REDUNDANCY OF MULTISET TOPOLOGICAL SPACES

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Abstract. In this paper, we show the redundancies of multiset topological spaces. It is proved that $(P^\star(U), \sqsubseteq)$ and $(Ds(\varphi(U)), \subseteq)$ are isomorphic. It follows that multiset topological spaces are superfluous and unnecessary in the theoretical view point.

1. Introduction

The rapid development of science has led to an urgent need for the sets theory development. For example, Zadeh [15] defined the fuzzy sets theory as an extension of the crisp sets theory. In a fuzzy set, a membership degree was given to each element of the universal set and not confined it to one of the two values 0 and 1. Later, Atanassov [1] extended the fuzzy sets by adding another degree of non-membership. The new structure given the name "intuitionistic" fuzzy sets. The two degrees are independent and their sum in $[0, 1]$. Despite to the controversy raised on "intuitionistic" fuzzy sets theory, it is still attract researchers attention, especially in computer science applications. The Attempts to offer generalizations of crisp sets theory did not stop. In 1999, Molodtsov [11] proposed the soft sets theory to deal with the uncertainty in a parametric behavior.

The repetition of elements are not permitted in the crisp sets theory and its generalizations. If repetition of any object is permitted in the crisp set, then a new sets theory, recognized as multisets theory [2] (or theory of bags in the sense of Yager [14]). In fact, the elements repetition is mandatory in several situations and cannot be ignored. Research works on multiset theory and its applications are progressing rapidly in various fields, including topology (see [7, 6, 5, 8, 10, 3, 4, 9]) and so on. Despite to the increasment of the research works in multiset topology, we must ask our self a principal question about its redundancy in theoretical view-point. If multiset topology is mathematically redundant, in which sense does its redundancy occur?

In this paper, we follow Shi and Pang [12, 13] to show the redundancies of multiset topologies. We will prove that multiset topologies are redundant and superfluous complicated in the theoretical view point.

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2. Preliminaries

Throughout this section, a concise survey of the notions of multisets and multiset topology are introduced. We refer to [7, 6, 5, 10] for more details.

Definition 2.1. [7] Let X be a crisp set. A multiset (briefly, an mset) U defined on X is a function $C_U : X \longrightarrow \mathbb{N}$, where $\mathbb{N} = \{0\} \cup \mathbb{Z}^+$ and $\mathbb{Z}^+$ denotes the positive integers.

The value $C_U(x)$ represents the repetition of x in U. An mset U defined on $X = \{x_1, x_2, x_3, ..., x_n\}$ can be presented by

$$U = \{k_1/x_1, k_2/x_2, k_3/x_3, \ldots, k_n/x_n\},$$

where k_i is the occurrences of x_i in the mset U and $i = 1, 2, 3, \ldots, n$. Nevertheless, the element $x \in X$ will have $C_U(x) = 0$ if it is not belonged to the mset U. If U is an mset with element x repeating l-times, then it is denotes by $x \in^l U$ and the negation of this statement is denoted by $x \notin^l U$. $[U]_x$ denoted that x belongs to U with neglecting to mention the occurrences of x in U. The set X of all elements which constructs the msets is called a domain set. By $[X]^w$, we denote the mset space, i.e., the family of all msets defined on X provided that the repetition of any element in the mset does not exceed w-times, where $w \in \mathbb{N}$.

For any mset U defined on the domain set X, the support set $U^\star$ of U is a subset of X such that $U^\star = \big\{x \in X : C_U(x) > 0\big\}$. The cardinality of U(denoted by $|U|$) is given by

$$|U| = \sum_{x \in X} C_U(x).$$

$|[U]_x|$ denotes the cardinality of an element x in U.

Definition 2.2. [7] For any two msets U and V defined on a domain set X, we have the following operations:

(1) $U = V$ if $C_U(x) = C_V(x)$ for each $x \in X$.
(2) $U \sqsubseteq V$ if $C_U(x) \leq C_V(x)$ for each $x \in X$.
(3) $W = U \sqcup V$ if $C_W(x) = \max\{C_U(x), C_V(x)\}$ for each $x \in X$.
(4) $W = U \sqcap V$ if $C_W(x) = \min\{C_U(x), C_V(x)\}$ for each $x \in X$.
(5) $W = U \oplus V$ if $C_W(x) = C_U(x) + C_V(x)$ for each $x \in X$.
(6) $W = U \ominus V$ if $C_W(x) = \max\{C_U(x) - C_V(x), 0\}$ for each $x \in X$.

The operations $\oplus$ and $\ominus$ are addition and subtraction of msets, respectively.

Theorem 2.3. *Binary msets relation $\sqsubseteq$ is in accordance with $\sqcup$ and $\sqcap$, i.e., $\sqsubseteq$, $\sqcup$ and $\sqcap$ satisfy the connecting lemma, i.e., for each $M \in [X]^w$, and U, $V \in P^*(M)$, then the following statements are equivalent:*

(1) $U \sqsubseteq V$.
(2) $U \sqcup V = V$.
(3) $U \sqcap V = U$.

Proof. Straightforward. $\square$

Definition 2.4. [7] For every $U \in [X]^w$, the mset $U^\Delta \in [X]^w$ is the complement of U where $C_{U^\Delta}(x) = w - C_U(x)$ for each $x \in X$.

Definition 2.5. [7] A submset V of U is called:

(1) a whole submset of U if and only if $C_U(x) = C_V(x)$ for all $x \in V^\star$.
(2) a partial whole submset of U if and only if there exists $x \in V^\star$ such that $C_U(x) = C_V(x)$.
(3) a full submset of U if and only if $U^\star = V^\star$ and $C_V(x) \leq C_U(x)$ for all $x \in V^\star$.

It is clear that $\emptyset$ could be a whole submset of each mset however it does not achieve both of the other two types in the case of mset is nonempty.

Definition 2.6. [7] Let $U \in [X]^w$. Then:

(1) The power whole mset $PW(U)$ of U is the family of each whole submsets of U and its cardinality is 2^α, where α is the cardinality of $U^\star$.
(2) The power full mset $PF(U)$ of U is the family of each full submsets of U.
(3) The power mset $P(U)$ of U is the family of all submsets of U. In the case of $V = \emptyset$, we have $V \in^1 P(U)$, otherwise $V \in^\alpha P(U)$ such that

$$\alpha = \prod_x \left(\begin{array}{c} |[U]_x| \\ |[V]_x| \end{array} \right)$$

the product $\prod_x$ is token over the distinct elements x of V, $|[U]_x| = n$ if and only if $x \in^n U$, and $|[V]_x| = r$ if and only if $x \in^r V$, then

$$\left(\begin{array}{c} |[U]_x| \\ |[V]_x| \end{array} \right) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}.$$

We will use $P^\star(U)$ to denote the support set of $P(U)$.

Definition 2.7. [7] Let U, U_1, $U_2 \in [X]^w$. The Cartesian product of U_1 and U_2 is given by

$$U_1 \times U_2 = \left\{ (k_1/x, k_2/y)/k_1k_2 : \ x \in^{k_1} U_1, \ y \in^{k_2} U_2 \right\}.$$

A submset R of $U \times U$ is called an mset relation on U if each element $(k_1/x, k_2/y)$ of the submset R has a count as the product of $C_1(x, y)$ and $C_2(x, y)$. We say k_1/x is related to k_2/y. An msets relation R on U is called:

(1) reflexive if $k_1/x \ R \ k_1/x$ for each $x \in^{k_1} U$.
(2) symmetric if $k_1/x \ R \ k_2/y$ implies $k_2/y \ R \ k_1/x$.
(3) transitive if $k_1/x \ R \ k_2/y$, $k_2/y \ R \ k_3/z$, then $k_1/x \, R \, k_3/z$.

An mset relation is said to be equivalence provided that it is reflexive, symmetric, and also transitive.

Definition 2.8. [7] An mset relation f is called an mset function if for every element k_1/x in domain of f, there is exactly one k_2/y in range of f provided that $(k_1/x, k_2/y)$ has pair occurring in f equal to the value of the product $C_1(x, y).C_2(x, y)$.

Let $\{U_1, U_2, U_3, \ldots, U_n\}$ be a family of msets defined on $[X]^w$. Then we have the following operations [7]:

(1) The msets union
$$\bigsqcup_{i\in I} U_i = \left\{ C_{\bigsqcup_{i\in I} U_i}(x)/x : C_{\bigsqcup_{i\in I} U_i}(x) = \max\{C_{U_i}(x)\} \right\}.$$
(2) The msets intersection
$$\bigsqcap_{i\in I} U_i = \left\{ C_{\bigsqcap_{i\in I} U_i}(x)/x : C_{\bigsqcap_{i\in I} U_i}(x) = \min\{C_{U_i}(x)\} \right\}.$$
(3) The mset complement
$$\begin{aligned} U^{\Delta} &= \mathbb{Z} - U \\ &= \left\{ C_{U^{\Delta}}(x)/x : C_{M^{\Delta}}(x) = C_{\mathbb{Z}}(x) - C_U(x), \quad \text{for all } x \in X \right\}. \end{aligned}$$

It is evident that every nonempty set of real numbers that has an upper bound has a supremum and that have a lower bound has an infimum. Therefore, the arbitrary union and intersection defined above are closed under the collection $\{U_i\}_{i\in I}$, since the collection $\{U_i\}_{i\in I}$ drawn from $[X]^w$ contains elements with finite cardinality and multiplicity of each element $x \in U_i^\star$ is always less than or equal to w.

Definition 2.9. [7] Let $U \in [X]^w$ and $\tau \subseteq P^\star(U)$. Then τ is called a multiset topology (briefly, m-topology) of U if τ satisfies the following properties:

(1) the mset U and the empty mset $\emptyset$ are in τ.
(2) the mset union of the elements of any subcollection of τ is in τ.
(3) the mset intersection of the elements of any finite subcollection of τ is in τ.

The pair (U, τ) is called a multiset topological space. A submset V of m-topological space (U, τ) is said to be closed mset iff $U \ominus V$ is open mset.

3. Main Results

It is clear that $\sqsubseteq$ is a partial order on $P^\star(U)$, i.e., $(P^\star(U), \sqsubseteq)$ is a poset. Moreover, by using Theorem 2.3, we conclude that $(P^\star(U), \sqsubseteq)$ is a lattice, where $U \in [X]^w$, the supremum $\bigvee = \bigsqcup$, and the infimum $\bigwedge = \bigsqcap$.

Theorem 3.1. *For an mset U in $[X]^w$, $(P^\star(U), \sqsubseteq)$ is a complete Lattice.*

Proof. Straightforward. □

Lemma 3.2. *Let U be an mset in $[X]^w$. Define $\varphi(U)$ as following:*
$$\varphi(U) = \bigcup_{x\in U^\star} \{x\} \times \{n \in \mathbb{N} : 0 < n \leq C_U(x)\}.$$
Then $(y, n) \in \varphi(U)$ if and only if $0 < n \leq C_U(y)$.

Proof. Straightforward. □

Let U be an mset in $[X]^\omega$. For arbitrary (x, n), $(y, m) \in \varphi(U)$, denote $(x, n) \leq (y, m)$ if $x = y$ and $n \leq m$. Thus "$\leq$" is a partial order of $\varphi(U)$ and the family of all down sets is denoted by $Ds(\varphi(U))$. It is easily to verify that $(Ds(\varphi(U)), \bigcap, \bigcup)$ is a complete lattice, where "$\bigcap$" and "$\bigcup$" are ordinary operators of sets.

In the following theorems, we will consider the restriction function $\varphi|_{P^\star(U)} : P^\star(U) \longrightarrow Ds(\varphi(U))$. For convenience, we will use φ instead of $\varphi|_{P^\star(U)}$.

Theorem 3.3. *Let U be an mset in $[X]^\omega$. Then*

$$\varphi : (P^\star(U), \bigsqcap, \bigsqcup) \longrightarrow (Ds(\varphi(U)), \bigcap, \bigcup)$$

is an isomorphism.

Proof. It is easy to verify that $\varphi(V) \in Ds(\varphi(U))$ for arbitrary $V \in P^\star(U)$ and φ is a bijection.

Let J be an index set and $V_j \in P^\star(U)$ for arbitrary $j \in J$. Then:

$$\begin{aligned}
(x,n) \in \varphi\left(\bigsqcup_{j\in J} V_j\right) &\Leftrightarrow x \in \left(\bigsqcup_{j\in J} V_j\right)^\star \quad \text{and} \quad C_{\left(\bigsqcup_{j\in J} V_j\right)}(x) \geq n > 0 \\
&\Leftrightarrow \exists j_0 \in J \text{ such that } x \in (V_{j_0})^\star \text{ and } C_{V_{j_0}}(x) \geq n > 0 \\
&\Leftrightarrow \exists j_0 \in J \text{ such that } (x,n) \in \varphi(V_{j_0}) \\
&\Leftrightarrow (x,n) \in \bigcup_{j\in J} \varphi(V_j).
\end{aligned}$$

So, $\varphi\left(\bigsqcup_{j\in J} V_j\right) = \bigcup_{j\in J} \varphi(V_j)$. Similarly, we can prove $\varphi\left(\bigsqcap_{j\in J} V_j\right) = \bigcap_{j\in J} \varphi(V_j)$. Thus $\varphi : (P^\star(U), \bigsqcap, \bigsqcup) \longrightarrow (Ds(\varphi(U)), \bigcap, \bigcup)$ is an isomorphism. $\square$

Corollary 3.4. *Let U be an mset in $[X]^\omega$ and $\tau \subseteq P^\star(U)$. Then (U,τ) is a multiset topological space if and only if $(\varphi(U), \varphi^{\rightarrow}(\tau))$ is a general topological space, where $\varphi^{\rightarrow}(\tau) = \{\varphi(V) : V \in \tau\}$.*

Corollary 3.5. *Let U be an mset in $[X]^\omega$ and $\tau \subseteq Ds(\varphi(U))$. Then $(\varphi(U),\tau)$ is a general topological space if and only if $(U, \varphi^{\leftarrow}(\tau))$ is a multiset topological space, where $\varphi^{\leftarrow}(\tau) = \{\varphi^{-1}(V) : V \in \tau\}$.*

The previous theorem and its corollaries lead us to the following main consequence:

Claim. The multiset topology defined on U is equivalent to the general topology defined on the family of all down sets $Ds(\varphi(U))$. Subsequently, all claims around general topology can be acclimatized to preclude any possible scheme based on multiset topology.

4. **Conclusion**

The rapid development of set theory has led to the emergence of many kinds of topologies which has been considered as a generalization of the general topology. Some of them may still be considered as a generalization of general topology and the others are redundant and unnecessary complicated in the theoretical sense. In this paper, we concluded that multiset topology is exactly a special subcase of general topology. Thus there is no benefit from continuing to study the theoretical aspects of multiset topology. Applied researchers also are advised to take caution when applying the multisets theory.

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