \subset E$. Here $\mathbb{R} \subset E$ is understood as $\mathbb{R} = \{\chi_{\{t\}} : t \text{ is an usual real number}\}$ [9]. For $0 < r \leqslant 1$, we define $r$-levels of fuzzy number $\widetilde{x}$ as $[\widetilde{x}]_r = \{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) \geqslant r\}$ and $[\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$. Also, we define the support of fuzzy number $\widetilde{x}$ as

$$supp\,(\widetilde{x}) = [\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}.$$

Then, from $(i)$-$(iv)$ it follows that $[\widetilde{x}]_r$ is a bounded closed interval for each $r \in [0,1]$ [17]. In this paper, we denote the $r$-levels of fuzzy number $\widetilde{x}$ as $[\widetilde{x}]_r = [\underline{x}(r), \overline{x}(r)]$, for each $r \in [0,1]$. Sometimes it is important to know whether the given intervals $[\underline{x}(r), \overline{x}(r)]$, $0 \leqslant r \leqslant 1$, are the $r$-levels of some fuzzy number in $E$. The following answer is presented in [15].

**Lemma 2.2.** *Let* $\{[\underline{x}(r), \overline{x}(r)] \ : \ 0 \leqslant r \leqslant 1\}$*, be a given family of non-empty sets in* $\mathbb{R}$*. If*

**(i):** $[\underline{x}(r), \overline{x}(r)]$ *is a bounded closed interval, for each* $r \in [0,1]$*,*
**(ii):** $[\underline{x}(r_1), \overline{x}(r_1)] \supseteq [\underline{x}(r_2), \overline{x}(r_2)]$ *for all* $0 \leqslant r_1 \leqslant r_2 \leqslant 1$*,*
**(iii):** $[\lim_{k\to\infty} \underline{x}(r_k), \lim_{k\to\infty} \overline{x}(r_k)] = [\underline{x}(r), \overline{x}(r)]$ *whenever* $\{r_k\}$ *is a non-decreasing sequence in* $[0,1]$ *converging to* $r$*,*

*then the family* $[\underline{x}(r), \overline{x}(r)]$ *represents the* $r$*-levels of a fuzzy number* $\widetilde{x}$ *in* $E$*.*

*Conversely, if* $[\underline{x}(r), \overline{x}(r)]$*,* $0 \leqslant r \leqslant 1$*, are the* $r$*-levels of a fuzzy number* $\widetilde{x} \in E$*, then the conditions* $(i)$*-*$(iii)$ *are satisfied.*

**Remark 2.3.** From Lemma 2.2 we conclude that if the family

$$\{[\underline{x}(r), \overline{x}(r)] \ : \ 0 \leqslant r \leqslant 1\},$$

are the $r$-levels of a fuzzy number, then:

**1):** The condition $(i)$ implies the functions $\underline{x}$ and $\overline{x}$ are bounded over $[0,1]$ and $\underline{x}(r) \leqslant \overline{x}(r)$ for each $r \in [0,1]$.
**2):** The condition $(ii)$ implies the functions $\underline{x}$ and $\overline{x}$ are non-decreasing and non-increasing over $[0,1]$, respectively.
**3):** The condition $(iii)$ implies that the functions $\underline{x}$ and $\overline{x}$ are left-continuous over $[0,1]$.

For $\widetilde{x}, \widetilde{y} \in E$, and $\lambda \in \mathbb{R}$, $r$-levels of the sum $\widetilde{x} + \widetilde{y}$ and the product $\lambda \cdot \widetilde{x}$ are defined based on interval arithmetic as

$$[\widetilde{x} + \widetilde{y}]_r = [\widetilde{x}]_r + [\widetilde{y}]_r = \{s + t : s \in [\widetilde{x}]_r,\ t \in [\widetilde{y}]_r\} = [\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r)],$$

$$[\lambda \cdot \widetilde{x}]_r = \lambda \cdot [\widetilde{x}]_r = \{\lambda t : t \in [\widetilde{x}]_r\} = \begin{cases} [\lambda \underline{x}(r), \lambda \overline{x}(r)], & \lambda \geqslant 0, \\ [\lambda \overline{x}(r), \lambda \underline{x}(r)], & \lambda < 0. \end{cases}$$

**Definition 2.4.** We say that the fuzzy number (fuzzy set) $\widetilde{x}$ is a subset of the fuzzy number (fuzzy set) $\widetilde{y}$ if and only if $\forall t \in \mathbb{R}, \quad \mu_{\widetilde{x}}(t) \leqslant \mu_{\widetilde{y}}(t)$ or, in other words $[\widetilde{x}]_r \subseteq [\widetilde{y}]_r$, for any $r \in [0,1]$.

**Definition 2.5.** Two fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$ are said to be equal, if and only if $\forall t \in \mathbb{R}, \quad \mu_{\widetilde{x}}(t) = \mu_{\widetilde{y}}(t)$ or in other words $[\widetilde{x}]_r = [\widetilde{y}]_r$, for any $r \in [0,1]$.

**Definition 2.6.** The $n \times n$ linear system

$$\begin{cases} a_{11}\,\widetilde{x}_1 + a_{12}\,\widetilde{x}_2 + \cdots + a_{1n}\,\widetilde{x}_n = \widetilde{b}_1, \\ a_{21}\,\widetilde{x}_1 + a_{22}\,\widetilde{x}_2 + \cdots + a_{2n}\,\widetilde{x}_n = \widetilde{b}_2, \\ \qquad\qquad \vdots \\ a_{n1}\,\widetilde{x}_1 + a_{n2}\,\widetilde{x}_2 + \cdots + a_{nn}\,\widetilde{x}_n = \widetilde{b}_n, \end{cases} \tag{3}$$

where the coefficient matrix $A = (a_{ij})_{n\times n}$ is an $n \times n$ crisp-valued matrix and $\widetilde{b}_i$, $i = 1, 2, \cdots, n$, are fuzzy numbers, is called a fuzzy linear system.

We denote the fuzzy linear system (3) as

$$A\widetilde{X} = \widetilde{B},$$

where $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)^T$ and $\widetilde{B} = (\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_n)^T$ are two fuzzy number-valued vectors.

**Definition 2.7.** A fuzzy number-valued vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)^T$ is called an algebraic solution of the fuzzy linear system (3) if

$$\sum_{j=1}^{n} a_{ij}\, \widetilde{x}_j = \widetilde{b}_i, \qquad \forall i = 1, 2, \dots, n,$$

or in other words

$$\sum_{j=1}^{n} a_{ij}\ [\widetilde{x}_j]_r = \left[\widetilde{b}_i\right]_r, \qquad \forall i = 1, 2, \dots, n, \quad \forall r \in [0, 1].$$

In this paper, we denote the algebraic solution of fuzzy linear system (3) as

$$\widetilde{X}_A = (\widetilde{x}_{1A}, \widetilde{x}_{2A}, \dots, \widetilde{x}_{nA})^T.$$

**Definition 2.8.** Let us suppose that $[x_1]$, $[x_2]$, ... , $[x_n]$ be bounded closed intervals in $\mathbb{R}$. Then, we denote cartesian product of these intervales as following

$$\prod_{i=1}^{n} [x_i] = \left\{ (t_1, t_2, \dots, t_n)^T \in \mathbb{R}^n \ : \ t_i \in [x_i], \quad \forall i = 1, 2, \dots, n \right\}.$$

In the next section, we propose a new class of fuzzy problems and present a detailed discussion about it's fuzzy solution set.

## 3. Fuzzy Inclusion Linear Systems

Here, we are going to propose a new concept, namely "Fuzzy inclusion linear systems".

### 3.1. Fuzzy set $\widetilde{\mathcal{SS}}$.

**Definition 3.1.** The $n \times n$ linear system

$$\begin{cases} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n \in \widetilde{b}_1, \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n \in \widetilde{b}_2, \\ \qquad \vdots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \cdots + a_{nn}\, x_n \in \widetilde{b}_n, \end{cases} \tag{4}$$

where the coefficient matrix $A = (a_{ij})_{n\times n}$ is an $n \times n$ real-valued matrix and $\widetilde{b}_i$, $i = 1, 2, \cdots, n$, are fuzzy numbers, is called a Fuzzy Inclusion Linear System (FILS).

The matrix form of the above equations is

$$A\,X \in \widetilde{B},$$

where $X = (x_1, x_2, \dots, x_n)^T$ is a real-valued vector and $\widetilde{B} = \left(\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_n\right)^T$ is a fuzzy number-valued vector.

We define the fuzzy solution set of FILS (4) as follows.

**Definition 3.2.** The fuzzy set

$$\widetilde{\mathcal{SS}} = \left\{ \left(Y, \mu_{\widetilde{\mathcal{SS}}}(Y)\right) \ : Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n, \sum_{j=1}^{n} a_{ij} y_j \in supp\left(\widetilde{b}_i\right), \forall i = 1, \ldots, n \right\}, \tag{5}$$

with the membership function

$$\mu_{\widetilde{\mathcal{SS}}}(Y) = \min_{i=1,2,\ldots,n} \left\{ \mu_{\widetilde{b}_i} \left( \sum_{j=1}^{n} a_{ij} y_j \right) \right\}, \tag{6}$$

is called as fuzzy solution set of the FILS (4).

We define $r$-levels of the fuzzy solution set $\widetilde{\mathcal{SS}}$, for any $r \in (0,1]$, as follows

$$\left[\widetilde{\mathcal{SS}}\right]_r = \left\{ Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{SS}}}(Y) \geqslant r \right\},$$

and for $r = 0$

$$\left[\widetilde{\mathcal{SS}}\right]_0 = supp\left(\widetilde{\mathcal{SS}}\right) = \overline{\left\{ Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{SS}}}(Y) > 0 \right\}}.$$

Then, we conclude that

$$\left[\widetilde{\mathcal{SS}}\right]_r = \left\{ Y \in \mathbb{R}^n \ : \ \sum_{j=1}^{n} a_{ij} y_j \in \left[\widetilde{b}_i\right]_r \quad \forall i = 1, 2, \ldots, n \right\}, \qquad \forall r \in (0,1], \tag{7}$$

and also

$$supp\left(\widetilde{\mathcal{SS}}\right) = \overline{\left\{ Y \in \mathbb{R}^n \ : \ \sum_{j=1}^{n} a_{ij} y_j \in supp\left(\widetilde{b}_i\right) \quad \forall i = 1, 2, \ldots, n \right\}}. \tag{8}$$

In the following example, we present an $2 \times 2$ FILS to illustrate the above fuzzy solution set $\widetilde{\mathcal{SS}}$.

**Example 3.3.** Consider the $2 \times 2$ FILS

$$\begin{cases} 3x_1 - x_2 \in \widetilde{b}_1, \\ x_1 + 2x_2 \in \widetilde{b}_2, \end{cases} \tag{9}$$

where the membership functions of fuzzy numbers $\widetilde{b}_1$ and $\widetilde{b}_2$ are defined as follows:

$$\mu_{\widetilde{b}_1}(t) = \begin{cases} \frac{t+11}{9}, & if \ t \in [-11,-2], \\ \frac{3-t}{5}, & if \ t \in [-2,3], \\ 0, & otherwise, \end{cases} \qquad \mu_{\widetilde{b}_2}(t) = \begin{cases} \frac{t-5}{6}, & if \ t \in [5,11], \\ \frac{18-t}{7}, & if \ t \in [11,18], \\ 0, & otherwise. \end{cases} \tag{10}$$

According to equations (7) and (8) we obtain the $r$-levels and the support of $\widetilde{\mathcal{SS}}$ as follows

$$\left[\widetilde{\mathcal{SS}}\right]_r = \left\{ (y_1, y_2)^T \in \mathbb{R}^2 \ : \ \begin{array}{c} -11 + 9r \leqslant 3y_1 - y_2 \leqslant 3 - 5r \\ 5 + 6r \leqslant y_1 + 2y_2 \leqslant 18 - 7r \end{array} \right\}, \qquad \forall r \in (0,1],$$

and

$$supp\left(\widetilde{\mathcal{SS}}\right) = \overline{\left\{ (y_1, y_2)^T \in \mathbb{R}^2 \ : \ \begin{array}{c} -11 \leqslant 3y_1 - y_2 \leqslant 3 \\ 5 \leqslant y_1 + 2y_2 \leqslant 18 \end{array} \right\}}.$$

A graphical representation of the fuzzy solution set $\widetilde{\mathcal{SS}}$ is presented in Figure 1.

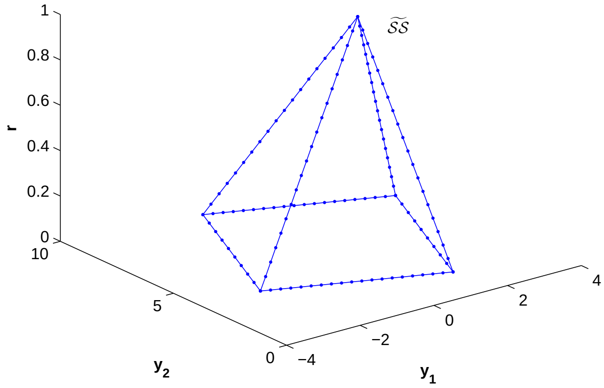


FIGURE 1. Graphical Representation of the Fuzzy Solution Set $\widetilde{\mathcal{SS}}$ of Example 3.3

## 3.2. Fuzzy set $\widetilde{\mathcal{X}}^*$.

Now, suppose that the matrix $A$ is nonsingular and $D = A^{-1} = (d_{ij})_{n\times n}$. Then, for any $i = 1, 2, \ldots, n$, we define the following fuzzy set

$$\widetilde{x}_i^* = \left\{ \left(y, \mu_{\widetilde{x}_i^*}(y)\right) \;:\; y \in \mathbb{R},\; \exists (b_1^*, b_2^*, \ldots, b_n^*)^T \in \prod_{i=1}^{n} supp\left(\widetilde{b}_i\right),\; \sum_{j=1}^{n} d_{ij} b_j^* = y \right\}, \tag{11}$$

where the membership function $\mu_{\widetilde{x}_i^*}$ is defined as follows

$$\mu_{\widetilde{x}_i^*}(y) = \sup_{b_j^*} \left\{ \min_{j=1,2,\ldots,n} \left\{ \mu_{\widetilde{b}_j}(b_j^*) \right\} \;:\; \sum_{j=1}^{n} d_{ij} b_j^* = y \right\}. \tag{12}$$

In a similar manner, for any $i = 1, 2, \ldots, n$ and for any $r \in (0, 1]$, we define $r$-levels of the fuzzy set $\widetilde{x}_i^*$ as follows

$$[\widetilde{x}_i^*]_r = \left\{ y \in \mathbb{R} \;:\; \mu_{\widetilde{x}_i^*}(y) \geqslant r \right\},$$

and for $r = 0$

$$[\widetilde{x}_i^*]_0 = supp\,(\widetilde{x}_i^*) = \overline{\left\{ y \in \mathbb{R} \;:\; \mu_{\widetilde{x}_i^*}(y) > 0 \right\}}.$$

By the fuzzy sets $\widetilde{x}_i^*$, $i = 1, 2, \ldots, n$, we can define another fuzzy set as follows:

$$\widetilde{\mathcal{X}}^* = \left\{ \left(Y, \mu_{\widetilde{\mathcal{X}}^*}(Y)\right) \;:\; Y = (y_1, y_2, \ldots, y_n)^T \in \prod_{i=1}^{n} supp\,(\widetilde{x}_i^*) \right\}, \tag{13}$$

with the membership function

$$\mu_{\widetilde{\mathcal{X}}^*}(Y) = \min_{i=1,2,\ldots,n} \left\{ \mu_{\widetilde{x}_i^*}(y_i) \right\}. \tag{14}$$

We define

$$\left[\widetilde{\mathcal{X}}^*\right]_r = \left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}^*}(Y) \geqslant r\right\}, \qquad r \in (0,1], \tag{15}$$

and also

$$\left[\widetilde{\mathcal{X}}^*\right]_0 = supp\left(\widetilde{\mathcal{X}}^*\right) = \overline{\left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}^*}(Y) > 0\right\}}. \tag{16}$$

**Theorem 3.4.** *For the fuzzy set $\widetilde{\mathcal{X}}^*$, we have*

$$\left[\widetilde{\mathcal{X}}^*\right]_r = \prod_{i=1}^{n} [\widetilde{x}_i^*]_r, \qquad r \in [0,1],$$

*specially*

$$supp\left(\widetilde{\mathcal{X}}^*\right) = \prod_{i=1}^{n} supp\left(\widetilde{x}_i^*\right).$$

*Proof.* From equations (12)-(16), for any $r \in (0,1]$, we have

$$\begin{aligned}
\left[\widetilde{\mathcal{X}}^*\right]_r &= \left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}^*}(Y) \geqslant r\right\} \\
&= \left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{x}_i^*}(y_i) \geqslant r \ \forall i = 1,2,\ldots,n\right\} \\
&= \{Y \in \mathbb{R}^n \ : \ y_i \in [\widetilde{x}_i^*]_r \ \forall i = 1,2,\ldots,n\} \\
&= \prod_{i=1}^{n} [\widetilde{x}_i^*]_r.
\end{aligned}$$

Also, for $r = 0$ we have

$$\begin{aligned}
\left[\widetilde{\mathcal{X}}^*\right]_0 = supp\left(\widetilde{\mathcal{X}}^*\right) &= \overline{\left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}^*}(Y) > 0\right\}} \\
&= \overline{\left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{x}_i^*}(y_i) > 0 \ \forall i = 1,2,\ldots,n\right\}} \\
&= \left\{Y \in \mathbb{R}^n \ : \ y_i \in supp\left(\widetilde{x}_i^*\right) \ \forall i = 1,2,\ldots,n\right\} \\
&= \prod_{i=1}^{n} supp\left(\widetilde{x}_i^*\right) = \prod_{i=1}^{n} [\widetilde{x}_i^*]_0.
\end{aligned}$$

□

**Theorem 3.5.** *In the fuzzy linear system (3), if we set $D = A^{-1} = (d_{ij})_{n\times n}$, then the $r$-levels of the fuzzy set $\widetilde{\mathcal{X}}^*$ are indicated as following:*

$$\left[\widetilde{\mathcal{X}}^*\right]_r = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} d_{ij}\left[\widetilde{b}_j\right]_r\right).$$

*In other words,*

$$\left[\widetilde{\mathcal{X}}^*\right]_r = \left\{(y_1,\ldots,y_n) \in \mathbb{R}^n \ : \ \forall i = 1,\ldots,n, \exists (b_{1_i}^*,\ldots,b_{n_i}^*) \in \prod_{j=1}^{n}\left[\widetilde{b}_j\right]_r; y_i = \sum_{j=1}^{n} d_{ij} b_{j_i}^*\right\}.$$

*Proof.* Suppose that $r \in [0,1]$ is fixed, then based on equations (12) and (14), we have

$$\begin{aligned}
Z = (z_1, z_2, \ldots, z_n) \in \left[\widetilde{\mathcal{X}}^*\right]_r \quad & \Leftrightarrow \quad \mu_{\widetilde{\mathcal{X}}^*}(Z) \geqslant r \\
& \Leftrightarrow \quad \min_{i=1,2,\ldots,n} \left\{\mu_{\widetilde{x}_i^*}(z_i)\right\} \geqslant r \\
& \Leftrightarrow \quad \forall i = 1, 2, \ldots, n, \quad \mu_{\widetilde{x}_i^*}(z_i) \geqslant r \\
& \Leftrightarrow \quad \forall i = 1, 2, \ldots, n, \exists (b^*_{1_i}, \ldots, b^*_{n_i}) \in \prod_{j=1}^{n} supp(\widetilde{b}_j); \\
& \quad\quad \min_{j=1,2,\ldots,n} \left\{\mu_{\widetilde{b}_j}(b^*_{j_i})\right\} \geqslant r, \quad z_i = \textstyle\sum_{j=1}^{n} d_{ij} b^*_{j_i} \\
& \Leftrightarrow \quad \forall i = 1, 2, \ldots, n, \exists (b^*_{1_i}, \ldots, b^*_{n_i}) \in \prod_{j=1}^{n} supp(\widetilde{b}_j); \\
& \quad\quad \forall j = 1, 2, \ldots, n, \ \mu_{\widetilde{b}_j}(b^*_{j_i}) \geqslant r, \quad z_i = \sum_{j=1}^{n} d_{ij} b^*_{j_i} \\
& \Leftrightarrow \quad \forall i = 1, 2, \ldots, n, \exists (b^*_{1_i}, \ldots, b^*_{n_i}) \in \prod_{j=1}^{n} [\widetilde{b}_j]_r; \\
& \quad\quad z_i = \sum_{j=1}^{n} d_{ij} b^*_{j_i} \\
& \Leftrightarrow \quad Z = (z_1, z_2, \ldots, z_n) \in \prod_{i=1}^{n} \left( \sum_{j=1}^{n} d_{ij} \left[\widetilde{b}_j\right]_r \right).
\end{aligned}$$

Therefore, the proof of theorem is completed. □

The following corollary is a straightforward result of Theorems 3.4 and 3.5.

**Corollary 3.6.** *For the fuzzy set $\widetilde{\mathcal{X}}^*$ we have*

1. $supp(\widetilde{\mathcal{X}}^*) = \prod_{i=1}^{n} \left( \sum_{j=1}^{n} d_{ij} \ supp(\widetilde{b}_j) \right)$.
2. *For each $r \in [0,1]$ and for any $i = 1, 2, \ldots, n$, $[\widetilde{x}_i^*]_r = \sum_{j=1}^{n} d_{ij} \left[\widetilde{b}_j\right]_r$ and consequently $supp(\widetilde{x}_i^*) = \sum_{j=1}^{n} d_{ij}\, supp(\widetilde{b}_j)$.*

In the following example, we are going to illustrate the fuzzy set $\widetilde{\mathcal{X}}^*$ for a $2 \times 2$ fuzzy linear system.

**Example 3.7.** Consider the $2 \times 2$ fuzzy linear system

$$\begin{cases} 3x_1 - x_2 = \widetilde{b}_1, \\ x_1 + 2x_2 = \widetilde{b}_2, \end{cases} \tag{17}$$

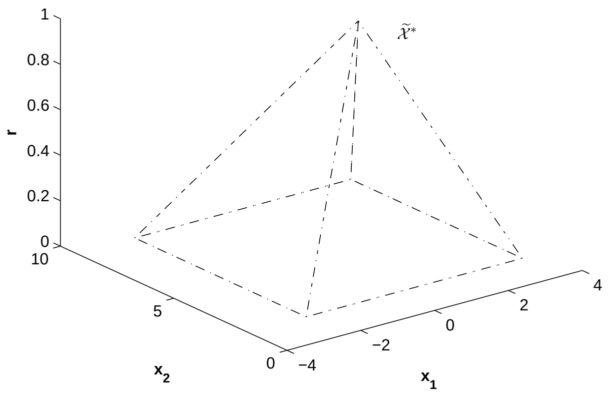


FIGURE 2. Graphical Representation of the Fuzzy Set $\widetilde{\mathcal{X}}^*$ of Example 3.7

where the membership functions of the fuzzy numbers $\widetilde{b}_1$ and $\widetilde{b}_2$ are defined by equation (10). According to Theorem 3.5, we conclude that

$$\left[\widetilde{\mathcal{X}}^*\right]_r = \left[\frac{24r-17}{7}, \frac{24-17r}{7}\right] \times \left[\frac{12+23r}{7}, \frac{65-30r}{7}\right], \qquad \forall r \in [0,1],$$

where $\times$ is denoted as the cartesian product of closed intervals (see Definition 2.8). The fuzzy set $\widetilde{\mathcal{X}}^*$ is showed in Figure 2.

## 3.3. **Fuzzy set $\widetilde{\mathcal{X}}_{\mathcal{A}}$.**

Now, if the fuzzy number-valued vector $\widetilde{X}_A = (\widetilde{x}_{1A}, \widetilde{x}_{2A}, \ldots, \widetilde{x}_{nA})^T$ is the algebraic solution of fuzzy linear system (3), then we can define an important fuzzy set as follows:

$$\widetilde{\mathcal{X}}_{\mathcal{A}} = \left\{ \left(Y, \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y)\right) \ : \ Y = (y_1, y_2, \ldots, y_n)^T \in \prod_{i=1}^{n} supp\,(\widetilde{x}_{iA}) \right\}, \tag{18}$$

with the membership function

$$\mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) = \min_{i=1,2,\ldots,n} \left\{\mu_{\widetilde{x}_{iA}}(y_i)\right\}. \tag{19}$$

As before, we define

$$\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r = \left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) \geqslant r\right\}, \qquad r \in (0,1], \tag{20}$$

and also

$$\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_0 = supp\left(\widetilde{\mathcal{X}}_{\mathcal{A}}\right) = \overline{\left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) > 0\right\}}. \tag{21}$$

**Theorem 3.8.** *For the fuzzy set $\widetilde{\mathcal{X}}_{\mathcal{A}}$, we have*

$$\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r = \prod_{i=1}^{n} [\widetilde{x}_{iA}]_r\,, \qquad r \in [0,1],$$

*specially*

$$supp\left(\widetilde{\mathcal{X}}_{\mathcal{A}}\right) = \prod_{i=1}^{n} supp\left(\widetilde{x}_{iA}\right).$$

*Proof.* From equations (18)-(21), for any $r \in (0,1]$, we have

$$\begin{aligned}\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r = \left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) \geqslant r\right\} &= \{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{x}_{iA}}(y_i) \geqslant r \ \forall i = 1,2,\dots,n\}\\ &= \{Y \in \mathbb{R}^n \ : \ y_i \in [\widetilde{x}_{iA}]_r \ \forall i = 1,2,\dots,n\}\\ &= \prod_{i=1}^{n} [\widetilde{x}_{iA}]_r.\end{aligned}$$

Also, for $r = 0$ we have

$$\begin{aligned}\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_0 = supp\left(\widetilde{\mathcal{X}}_{\mathcal{A}}\right) &= \overline{\left\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) > 0\right\}}\\ &= \overline{\{Y \in \mathbb{R}^n \ : \ \mu_{\widetilde{x}_{iA}}(y_i) > 0 \ \forall i = 1,2,\dots,n\}}\\ &= \{Y \in \mathbb{R}^n \ : \ y_i \in supp\left(\widetilde{x}_{iA}\right) \ \forall i = 1,2,\dots,n\}\\ &= \prod_{i=1}^{n} supp\left(\widetilde{x}_{iA}\right) = \prod_{i=1}^{n} [\widetilde{x}_{iA}]_0.\end{aligned}$$

□

In the following example, we display the fuzzy set $\widetilde{\mathcal{X}}_{\mathcal{A}}$ for an $2 \times 2$ fuzzy linear system.

**Example 3.9.** Let us consider the fuzzy linear system presented in Example 3.7. We can easily obtain the membership functions of the algebraic solution as following

$$\mu_{\widetilde{x}_{1A}}(t) = \begin{cases} \frac{t+1}{2}, & if\ t \in [-1,1],\\ 2-t, & if\ t \in [1,2],\\ 0, & otherwise,\end{cases} \qquad \mu_{\widetilde{x}_{2A}}(t) = \begin{cases} \frac{t-3}{2}, & if\ t \in [3,5],\\ \frac{8-t}{3}, & if\ t \in [5,8],\\ 0, & otherwise,\end{cases}$$

and consequently by Theorem 3.8 we obtain

$$\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r = [-1+2r, 2-r] \times [3+2r, 8-3r]\,, \qquad \forall r \in [0,1],$$

where $\times$ is denoted as the cartesian product of the closed intervals. In Figure 3, the graphical representation of the fuzzy set $\widetilde{\mathcal{X}}_{\mathcal{A}}$ is given.

**Theorem 3.10.** *If the matrix $A$ is nonsingular, then for the fuzzy sets $\widetilde{\mathcal{X}}_{\mathcal{A}}$, $\widetilde{\mathcal{SS}}$ and $\widetilde{\mathcal{X}}^*$, we have*

$$\widetilde{\mathcal{X}}_{\mathcal{A}} \subseteq \widetilde{\mathcal{SS}} \subseteq \widetilde{\mathcal{X}}^*.$$

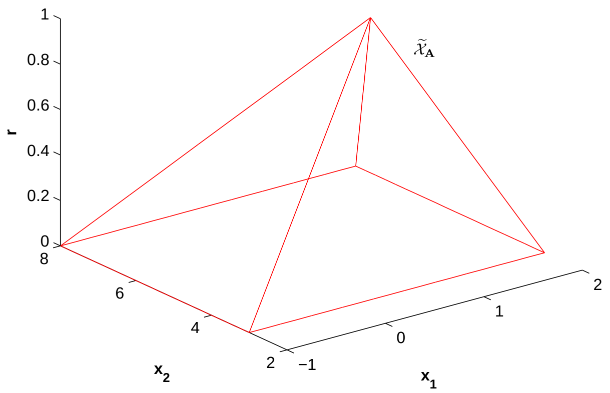


FIGURE 3. Graphical Representation of the Fuzzy Set $\widetilde{\mathcal{X}}_{\mathcal{A}}$ of Example 3.9

*Proof.* According to the Definition 2.4, it is sufficient to show

$$\forall Y \in \mathbb{R}^n \Rightarrow \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) \leqslant \mu_{\widetilde{\mathcal{SS}}}(Y) \leqslant \mu_{\widetilde{\mathcal{X}}^*}(Y).$$

We first prove that

$$\forall Y \in \mathbb{R}^n \Rightarrow \mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) \leqslant \mu_{\widetilde{\mathcal{SS}}}(Y).$$

Suppose that $Y = (y_1, y_2, \ldots, y_n)^T \in \prod_{i=1}^n supp\,(\widetilde{x}_{iA})$. Otherwise the equations (18) and (19) imply $\mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) = 0$ and thus the proof is obvious. Also, suppose that

$$\mu_{\widetilde{x}_{iA}}(y_i) = r_i, \qquad \forall i = 1, 2, \ldots, n, \tag{22}$$

and

$$r^* = \min_{i=1,2,\ldots,n} \{r_i\}. \tag{23}$$

Then, we will have

$$\mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) = \min_{i=1,2,\ldots,n} \{\mu_{\widetilde{x}_{iA}}(y_i)\} = \min_{i=1,2,\ldots,n} \{r_i\} = r^*. \tag{24}$$

From equations (22) and (23) we have

$$\mu_{\widetilde{x}_{iA}}(y_i) \geqslant r^* \quad \Rightarrow \quad y_i \in [\widetilde{x}_{iA}]_{r^*}, \qquad \forall i = 1, 2, \ldots, n. \tag{25}$$

On the other hand, since $\widetilde{X}_A = (\widetilde{x}_{1A}, \widetilde{x}_{2A}, \ldots, \widetilde{x}_{nA})^T$ is an algebraic solution of fuzzy linear system (3), then $A\,\widetilde{X}_A = \widetilde{B}$ and consequently

$$\sum_{j=1}^{n} a_{ij}[\widetilde{x}_{jA}]_{r^*} = \left[\widetilde{b}_i\right]_{r^*}.$$

By equations (25) and (26), for any $i = 1, 2, \ldots, n$, we conclude that

$$\sum_{j=1}^{n} a_{ij} y_j \in \left[\widetilde{b}_i\right]_{r^*} \quad \Rightarrow \quad \mu_{\widetilde{b}_i}\left(\sum_{j=1}^{n} a_{ij} y_j\right) \geqslant r^*. \tag{26}$$

From other point of view, it is obvious that $\sum_{j=1}^{n} a_{ij} y_j \in supp\left(\widetilde{b}_i\right)$ and from equation (6) we have

$$\mu_{\widetilde{\mathcal{SS}}}(Y) = \min_{i=1,2,\ldots,n}\left\{\mu_{\widetilde{b}_i}\left(\sum_{j=1}^{n} a_{ij} y_j\right)\right\} \geqslant r^*. \tag{27}$$

From equations (24) and (27) we obtain

$$\mu_{\widetilde{\mathcal{X}}_{\mathcal{A}}}(Y) \leqslant \mu_{\widetilde{\mathcal{SS}}}(Y),$$

that implies

$$\widetilde{\mathcal{X}}_{\mathcal{A}} \subseteq \widetilde{\mathcal{SS}}. \tag{28}$$

Therefore, the proof of the first part is completed. Now, we present the proof of the second part, i.e.,

$$\forall Y \in \mathbb{R}^n \Rightarrow \mu_{\widetilde{\mathcal{SS}}}(Y) \leqslant \mu_{\widetilde{\mathcal{X}}^*}(Y).$$

Suppose that $Y \in \mathbb{R}^n$ and also for any $i = 1, 2, \ldots, n$

$$\sum_{j=1}^{n} a_{ij} y_j \in supp\left(\widetilde{b}_i\right). \tag{29}$$

Otherwise the equations (5) and (6) imply $\mu_{\widetilde{\mathcal{SS}}}(Y) = 0$ and thus the proof is obvious. From equation (29) we conclude that

$$\exists b_i^* \in ssup\left(\widetilde{b}_i\right); \qquad \sum_{j=1}^{n} a_{ij} y_j = b_i^*, \qquad i = 1, 2, \ldots, n.$$

Therefore

$$\mu_{\widetilde{\mathcal{SS}}}(Y) = \min_{i=1,2,\ldots,n}\left\{\mu_{\widetilde{b}_i}\left(\sum_{j=1}^{n} a_{ij} y_j\right)\right\} = \min_{i=1,2,\ldots,n}\left\{\mu_{\widetilde{b}_i}\left(b_i^*\right)\right\}. \tag{30}$$

On the other hand, if $D = A^{-1} = (d_{ij})_{n \times n}$, then

$$y_i = \sum_{j=1}^{n} d_{ij} b_j^*, \qquad i = 1, 2, \ldots, n.$$

Hence, from equation (11) we conclude that $y_i \in supp\left(\widetilde{x}_i^*\right)$ and consequently, according to Theorem 3.4, $Y \in supp\left(\widetilde{\mathcal{X}}^*\right)$ and thus

$$\begin{aligned} \mu_{\widetilde{\mathcal{X}}^*}(Y) &= \min_{i=1,2,\ldots,n}\left\{\mu_{\widetilde{x}_i^*}(y_i)\right\} \\ &= \min_{i=1,2,\ldots,n}\left\{\sup_{b'_j}\left\{\min_{j=1,2,\ldots,n}\left\{\mu_{\widetilde{b}_j}(b'_j)\right\} : \sum_{j=1}^{n} d_{ij} b'_j = y\right\}\right\}. \end{aligned} \tag{31}$$

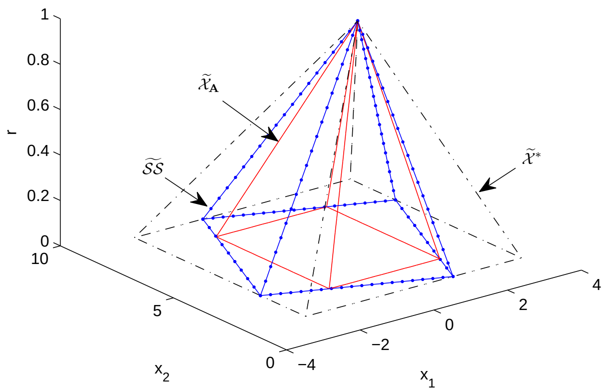


FIGURE 4. Graphical Representation of the Fuzzy Sets $\widetilde{\mathcal{SS}}$, $\widetilde{\mathcal{X}}^*$ and $\widetilde{\mathcal{X}}_{\mathcal{A}}$ for Example 3.11

On the other hand, it is obvious that for any $i = 1, 2, \ldots, n$

$$\sup_{b'_j} \left\{ \min_j \left\{ \mu_{\widetilde{b}_j}(b'_j) \right\} \ : \ \sum_{j=1}^{n} d_{ij} b'_j = y \right\} \geqslant \min_j \left\{ \mu_{\widetilde{b}_j}(b^*_j) \ : \ \sum_{j=1}^{n} d_{ij} b^*_j = y \right\}. \qquad (32)$$

From equations (32)-(30), we conclude that

$$\mu_{\widetilde{\mathcal{SS}}}(Y) \leqslant \mu_{\widetilde{\mathcal{X}}^*}(Y),$$

that implies

$$\widetilde{\mathcal{SS}} \subseteq \widetilde{\mathcal{X}}^*. \qquad (33)$$

Regarding to equations (28) and (33), the proof of theorem is completed. □

In the following, we present an example in order to Theorem 3.10.

**Example 3.11.** Consider the FILS (9) presented in Example 3.3 and the fuzzy linear system (17) presented in Example 3.7. In Figure 4, we present simultaneously the graphical representation of the fuzzy solution set $\widetilde{\mathcal{SS}}$ of FILS (9) and the fuzzy sets $\widetilde{\mathcal{X}}^*$ and $\widetilde{\mathcal{X}}_{\mathcal{A}}$ of fuzzy linear system (17). Obviously we have

$$\widetilde{\mathcal{X}}_{\mathcal{A}} \subseteq \widetilde{\mathcal{SS}} \subseteq \widetilde{\mathcal{X}}^*.$$

**Definition 3.12.** [5] Let $\Gamma$ be an $n$-dimensional rectangle, that is $\Gamma = \prod_{i=1}^{n}[\underline{\gamma_i}, \overline{\gamma_i}]$, where $[\underline{\gamma_i}, \overline{\gamma_i}]$, $1 \leqslant i \leqslant n$ are the closed intervals in $\mathbb{R}$. The point $(x_1, x_2, \ldots, x_n)^T$ is called a vertex point of $n$-dimensional rectangle $\Gamma$ if

$$x_i \in \{\underline{\gamma_i}, \overline{\gamma_i}\}, \qquad \forall i = 1, 2, \ldots, n.$$

It is clear that $\Gamma$ has $2^n$ vertex points.

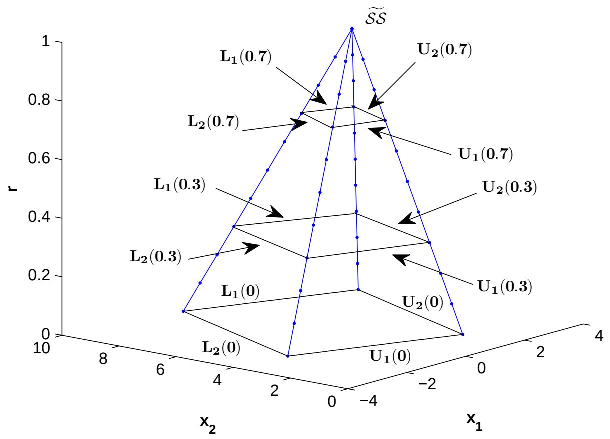


FIGURE 5. Graphical Representation of Some Lower and Upper $r$-boundaries of $\widetilde{\mathcal{SS}}$

**Definition 3.13.** For the fuzzy solution set $\widetilde{\mathcal{SS}}$ and for any $i = 1, 2, \ldots, n$, we define the set

$$L_i(r) = \left\{ (x_1, x_2, \ldots, x_n)^T \in \left[\widetilde{\mathcal{SS}}\right]_r \ : \ \sum_{j=1}^{n} a_{ij} x_j = \underline{b_i}(r) \right\}, \qquad \forall r \in [0,1],$$

as lower $r$-boundary and the set

$$U_i(r) = \left\{ (x_1, x_2, \ldots, x_n)^T \in \left[\widetilde{\mathcal{SS}}\right]_r \ : \ \sum_{j=1}^{n} a_{ij} x_j = \overline{b_i}(r) \right\}, \qquad \forall r \in [0,1],$$

as upper $r$-boundary of $\widetilde{\mathcal{SS}}$. It is clear that for each $r \in [0,1]$, the fuzzy solution set $\widetilde{\mathcal{SS}}$ has $n$ lower $r$-boundaries and $n$ upper $r$-boundaries.

For an illustration of the above Definition 3.13, consider the fuzzy solution set $\widetilde{\mathcal{SS}}$ of Example 3.3. In Figure 5, we indicated some lower $r$-boundaries and upper $r$-boundaries of $\widetilde{\mathcal{SS}}$ for $r = 0, 0.3$ and $0.7$.

In the following theorem, we are going to answer a question: What subset of $\widetilde{\mathcal{SS}}$ is an algebraic solution of a fuzzy linear system (3).

**Theorem 3.14.** *Suppose that the fuzzy set* $\widetilde{\Gamma}$ *be an arbitrary subset of fuzzy set* $\widetilde{\mathcal{SS}}$, *i.e.,* $\widetilde{\Gamma} \subseteq \widetilde{\mathcal{SS}}$, *such that*

(i) *For any* $r \in [0,1]$, $\left[\widetilde{\Gamma}\right]_r$ *be an $n$-dimensional rectangle in* $\mathbb{R}^n$, *i.e.,* $\left[\widetilde{\Gamma}\right]_r = \prod_{i=1}^{n} [\widetilde{\gamma}_i]_r$, *such that for each* $i = 1, 2, \ldots, n$, $\widetilde{\gamma}_i$ *is a fuzzy number.*

(ii) *For any* $r \in [0,1]$, *each lower* $r$*-boundary and each upper* $r$*-boundary of* $\widetilde{\mathcal{SS}}$ *at least contain one point of* $\left[\widetilde{\Gamma}\right]_r$, *i.e.,* $\forall r \in [0,1]$, *there exists two points* $\alpha(r) = (\alpha_1^r, \alpha_2^r, \ldots, \alpha_n^r)^T$, $\beta(r) = (\beta_1^r, \beta_2^r, \ldots, \beta_n^r)^T \in \left[\widetilde{\Gamma}\right]_r$, *such that* $\alpha(r) \in L_i(r)$ *and* $\beta(r) \in U_i(r)$.

*Then, the fuzzy number-valued vector* $\widetilde{X} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_n)^T$ *is the algebraic solution of fuzzy linear system (3).*

*Proof.* We must show

$$A\,\widetilde{X} = \widetilde{B}.$$

To this end, it is sufficient to show

$$\sum_{j=1}^{n} a_{ij}[\widetilde{\gamma}_j]_r = \left[\widetilde{b}_i\right]_r, \qquad \forall i = 1, 2, \ldots, n. \tag{34}$$

In order to the proof of equation (34), it is obvious that

$$\sum_{j=1}^{n} a_{ij}\,[\widetilde{\gamma}_j]_r = \left\{ \sum_{j=1}^{n} a_{ij} y_j \;:\; Y = (y_1, y_2, \ldots, y_n)^T \in \left[\widetilde{\Gamma}\right]_r \right\}, \quad \forall r \in [0,1].$$

Suppose that $r \in [0,1]$ and $Z \in \sum_{j=1}^{n} a_{ij}[\widetilde{\gamma}_j]_r$, then there exists $(z_1, z_2, \ldots, z_n)^T \in \left[\widetilde{\Gamma}\right]_r$ such that

$$Z = \sum_{j=1}^{n} a_{ij} z_j. \tag{35}$$

On the other hand, since $\widetilde{\Gamma} \subseteq \widetilde{\mathcal{SS}}$ then

$$\left[\widetilde{\Gamma}\right]_r \subseteq \left[\widetilde{\mathcal{SS}}\right]_r, \qquad r \in [0,1].$$

Hence

$$(z_1, z_2, \ldots, z_n)^T \in \left[\widetilde{\mathcal{SS}}\right]_r,$$

and from equation (7) we conclude that

$$\sum_{j=1}^{n} a_{ij} z_j \in \left[\widetilde{b}_i\right]_r. \tag{36}$$

equations (35) and (36) imply $Z \in \left[\widetilde{b}_i\right]_r$ and therefore

$$\sum_{j=1}^{n} a_{ij}[\widetilde{\gamma}_j]_r \subseteq \left[\widetilde{b}_i\right]_r, \qquad \forall i = 1, 2, \ldots, n, \qquad \forall r \in [0,1]. \tag{37}$$

Now, suppose that $r \in [0,1]$ and $Z \in \left[\widetilde{b}_i\right]_r$, then

$$\exists \lambda \in [0,1]; \qquad Z = \underline{b}_i(r) + \lambda \left(\overline{b}_i(r) - \underline{b}_i(r)\right). \tag{38}$$

On the other hand, regarding to the assumption of theorem, since any lower $r$-boundary and upper $r$-boundary of $\widetilde{\mathcal{SS}}$ at least contain one point of $\left[\widetilde{\Gamma}\right]_r$, then for any $i = 1, 2, \ldots, n$ we have

$$\exists(\alpha_1^r, \alpha_2^r, \ldots, \alpha_n^r)^T \in \left[\widetilde{\Gamma}\right]_r; \quad \sum_{j=1}^{n} a_{ij}\alpha_j^r \in L_i(r) \;\Rightarrow\; \sum_{j=1}^{n} a_{ij}\alpha_j^r = \underline{b}_i(r), \tag{39}$$

$$\exists(\beta_1^r, \beta_2^r, \ldots, \beta_n^r)^T \in \left[\widetilde{\Gamma}\right]_r; \quad \sum_{j=1}^{n} a_{ij}\beta_j^r \in U_i(r) \;\Rightarrow\; \sum_{j=1}^{n} a_{ij}\beta_j^r = \overline{b}_i(r). \tag{40}$$

From equations (38)-(40) we conclude that

$$Z = \sum_{j=1}^{n} a_{ij}\left(\alpha_j^r + \lambda\left(\beta_j^r - \alpha_j^r\right)\right).$$

Also, since $\alpha_j^r,\ \beta_j^r \in [\widetilde{\gamma}_j]_r,\ j = 1, 2, \ldots, n$, then $\left(\alpha_j^r + \lambda\left(\beta_j^r - \alpha_j^r\right)\right) \in [\widetilde{\gamma}_j]_r$ and therefore

$$Z \in \left\{\sum_{j=1}^{n} a_{ij}y_j \;:\; Y = (y_1, y_2, \ldots, y_n)^T \in \left[\widetilde{\Gamma}\right]_r\right\}. \tag{41}$$

From equation (41) we obtain

$$Z \in \sum_{j=1}^{n} a_{ij}[\widetilde{\gamma}_j]_r \;\Rightarrow\; \sum_{j=1}^{n} a_{ij}[\widetilde{\gamma}_j]_r \subseteq \left[\widetilde{b}_i\right]_r, \qquad \forall i = 1, 2, \ldots, n, \quad \forall r \in [0, 1]. \tag{42}$$

equations (37) and (42) complete the proof of theorem. □

In the following theorem, we present a necessary and sufficient condition for a fuzzy number-valued vector to be an algebraic solution of a fuzzy linear system.

**Theorem 3.15.** *Suppose that $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ be an arbitrary fuzzy number-valued vector and the fuzzy set $\widetilde{\Gamma}$ is constructed as follows:*

$$\widetilde{\Gamma} = \left\{\left(Y, \mu_{\widetilde{\Gamma}}(Y)\right) \;:\; Y = (y_1, y_2, \ldots, y_n)^T \in \prod_{i=1}^{n} suup\left(\widetilde{x}_i\right)\right\},$$

*with the membership function*

$$\mu_{\widetilde{\Gamma}}(Y) = \min_{i=1,2,\ldots,n}\left\{\mu_{\widetilde{x}_i}(y_i)\right\}.$$

*Then:*

(1) *For any $r \in [0, 1]$, we have $\left[\widetilde{\Gamma}\right]_r = \prod_{i=1}^{n}[\widetilde{x}_i]_r$.*

(2) *The fuzzy number-valued vector $\widetilde{X}$ is the algebraic solution of fuzzy linear system (3) if and only if*
   - (i) $\widetilde{\Gamma} \subseteq \widetilde{\mathcal{SS}}$.
   - (ii) *For any $r \in [0, 1]$, each lower $r$-boundary and each upper $r$-boundary of $\widetilde{\mathcal{SS}}$ at least contain one vertex point of $\left[\widetilde{\Gamma}\right]_r$.*

*Proof.* The proof of the first part is exactly the same as that of Theorem 3.8. Thus we only prove the second part of theorem. Firstly, we suppose that the fuzzy number-valued vector $\widetilde{X}$ be an algebraic solution of fuzzy linear system (3). Then from equations (18) and (19) we find out $\widetilde{\Gamma} = \widetilde{\mathcal{X}}_{\mathcal{A}}$ and according to Theorem 3.10

$$\widetilde{\Gamma} \subseteq \widetilde{\mathcal{SS}}.$$

Then property $(i)$ holds. On the other hand, since $\widetilde{X}$ is an algebraic solution, then

$$\sum_{j=1}^{n} a_{ij}[\widetilde{x}_j]_r = \left[\widetilde{b}_i\right]_r, \qquad \forall i = 1, 2, \ldots, n, \qquad \forall r \in [0, 1].$$

By the first part of theorem we obtain

$$\left[\widetilde{b}_i\right]_r = [\underline{b}_i(r), \overline{b}_i(r)] = \left\{ \sum_{j=1}^{n} a_{ij} z_j \ : \ (y_1, y_2, \ldots, y_n)^T \in \left[\widetilde{\Gamma}\right]_r \right\}, \qquad \forall r \in [0, 1].$$

Therefore, for any $i = 1, 2, \ldots, n$, we have

$$\exists \left(z_1^i(r), z_2^i(r), \ldots, z_n^i(r)\right)^T \in \left[\widetilde{\Gamma}\right]_r; \quad \sum_{j=1}^{n} a_{ij} z_j^i(r) = \underline{b}_i(r). \qquad \forall r \in [0, 1]$$

Obviously, the point $\left(z_1^i(r), z_2^i(r), \ldots, z_n^i(r)\right)^T$ is belong to lower $r$-boundary of $\widetilde{\mathcal{SS}}$. We show that this point is a vertex point of $\left[\widetilde{\Gamma}\right]_r$, i.e., $z_j^i(r) = \underline{x}_j(r)$ or $z_j^i(r) = \overline{x}_j(r)$. Suppose on contrary, there exist $1 \leqslant k \leqslant n$ and $0 \leqslant r^* \leqslant 1$ such that $z_k^i(r^*) \neq \underline{x}_k(r^*)$ and $z_k^i(r^*) \neq \overline{x}_k(r^*)$. Thus we find out

$$\underline{x}_k(r^*) < z_k^i(r^*) < \overline{x}_k(r^*). \tag{43}$$

On the other hand, it is clear that

$$\underline{x}_j(r^*) \leqslant z_j^i(r^*) \leqslant \overline{x}_j(r^*), \qquad \forall j = 1, 2, \ldots, k-1, k+1, \ldots, n. \tag{44}$$

From equations (43) and (44) we conclude that

$$\sum_{j \in \Omega_i^+} a_{ij} \underline{x}_j(r^*) + \sum_{j \in \Omega_i^-} a_{ij} \overline{x}_j(r^*) < \sum_{j=1}^{n} a_{ij} z_j^i(r^*) < \sum_{j \in \Omega_i^+} a_{ij} \overline{x}_j(r^*) + \sum_{j \in \Omega_i^-} a_{ij} \underline{x}_j(r^*), \tag{45}$$

where $\Omega_i^+$ and $\Omega_i^-$, $i = 1, 2, \ldots, n$, are defined as follows

$$\Omega_i^+ = \{j \ : \ 1 \leqslant j \leqslant n, \ a_{ij} \geqslant 0\}, \qquad \Omega_i^- = \{j \ : \ 1 \leqslant j \leqslant n, \ a_{ij} < 0\}.$$

Finally, by equation (45), we obtain

$$\underline{b}_i(r^*) < \underline{b}_i(r^*) < \overline{b}_i(r^*).$$

Obviously, this is a contradiction. Hence, for any $j = 1, 2 \ldots, n$, $z_j^i(r) = \underline{x}_j(r)$ or $z_j^i(r) = \overline{x}_j(r)$ and so the point $\left(z_1^i(r), z_2^i(r), \ldots, z_n^i(r)\right)^T$ is a vertex point of $\left[\widetilde{\Gamma}\right]_r$. In a similar manner, we can show that any upper $r$-boundary of $\widetilde{\mathcal{SS}}$ at least contains one vertex point of $\left[\widetilde{\Gamma}\right]_r$, for each $r \in [0, 1]$. Then property $(ii)$ holds, too.

Conversely, suppose that the properties $(i)$ and $(ii)$ hold, then based on the first part of theorem we conclude that for any $r \in [0,1]$, $[\Gamma]_r$ is an $n$-dimensional rectangle in $\mathbb{R}^n$ and consequently according to Theorem 3.14 we conclude that the fuzzy number-valued vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ is the algebraic solution of fuzzy linear system (3). □

**Remark 3.16.** According to Theorems 3.14 and 3.15, if the fuzzy solution set $\widetilde{\mathcal{SS}}$ does not have any fuzzy subset such that satisfies the properties $(i)$ of Theorem 3.14 and $(ii)$ of Theorem 3.15, then fuzzy linear system (3) does not have any algebraic solution. Also, if the fuzzy solution set $\widetilde{\mathcal{SS}}$ has infinite subsets such that satisfies the properties $(i)$ of Theorem 3.14 and $(ii)$ of Theorem 3.15, then fuzzy linear system (3) has infinite algebraic solutions.

**Remark 3.17.** Theorems 3.14 and 3.15 present a necessary and sufficient condition for the existence and uniqueness of the algebraic solution of fuzzy linear system (3) as follows: "The fuzzy linear system (3) has an algebraic solution if and only if the solution set $\widetilde{\mathcal{SS}}$ of FILS (4) has a subset such that satisfies the properties $(i)$ of Theorem 3.14 and $(ii)$ of Theorem 3.15, and if such a subset is unique then the algebraic solution is unique, too."

**Remark 3.18.** Let us suppose that $\mathcal{A}$ be an $n$-dimensional rectangle in $\mathbb{R}^n$ and be embedded in an $2n$-sided pyramid $\Delta$, i.e., $\mathcal{A} \subseteq \Delta$. Also, suppose that each boundary of $\Delta$ at least contains a vertex point of $\mathcal{A}$. Then, we can specify $n$-dimensional rectangle $\mathcal{A}$ only by $2n$ it's vertex points. In other words, if the point $A_i = (a_1^i, a_2^i, \ldots, a_n^i)^T \in \mathbb{R}^n$, $(i = 1, 2, \ldots, 2n)$ is the vertex point of $\mathcal{A}$ and is located on $i$th boundary of $\Delta$, then we can specify $n$-dimensional rectangle $\mathcal{A}$ as following:

$$\mathcal{A} = \left\{ (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n \ : \ \min_{i=1,2,\ldots,n} \left\{ a_j^i \right\} \leqslant y_j \leqslant \max_{i=1,2,\ldots,n} \left\{ a_j^i \right\} \right\}.$$

It should be noted that the $n$-dimensional rectangle $\mathcal{A}$ has $2^n$ vertex points, but by the above assumptions we can specify $\mathcal{A}$ only by $2n$ it's vertex points.

In the following, we present the membership function of the algebraic solution of a fuzzy linear system only by some points which are located on lower $r$-boundaries and upper $r$-boundaries of $\widetilde{\mathcal{SS}}$.

**Theorem 3.19.** *Suppose that $\widetilde{\Gamma} \subseteq \widetilde{\mathcal{SS}}$ and $\widetilde{\Gamma}$ satisfies the following properties:*

(i) $\left[\widetilde{\Gamma}\right]_r$ *be an $n$-dimensional rectangle for any $r \in [0,1]$, i.e., for any $r \in [0,1]$,*

$$\left[\widetilde{\Gamma}\right]_r = \prod_{i=1}^{n} [\widetilde{\gamma}_i]_r = \prod_{i=1}^{n} [\underline{\gamma}_i(r), \overline{\gamma}_i(r)],$$

*where $\widetilde{\gamma}_i$ is a fuzzy number.*

(ii) *There exist the vertex points*

$$\left(\alpha_1^i(r), \alpha_2^i(r), \ldots, \alpha_n^i(r)\right)^T, \qquad \left(\beta_1^i(r), \beta_2^i(r), \ldots, \beta_n^i(r)\right)^T,$$

*of $[\Gamma]_r$ such that are located on ith lower r-boundaries and ith upper r-boundaries of $\widetilde{\mathcal{SS}}$, respectively, for any $i = 1, 2, \ldots, n$.*

*Then:*

1. *The fuzzy linear system (3) has an algebraic solution and the fuzzy number-valued vector*
$$\widetilde{X} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_n)^T,$$
*is it's algebraic solution.*
2. *The membership function of the algebraic solution of the fuzzy linear system (3) is as following:*
$$\mu_{\widetilde{x}_{jA}}(y) = \sup\left\{r \ : \ \min_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\} \leqslant y \leqslant \max_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}\right\},$$
*for any $j = 1, 2, \ldots, n$.*

*Proof.* 1. The assumption $(ii)$ implies that for any $r \in [0,1]$, each lower $r$-boundary and each upper $r$-boundary of $\widetilde{\mathcal{SS}}$ at least contain one vertex point of $\left[\widetilde{\Gamma}\right]_r$. Therefore, based on the assumption $(i)$ and Theorems 3.14 and 3.15 it is clear that the fuzzy linear system (3) has an algebraic solution and the fuzzy number-valued vector $(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_n)^T$ is it's algebraic solution.

2. Based on the assumptions of theorem, for any $r \in [0,1]$ we have $\left[\widetilde{\Gamma}\right]_r \subseteq \left[\widetilde{\mathcal{SS}}\right]_r$. Therefore, from Remark 3.18 we obtain

$$\left[\widetilde{\Gamma}\right]_r = \left\{(y_1, y_2, \ldots, y_n) \ : \ \min_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\} \leqslant y_j \leqslant \max_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}\right\}.$$

for any $r \in [0,1]$. On the other hand, regarding to the first part of theorem and Theorem 3.8 we conclude that

$$\left[\widetilde{\Gamma}\right]_r = \left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r, \qquad r \in [0,1].$$

Hence we obtain

$$\left[\widetilde{\mathcal{X}}_{\mathcal{A}}\right]_r = \left\{(y_1, y_2, \ldots, y_n) \ : \ \min_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\} \leqslant y_j \leqslant \max_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}\right\}.$$

Therefore, according to Theorem 3.8 we can present the $r$-levels of algebraic solution of fuzzy linear system (3) as following:

$$[\widetilde{x}_{jA}]_r = \left[\min_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}, \max_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}\right], \tag{46}$$

for any $j = 1, 2, \ldots, n$ and $r \in [0,1]$. Finally, by equation (46) the membership function of algebraic solution of fuzzy linear system (3) can be presented as following:

$$\mu_{\widetilde{x}_{jA}}(y) = \sup\left\{r \ : \ \min_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\} \leqslant y \leqslant \max_{i=1,2,\ldots,n}\left\{\alpha_j^i(r), \beta_j^i(r)\right\}\right\},$$

for any $j = 1, 2, \ldots, n$. This completes the proof of theorem. □

We end the paper with the following remark.

**Remark 3.20.** It should be noted that:

**(i)** The inclusion setting adopted in this paper can be applied to fuzzy coefficients in the matrix $A$. In this case, we require that the coefficient matrix $\widetilde{A}$ of fuzzy numbers is regular in the sense that the matrix $A^{-1}$ exists for all $a_{ij} \in supp(\widetilde{a}_{ij})$, which will be verified in our future work.

**(ii)** If the coefficient matrix $A$ is singular, then according to subsection 3.1, the fuzzy set $\widetilde{\mathcal{SS}}$ is either an empty fuzzy set or a fuzzy set with infinite support. Also, if the matrix $A$ is singular, by subsection 3.2, we cannot define the fuzzy set $\widetilde{\mathcal{X}}^*$. Then, obviously the results cannot hold for this case. It should be emphasized that in this paper, for holding the results, the matrix $A$ must be nonsingular.

**(iii)** For obtaining the fuzzy set $\widetilde{\mathcal{X}}^*$, we can use various methods such as Cholesky decomposition method.

## 4. Conclusion

This paper was an extension of our previous works. Here, we defined a new class of fuzzy problems, namely "fuzzy inclusion linear systems", and proposed it's fuzzy solution set. Then, we compared the fuzzy solution set of a fuzzy inclusion linear system with the algebraic solution of a fuzzy linear system. The results reveal that the former is a subset of the latter and also both are subsets of a larger fuzzy set. Also, a new necessary and sufficient condition is derived for obtaining the unique algebraic solution of a fuzzy linear system. Some important definitions were illustrated by numerical examples.

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