t-BEST APPROXIMATION IN FUZZY NORMED SPACES

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Abstract. The main purpose of this paper is to find t-best approximations in fuzzy normed spaces. We introduce the notions of t-proximinal sets and F-approximations and prove some interesting theorems. In particular, we investigate the set of all t-best approximations to an element from a set.

1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh [9] in 1965. since then, many mathematicians have studied fuzzy normed spaces from several angles ([6], [1], [2]) and, in 2001, Veeramani introduced the concept of t-best approximations in fuzzy metric spaces. In this paper we consider the set of all t-best approximations on fuzzy normed spaces and prove several theorems pertaining to this set.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if $([0,1], *)$is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

Definition 1.2. [6] The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$,

(i) $N(x, t) > 0$,
(ii) $N(x, t) = 1 \Leftrightarrow x = 0$,
(iii) $N(\alpha x, t) = N(x, t/ \mid \alpha \mid)$, for all $\alpha \neq 0$,
(iv) $N(x, t) * N(y, s) \leq N(x + y, t + s)$,
(v) $N(x, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous,
(vi) $\lim_{t\to\infty} N(x, t) = 1$.

Lemma 1.3. [6] *Let N be a fuzzy norm. Then:*
(i) $N(x, t)$ *is nondecreasing with respect to* t *for each* $x \in X$,
(ii) $N(x - y, t) = N(y - x, t)$.

Remark 1.4. As was shown in [6], every fuzzy normed space induces a fuzzy metric space on it and is therefore a topological space.

Definition 1.5. [6] Let $(X, N, *)$ be a fuzzy normed space. The open ball $B(x, r, t)$ and the closed ball B[x,r,t] with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are

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defined as follows:

$$B(x,r,t) = \{y \in X : N(x-y,t) > 1-r\},$$
$$B[x,r,t] = \{y \in X : N(x-y,t) \geqslant 1-r\}.$$

Lemma 1.6. [6] *If $(X, N, *)$ is a fuzzy normed space. Then:*
(i) the function $(x,y) \longrightarrow x+y$ is continuous,
(ii) the function $(\alpha, x) \longrightarrow \alpha x$ is continuous.

2. t-best Approximation

Definition 2.1. [8] Let A be a nonempty subset of a fuzzy normed space $(X, N, *)$. For $x \in X$, $t > 0$, let

$$d(A,x,t) = sup\{N(y-x,t) : y \in A\}.$$

An element $y_0 \in A$ is said to be a t-best approximation of x from A if

$$N(y_0 - x, t) = d(A,x,t)$$

Definition 2.2. Let A be a nonempty set of a fuzzy normed space $(X, N, *)$. For $x \in X$, $t > 0$, we shall denote the set of all elements of t-best approximation of x from A by $P_A^t(x)$; i.e.,

$$P_A^t(x) = \{y \in A : d(A,x,t) = N(y-x,t)\}.$$

If each $x \in X$ has at least (respectively exactly) one t-best approximation in A, then A is called a t-proximinal (respectively t-Chebyshev) set.

Definition 2.3. For $t > 0$, a nonempty closed subset A of a fuzzy normed space $(X, N, *)$ is said to be t-boundedly compact if for each x in X and $0 < r < 1, B[x,r,t] \cap A$ is a compact subset of X.

Theorem 2.4. *(Invariance by translation and scalar multiplication)*
Let A be a nonempty subset of a fuzzy normed space $(X, N, *)$. Then:
(i) $d(A+y, x+y, t) = d(A,x,t)$, for every $x, y \in X$ and $t > 0$,
(ii) $P_A^t(x+y) = P_A^t(x) + y$, for every $x, y \in X$ and $t > 0$,
(iii) $d(\alpha A, \alpha x, t) = d(A, x, t/ \mid \alpha \mid)$ for every $x \in X$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$,
(iv) $P_{\alpha A}^{|\alpha| t}(\alpha x) = \alpha P_A^t(x)$, for every $x \in X$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$,
(v) A is $t-$proximinal (respectively $t-$chebyshev) if and only if $A+y$ is $t-$proximinal (respectively $t-$chebyshev), for any given $y \in X$,
(vi) A is $t-$proximinal (respectively $t-$chebyshev) if and only if αA is $\mid \alpha \mid t$-proximinal (respectively $\mid \alpha \mid t$-chebyshev), for any given $\alpha \in \mathbb{R} \setminus \{0\}$.

Proof. (i) For any $x, y \in X$ and $t > 0$,

$$\begin{aligned} d(A+y, x+y, t) &= \sup\{N((z+y)-(x+y), t) : z \in A\} \\ &= \sup\{N(z-x,t) : z \in A\} \\ &= d(A,x,t). \end{aligned}$$

(ii) Using (i), $y_0 \in P^t_{A+y}(x+y)$ if and only if $y_0 \in A+y$ and $d(A+y, x+y, t) = N(x+y-y_0, t)$ if and only if $y_0 - y \in A$ and $d(A, x, t) = N(x-(y_0-y), t)$ if and only if $y_0 - y \in P^t_A(x)$; i.e., $y_0 \in P^t_A(x) + y$.
(iii) We have,

$$\begin{aligned} d(\alpha A, \alpha x, t) &= \sup\{N(\alpha x - \alpha z, t) : z \in A\} \\ &= \sup\{N(\alpha(x-z), t) : z \in A\} \\ &= \sup\{N(x-z, t/\mid \alpha \mid) : z \in A\} \\ &= d(A, x, t/\mid \alpha \mid). \end{aligned}$$

(iv) From (iii) it follows that $y_0 \in P^{|\alpha|t}_{\alpha A}(\alpha x)$ if and only if $y_0 \in \alpha A$ and $d(\alpha A, \alpha x, \mid \alpha \mid t) = N(\alpha x - y_0, \mid \alpha \mid t)$ if and only if $y_0/\alpha \in A$ and $N(x - y_0/\alpha, t) = d(A, x, t)$. However, this is equivalent to $y_0/\alpha \in P^t_A(x)$; i.e., $y_0 \in \alpha P^t_A(x)$.
(v) is an immediate consequence of (ii), and (vi) follows from (iv). □

Corollary 2.5. *Let M be a nonempty subspace of x. Then:*
(i) $d(M, x+y, t) = d(M, x, t)$, for every $t > 0$, $x \in X$ and $y \in M$,
(ii) $P^t_M(x+y) = P^t_M(x) + y$, for every $t > 0$, $x \in X$ and $y \in M$,
(iii) $d(M, \alpha x, \mid \alpha t \mid) = d(M, x, t)$, for every $t > 0$, $x \in X$ and $\alpha \in \mathbb{R} \setminus \{0\}$,
(iv) $P^{|\alpha|t}_M(\alpha x) = \alpha P^t_M(x)$, for every $t > 0$, $x \in X$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

Proof. (i) and (ii) follow from theorem (2.4)(i) and (2.4)(ii), and the fact that if M is a subspace and $y \in M$, then $M + y = M$.
(iii) and (iv) follow from Theorem 2.4 ((iii) and (iv)), and the fact that if M is a subspace and $\alpha \neq 0$, then $\alpha M = M$.

□

Definition 2.6. For $x \in X,\ 0 < r < 1,\ t > 0$,

$$S[x, r, t] = \{y \in X : N(x-y, t) = 1 - r\},$$
$$e^t_A(x) = 1 - d(A, x, t).$$

Theorem 2.7. *Let $(X, N, *)$ be a fuzzy normed space, A be a subset of X, $x \in X\backslash\overline{A}$ and $t > 0$. Then we have,*

$$\begin{aligned} P^t_A(x) &= A \cap B[x, e^t_A(x), t] \\ &= A \cap S[x, e^t_A(x), t]. \end{aligned} \tag{1}$$

Proof. The inclusions

$$P^t_A(x) \subseteq A \cap S[x, e^t_A(x), t] \subseteq A \cap B[x, e^t_A(x), t] \tag{2}$$

are obvious by the definitions of $P^t_A(x)$ and $e^t_A(x)$.
Conversely, let $y \in A \cap B[x, e^t_A(x), t]$, then we have, $y \in A$ and

$$N(y-x, t) \geq 1 - e^t_A(x) = d(A, x, t) \geq N(y-x, t).$$

Therefore $y \in A$ and

$$N(y-x, t) = d(A, x, t),$$

which implies that $y \in P^t_A(x)$. So, $A \cap B[x, e^t_A(x), t] \subset P^t_A(x)$, whence, by (2), we have (1), which completes the proof.

□

Remark 2.8. Let $(X, N, *)$ be a fuzzy normed space, A be a subset of X, $x \in X\backslash\overline{A}$ and $t > 0$. Then we have

$$A \cap B(x, e_A^t(x), t) = \emptyset, \tag{3}$$

because, if $y_0 \in A \cap B(x, e_A^t(x), t)$ then $d(A, x, t) \geq N(x - y_0, t) > d(A, x, t)$, which is absurd.

Corollary 2.9. *Let $(X, N, *)$ be a fuzzy normed space, A be a subset of X, $x \in X\backslash\overline{A}$ with $P_A^t(x) \neq \emptyset$ and $0 < r < 1$ such that :*

$$\emptyset \neq A \cap B[x, r, t] \subseteq S[x, r, t]. \tag{4}$$

Then we have

$$r = e_A^t(x),$$

and consequently $A \cap B[x, r, t] = P_A^t(x)$.

Proof. If $r < e_A^t(x)$, then by the definition of $e_A^t(x)$ we have $A \cap B[x, r, t] = \emptyset$, which contradicts (4). If $r > e_A^t(x)$, since $P_A^t(x) \neq \emptyset$, then by (1) we have

$$\emptyset \neq P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x, r, t),$$

which contradicts (4), and this completes the proof. □

Definition 2.10. Let $(X, N, *)$ be a fuzzy normed space, $0 < r < 1$ and $t > 0$. We shall say that a set $A \subset X$ supports the cell $B[x, r, t]$, or that A is a support set of the cell $B[x, r, t]$, if we have $d(A, B[x, r, t], t) = 1$ and $A \cap B(x, r, t) = \emptyset$.

Theorem 2.11. *Let $(X, N, *)$ be a fuzzy normed space, A a non-void set in X, $x \in X\backslash\overline{A}$, $a_0 \in A$ and $t > 0$. We have $a_0 \in P_A^t(x)$ if and only if the set A supports the cell $B = B[x, 1 - N(a_0 - x, t), t]$.*

Proof. Assume that $a_0 \in P_A^t(x)$. Hence $N(a_0 - x, t) = d(A, x, t)$. Then by (3), we have $A \cap B(x, 1 - N(a_0 - x, t), t) = \emptyset$, on the other hand, since $a_0 \in A \cap B[x, 1 - N(a_0 - x, t), t]$, we have $d(A, B, t) = 1$. Consequently, the set A supports the cell B. Conversely, suppose $a_0 \notin P_A^t(x)$, hence $N(a_0 - x, t) < d(A, x, t)$, and let $0 < \varepsilon < 1$ be such that $N(a_0 - x, t) < d(A, x, t) - \varepsilon$. Then there exists an $a \in A$ such that $N(a_0 - x, t) < d(A, x, t) - \varepsilon < N(a - x, t)$, hence $a \in B(x, 1 - N(a_0 - x, t), t)$. Consequently, A does not support the cell B, which completes the proof. □

Remark 2.12. We recall that a set A in a topological space τ is said to be countably compact, if every countable open cover of A has a finite subcover, or, what is equivalent, if for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ of non-void closed subsets of A we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Theorem 2.13. *Let $(X, N, *)$ be a fuzzy normed space, τ be an arbitrary topology on X and $t > 0$. If A is a non-void set of X such that for $A \cap B[x, r, t]$ is $\tau-$countably compact, then A is t-proximinal.*

Proof. For every $n \in \mathbb{N}, 0 < 1 - d(A,x,t) + \frac{d(A,x,t)}{n+1} < 1$. Put

$$A_n^t = A \cap B[x, 1 - d(A,x,t) + \frac{d(A,x,t)}{n+1}, t] \ (n = 1, 2, \ldots).$$

Since for every $n \in \mathbb{N}$, $d(A,x,t)(1 - \frac{1}{n+1}) < d(A,x,t)$, obviously $A_1^t \supset A_2^t \supset \cdots$ and each A_n^t is non-void. Hence there exists $a_n^t \in A$ such that

$$d(A,x,t)(1 - \frac{1}{n+1}) < N(a_n^t - x, t)$$

It follows that $a_n^t \in A_n^t$. Now, since each A_n^t is $\tau-$countably compact and $\tau-$closed, we conclude that there exists an $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$. Then we have

$$d(A,x,t) \geq N(a_0 - x, t) \geq d(A,x,t)(1 - \frac{1}{n+1})(n = 1, 2, \ldots),$$

whence $a_0 \in P_A^t(x)$ which completes the proof.

□

Definition 2.14. Let A be a nonempty subset of a fuzzy normed space $(X, N, *)$. An element $y_0 \in A$ is said to be an $F-$best approximation of $x \in X$ from A if it is a t-best approximation of x from A, for every $t > 0$, i.e.,

$$y_0 \in \bigcap_{t \in (0,\infty)} P_A^t(x).$$

The set of all elements of $F-$best approximations of X from A is denoted by $FP_A(x)$, i.e.,

$$FP_A(x) = \bigcap_{t \in (0,\infty)} P_A^t(x).$$

If each $x \in X$ has at least (respectively exactly) one $F-$best approximation in A, then A is called a $F-$proximinal (respectively $F-$chebyshev) set.

Example 2.15. Let $X = \mathbb{R}^2$. For $a, b \in [0,1]$, let $a * b = ab$. Define $N : \mathbb{R}^2 \times (0, \infty) \to [0,1]$ by

$$N((x_1, x_2), t) = (\exp \frac{\sqrt{x_1^2 + x_2^2}}{t})^{-1}.$$

Then $(X, N, *)$ is a fuzzy normed space. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\}$ and $x = (0, 2)$. Then for every $t > 0$,

$$N((-1,1) - (0,3), t) = N((1,1) - (0,3), t) = (\exp \frac{\sqrt{5}}{t})^{-1}.$$

On the other hand,

$$\begin{aligned} d(A, (0,3), t) &= \sup\{N((x_1, x_2) - (0,3), t) \mid -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\} \\ &= \sup\{(\exp \frac{\sqrt{x_1^2 + (x_2 - 3)^2}}{t})^{-1} \mid -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\} \\ &= (\exp \frac{\sqrt{5}}{t})^{-1}. \end{aligned}$$

So, for every $t > 0$, $y_0 = (-1, 1)$ and $y_1 = (1, -1)$ are t-best approximations of $(0, 3)$ from A. Therefore $y_0 = (-1, 1)$ and $y_1 = (1, -1)$ are $F-$best approximations of $x = (0, 3)$ from A. Therefore A is not an $F-$Chebyshev set.

Example 2.16. Let $X = \mathbb{R}$. For $a, b \in [0, 1]$, let $a * b = ab$. Define

$$N : \mathbb{R} \times (0, \infty) \to [0, 1],$$

by

$$N(x, t) = \frac{t}{t + |x|}.$$

Then $(X, N, *)$ is a fuzzy normed space. Let $A = [0, 1]$. Then, for every $x > 1$, 1 is an $F-$best approximation of x from A and for every $x < 0$, 0 is an $F-$best approximation of x from A. So A is an $F-$proximinal set.

Remark 2.17. For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of A, by $IntA$ the interior of A (hence $\partial A = \overline{A} \backslash IntA$), and by $\mathcal{M}_A$ the set of all elements of the $F-$best approximation of the elements $x \in X$ from A, i.e.

$$\mathcal{M}_A = \bigcup_{x \in X} FP_A(x).$$

Theorem 2.18. *Let $(X, N, *)$ be a fuzzy normed space and A be a $F-$best proximinal set in X. Then*

$$\partial A \subset \overline{\mathcal{M}}_A.$$

Proof. If $\partial A = \emptyset$, the statement is obvious. If $\partial A \neq \emptyset$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $(1 - \varepsilon') * (1 - \varepsilon') > 1 - \varepsilon$ and the cell $B(a_0, \varepsilon', t/2)$ contains at least one element $x \in X \backslash A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, A is $F-$proximinal). Then we have,

$$\begin{aligned} N(a_0 - \pi_A(x), t) &\geq N(a_0 - x, t/2) * N(x - \pi_A(x), t/2) \\ &= N(a_0 - x, t/2) * N(A, x, t/2) \\ &\geq N(a_0 - x, t/2) * N(a_0 - x, t/2) \\ &\geq (1 - \varepsilon') * (1 - \varepsilon') \\ &> 1 - \varepsilon. \end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathcal{M}_A \neq \emptyset$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathcal{M}}_A$ which completes the proof.

□

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