ADAPTIVE FUZZY OUTPUT FEEDBACK TRACKING CONTROL FOR A CLASS OF NONLINEAR TIME-VARYING DELAY SYSTEMS WITH UNKNOWN BACKLASH-LIKE HYSTERESIS

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Abstract. This paper considers the problem of adaptive output feedback tracking control for a class of nonstrict-feedback nonlinear systems with unknown time-varying delays and unknown backlash-like hysteresis. Fuzzy logic systems are used to estimate the unknown nonlinear functions. Based on the LyapunovKrasovskii method, the control scheme is constructed by using the backstepping and adaptive technique. The proposed adaptive controller guarantees that all the closed-loop signals are semiglobally uniformly ultimately bounded and the tracking error can converge to a small neighborhood of the origin. Finally, Simulation results further show the effectiveness of the proposed approach.

1. Introduction

In recent past decades, adaptive backstepping control for nonlinear systems with time delays (or time delays free) has been abundantly investigated, and some considerable outcomes have been stated in [2–4, 6, 13, 22, 26–29]. Nonlinear Strict-feedback systems have been studied more precisely in [2–4,13,26–28], and nonlinear systems with immeasurable states are presented in [6, 22, 23, 29]. In these control structures, unknown nonlinear functions in system dynamics are approximated by the fuzzy-logic systems or neural networks, and adaptive method combining with backstepping is utilized to construct adaptive controllers. In [20], Tong and Li proposed an adaptive fuzzy output feedback backstepping control approach for a class of nonlinear systems with unknown functions, unknown dead zones, and immeasurable states. Based on the backstepping recursive design technique and by constructing the dead zone inverse, an adaptive fuzzy backstepping control approach is synthesized. In [11] an adaptive fuzzy output feedback control approach has been proposed for a class of uncertain nonlinear strict-feedback systems with the unknown dead-zone input, dynamic uncertainties, and without the direct measurements of state variables. Problem of dynamic surface control (DSC) strategy and Adaptive fuzzy output feedback controller for strict-feedback nonlinear systems with unknown time-delay functions was studied in [29]. Hysteresis characteristic

Received: June 2016; Revised: October 2016; Accepted: February 2017
Key words and phrases: Adaptive fuzzy control, Backstepping design technique, Backlash-like hysteresis, Nonstrict-feedback form, Nonlinear control.

makes the system be nonlinear, such as biology optics, mechanical actuators, and electronic relay circuits [18]. Therefore, it is important to control the systems with hysteresis nonlinearities. In [16], non-differentiable nonlinearities backlash was first proposed to describe hysteresis phenomenon in the actuator. In [30], adaptive backstepping control was investigated for nonlinear systems with unknown backlash-like hysteresis. In [14,15,17–19,31] , several adaptive control schemes suggested to deal with unknown backlash-like hysteresis. All the aforementioned results are only suitable for strict-feedback nonlinear systems, but many practical systems such as biochemical process [8] and aircraft flight control system [7], are not in strict-feedback form. To overcome this difficulty, some adaptive fuzzy/neural control methods proposed to control these nonstrict-feedback systems. Nonstrict feedback system cannot be straightly compensated by the estimators with current states, Moreover, the virtual control signals need to be the functions of current states to guarantee their existence [12]. To deal with these difficulties, a variable separation method Based on the Lyapunov synthesis and the FLSs method have proposed in [3]. In [21] an adaptive fuzzy tracking control design problem for single-input and single-output (SISO) uncertain non-strict feedback nonlinear systems were developed. For the cases of states measurable and the states immeasurable, fuzzy logic systems are separately adopted to approximate the unknown nonlinear functions or model the uncertain nonlinear systems. Tong and Li were extended the results of this survey to switched systems [10]. In [12], Z. Liu et al. proposed an adaptive neural control for time-delayed continuous nonstrict-feedback systems with unknown direction modified BoucWen hysteresis model. In [1], the problem of fuzzy adaptive output feedback control based on observer was considered for time delayed nonlinear systems without strict-feedback form and with input saturation. Based on the above observation, results in [1,12,24] will be generalized in this study. With [1, 12, 24] different, we will investigate adaptive fuzzy output feedback tracking control design for nonstrict-feedback nonlinear time-varying delay systems with unknown backlash-like hysteresis. In this work, appropriate LyapunovKrasovskii functional is used to analyses the stability of systems and compensate the effect of delay terms. Control design is achieved by using the backstepping method, adaptive fuzzy logic system is employed to approximate the unknown nonlinear functions encountered. The proposed adaptive controller guarantees that all the closed-loop signals are semiglobally uniformly ultimately bounded and the tracking error can converge to a small neighborhood of the origin. Compared with the existing results, the main contributions of the paper are listed as follows: The time-varying delay systems addressed in this paper are much more general than the ones studied in [1, 12, 24] The main differences among them lies in the following aspects: although the stochastic nonlinear system with unknown backlash-like hysteresis was considered in [24], the time-delay problem was not taken into account. In [1], an adaptive fuzzy output feedback backstepping approach is proposed for nonlinear systems in nonstrict-feedback form. However, in this case, actuator does not have hysteresis, time delay is constant, and scholar does not study the reference signal tracking. The proposed controller design is independent of any prior knowledge of fuzzy approximations. With [1, 12, 24] different, the proposed controller design

requires only two adaptive laws for a nth-order system, which significantly reduces the online computation burden and with [14, 15, 17, 31] different, it is not needed to construct the hysteresis inverse. The rest of the paper is organized as follows. The problem formulation and preliminaries are introduced in Section 2. The adaptive fuzzy output feedback control design is presented in Section 3. In Section 4, the simulation examples are given to demonstrate the validity proposed control schemes. Finally, the conclusion is given in Section 5.

2. Problem Formulation and Preliminaries

2.1. **Problem Formulation.** Consider the single-input single-output (SISO) nonlinear time-delay system with non-strict feedback form described by

$$\begin{cases} \dot{x_i} = x_{i+1} + f_i(x) + h_i\Big(x\big(t-\tau_i(t)\big)\Big) + \omega_i(t,x) \qquad 1 \le i \le n-1 \\ \dot{x_n} = u(v) + f_n(x) + h_n\Big(x\big(t-\tau_n(t)\big)\Big) + \omega_n(t,x) \\ y = x_1 \end{cases} \tag{1}$$

where $x = [x_1, x_2, ..., x_n]^T$, $x \in R^n$ is the state variable, $v \in R$ and $y \in R$ is system input and system output, respectively, $f_i(.)$, $h_i(.)$, for $(1 \le i \le n)$, are unknown continuous smooth nonlinear functions, satisfying $f_i(0) = 0$ and $h_i(0) = 0$, In addition, are the external disturbance. In addition, $\omega_i(t,x)$ only the output $y(t)$ is available for measurement. Throughout this paper, we make the following assumptions.

Assumption 1. Backstepping controller design method requires successive derivative of reference signal. For this reason the reference signal $y_d(t)$ and its time derivatives up to the nth order $y_d^{(n)}(t)$ are bounded and continuous and assumed that there exist a positive constant d such that $|y_d| \le d$.

Assumption 2. For $1 \le i \le n$, consider $\tau_i(t)$ is an unknown time-varying delay that needs to satisfy the following inequalities:

$0 \le \tau_i(t) \le \tau \qquad \dot{\tau}_i(t) \le \bar{\tau}_{max} \le 1$

τ and $\bar{\tau}_{max}$ are unknown constants, and respectively are the upper bound of $\tau_i(t)$ and $\dot{\tau}_i(t)$.

Assumption 3. There exist unknown positive smooth functions $\varpi_i(x) : 1 \le i \le n$, such that $|\omega_i(t,x)| \le \varpi_i(x)$. This assumption is a relaxed version of the restriction $|\omega_i(t,x)| \le \varpi_i(\bar{x}_i)$, $\bar{x}_i = [x_1, ..., x_i]^T$ in [25].

Assumption 4. It is assumed that, there exist positive constants m_i and d_i, such that the nonlinear functions $f_i(.)$ and $h_i(.)$, $1 \le i \le n$ satisfies the constraints $|f_i(x) - f_i(\hat{x})| \le m_i\|x - \hat{x}\|$ and $|h_i(x) - h_i(\hat{x})| \le d_i\|x - \hat{x}\|$.

Remark 2.1. By putting $\hat{x} = 0$ in Assumption 4, one has $|f_i(x)| \le m_i\|x\|$, it means that there exist $\varphi_i(s) = m_i s$ with $s \in R$, which are bounding functions $f_i(.)$. In a similar way, there also exists $\phi_i(s) = d_i s$ with $s \in R$, which are bounding functions $h_i(.)$. $\big(\varphi_i(s)$ and $\phi_i(s)$ are strictly increasing smooth functions$\big)$.

2.2. **Backlash-like Hysteresis Nonlinearity.** Backlash-like hysteresis, as another actuator nonlinearity exists in a wide range of physical systems and devices, such as electromagnetism, which is formulated as

$$\frac{du}{dt} = \alpha|\frac{dv}{dt}|(cv - u) + B\frac{dv}{dt} \tag{2}$$

where α, c, B are unknown constants, and v, u are the input and output of the backlash-like hysteresis, respectively, and $c > 0$ is the slope of the line, which satisfies $c > B$. Based on the analysis in [15] this equation can be solved explicitly for v piecewise monotone

$$u(t) = cv(t) + d(v) \tag{3}$$

$$d(v) = \big[u_0 - cv_0\big]e^{-\alpha(v-v_0)sgn\dot{v}} + e^{-\alpha vsgn\dot{v}}\int_{v_0}^{v}\big[B - c\big]e^{\alpha\xi(sgn\dot{v})}d\xi \tag{4}$$

where u_0 and v_0 are the initial conditions of u and v respectively and is there exist an unknown positive constant d^* such that

$$|d(v)| \leq d^* \tag{5}$$

2.3. **Fuzzy Logic Systems.** In this paper fuzzy logic systems are used to approximate the completely unknown nonlinear function and the need for exact knowledge of system nonlinearities is removed. A fuzzy logic system (FLS) consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine working on fuzzy rules, and the defuzzifier. The knowledge base for FLS comprises a collection of fuzzy IF-THEN rules of the following form [2]:

R_i : IF x_1 is F_1^i and, x_n is F_n^i THEN y is B^i, $1 \leq i \leq n$
where $x = [x_1, x_2, ..., x_n]^T$, $x \in R^n$, and y are the FLS input signal and output signal, respectively. $\mu_{F_j^i}(x_j)$ and $\mu_{B^i}(y)$ are membership function of F_j^i and B^i , respectively. N is the fuzzy rule number. The fuzzy logic systems with the singleton fuzzifier, product inference, and center average defuzzifier can be expressed as

$$y(x) = \frac{\sum_{i=1}^{N}\bar{W}_i\prod_{j=1}^{n}\mu_{F_j^i}(x_j)}{\sum_{i=1}^{N}\prod_{j=1}^{n}\mu_{F_j^i}(x_j)}$$

where $\bar{W}_i = max_{y\in R}\mu_{B^i}(y)$. Define $S(x) = \big[S_1(x), S_2(x), ..., S_N(x)\big]$ and $W = \big[\bar{W}_1, \bar{W}_2, ..., \bar{W}_N\big]^T$ with the fuzzy basis function $S_i(x)$ given by

$$S_i(x) = \frac{\prod_{j=1}^{n}\mu_{F_j^i}(x_j)}{\sum_{i=1}^{N}\prod_{j=1}^{n}\mu_{F_j^i}(x_j)}$$

Then, the FLS can be rewritten as

$$y(x) = W^T S(x) \tag{6}$$

Lemma 2.2. *[9]: Let $f(x)$ be a continuous function defined on a compact set Ω; then, for any given constant $\bar{\varepsilon} > 0$ there exists a FLS $y(x)$ in the form of (6) such that: $SUP_{x\in\Omega}|f(x) - y(x)| \leq \bar{\varepsilon}$*

3. Output Feedback and Adaptive Fuzzy Control Design

In this section, an adaptive fuzzy output-feedback control method via backstepping is proposed for system (1).

3.1. **Observer Design.** For system (1), each of states x_i is not available for feedback control design; thus, a full-order observer is needed to estimate $x_1, ..., x_n$ and generate some signals for controller design. Let us consider the linear observer [1, 6, 29] for system (1):

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} - l_i(y - \hat{x}_1) & 1 \le i \le n-1 \\ \dot{\hat{x}}_n = u(v) - l_n(y - \hat{x}_1) \end{cases} \tag{7}$$

where $\hat{x}_i$ is the estimation of x_i, $1 \le i \le n$ the design parameter l_i is chosen such that the matrix

$$\mathbf{A} = \begin{bmatrix} l_1 & & I_n \\ \vdots & & \\ l_n & \cdots & 0 \end{bmatrix}$$

is strict Hurwitz matrix, which means for a given matrix $Q > 0$, there exists a matrix $P > 0$ satisfying:

$$A_c^T P + P A_c = -Q \tag{8}$$

After defining $e_i = x_i - \hat{x}_i$. From (1) and (7), the error dynamic can be expressed as

$$\dot{e} = A_c e + F(x) + H\Big(x\big(t - \tau(t)\big)\Big) + \omega(t, x) \tag{9}$$

where $e = \big[e_1, e_2, ..., e_n\big]^T$, $F(x) = \big[f_1(x), ..., f_n(x)\big]^T$, $\omega = \big[\omega_1(t, x), ..., \omega_n(t, x)\big]^T$ and $H\Big(x\big(t - \tau(t)\big)\Big) = \Big[h_1\Big(x\big(t - \tau_1(t)\big)\Big), ..., h_n\Big(x\big(t - \tau_1(t)\big)\Big)\Big]^T$.

3.2. **Adaptive control design.** In this section, we will develop the backstepping-based fuzzy adaptive output feedback controller and stability analysis procedure. The backstepping design procedure contains n steps, in step i, a virtual control function α_i is proposed, and the real tracking control law v is designed at the last step. The virtual control function α_i and the real control law v will be constructed as:

$$\alpha_i(Z_i) = -(\lambda_i + 0.5)z_i - \frac{1}{2a_i^2} z_i \theta \tag{10}$$

$$v = -(\lambda_n + 0.5)z_n - \frac{1}{2a_n^2} z_n \theta_n \tag{11}$$

where λ_i, $a_i : 1 \le i \le n$ are positive design parameters. Satisfies the following variable transformation:

$$z_i = \hat{x}_i - \alpha_{i-1} \tag{12}$$

with: $Z_i = \big[\bar{\hat{x}}_i^T, \theta, y_{di}^T\big]^T : 1 \le i \le n-1$, $Z_n = \big[\bar{\hat{x}}_n^T, \theta, \theta_n, y_{dn}^T\big]^T$, $\alpha_0 = y_d$, $\bar{\hat{x}}_i^T = \big[\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\big]$ and $y_{di} = \big[y_d, \dot{y}_d, ..., y_d^{(i)}\big]^T : 1 \le i \le n$,

$y_d^{(i)}$, being the i th derivative of y_d. θ^* and θ_n^* are unknown constants, where defines as:

$$\theta^* = max\Big(\|W_i\|^2\Big) \quad i = 0, 1, ..., n-1$$

$$\theta_n^* = \frac{1}{c}\|W_n\|^2$$

θ and θ_n the estimation of θ^* and θ_n^* respectively and satisfies the following differential equations:

$$\begin{cases} \dot{\theta} = \sum_{i=1}^{n-1} \frac{r}{2a_i^2} z_i^2 - \sigma\theta \\ \dot{\theta}_n = \frac{r}{2a_n^2} z_n^2 - \sigma_n \theta_n \end{cases} \tag{13}$$

with r, σ and r_n, σ_n being positive design parameters.

Remark 3.1. Apparently, (13) means that for any initial condition $\theta(t_0) \geq 0, \theta_n(t_0) \geq 0$ the solution $\theta(t) \geq 0, \theta_n(t) \geq 0$ holds for $t \geq 0$ thus throughout this paper, it is assumed that $\theta(t_0) \geq 0, \theta_n(t_0) \geq 0$.

Under coordinate transformation (12) linear observer (7) is transformed into the following equivalent form, for $i = 1, 2, ..., n$:

$$\dot{z}_i = \dot{\hat{x}}_i - \dot{\alpha}_{i-1} \tag{14}$$

with $\alpha_0 = y_d$, where:

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_k} (\hat{x}_{k+1} - l_k e_1) + \sum_{k=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d{}^{(k)}} y_d^{(k+1)} + \frac{\partial \alpha_{i-1}}{\partial \theta} \dot{\theta} \tag{15}$$

and in (15), for $i = 1$ we have

$$\frac{\partial \alpha_0}{\partial \theta} \dot{\theta} = 0 \qquad \sum_{k=1}^{0} \frac{\partial \alpha_0}{\partial \hat{x}_k} (\hat{x}_{k+1} - l_k e_1) = 0$$

Lemma 3.2. *[3]: for $z_i = \hat{x}_i - \alpha_{i-1}, i = 1, 2, ..., n$ the following holds:*

$$\|\hat{x}\| \leq \sum_{i=1}^{n} |z_i| \varphi_i(\theta)$$

with $\varphi_i(\theta) = (\lambda_i + 1.5) + \frac{1}{2a_i^2}\theta$ for $i = 1, 2, ..., n-1$, and $\varphi_n(\theta) = 1$

Theorem 3.3. *Under Assumptions 1-4, consider system (1) and construct linear observer (3) with the control law (11), the intermediate virtual-control function α_i for $1 \leq i \leq n-1$, described as (10), and the adaptive laws (13). Then, for bounded initial conditions, guarantee that the closed-loop system is semi-globally bounded, and for any given scalar $\varepsilon > 0$, the tracking error satisfies $lim_{t\to\infty} sup(z_1^2(t)) \leq \varepsilon^2$ by choosing appropriate design parameters.*

Proof. Consider the Lyapunov function candidate as:

Step 1.

$$\begin{aligned} V_1 = e^T P e + \frac{1}{1 - \bar{\tau}_{max}} \int_{t-\tau_1(t)}^{t} \eta^{-2} \|P\|^2 d_1^2 \|e(s)\|^2 ds + \frac{1}{2} z_1^2 \\ + \frac{\gamma}{1 - \bar{\tau}_{max}} \int_{t-\tau_1(t)}^{t} z_1^2(s) \phi_1^2(\theta(s)) ds \end{aligned} \tag{16}$$

where the constant γ will be specified later. According to (9) one has

$$\begin{aligned}\dot{V}_1 &= -e^T Qe + 2e^T P\big(F(x) - F(\hat{x})\big) + 2e^T PF(\hat{x}) \\ &+ 2e^T P\Big(H\Big(x\big(t-\tau(t)\big)\Big) - H\Big(\hat{x}\big(t-\tau(t)\big)\Big)\Big) + 2e^T PH\Big(\hat{x}\big(t-\tau(t)\big)\Big) \\ &+ 2e^T P\omega(t,x) + \frac{1}{1-\bar{\tau}_{max}}\eta^{-2}\|P\|^2 d_1^2\|e(t)\|^2 \\ &- \frac{1-\dot{\tau}(t)}{1-\bar{\tau}_{max}}\eta^{-2}\|P\|^2 d_1^2\|e(t-\tau_1(t))\|^2 + z_1(\dot{\hat{x}}_1 - \dot{y}_d) \\ &+ \gamma\Big(\frac{1}{1-\bar{\tau}_{max}} z_1^2\phi_1^2(\theta) - \frac{1-\dot{\tau}_1(t)}{1-\bar{\tau}_{max}} z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big)\Big)\end{aligned} \tag{17}$$

Invoking Assumption 2, one has

$$\big(1-\dot{\tau}(t)\big) \le -(1-\bar{\tau}_{max}) \tag{18}$$

Substituting (17) into (16) produces

$$\begin{aligned}\dot{V}_1 &\le -e^T Qe + 2e^T P(F(x) - F(\hat{x})) + 2e^T PF(\hat{x}) \\ &+ 2e^T P\Big(H\Big(x\big(t-\tau(t)\big)\Big) - H\Big(\hat{x}\big(t-\tau(t)\big)\Big)\Big) + 2e^T PH\Big(\hat{x}\big(t-\tau(t)\big)\Big) \\ &+ 2e^T P\omega(t,x) + \frac{1}{1-\bar{\tau}_{max}}\eta^{-2}\|P\|^2 d_1^2\|e(t)\|^2 \\ &- \eta^{-2}\|P\|^2 d_1^2\|e\big(t-\tau_1(t)\big)\|^2 + z_1\big((\hat{x}_2 - l_1 e_1) - \dot{y}_d\big) \\ &+ \gamma\Big(\frac{1}{1-\bar{\tau}_{max}} z_1^2\phi_1^2(\theta) - z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big)\Big)\end{aligned} \tag{19}$$

We define,

$$\begin{aligned}* &= -e^T Qe + 2e^T P\big(F(x) - F(\hat{x})\big) + 2e^T PF(\hat{x}) \\ &+ 2e^T P\Big(H\Big(x\big(t-\tau(t)\big)\Big) - H\Big(\hat{x}\big(t-\tau(t)\big)\Big)\Big) + 2e^T PH\Big(\hat{x}\big(t-\tau(t)\big)\Big) \\ &+ 2e^T P\omega(t,x) + \frac{1}{1-\bar{\tau}_{max}}\eta^{-2}\|P\|^2 d_1^2\|e(t)\|^2 \\ &- \eta^{-2}\|P\|^2 d_1^2\|e\big(t-\tau_1(t)\big)\|^2\end{aligned} \tag{20}$$

Then

$$\begin{aligned}\dot{V}_1 \le * &+ z_1(\alpha_1 - l_1 e_1) + \frac{\gamma}{1-\bar{\tau}_{max}} z_1^2\phi_1^2(\theta) \\ &- \gamma z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big) + z_2 z_1 - z_1\dot{y}_d\end{aligned} \tag{21}$$

with: $\hat{x}_2 = (z_2 + \alpha_1)$

Furthermore, $\dot{V}_1$ can be rewritten as

$$\begin{aligned}\dot{V}_1 \le * &+ z_1(\alpha_1 - \dot{y}_d) + \frac{\gamma}{1-\bar{\tau}_{max}} z_1^2\phi_1^2(\theta) \\ &- \gamma z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big) - z_1 l_1 e_1 + z_2 z_1\end{aligned} \tag{22}$$

Using Young's inequality, we have

$$-z_1 l_1 e_1 \leq \frac{1}{2}\beta_1 e_1^2 + \frac{1}{2\beta_1} l_1^2 z_1^2 \tag{23}$$

that β_1 is a positive constant. Then, Substituting (23) into (22) yields

$$\begin{aligned} \dot{V}_1 \leq *+ z_1(\alpha_1 - \dot{y}_d) + \frac{\gamma}{1-\bar{\tau}_{max}} z_1^2 \phi_1^2(\theta) \\ - \gamma z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big) + \frac{1}{2}\beta_1 e_1^2 + \frac{1}{2\beta_1} l_1^2 z_1^2 + z_2 z_1 \end{aligned} \tag{24}$$

and define

$$\bar{f}_1 = -\dot{y}_d + \frac{\gamma}{1-\bar{\tau}_{max}} z_1 \phi_1^2(\theta) + \frac{1}{2\beta_1} l_1^2 z_1 + \kappa z_1 \phi_1^2(\theta) \tag{25}$$

κ is defined in the following.

Replacing (25) into(24) gives

$$\begin{aligned} \dot{V}_1 \leq *+ z_1(\alpha_1 + \bar{f}_1) - \gamma z_1^2\big(t-\tau_1(t)\big)\phi_1^2\Big(\theta\big(t-\tau_1(t)\big)\Big) \\ + \frac{1}{2}\beta_1 e_1^2 + z_1 z_2 - \kappa z_1^2 \phi_1^2(\theta) \end{aligned} \tag{26}$$

Now we compute the bound of *:

By Assumption 4, one has

$$2e^T P\big(F(x) - F(\hat{x})\big) \leq \eta^2 \|e(t)\|^2 + \eta^{-2}\|P\|^2 \sum_{i=1}^{n} m_i^2 \|e(t)\|^2 \tag{27}$$

By using Remark 2.1 and Lemma 3.2, one can get

$$\begin{aligned} 2e^T P F(\hat{x}) &\leq \varepsilon_0 \|e(t)\|^2 + \frac{1}{\epsilon_0}\|P\|^2 \|F(\hat{x})\|^2 \leq \varepsilon_0 \|e(t)\|^2 + \kappa \|\hat{x}\|^2 \\ &\leq \varepsilon_0 \|e(t)\|^2 + \kappa\Big(\sum_{i=1}^{n} z_i^2 \varphi_i^2(\theta) + \frac{d^2}{n}\Big) \end{aligned} \tag{28}$$

with: $\kappa = \frac{2n}{\varepsilon_0}\|P\|^2 \sum_{i=1}^{n} m_i^2$

By using the same line in (27) and (28) one has

$$2e^T P\Big(H\big(x(t-\tau(t))\big) - H\big(\hat{x}(t-\tau(t))\big)\Big) \leq \eta^{-2}\|P\|^2 \sum_{i=1}^{n} d_i^2 \|e\big(t-\tau_i(t)\big)\|^2 + \eta^2 \|e(t)\|^2 \tag{29}$$

$$2e^T P H\Big(\hat{x}\big(t-\tau(t)\big)\Big) \leq \gamma\Big(\sum_{i=1}^{n} z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big) + \frac{d^2}{n}\Big) + \varepsilon_0 \|e(t)\|^2 \tag{30}$$

with: $\gamma = \frac{2n}{\varepsilon_0}\|P\|^2 \sum_{i=1}^{n} d_i^2$

Using Assumption 3 and Young's inequality, one has

$$2e^T P\omega(t,x) \leq \varepsilon_0 \|e(t)\|^2 + \varepsilon_0^{-1}\|P\|^2 \bar{\varpi}(x) \tag{31}$$

with: $\bar{\varpi}(x) = \sum_{i=1}^{n} \varpi_i^2(x)$

As $\bar{\varpi}(x)$ is an unknown smooth function, by Lemma 2.2, for any $\bar{\varepsilon}_0 > 0$, there exists a fuzzy logic system $W_0^T S_0(Z_0)$ such that:

$$\bar{\varpi}(x) = W_0^T S_0(Z_0) + \delta_0(Z_0) \quad |\delta_0(Z_0)| \leq \bar{\varepsilon}_0$$

where $Z_0 = x$.

As $S_0^T S_0 \leq 1$, and based on the definition of θ^*, we know $\|W_0\|^2 \leq \theta^*$.

Thus, the following inequality holds:

$$\bar{\varpi}(x) \leq \frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\epsilon}_0 \tag{32}$$

Thus, replacing (32) into (31) gives

$$2e^T P\omega(t,x) \leq \varepsilon_0 \|e(t)\|^2 + \varepsilon_0^{-1}\|P\|^2\left(\frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\varepsilon}_0\right) \tag{33}$$

Consequently, substituting (27)-(30) and (32) into (20) yields

$$\begin{aligned} * \leq & -e^T Qe + 2\eta^2\|e(t)\|^2 + \eta^{-2}\|P\|^2\left(\sum_{i=1}^{n} m_i^2 + \frac{d_1^2}{1-\bar{\tau}_{max}}\right)\|e(t)\|^2 \\ & + \varepsilon_0^{-1}\|P\|^2\left(\frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\varepsilon}_0\right) + 3\varepsilon_0\|e(t)\|^2 \\ & + \gamma\left(\sum_{i=1}^{n} z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big) + \frac{d^2}{n}\right) \\ & + \kappa\left(\sum_{i=1}^{n} z_i^2(t)\varphi_i^2\big(\theta(t)\big) + \frac{d^2}{n}\right) - \sum_{i=2}^{n}\eta^{-2}\|P\|^2 d_i^2\|e\big(t-\tau_i(t)\big)\|^2 \end{aligned} \tag{34}$$

Thus, replacing (34) into (26) gives

$$\begin{aligned} \dot{V}_1 \leq & -e^T Qe + 2\eta^2\|e(t)\|^2 + \eta^{-2}\|P\|^2\left(\sum_{i=1}^{n} m_i^2 + \frac{d_1^2}{1-\bar{\tau}_{max}}\right)\|e(t)\|^2 \\ & + \varepsilon_0^{-1}\|P\|^2\left(\frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\varepsilon}_0\right) + 3\varepsilon_0\|e(t)\|^2 \\ & + \gamma\left(\sum_{i=2}^{n} z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big) + \frac{d^2}{n}\right) \\ & + \kappa\left(\sum_{i=2}^{n} z_i^2(t)\varphi_i^2\big(\theta(t)\big) + \frac{d^2}{n}\right) + z_1(\alpha_1 + \bar{f}_1) \\ & + \frac{1}{2}\beta_1 e_1^2 + z_1 z_2 - \sum_{i=2}^{n}\eta^{-2}\|P\|^2 d_i^2\|e\big(t-\tau_i(t)\big)\|^2 \end{aligned} \tag{35}$$

From Lemma 2.2, there exists a fuzzy logic system $W_1^T S_1(Z_1)$ such that $\bar{f}_1 = W_1^T S_1(Z_1) + \delta_1(Z_1)$ where $\delta_1(Z_1)$ denotes the approximation error and satisfies $|\delta_1(Z_1)| \leq \bar{\varepsilon}_1$, Thus:

$$\begin{aligned} z_1\bar{f}_1 &= z_1 W_1^T S_1(Z_1) + z_1\delta_1(Z_1) \\ &\leq \frac{1}{2a_1^2} z_1^2 \|W_1\|^2 S_1^T(Z_1)S_1(Z_1) + \frac{1}{2}a_1^2 + \frac{1}{2}z_1^2 + \frac{1}{2}\bar{\varepsilon}_1^2 \\ &\leq \frac{1}{2a_1^2} z_1^2\theta^* + \frac{1}{2}a_1^2 + \frac{1}{2}z_1^2 + \frac{1}{2}\bar{\varepsilon}_1^2 \end{aligned} \tag{36}$$

We use the inequality $S_1^T(Z_1)S_1(Z_1) \leq 1$. In addition, according to (10) one has

$$z_1\alpha_1 = -(\lambda_1 + 0.5)z_1^2 - \frac{1}{2a_1^2} z_1^2\theta \tag{37}$$

Substituting (36) and (37) into (35) yields

$$\begin{aligned} \dot{V}_1 \leq e^T\big(-Q + 2\eta^2 I\big)e + 3\varepsilon_0 e^T e + \eta^{-2}\|P\|^2\bigg(\sum_{i=1}^{n} m_i^2 + \frac{d_1^2}{1-\bar{\tau}_{max}}\bigg)e^T e \\ - \lambda_1 z_1^2 + z_1 z_2 + \frac{1}{2}(a_1^2 + \bar{\varepsilon}_1^2) + \frac{1}{2}\beta_1 e_1^2 + \frac{1}{2a_1^2} z_1^2\tilde{\theta} + \chi_1 + \psi_1 \end{aligned} \tag{38}$$

with

$$\begin{aligned} \chi_1 &= \frac{d^2}{n}(\gamma + \kappa) + \varepsilon_0^{-1}\|P\|^2\bigg(\frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\varepsilon}_0\bigg) \\ \psi_1 &= \gamma\sum_{i=2}^{n} z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big) \\ &\quad + \kappa\sum_{i=2}^{n} z_i^2(t)\varphi_i^2\big(\theta(t)\big) - \sum_{i=2}^{n}\eta^{-2}\|P\|^2 d_i^2\|e\big(t-\tau_i(t)\big)\|^2 \end{aligned}$$

Step $i : 2 \leq i \leq n-1$**.**

$$\begin{aligned} V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{\gamma}{1-\bar{\tau}_{max}}\int_{t-\tau_i(t)}^{t} z_i^2(s)\phi_i^2\big(\theta(s)\big)ds \\ + \frac{1}{1-\bar{\tau}_{max}}\int_{t-\tau_i(t)}^{t}\eta^{-2}\|P\|^2 d_i^2\|e(s)\|^2 ds \end{aligned} \tag{39}$$

with differentiation of (39), $\dot{V}_i$ gives

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} + z_i(\dot{\hat{x}}_i - \dot{\alpha}_{i-1}) \\ &\quad + \gamma\bigg(\frac{1}{1-\bar{\tau}_{max}} z_i^2\phi_i^2(\theta) - z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big)\bigg) \\ &\quad + \frac{1}{1-\bar{\tau}_{max}}\Big(\eta^{-2}\|P\|^2 d_i^2\|e(t)\|^2 - \eta^{-2}\|P\|^2 d_i^2\|e\big(t-\tau_i(t)\big)\|^2\Big) \end{aligned} \tag{40}$$

Thus, replacing (15) and (7) into (40) gives

$$\begin{aligned}\dot{V}_i &\le \dot{V}_{i-1} + z_i(z_{i+1} + \alpha_i - l_i e_1) \\ &- z_i\Big(\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_k}(\hat{x}_{k+1} - l_k e_1) + \sum_{k=0}^{i-1}\frac{\partial \alpha_{i-1}}{\partial y_d{}^{(k)}} y_d^{(k+1)} + \frac{\partial \alpha_{i-1}}{\partial \theta}\dot{\theta}\Big) \\ &+ \gamma\Big(\frac{1}{1-\bar{\tau}_{max}} z_i^2\phi_i^2(\theta) - z_i^2\big(t-\tau_i(t)\big)\phi_i^2\Big(\theta\big(t-\tau_i(t)\big)\Big)\Big) \\ &+ \frac{1}{1-\bar{\tau}_{max}}\Big(\eta^{-2}\|P\|^2 d_i^2\|e(t)\|^2 - \eta^{-2}\|P\|^2 d_i^2\|e\big(t-\tau_i(t)\big)\|^2\Big)\end{aligned} \tag{41}$$

with: $\hat{x}_2 = (z_2 + \alpha_1)$

Using Young's inequality, we have

$$z_i\Big(\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_k} l_k e_1 - l_i e_1\Big) \le \frac{1}{2}\beta_i e_1^2 + \frac{1}{2\beta_i} z_i^2\Big(\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_k} l_k - l_i\Big)^2 \tag{42}$$

that β_i is a positive constant. and define

$$\begin{aligned}\bar{f}_i &= -\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_k}\hat{x}_{k+1} - \sum_{k=0}^{i-1}\frac{\partial \alpha_{i-1}}{\partial y_d{}^{(k)}} y_d^{(k+1)} - \frac{\partial \alpha_{i-1}}{\partial \theta}\dot{\theta} + z_{i-1} \\ &+ \frac{\gamma}{1-\bar{\tau}_{max}} z_i^2\phi_i^2(\theta) + \kappa z_i\varphi_i^2(\theta) + \frac{1}{2\beta_i} z_i^2\Big(\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_k} l_k - l_i\Big)^2\end{aligned} \tag{43}$$

Thus, replacing (42) and (43) into (41) $\dot{V}_i$ gives

$$\begin{aligned}\dot{V}_i &\le -e^T Q e + 2\eta^2\|e(t)\|^2 + \eta^{-2}\|P\|^2\Big(\sum_{i=1}^{n} m_i^2 + \sum_{k=1}^{i}\frac{d_k^2}{1-\bar{\tau}_{max}}\Big)\|e(t)\|^2 \\ &+ \varepsilon_0^{-1}\|P\|^2\Big(\frac{1}{2a_0^2}\theta^* + \frac{1}{2}a_0^2 + \bar{\varepsilon}_0\Big) + 3\varepsilon_0\|e(t)\|^2 + z_i(\alpha_i + \bar{f}_i) + z_i z_{i+1} \\ &+ \kappa\Big(\sum_{k=i+1}^{n} z_k^2(t)\phi_k^2(\theta(t)) + \frac{d^2}{n}\Big) - \sum_{k=i+1}^{n}\eta^{-2}\|P\|^2 d_k^2\|e\big(t-\tau_k(t)\big)\|^2 \\ &+ \gamma\Big(\sum_{k=i+1}^{n} z_k^2\big(t-\tau_k(t)\big)\phi_k^2\Big(\theta\big(t-\tau_k(t)\big)\Big) + \frac{d^2}{n}\Big) + \sum_{k=1}^{i}\frac{1}{2}\beta_k e_1^2 \\ &+ \sum_{k=1}^{i-1}\Big(\frac{1}{2a_k^2} z_k^2\theta^* + \frac{1}{2}a_k^2 + \frac{1}{2}z_k^2 + \frac{1}{2}\bar{\varepsilon}_k^2\Big)\end{aligned} \tag{44}$$

From Lemma 2.2, there exists a fuzzy logic system $W_i^T S_i(Z_i)$ such that $\bar{f}_i = W_i^T S_i(Z_i) + \delta_i(Z_i)$ where $\delta_i(Z_i)$ denotes the approximation error and satisfies $|\delta_i(Z_i)| \le \bar{\varepsilon}_i$, Thus:

$$z_i\bar{f}_i = z_i W_i^T S_i(Z_i) + z_i\delta_i(Z_i) \le \frac{1}{2a_i^2} z_i^2\theta^* + \frac{1}{2}a_i^2 + \frac{1}{2}z_i^2 + \frac{1}{2}\bar{\varepsilon}_i^2 \tag{45}$$

We use the inequality $S_i^T(Z_i)S_i(Z_i) \leq 1$.

In addition, according to (10) one has

$$z_i\alpha_i = -(\lambda_i + 0.5)z_i^2 - \frac{1}{2}a_i^2 + \frac{1}{2}z_i^2 + \frac{1}{2}\bar{\varepsilon}_i^2 \tag{46}$$

Substituting (45) and (46) into (44) yields

$$\begin{aligned}\dot{V}_i \leq e^T\big(-Q + 2\eta^2 I\big)e + 3\varepsilon_0 e^T e - \sum_{k=1}^{i}\lambda_k z_k^2 + z_i z_{i+1} + \frac{1}{2}\sum_{k=1}^{i}\Big(a_k^2 + \bar{\varepsilon}_k^2 + \beta_k e_1^2\Big)\\ + \chi_1 + \psi_i + \sum_{k=1}^{i}\frac{1}{2a_k^2}z_k^2\tilde{\theta} + \eta^{-2}\|P\|^2\Big(\sum_{i=1}^{n}m_i^2 + \sum_{k=1}^{i}\frac{d_k^2}{1-\bar{\tau}_{max}}\Big)e^T e\end{aligned} \tag{47}$$

with

$$\begin{aligned}\psi_i = \gamma\sum_{k=i+1}^{n} z_k^2(t-\tau_k(t))\phi_k^2\Big(\theta\big(t-\tau_k(t)\big)\Big)\\ + \kappa\sum_{k=i+1}^{n} z_k^2(t)\phi_k^2\big(\theta(t)\big) - \sum_{k=i+1}^{n}\eta^{-2}\|P\|^2 d_k^2\|e\big(t-\tau_k(t)\big)\|^2\end{aligned}$$

Step n.

$$V_n = V_{n_1} + V_{n_2} \tag{48}$$

$$\begin{aligned}V_{n_1} = V_{n-1} + \frac{1}{1-\bar{\tau}_{max}}\int_{t-\tau_n(t)}^{t}\eta^{-2}\|P\|^2 d_n^2\|e(s)\|^2 ds + \frac{1}{2}z_n^2\\ - \frac{\gamma}{1-\bar{\tau}_{max}}\int_{t-\tau_n(t)}^{t} z_n^2(s)\phi_n^2\big(\theta(s)\big)ds\end{aligned} \tag{49}$$

$$V_{n_2} = \frac{1}{2r}\tilde{\theta}^2 + \frac{c}{2r_n}\tilde{\theta}_n^2 \tag{50}$$

With regard to the step i, it can be written

$$\begin{aligned}\dot{V}_{n_1} \leq \dot{V}_{n-1} + z_n\big(u(v) + \bar{f}_n\big) + \gamma z_n^2\big(t-\tau_n(t)\big)\phi_n^2\Big(\theta\big(t-\tau_n(t)\big)\Big) - \kappa z_n^2\phi_n^2(\theta)\\ + \frac{1}{2}\beta_n e_1^2 + \frac{1}{1-\bar{\tau}_{max}}\Big(\eta^{-2}\|P\|^2 d_n^2\|e(t)\|^2 - \eta^{-2}\|P\|^2 d_n^2\|e(t-\tau(t))\|^2\Big)\end{aligned} \tag{51}$$

that $\bar{f}_n = \bar{f}_i|_{i=n}$

$$z_n u(v) = z_0\big(cv(t) + d(v)\big) \leq \big(-c(\lambda_n + 0.5) + 0.5\big)z_n^2 - \frac{c}{2a_n^2}z_n^2\theta_n + d_n^{*2} \tag{52}$$

$$z_n\bar{f}_n \leq \frac{c}{2a_n^2}z_n^2\theta_n^* + \frac{1}{2}a_n^2 + \frac{1}{2}z_n^2 + \frac{1}{2}\bar{\varepsilon}_n^2 \tag{53}$$

Thus, replacing (52) and (53) into (51) $\dot{V}_i$ gives

$$\begin{aligned}\dot{V}_{n_1} \leq\ & e^T\big(-Q+2\eta^2 I\big)e+3\varepsilon_0 e^T e-\sum_{k=1}^{n-1}\lambda_k z_k^2+\frac{1}{2}\sum_{k=1}^{n}\left(a_k^2+\bar{\varepsilon}_k^2\right)+\chi_1\\ &+\frac{d^{*2}}{2}+\big(-c(\lambda_n+0.5)+1\big)z_n^2+e^T\beta e+\sum_{k=1}^{n-1}\frac{1}{2a_k^2}z_k^2\tilde{\theta}+\frac{c}{2a_n}z_n^2\tilde{\theta}_n\\ &+\eta^{-2}\|P\|^2\left(\sum_{i=1}^{n}\left(m_i^2+\frac{d_i^2}{1-\bar{\tau}_{max}}\right)\right)e^T e\end{aligned}\tag{54}$$

with differentiation of (50), $\dot{V}_{n_2}$ gives

$$\dot{V}_{n_2}=-\tfrac{1}{r}\tilde{\theta}\dot{\theta}-\tfrac{c}{r_n}\tilde{\theta}_n\dot{\theta}_n\tag{55}$$

Substituting (13) into (55) yields

$$\dot{V}_{n_2}\leq-\textstyle\sum_{i=1}^{n-1}\frac{1}{2a_i^2}z_i^2\tilde{\theta}+\frac{1}{r}\sigma\tilde{\theta}\theta-\frac{c}{2a_n^2}z_n^2\tilde{\theta}_n\frac{c}{r_n}\sigma_n\tilde{\theta}_n\theta_n\tag{56}$$

As a result

$$\begin{aligned}\dot{V}_n \leq\ & e^T\big(-Q+2\eta^2 I\big)e+3\varepsilon_0 e^T e+e^T\beta e-\sum_{k=1}^{n-1}\lambda_k z_k^2+\frac{1}{2}\sum_{k=1}^{n}\left(a_k^2+\bar{\varepsilon}_k^2\right)\\ &+\chi_1+\frac{d^{*2}}{2}+\big(-c(\lambda_n+0.5)+1\big)z_n^2+\frac{1}{r}\sigma\tilde{\theta}\theta+\frac{c}{r_n}\sigma_n\tilde{\theta}_n\theta_n\\ &+\eta^{-2}\|P\|^2\left(\sum_{i=1}^{n}\left(m_i^2+\frac{d_i^2}{1-\bar{\tau}_{max}}\right)\right)e^T e\end{aligned}\tag{57}$$

with

$$\beta=diag\left[\sum_{i=1}^{n}\frac{1}{2}\beta_i,0,\cdots,0\right].$$

We consider

$$\dot{V}_n=\dot{V}\tag{58}$$

Thus, replacing (58) into (57) $\dot{V}$ gives

$$\begin{aligned}\dot{V}\leq\ & e^T\left(-Q+\left(2\eta^2+3\varepsilon_0+\eta^{-2}\|P\|^2\sum_{i=1}^{n}\left(m_i^2+\frac{d_i^2}{1-\bar{\tau}_{max}}\right)\right)I+\beta\right)e+\frac{1}{r}\sigma\theta\tilde{\theta}\\ &+\chi+\sum_{i=1}^{n}\frac{1}{2}(a_i^2+\bar{\varepsilon}_i^2)-\sum_{i=1}^{n-1}\lambda_i z_i^2+\big(-c(\lambda_n+0.5)+1\big)z_n^2+\frac{c}{r_n}\sigma_n\theta_n\tilde{\theta}_n\end{aligned}\tag{59}$$

with $\chi=\frac{d^{*2}}{2}+\frac{d^2}{n}(\gamma+\kappa)+\varepsilon_0^{-1}\|P\|^2\left(\frac{1}{2a_0^2}\theta^*+\frac{1}{2}a_0^2+\bar{\varepsilon}_0\right)$

Let

$$\left(-Q+\left(2\eta^2+3\varepsilon_0+\eta^{-2}\|P\|^2\sum_{i=1}^{n}\left(m_i^2+\frac{d_i^2}{1-\bar{\tau}_{max}}\right)\right)I+\beta\right)<0\tag{60}$$

Notice that

$$\tilde{\theta}\theta = \tilde{\theta}(\theta^* - \tilde{\theta}) \leq -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^{*2}$$

$$\tilde{\theta}_n\theta_n = \tilde{\theta}_n(\theta_n^* - \tilde{\theta}_n) \leq -\frac{1}{2}\tilde{\theta}_n^2 + \frac{1}{2}\theta_n^{*2}$$

Thus, inequality (59) can be expressed as

$$\begin{aligned}\dot{V} \leq e^T\Bigg(-Q + \Bigg(2\eta^2 + 3\varepsilon_0 + \eta^{-2}\|P\|^2\sum_{i=1}^{n}\Bigg(m_i^2 + \frac{d_i^2}{1-\bar{\tau}_{max}}\Bigg)\Bigg)I + \beta\Bigg)e \\ + \sum_{i=1}^{n}\frac{1}{2}(a_i^2 + \bar{\varepsilon}_i^2) - \sum_{i=1}^{n-1}\lambda_i z_i^2 + \big(-c(\lambda_n + 0.5\big) + 1)z_n^2 - \frac{\sigma}{2r}\tilde{\theta}^2 + \frac{\sigma}{2r}\theta^{*2} \\ - \frac{c}{2r_n}\sigma_n\tilde{\theta}_n^2 + \frac{c}{2r_n}\sigma_n\theta_n^{*2} + \chi\end{aligned} \tag{61}$$

Inequality (60) implies that there exists a constant $\alpha > 0$ such that

$$\Bigg(-Q + \Bigg(2\eta^2 + 3\varepsilon_0 + \eta^{-2}\|P\|^2\sum_{i=1}^{n}\Bigg(m_i^2 + \frac{d_i^2}{1-\bar{\tau}_{max}}\Bigg)\Bigg)I + \beta\Bigg) < -\alpha I \tag{62}$$

This implies that:

$$e^T\Bigg(-Q + \Bigg(2\eta^2 + 3\varepsilon_0 + \eta^{-2}\|P\|^2\sum_{i=1}^{n}\Bigg(m_i^2 + \frac{d_i^2}{1-\bar{\tau}_{max}}\Bigg)\Bigg)I + \beta\Bigg)e \leq \frac{-\alpha}{\lambda_M(P)}\lambda_M(P)\|e\|^2$$

$$\leq \frac{-\alpha}{\lambda_M(P)}e^T P e$$

where $\lambda_M(P)$ denotes the maximum eigenvalue of matrix P.

Let $\bar{a}_0 = min\Big\{\frac{\alpha}{\lambda_M(P)}, 2\lambda_i : 1 \leq i \leq n-1, 2\big(c(\lambda_n + 0.5) + 1\big), \sigma_n, \sigma\Big\}$

and $\bar{b}_0 = \sum_{i=1}^{n}\frac{1}{2}(a_i^2 + \bar{\varepsilon}_i^2) + \frac{\sigma}{2r}\theta^{*2} + \frac{c}{2r_n}\sigma_n\theta_n^{*2} + \chi$,

Then (61) can be rewritten as

$$\dot{V} \leq -\bar{a}_0(V - V_H) + \bar{b}_0$$

where

$$V_H = \frac{1}{1-\bar{\tau}_{max}}\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\eta^{-2}\|P\|^2 d_i^2\|e(s)\|^2 ds + \frac{\gamma}{1-\bar{\tau}_{max}}\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} z_i^2(s)\phi_i^2\big(\theta(s)\big)ds$$

. Let us define a set as

$$M = \Big\{(z, \tilde{\theta}, e) : \bar{a}_0(V - V_H) \leq \bar{b}_0\Big\}$$

It is obvious that when $(z, \tilde{\theta}, e)$ is outside of M, i.e., $\bar{a}_0(V - V_H) > \bar{b}_0$, then $\dot{V} < 0$ holds. As a result, any trajectory of the closed loop, which starts in M, will always stay inside this compact set. It concludes that all the signals in the closed-loop system are SGUUB, and the proof is completed.

□

4. **Simulation Results**

Example 4.1. Consider the following nonlinear time-delay system in [5].

$$\begin{cases} \dot{x_1} = x_2 + 0.25x_1x_2^2 + h_1\Big(x\big(t-\tau_1(t)\big)\Big) \\ \dot{x_2} = u(v) + h_1\Big(x\big(t-\tau_2(t)\big)\Big) \\ y = x_1 \end{cases} \tag{63}$$

To consider the effect of delays, we add the delay terms, i.e.,

$$h_1\Big(x\big(t-\tau_1(t)\big)\Big) = \frac{0.5x_1\big(t-\tau_1(t)\big)x_2\big(t-\tau_1(t)\big)}{1+x_1^2\big(t-\tau_1(t)\big)+x_2^2\big(t-\tau_1(t)\big)}$$

and $h_2\Big(x\big(t-\tau_2(t)\big)\Big) = 0.5x_1^2\big(t-\tau_2(t)\big)sin\Big(x_2\big(t-\tau_2(t)\big)\Big)$, with $\tau_1(t) = 1 - 0.5cost, \tau_2(t) = e^{-0.5t}$ to the original system, Apparently, this system is a nonstrict-feedback system because of the nonlinear term $x_1x_2^2$ in the first subsystem. In addition, in this case (63) seen as a pure-feedback system by selecting $f_1 = x_2 + 0.25x_1x_2^2$. However, since $\frac{\partial f_1}{\partial x_2} = 1 + 0.5x_1x_2$, the condition $\frac{\partial f_1}{\partial x_2} > 0$ cannot be guaranteed for all x_2. Therefore, the existing control approaches for pure-feedback systems are not feasible. The simulation is run under the initial conditions $\big[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0), \theta(0), \theta_2(0)\big]^T = \big[1, 0.5, 0, 0, 0, 0\big]^T$ and design parameters $a_1 = a_2 = 3, r = 1.5, r_2 = 3, \lambda_1 = 10, \lambda_2 = 5$ and $\sigma = 0.05, \sigma_2 = 0.15$ the observer parameters are selected as $l_1 = l_2 = -10$. The reference signal is $y_d = sin(0.5t) + 0.5sin(1.5t)$ and the backlash-like hysteresis parameters are presented are selected as $\alpha = 1, c = 3.1635, B = 0.345, u(0) = 0$. The control objective of this simulation is to construct an adaptive fuzzy output feedback control scheme to guarantee that all the closed-loop signals remain bounded, and the tracking error converge to an neighborhood of the origin. The simulation results are shown in Figures. 1-6. Figure. 1 shows the tracking performance of control system, as it seen a desired tracking performance has been achieved. Figure. 2 shows that the trajectories of state x_2 and $\hat{x}_2$, it is clear, state estimation has been done well. Figures. 3-5 shows that the adaptive laws, the control input signal and output of backlash-like hysteresis respectively. Figure. 6 displays the phase portrait of backlash-like hysteresis. Clearly, the simulation results can further demonstrate the validity proposed control schemes in this paper.

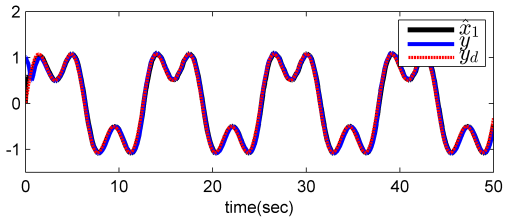


FIGURE 1. The System Output y, the Reference Signal y_d and the Estimate of x_1

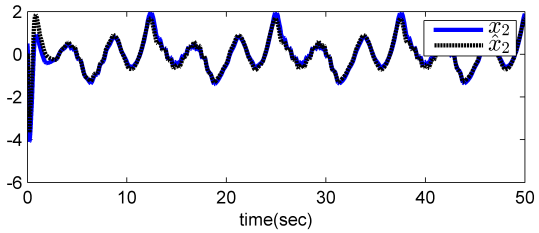


FIGURE 2. The Trajectories of x_2 and $\hat{x}_2$

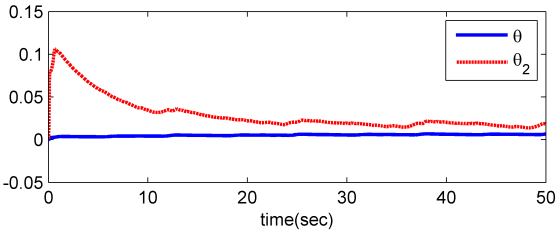


FIGURE 3. The Adaptive Laws

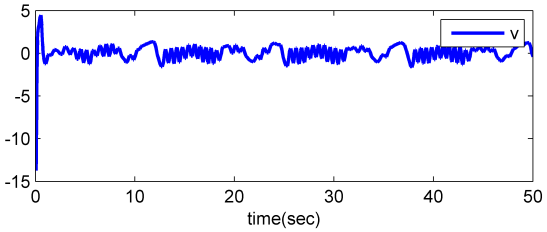


FIGURE 4. The Control Input Signal $v(t)$

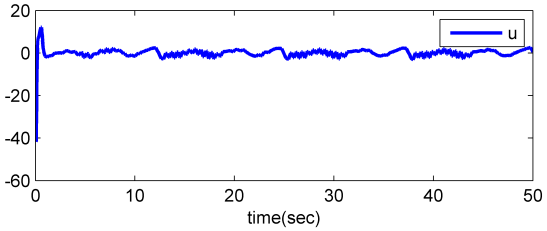


FIGURE 5. The Output of the Backlash-like Hysteresis

Example 4.2. Consider the following three-order nonlinear dynamic system:

$$\begin{cases} \dot{x_1} = x_2 + x_1 x_2^2 x_3 + sin(x_1 x_2^2 x_3) \\ \dot{x_3} = x_3 + x_1^2 x_2 x_3^2 + x_2^2\big(t-\tau_2(t)\big)x_3\big(t-\tau_2(t)\big) + \omega_2(t,x) \\ \dot{x_3} = u(v) + sin(x_1 - x_2) + 0.5x_2^2\big(t-\tau_3(t)\big)cos\Big(x_3\big(t-\tau_3(t)\big)\Big) \\ y = x_1 \end{cases} \tag{64}$$

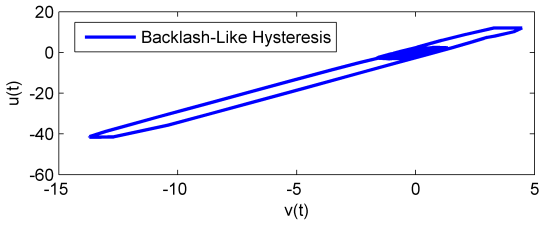


FIGURE 6. The Phase Portrait of Backlash-like Hysteresis

where $\omega_2(t,x) = sin(0.5tx_1x_2^4x_3)$ is the external disturbance, and u is the output of the backlash-like hysteresis defined in (2) with the parameters α, c and B being the same values, as those in Example 4.1. The reference signal is assumed to be $y_d = 0.5sin(1.5t)$. To control this system, according to Theorem 3.3, the true control law $v(t)$, the virtual control law $\alpha_i(t) : i = 1, 2$ and adaption laws $\dot{\theta}, \dot{\theta_3}$ are designed as follows:

$$\alpha_1 = -(\lambda_1 + 0.5)z_1 - \frac{1}{2a_1^2}z_1\theta$$

$$\alpha_2 = (\lambda_2 + 0.5)z_2 - \frac{1}{2a_2^2}z_2\theta$$

$$v = -(\lambda_3 + 0.5)z_3 - \frac{1}{2a_3^2}z_3\theta_3$$

$$\dot{\theta} = \sum_{i=1}^{2} \frac{r}{2a_i^2}z_i^2 - \sigma\theta$$

$$\dot{\theta}_3 = \frac{r_3}{2a_3^2}z_i^2 - \sigma_3\theta_3$$

where $z_1 = \hat{x}_1 - y_d, z_2 = \hat{x}_2 - \alpha_1, z_3 = \hat{x}_3 - \alpha_2,$

In the simulation, the design parameters are chosen as $a_1 = 3, a_2 = a_3 = 10, r = r_3 = 5, \lambda_1 = 4, \lambda_2 = 6, \lambda_3 = 10$ and $\sigma = 0.05, \sigma_2 = 0.15$ and $\sigma = \sigma_3 = 0.15$. The time delays are selected as $\tau_2(t) = 0.5 - 0.2sint$, $\tau_3(t) = 1 - 0.5sint$. The simulation is run under the initial conditions $\left[x_1(0), x_2(0), x_3(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), \theta(0), \theta_3(0)\right]^T = \left[0.5, 0.5, 0, 0, 0, 0, 0, 0\right]^T$ In addition, the observer parameters are selected as $l_1 = -7, l_2 = -18.2924, l_3 = -16.9207$, Figures. 7-13 show the simulation results. Figure. 7 shows the tracking performance of control system and the estimate of x_1, as it seen a desired tracking performance has been achieved Figures. 8 and 9 shows that the trajectories of states $x_2, \hat{x}_2$ and $x_3, \hat{x}_3$ respectively, it is clear, state estimation has been done well. Figures. 10-12 shows that the adaptive laws, the control input signal and output of backlash-like hysteresis respectively. Figure. 13 presents the phase portrait of backlash-like hysteresis. Clearly, the simulation results can further demonstrate the validity proposed control schemes in this paper.

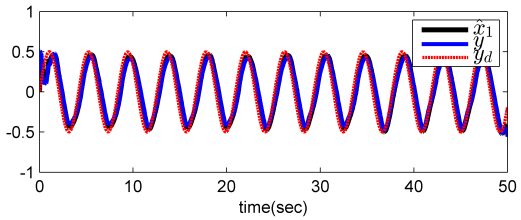


FIGURE 7. The System Output y, the Reference Signal y_d and the Estimate of x_1

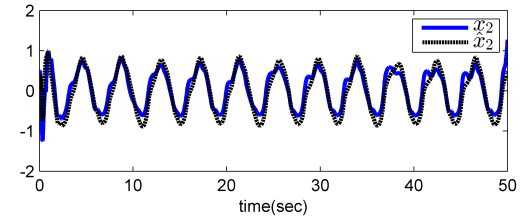


FIGURE 8. The Trajectories of x_2 and $\hat{x}_2$

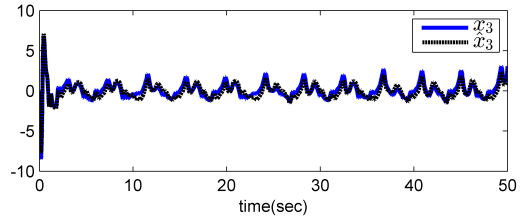


FIGURE 9. The Trajectories of x_3 and $\hat{x}_3$

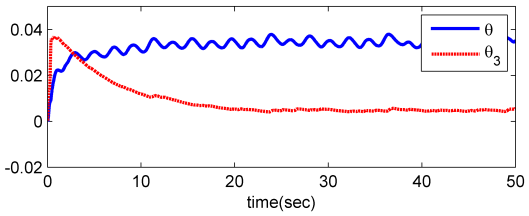


FIGURE 10. The Adaptive Laws

Example 4.3. Consider the following piezo-positioning mechanism in [31] and [12] as follows:

$$M\ddot{y}(t) + D\dot{y}(t) + Fy(t) = u(v) \tag{65}$$

where y, $\dot{y}$ and $\ddot{y}$ represent the position, the velocity, and the acceleration, respectively. v is the voltage signal applied to the piezo-positioning mechanic system. M, D, and F denote the mass, damping, and stiffness coefficients, respectively.

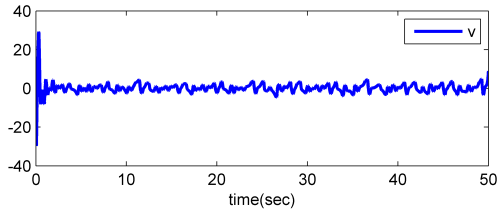


FIGURE 11. The Control Input Signal $v(t)$

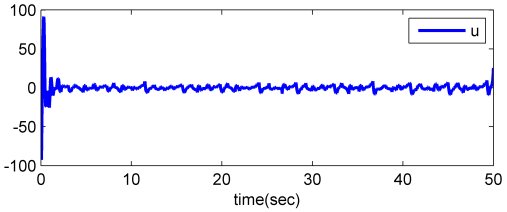


FIGURE 12. The Output of the Backlash-like Hysteresis

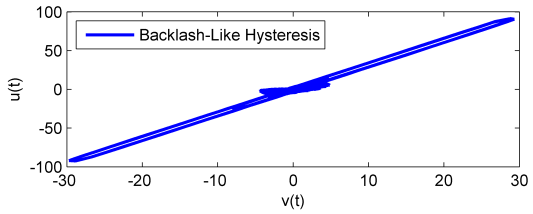


FIGURE 13. The Phase Portrait of Backlash-like Hysteresis

The simulation is carried out with the system parameters $M = 1kg, D = 0.15Ns/m, F = 1M/m$. The backlash-like hysteresis parameters are presented in Example 4.1 and the reference signal is $y_d = 2sin(0.5\pi t)$. The simulation is run under the initial conditions $\left[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0), \theta(0), \theta_2(0)\right]^T = \left[1.5, 0.01, 0, 0, 0, 0\right]^T$ (where $y = x_1$ and $\dot{y} = x_2$) and the design parameters $a_1 = a_2 = 3, r = r_2 = 9, \lambda_1 = 10, \lambda_2 = 3$ and $\sigma = \sigma_2 = 0.15$. Furthermore, the observer parameters are chosen as $l_1 = -10, l_2 = -20$. Figures. 14-19 show the simulation results. Clearly, the simulation results can further demonstrate the validity proposed control schemes in this paper.

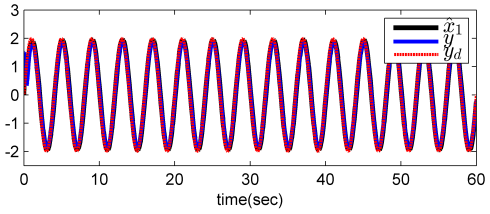


FIGURE 14. The System Output y, the Reference Signal y_d and the Estimate of x_1

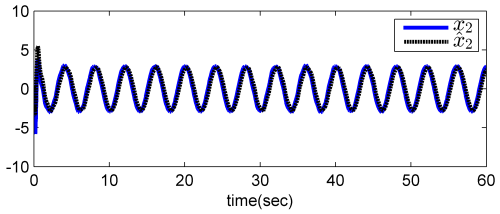


FIGURE 15. The Trajectories of x_2 and $\hat{x}_2$

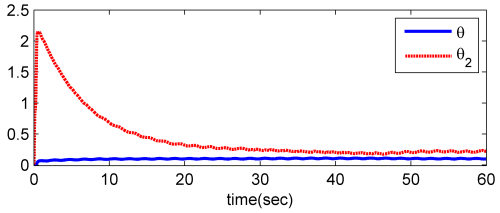


FIGURE 16. The Adaptive Laws

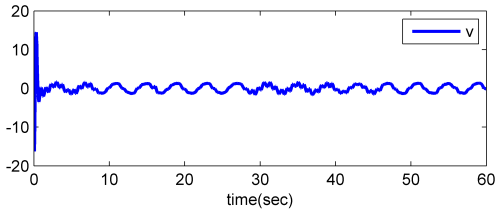


FIGURE 17. The Control Input Signal $v(t)$

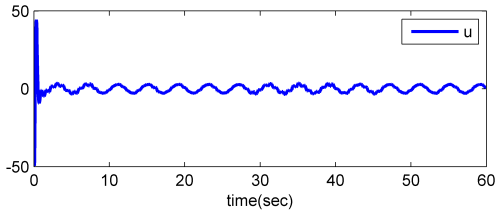


FIGURE 18. The Output of the Backlash-like Hysteresis

5. **Conclusion**

In this paper, the problem of adaptive fuzzy output feedback control scheme was investigated for a class of nonlinear time-varying delayed systems in nonstrict-feedback structure with unknown backlash-like hysteresis. The adaptive fuzzy output feedback controller has been constructed by combining LyapunovKrasovskii functional and the backstepping technique. The proposed control method guarantees that the closed-loop system is stable in the sense of semiglobally uniformly ultimately bounded and tracking error converge to an adjustable neighborhoods of the origin.

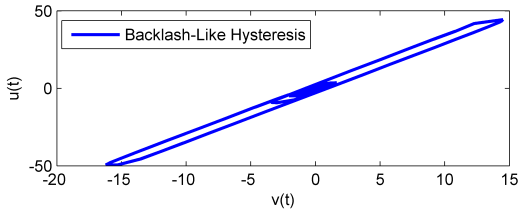


FIGURE 19. The Phase Portrait of Backlash-like Hysteresis

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