L_K-INTERIOR SYSTEMS AS SYSTEMS OF "ALMOST OPEN" L-SETS

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ABSTRACT. We study interior operators and interior structures in a fuzzy setting. We investigate systems of "almost open" fuzzy sets and the relationships to fuzzy interior operators and fuzzy interior systems.

1. Introduction

Interior and closure operators on ordinary sets belong to fundamental mathematical structures with a number of applications, both in mathematics (e.g. topology, logic) and other areas (e.g. data mining, knowledge representation, deductive reasoning). In fuzzy set theory, both general interior operators, which operate with fuzzy sets (so called fuzzy interior operators) and several particular interior operators have been studied, e.g. operators in fuzzy topology, formal concept formation operators in formal concept analysis and operators induces by binary fuzzy relations, see e.g. [1, 2, 3, 4, 5, 6, 13, 14].

Recall that an ordinary (crisp) interior operator on X is a mapping $I : \mathbf{2}^X \to \mathbf{2}^X$ satisfying the following conditions: (I1) $I(A) \subseteq A$, (I2) if $A \subseteq B$ then $I(A) \subseteq I(B)$, and (I3) $I(A) = I(I(A))$, for any $A, B \in \mathbf{2}^X$. An ordinary (crisp) closure operator on X is a mapping $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfying the following conditions: (C1) $A \subseteq C(A)$, (C2) if $A \subseteq B$ then $C(A) \subseteq C(B)$, and (C3) $C(A) = C(C(A))$, for any $A, B \in \mathbf{2}^X$. It is a well known fact that given an interior operator I and a closure operator C, putting $C_I(A) = \overline{I(\overline{A})}$ and $I_C(A) = \overline{C(\overline{A})}$, C_I is a closure operator and I_C is an interior operator. Moreover, the mappings thus defined are bijective. That is, having developed the theory of interior operators, one can automatically obtain the theory of closure operators. Then we can easily transfer true statements about interior operators to corresponding true statements about closure operators and vice versa. As an easy observation shows, this is possible due to the law of double negation (which says that for each set A we have $\overline{\overline{A}} = A$ with $\overline{B}$ denoting the complement of B) which is true in ordinary set theory. In general however, the law of double negation does not hold in fuzzy set theory. This means that the easy one–to–one relation between closure and interior operators is not applicable in fuzzy set theory, and therefore fuzzy interior operators and fuzzy interior systems have to be studied separately from fuzzy closure operators and fuzzy closure systems.

In earlier studies, monotony for fuzzy interior operators meant just (I2) with A and B being fuzzy sets and $A \subseteq B$ meaning that $A(x) \le B(x)$ for each element x of the universe set. As shown e.g. in [2], several fuzzy interior operators satisfy stronger conditions of monotony which enable us to tell more about the interior operator. For instance, one can obtain generalizations of theorems from the bivalent case for which the above weaker monotony is not sufficient.

A natural idea is to consider the property "to be closed (w.r.t. a given fuzzy closure operator C)" resp. "to be open (w.r.t. a given fuzzy interior operator I)" a graded property. In [5] the author studied the so called $\mathbf{L}_K$-closure systems as systems of "almost closed" $\mathbf{L}$-sets. (An $\mathbf{L}$-set A can be considered to be "almost closed w.r.t. C" iff "A almost equals C(A)", and then fuzzy closure systems can be defined as systems of "almost closed" fuzzy sets).

Naturally, we may ask if analogous statements hold for $\mathbf{L}_K$-interior systems. There are several papers dealing with the correspondence between closure and interior concepts, so it worthwhile to solve this problem as well and because of the the missing one-to-one correspondence between fuzzy closure and interior operators this has to be studied separately and that is what this paper aims to do.

Here an $\mathbf{L}$-set A can be considered to be "almost open w.r.t. I" iff "A almost equals I(A)", and the question is whether fuzzy interior systems can be defined as systems of "almost open" fuzzy sets.

2. Preliminaries

Complete residuated lattices, first introduced in the 1930s in ring theory, were defined in the context of fuzzy logic by Goguen [7, 8]. Various logical calculi were investigated using residuated lattices or particular types of residuated lattices. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is a commutative and associative binary operation on L satisfying $a \otimes 1 = a$), and $\otimes, \rightarrow$ form an adjoint pair, i.e. $a \otimes b \le c$ if and only if $a \le b \rightarrow c$ is valid for each $a, b, c \in L$. In the following, $\mathbf{L}$ denotes an arbitrary complete residuated lattice (with L being the universe set of $\mathbf{L}$). All properties of complete residuated lattices used in the sequel are well-known and can be found e.g. in [11]. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [5, 10, 12]).

We recall that the most studied and applied residuated lattices are those defined on the real unit interval $[0, 1]$ or on some subchain. It can be shown (see e.g. [11]) that $\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1 \rangle$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $\rightarrow$ is defined by $a \rightarrow b = \max\{c \mid a \otimes c \le b\}$. A t-norm is a binary operation on $[0, 1]$ which is associative, commutative, monotone, and has 1 as its neutral element, hence, capturing the basic properties of conjunction. A t-norm is called left-continuous if, as a real function, it is left-continuous in both arguments. Most commonly used t-norms are continuous. The

basic three of these are Łukasiewicz t-norm (given by $a \otimes b = \max(a + b - 1, 0)$ with the corresponding residuum $a \to b = \min(1 - a + b, 1)$), minimum (also called Gödel) t-norm ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \le b$ and $= b$ else), and product t-norm ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \le b$ and $= b/a$ else). A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$.

A fuzzy set with truth degrees from a complete residuated lattice $\mathbf{L}$ (also simply an $\mathbf{L}$-set) in a universe set X is any mapping $A : X \to L$, $A(x) \in L$ being interpreted as the truth value of "x belongs to A".

For $\mathbf{L}$-sets A and B in X, we define $E(A, B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$ (degree of equality of A and B) and $S(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$ (degree of subsethood of A in B). Note that $\leftrightarrow$ is defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. Clearly, $E(A, B) = S(A, B) \wedge S(B, A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A, B) = 1$, i.e. for each $x \in X$, $A(x) \le B(x)$, $A = B$ iff $E(A, B) = 1$, and $A \ne B$ iff $E(A, B) < 1$. $A \subset B$ means $A \subseteq B$ and $A \ne B$. It is easy to see that if $A_1 \subseteq A_2$, then $S(A_2, B) \le S(A_1, B)$ for any $A_1, A_2, B \in \mathbf{L}^X$. The set of all $\mathbf{L}$-sets in X will be denoted by $\mathbf{L}^X$. Note that the operations of $\mathbf{L}$ induce corresponding operations on L^X. For example, we have union $\bigcup$ on $\mathbf{L}^X$ induced by the supremum $\bigvee$ of L by $(\bigcup_{i \in I} A_i)(x) = \bigvee_{i \in I} A_i(x)$, etc.

A nonempty subset $K \subseteq L$ is called a $\le$-filter if for every $a, b \in L$ such that $a \le b$ we have $b \in K$ whenever $a \in K$. Then a $\le$-filter K is a filter if $a, b, \in K$ implies $a \otimes b \in K$. In the following, we denote by K an arbitrary $\le$-filter in $\mathbf{L}$, and by X we denote some fixed nonempty set.

Recall that an $\mathbf{L}_K$-interior operator (fuzzy interior operator) on X is a mapping $\mathrm{I} : \mathbf{L}^X \to \mathbf{L}^X$ satisfying: (FI1): $\mathrm{I}(A) \subseteq A$, (FI2): $S(A_1, A_2) \le S(\mathrm{I}(A_1), \mathrm{I}(A_2))$ whenever $S(A_1, A_2) \in K$, and (FI3): $\mathrm{I}(A) = \mathrm{I}(\mathrm{I}(A))$ for every $A, A_1, A_2 \in \mathbf{L}^X$. A system $\mathcal{S} = \{A_i \in \mathbf{L}^X \mid i \in I\}$ is called closed under S_K-unions iff for each $A \in \mathbf{L}^X$ we have $\bigcup_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i \in \mathcal{S}$, where

$$(\bigcup_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i)(x) = \bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i(x)$$

for each $x \in X$. A system closed under S_K-unions is called an $\mathbf{L}_K$-interior system. Loosely speaking, $\mathcal{S}$ is closed under S_K-unions iff for each fuzzy set A in X, the union of all $A_i \in \mathcal{S}$ which are almost included in A, belongs to $\mathcal{S}$.

Given an $\mathbf{L}_K$-interior operator I, and an $\mathbf{L}_K$-interior system $\mathcal{S}$, we put $\mathcal{S}_{\mathrm{I}} = \{\mathrm{I}(A) \mid A \in \mathbf{L}^X\}$, and define $\mathrm{I}_{\mathcal{S}} : \mathbf{L}^X \to \mathbf{L}^X$, by $\mathrm{I}_{\mathcal{S}}(A)(x) = \bigvee_{i \in I, S(A_i, A) \in K}(S(A_i, A) \otimes A_i(x))$ for any $A \in \mathbf{L}^X$. Then $\mathcal{S}_{\mathrm{I}}$ is an $\mathbf{L}_K$-interior system, and if K i a filter then $\mathrm{I}_{\mathcal{S}}$ is an $\mathbf{L}_K$-interior operator. Moreover we have the following Theorem (see [2]):

Proposition 2.1. *Let* I *be an* $\mathbf{L}_K$-*interior operator on* X, $\mathcal{S}$ *be an* $\mathbf{L}_K$-*interior system on* X *and* K *be a filter in* $\mathbf{L}$. *Then* $\mathcal{S}_{\mathrm{I}}$ *is an* $\mathbf{L}_K$-*interior system,* $\mathrm{I}_{\mathcal{S}}$ *is an* $\mathbf{L}_K$-*interiors operator on* X, *and we have* $\mathrm{I} = \mathrm{I}_{\mathcal{S}_{\mathrm{I}}}$ *and* $\mathcal{S} = \mathcal{S}_{\mathrm{I}_{\mathcal{S}}}$, *i.e. the mappings* $\mathrm{I} \to \mathcal{S}_{\mathrm{I}}$ *and* $\mathcal{S} \to \mathrm{I}_{\mathcal{S}}$ *are mutually inverse.*

Remark 2.2. From the proof of Proposition 2.1 (see [2]) one may verify that $\mathrm{I} = \mathrm{I}_{\mathcal{S}_{\mathrm{I}}}$ holds even if K is just a $\le$-filter.

3. $\mathbf{L}_K$-interior Systems as Systems of "Almost Open" L-sets

Definition 3.1. An **L**-system $M \in \mathbf{L}^{\mathbf{L}^X}$ is called an $\mathbf{L}_K$-interior **L**-system in X if for every $A, B \in \mathbf{L}^X$ we have

$$M(\bigcup_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i) = 1, \tag{1}$$

$$M(A) \otimes S(A, B) \otimes S(B, A) \leq M(B) \text{ whenever } S(A, B) \in K. \tag{2}$$

Remark 3.2. (i) The **L**-set $\bigcup_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i$ in X is defined by

$$(\bigcup_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i)(x) =$$

$$= \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i(x).$$

(ii) An $\mathbf{L}_K$-interior **L**-system is therefore an **L**-set of **L**-sets in X. We interpret $M(A)$ as the degree to which $A \in \mathbf{L}^X$ is open. Condition (2) is naturally interpreted as the requirement that an **L**-set that is both a subset and a superset of an "almost open" **L**-set is itself "almost open".

Example 3.3. Let **L** be a residuated lattice where $L = \{0, 0.5, 1\}$ with Łukasiewicz structure. Take $X = \{x_1, x_2\}$, and define M by $M(\{0/x_1, 0/x_2\}) = M(\{0.5/x_1, 1/x_2\}) = 1$, $M(\{1/x_1, 1/x_2\}) = 0.5$, and $= 0$ otherwise. An easy inspection shows that M is an $\mathbf{L}_{\{1\}}$-interior **L**-system in X. Now take $K = \{0.5, 1\}$, we have

$$M(\bigcup_{A_i \in \mathbf{L}^X, S(A_i, \{0.5/x_1, 0.5/x_2\}) \in K} M(A_i) \otimes S(A_i, \{0.5/x_1, 0.5/x_2\}) \otimes A_i) =$$

$M(\{0/x_1, 0.5/x_2, \}) = 0$, i.e. M is not an $\mathbf{L}_{\{0.5, 1\}}$-interior **L**-system in X.

In the following we shall investigate the relationship between $\mathbf{L}_K$-interior **L**-systems, $\mathbf{L}_K$-interior operators, and $\mathbf{L}_K$-interior systems. To this end we define the following mappings:

For an $\mathbf{L}_K$-interior operator I in X and an $\mathbf{L}_K$-interior system $\mathcal{S}$ in X we define **L**-sets M_{I} and $M_{\mathcal{S}}$ in $\mathbf{L}^X$ by

$$M_{\mathrm{I}}(A) = E(A, \mathrm{I}(A)), \tag{3}$$

$$M_{\mathcal{S}}(A) = E(A, \mathrm{I}_{\mathcal{S}}(A)). \tag{4}$$

Hence we have $M_{\mathrm{I}}(A) = S(A, \mathrm{I}(A))$ and $M_{\mathcal{S}}(A) = S(A, \mathrm{I}_{\mathcal{S}}(A))$.

For an $\mathbf{L}_K$-interior $\mathbf{L}$-system M in X we define a mapping $\mathrm{I}_M : \mathbf{L}^X \to \mathbf{L}^X$ and a set $\mathcal{S}_M \subseteq \mathbf{L}^X$ by

$$(\mathrm{I}_M(A))(x) = \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i(x) \tag{5}$$

$$\mathcal{S}_M = \{ A \in \mathbf{L}^X \mid M(A) = 1 \}. \tag{6}$$

Lemma 3.4. *For an $\mathbf{L}_K$-interior operator I in X we have $\mathrm{I}_{M_{\mathrm{I}}} = \mathrm{I}_{\mathcal{S}_{\mathrm{I}}}$.*

Proof. Take any $A \in \mathbf{L}^X, x \in X$. We have to show $(\mathrm{I}_{M_{\mathrm{I}}}(A))(x) = (\mathrm{I}_{\mathcal{S}_{\mathrm{I}}}(A))(x)$.

"$\geq$":

$$(\mathrm{I}_{M_{\mathrm{I}}}(A))(x) = \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M_{\mathrm{I}}(A_i) \otimes S(A_i, A) \otimes A_i(x) \geq$$

$$\geq \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K, M_{\mathrm{I}}(A_i) = 1} M_{\mathrm{I}}(A_i) \otimes S(A_i, A) \otimes A_i(x) =$$

$$= \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K, A_i = \mathrm{I}(A_i)} S(A_i, A) \otimes A_i(x) = (\mathrm{I}_{\mathcal{S}_{\mathrm{I}}}(A))(x).$$

"$\leq$": By definitions, the inequality holds iff

$$M_{\mathrm{I}}(A_j) \otimes S(A_j, A) \otimes A_j(x) \leq \mathrm{I}_{\mathcal{S}_{\mathrm{I}}}(x)$$

for any j such that $S(A_j, A) \in K$. Since K is a $\leq$-filter in $\mathbf{L}$, $S(A_j, A) \in K$ and $S(A_j, A) \leq S(\mathrm{I}(A_j), A)$ imply $S(\mathrm{I}(A_j), A) \in K$. Therefore

$$\mathrm{I}_{\mathcal{S}_{\mathrm{I}}}(x) = \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K, A_i = \mathrm{I}(A_i)} S(A_i, A) \otimes A_i(x) \geq$$

$$\geq S(\mathrm{I}(A_j), A) \otimes \mathrm{I}(A_j)(x).$$

Hence it suffices to show that

$$S(\mathrm{I}(A_j), A) \otimes \mathrm{I}(A_j)(x) \geq M_{\mathrm{I}}(A_j) \otimes S(A_j, A) \otimes A_j(x),$$

. Indeed,

$$M_{\mathrm{I}}(A_j) \otimes S(A_j, A) \otimes A_j(x) = S(A_j, \mathrm{I}(A_j)) \otimes S(A_j, A) \otimes A_j(x) \leq$$

$$\leq \mathrm{I}(A_j)(x) \otimes S(A_j, A) \leq \mathrm{I}(A_j)(x) \otimes S(\mathrm{I}(A_j), A),$$

. However, $A_j(x) \otimes S(A_j, \mathrm{I}(A_j)) = A_j(x) \otimes \bigwedge_{y \in X}(A_j(y) \to \mathrm{I}(A_j)(y)) \leq$ $\leq A_j(x) \otimes (A_j(x) \to \mathrm{I}(A_j)(x)) \leq \mathrm{I}(A_j)(x)$, and the claim follows. $\qquad\square$

Lemma 3.5. *For any $\mathbf{L}_K$-interior operator I in X, M_{I} is an $\mathbf{L}_K$-interior $\mathbf{L}$-system in X.*

Proof. We show M_I satisfies (1) and (2).

(1): We have to show that for any $A \in \mathbf{L}^X$ we have

$$M_\mathrm{I}(\bigcup_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M_\mathrm{I}(A_i) \otimes S(A_i, A) \otimes A_i) = 1$$

i.e. $M_\mathrm{I}(\mathrm{I}_{M_\mathrm{I}}(A)) = 1$, i.e. $\mathrm{I}_{M_\mathrm{I}}(A) = \mathrm{I}(\mathrm{I}_{M_\mathrm{I}}(A))$. The last equality follows from the idempotency of I by observing that $\mathrm{I}_{M_\mathrm{I}} = \mathrm{I}_{\mathcal{S}_\mathrm{I}} = \mathrm{I}$ (Lemma 3.4 and Remark 2.2).

(2): We have to show that $M_\mathrm{I}(A) \otimes S(A, B) \otimes S(B, A) \leq M_\mathrm{I}(B)$, whenever $S(A, B) \in K$, i.e.

$$S(A, \mathrm{I}(A)) \otimes S(A, B) \otimes S(B, A) \leq S(B, \mathrm{I}(B))$$

which holds iff for each $x \in X$ we have

$$B(x) \otimes S(A, \mathrm{I}(A)) \otimes S(A, B) \otimes S(B, A) \leq \mathrm{I}(B)(x).$$

The last inequality is true because

$$B(x) \otimes S(A, \mathrm{I}(A)) \otimes S(A, B) \otimes S(B, A) \leq$$
$$\leq B(x) \otimes S(B, A) \otimes S(A, \mathrm{I}(A)) \otimes S(\mathrm{I}(A), \mathrm{I}(B)) \leq \mathrm{I}(B)(x).$$

$\square$

Lemma 3.6. *Let K be a filter in $\mathbf{L}$. For any $\mathbf{L}_K$-interior $\mathbf{L}$-system M in X, I_M is an $\mathbf{L}_K$-interior operator in X.*

Proof. We verify (FI1)–(FI3)

(FI1): $\mathrm{I}_M(A) \subseteq A$ holds iff $M(A_i) \otimes S(A_i, A) \otimes A_i(x) \leq A(x)$ for any $x \in X$, and i such that $S(A_i, A) \in K$ which is true because

$$M(A_i) \otimes S(A_i, A) \otimes A_i(x) \leq A_i(x) \otimes S(A_i, A) \leq A(x).$$

(FI2): Let $S(A, B) \in K$. $S(A, B) \leq S(\mathrm{I}_M(A), \mathrm{I}_M(B))$ is true iff for each $x \in X$ we have $S(A, B) \otimes \mathrm{I}_M(A)(x) \leq \mathrm{I}_M(B)(x)$ iff for any $A_j \in \mathbf{L}^X$ such that $S(A_j, A) \in K$ we have

$$S(A, B) \otimes M(A_j) \otimes S(A_j, A) \otimes A_j(x) \leq \mathrm{I}_M(B)(x),$$

but because $S(A_j, A) \in K, S(A, B) \in K$ yield $S(A_j, B) \in K$ ($S(A_j, A) \otimes S(A, B) \leq S(A_j, B)$, and K is a filter) we have

$$S(A, B) \otimes M(A_j) \otimes S(A_j, A) \otimes A_j(x) \leq M(A_j) \otimes S(A_j, B) \otimes A_j(x) \leq$$
$$\leq \bigvee_{A_i \in \mathbf{L}^X, S(A_i, B) \in K} M(A_i) \otimes S(A_i, B) \otimes A_i(x) = \mathrm{I}_M(B).$$

(FI3): It suffices to prove $\mathrm{I}_M(A) \subseteq \mathrm{I}_M(\mathrm{I}_M(A))$.

$$\mathrm{I}_M(\mathrm{I}_M(A)) = \bigvee_{A_i \in \mathbf{L}^X, S(A_i, \mathrm{I}_M(A)) \in K} M(A_i) \otimes S(A_i, \mathrm{I}_M(A)) \otimes A_i(x) \geq$$
$$\geq M(\mathrm{I}_M(A)) \otimes S(\mathrm{I}_M(A), \mathrm{I}_M(A)) \otimes \mathrm{I}_M(A)(x) = 1 \otimes \mathrm{I}_M(A)(x).$$

$\square$

The relationship between $\mathbf{L}_K$-interior operators, $\mathbf{L}_K$-interior systems, and $\mathbf{L}_K$-interior $\mathbf{L}$-systems is the subject of the following theorems.
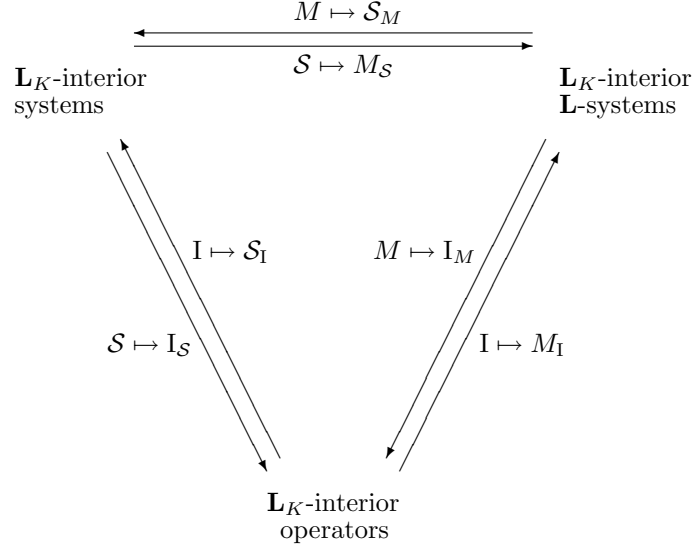
FIGURE 1. Commuting Diagram of Corollary 3.9

Theorem 3.7. *Let* I *be an* $\mathbf{L}_K$-*interior operator in* X, M *be an* $\mathbf{L}_K$-*interior* **L**-*system,* K *be a filter in* **L**. *Then* M_{I} *is an* $\mathbf{L}_K$-*interior* **L**-*system in* X, I_M *is an* $\mathbf{L}_K$-*interior operator in* X, *and* $\mathrm{I} = \mathrm{I}_{M_{\mathrm{I}}}$ *and* $M = M_{\mathrm{I}_M}$, *i.e. the mappings* $\mathrm{I} \to M_{\mathrm{I}}$ *and* $M \to \mathrm{I}_M$ *are mutually inverse.*

Proof. By Lemmas 3.5 and 3.6 it remains to verify $\mathrm{I} = \mathrm{I}_{M_{\mathrm{I}}}$ and $M = M_{\mathrm{I}_M}$. By Lemma 3.4 and Proposition 2.1 $\mathrm{I}_{M_{\mathrm{I}}} = \mathrm{I}_{\mathcal{S}_{\mathrm{I}}} = \mathrm{I}$. Since $M_{\mathrm{I}_M}(A) = S(A, \mathrm{I}_M(A))$, it remains to prove $M(A) = S(A, \mathrm{I}_M(A))$:

On the one hand, $M(A) \leq S(A, \mathrm{I}_M(A))$ iff for each $x \in X$ we have $M(A) \otimes A(x) \leq \mathrm{I}_M(A)(x)$, i.e.

$$M(A) \otimes A(x) = M(A) \otimes S(A, A) \otimes A(x) \leq$$

$$\leq \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i(x) = \mathrm{I}_M(A)(x).$$

On the other hand,

$$S(A, \mathrm{I}_M(A)) = M(\mathrm{I}_M(A)) \otimes S(A, \mathrm{I}_M(A)) \otimes S(\mathrm{I}_M(A), A) \leq M(A),$$

by (2). □

Theorem 3.8. *Let* $\mathcal{S}$ *be an* $\mathbf{L}_K$-*interior system in* X, M *be an* $\mathbf{L}_K$-*interior* **L**-*system,* K *be a filter in* **L**. *Then* $M_{\mathcal{S}}$ *is an* $\mathbf{L}_K$-*interior* **L**-*system in* X, $\mathcal{S}_M$ *is an* $\mathbf{L}_K$-*interior system in* X, *and* $\mathcal{S} = \mathcal{S}_{M_{\mathcal{S}}}$ *and* $M = M_{\mathcal{S}_M}$, *i.e. the mappings* $\mathcal{S} \to M_{\mathcal{S}}$ *and* $M \to \mathcal{S}_M$ *are mutually inverse.*

Proof. By definition $M_{\mathcal{S}} = M_{\mathrm{I}_{\mathcal{S}}}$. Therefore, by Lemma 3.5, $M_{\mathcal{S}}$ is an $\mathbf{L}_K$-interior system. To see that $\mathcal{S}_M$ is an $\mathbf{L}_K$-interior system it suffices to show that $\mathcal{S}_M = \mathcal{S}_{\mathrm{I}_M}$ (by Proposition 2.1 and by Lemma 3.6), i.e.

$$\{A \in \mathbf{L}^X \mid M(A) = 1\} = \{A \in \mathbf{L}^X \mid A = \mathrm{I}_M(A)\}.$$

Now, $M(A) = 1$ implies

$$\mathrm{I}_M(A)(x) = \bigvee_{A_i \in \mathbf{L}^X, S(A_i, A) \in K} M(A_i) \otimes S(A_i, A) \otimes A_i(x) \geq$$

$$\geq M(A) \otimes S(A, A) \otimes A(x) = A(x)$$

i.e. $A = \mathrm{I}_M(A)$. On the other hand, $A = \mathrm{I}_M(A)$ implies (using (1) $1 = M(\mathrm{I}_M(A)) = M(A)$.

We show that $\mathcal{S} = \mathcal{S}_{M_{\mathcal{S}}}$: We have $A \in \mathcal{S}$ iff $A = \mathrm{I}_{\mathcal{S}}(A)$ iff $M_{\mathcal{S}}(A) = 1$ iff $A \in \mathcal{S}_{M_{\mathcal{S}}}$. It remains to Show that $M(A) = M_{\mathcal{S}_M}(A)$: We have $\mathrm{I}_M = \mathrm{I}_{\mathcal{S}_{\mathrm{I}_M}}$ (see Proposition 2.1) and (by the above observation) $\mathcal{S}_{\mathrm{I}_M} = \mathcal{S}_M$. Therefore, $\mathrm{I}_M = \mathrm{I}_{\mathcal{S}_M}$. Using $M(A) = S(A, \mathrm{I}_M(A))$ (see the end of the proof of Theorem 3.7), we conclude $M(A) = S(A, \mathrm{I}_M(A)) = S(A, \mathrm{I}_{\mathcal{S}_M}(A)) = M_{\mathcal{S}_M}(A)$ completing the proof. $\qquad\square$

Corollary 3.9. *Under the above notation, the diagram in Figure 1 commutes.*

Proof. Each oriented path in the diagram in Figure 1 defines a mapping (the one composed of the mappings represented by the arrows). The assertion says that any two mappings corresponding to oriented paths with common starting and final node are equal. The coro;ary follows from the definitions, Proposition 2.1, Theorems 3.7, and 3.8, and the proof of Theorem 3.8. $\qquad\square$

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