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L-CONVEX SYSTEMS AND THE CATEGORICAL ISOMORPHISM TO SCOTT-HULL OPERATORS

C. SHEN AND F. G. SHI

ABSTRACT. The concepts of L-convex systems and Scott-hull spaces are proposed on frame-valued setting. Also, we establish the categorical isomorphism between L-convex systems and Scott-hull spaces. Moreover, it is proved that the category of L-convex structures is bireflective in the category of L-convex systems. Furthermore, the quotient systems of L-convex systems are studied.

1. Introduction

A convexity on a set is a family of subsets closed under arbitrary intersections and under directed unions, which contains the empty set as a member. Convexity theory (also called abstract convexity theory in [23]), which can be regarded as an axiomatization of the properties that usual convex sets fulfill, takes an important part in mathematics. In fact, convexity exists in so many mathematical research areas, such as lattices [4, 21], algebras [8, 12], metric spaces [11], graphs [2, 3, 6] and topological spaces [7, 22].

As we know, matroid theory plays an important role in combinatorial optimization problems [13]. Many real-world problems can be defined and solved by making use of matroid theory. In the classical situation, it is assumed that all weights are precisely known. However, this assumption may be a serious restriction, since in many practical applications the exact values of the weights are unknown in advance. To solve this, Shi proposed the concept of L-fuzzifying matroids in [18, 19], which is a generalization of matroids. In fact, a matroid is a convex structure that satisfies the Exchange Law (see [23] for detail). In this case, extending the theory of convex structures to fuzzy setting is a particularly meaningful topic in areas of both theoretical research and practical application. In 1994, the notions of fuzzy convex structures and convex hull operators were first proposed by Rosa [16]. Based on a completely distributive lattice L, Maruyama [10] generalized Rosa's definition to L-fuzzy setting, and investigated some combinatorial properties of lattice-valued fuzzy convex sets in Euclidean spaces. Recently, Pang and Shi [14] proposed several types of L-convex structures and established their mutual categorical relations. In a completely different way, Shi and Xiu [20] provided a new approach to fuzzification of convex structures and presented the notion of M-fuzzifying convex structures.

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Further, they defined M-fuzzifying hull operators to characterize M-fuzzifying convex structures.

This paper focuses on the fuzzification of convex systems. In the classical setting, convex systems are closely connected with convex structures. Many concepts and results in convex structures can actually be formulated in terms of convex systems, such as convexity preserving mappings, quotient spaces, product spaces and so on. As we know, convex hull operators play an important role in the description of fuzzy convex structures[16]. Precisely, there exists an categorical isomorphism between fuzzy convex structures and convex hull operators. For convex systems, the universal set is not required to be convex, which leads to that some sets don't have a convex hull, and meanwhile makes the admissible sets (the sets having a convex hull) particularly important. This is the key reason that we cannot define an operator in convex systems like the hull operator directly, but it will become possible if we make some changes in the domain of the hull operator. Obviously, the collection of admissible sets is a good choice. Motivated by this idea, we firstly introduce the concept of L-convex systems. Furthermore, by using the collection of all admissible sets as a domain of the hull operator, the notion of Scott-hull spaces is proposed, and also its categorical isomorphism to L-convex systems is constructed.

This paper is organized as follows. In Section 2, we recall some preliminaries on lattices and L-convexities. We devote Section 3 to introducing the concept of L-convex systems, while the definition of Scott-hull spaces and the categorical isomorphism to L-convex systems are presented in Section 4. The relationship between L-convex systems and L-convex structures is constructed in Section 5, which shows that the category of L-convex structures is a bireflective subcategory of the category of L-convex systems, but not a coreflective subcategory. In the last section, we investigate the quotient systems of L-convex systems.

2. **Preliminaries**

For convenience, let's review some basic notions related to fuzzy sets, lattices and L-convexities. For undefined notions in this paper, the reader can refer to [1, 5].

Definition 2.1. [5] A *poset* is a non-empty set L equipped with a reflexive, anti-symmetric and transitive relation $\leq$. We say a subset D of L is *directed* provided it is non-empty and every finite subset of D has an upper bound in D. We say a subset C of L is *totally ordered*, or a *chain*, if it is non-empty and all elements of C are comparable under $\leq$ (that is, either $x \leq y$ or $y \leq x$ for all elements $x, y \in L$). Clearly, each chain is directed.

Suppose L is a complete lattice. The greatest element of L is denoted by $\top$ and the least element of L is denoted by $\bot$. For $S \subseteq L$, write $\bigvee S$ for the least upper bound of S and $\bigwedge S$ for the greatest lower bound of S. In particular, we use the convenient notation $x = \bigvee^{\uparrow} D$ to express that the set D is directed and x is its least upper bound. Also, we adopt the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$.

Definition 2.2. [5] A complete lattice L is called a *complete Heyting algebra*, or a *frame* if L satisfies the infinite distributive law of finite meets over arbitrary joins, that is,

$$a \wedge \bigvee B = \bigvee_{a \in B} a \wedge b$$

for all $a \in L$ and $B \subseteq L$.

Throughout this paper, if there is no further statement, L always denotes a complete Heyting algebra.

For a non-empty set X, 2^X denotes the powerset of X, and 2^X_{fin} denotes the set of all finite subsets of X. Each mapping $A : X \longrightarrow L$ is called an L-subset of X, and we denote the collection of all L-subsets of X by L^X. L^X is also a complete lattice by defining $\leq$ on L^X pointwisely. We call an L-subset A finite if the support set of A is finite, that is, $\mathrm{supp}A = \{x \in X \mid A(x) \neq \bot\}$ is finite. Let L^X_{fin} denote the set of all finite L-subsets of X. For each $A \subseteq X$, its characteristic function $\chi_A \in L^X$ is defined as follows:

$$\chi_A(x) = \begin{cases} \top, & x \in A, \\ \bot, & x \notin A. \end{cases}$$

Given a mapping $f : X \longrightarrow Y$, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ (see [15]) by

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x), \quad f_L^{\leftarrow}(B) = B \circ f$$

for all $A \in L^X$, $y \in Y$ and $B \in L^Y$.

Definition 2.3. [10, 16, 17] A subset $\mathcal{C}$ of L^X is called an *L-convexity*, if it satisfies the following conditions:

(C1) $\chi_X, \chi_\emptyset \in \mathcal{C}$;
(C2) if $\{A_i \mid i \in I\} \subseteq \mathcal{C}$, then $\bigwedge_{i \in I} A_i \in \mathcal{C}$;
(C3) if $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ is directed, then $\bigvee^{\uparrow}_{i \in I} D_i \in \mathcal{C}$.

In this case, we call the pair $(X, \mathcal{C})$ an *L-convex structure*, or an *L-convexity space*, and each $A \in \mathcal{C}$ an *L-convex set*.

Definition 2.4. [10, 16, 17] Let $(X, \mathcal{C})$ be an L-convex structure. For each $A \in L^X$, we define

$$co(A) = \bigwedge\{B \in \mathcal{C} \mid A \leq B\},$$

that is, $co(A)$ is the least element of $\mathcal{C}$ that contains A, called the *L-convex hull* of A. The operator co is called the *L-convex hull* on $(X, \mathcal{C})$.

Definition 2.5. [10, 16, 17] A mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between two L-convex structures is called *L-convexity preserving* (*CP*, for short) provided that $B \in \mathcal{C}_Y$ implies $f_L^{\leftarrow}(B) \in \mathcal{C}_X$.

The category whose objects are L-convex structures and whose morphisms are CP mappings will be denoted by L-**CS**.

Definition 2.6. [16, 17, 24] Let R be an equivalence relation on an L-convex structure $(X, \mathcal{C})$, that is, $R \subseteq X \times X$ is a reflexive, symmetric and transitive binary relation. The quotient set X/R consists of all R-equivalence classes (usually, we use the notations $\widetilde{A}, \widetilde{B}, \widetilde{C} \cdots$ to represent the elements of $L^{X/R}$), and the corresponding quotient mapping $q : X \longrightarrow X/R$ assigns a point of X to its R-equivalence class, that is, $q(x) = \{y \in X \mid (x, y) \in R\}$ for all $x \in X$. The collection $\mathcal{C}/R$ on X/R defined by

$$\mathcal{C}/R = \left\{ \widetilde{C} \in L^{X/R} \middle| q_L^{\leftarrow}\left(\widetilde{C}\right) \in \mathcal{C} \right\}$$

forms an L-convex structure. We call $(X/R, \mathcal{C}/R)$ the *quotient space* of $(X, \mathcal{C})$, and the L-convexity $\mathcal{C}/R$ is called a *quotient L-convexity.*

3. L-convex Systems

In this section, we will mainly introduce the concept of L-convex systems. First, the properties of admissible sets and partially L-convex sets are studied. Besides, two types of partial L-convexity preserving mappings between L-convex systems are proposed. Also, their relationships are investigated.

Definition 3.1. A subset $\mathcal{E}$ of L^X is called a *partial L-convexity*, if it satisfies the following conditions:

(PC1) $\chi_\emptyset \in \mathcal{E}$;

(PC2) if $\{A_i \mid i \in I\} \subseteq \mathcal{E}$ is non-empty, then $\bigwedge_{i \in I} A_i \in \mathcal{E}$;

(PC3) if $\{D_i \mid i \in I\} \subseteq \mathcal{E}$ is directed, then $\bigvee^{\uparrow}_{i \in I} D_i \in \mathcal{E}$.

In this case, we call the pair $(X, \mathcal{E})$ an *L-convex system*, and each $A \in \mathcal{E}$ a *partially L-convex set.*

For a poset, Markowsky's Theorem (see [9]) states that the dcpos are exactly the chain-complete posets, that is, a poset P is closed under directed unions iff each chain in P has a supremum. Hence, we obtain the following proposition.

Proposition 3.2. *A family $\mathcal{E} \subseteq L^X$ is a partial L-convexity iff it satisfies* (PC1), (PC2) *and the following condition:*

(PC3)* *if $\{D_i \mid i \in I\} \subseteq \mathcal{E}$ is totally ordered, then $\bigvee^{\uparrow}_{i \in I} D_i \in \mathcal{E}$.*

Example 3.3. Let V be a vector space over a totally ordered field K. A set $W \subseteq V$ is called a subspace of V provided for all $x, y \in W$ and for each $k, l \in K$,

$$kx + ly \in W.$$

We write $sub(V)$ for the collection of all subspaces of V. It is easy to check that $sub(V)$ is a convex structure, and hence a convex system. Next, take a subset $S \subseteq V$, and define $\Phi \subseteq L^V$ as follows:

$$\Phi = \left\{ \mu \in L^S \mid \forall \alpha \in L,\ \mu_{[\alpha]} \in sub(V) \right\},$$

where $\mu_{[\alpha]} = \{x \in S \mid \mu(x) \geq \alpha\}$. Then it is easy to verify that Φ is an L-convex system on S, but not an L-convex structure in general. Moreover, the family Φ is an L-convex structure iff S is a subspace of V.

For L-convex systems, the difference with L-convex structures is that the greatest L-subset does not need to be L-convex, which results in a rather fundamental consequence: not every L-subset has an L-convex hull. Precisely, for an L-convex system $(X, \mathcal{E})$ and $A \in L^X$, since the family $\{B \in \mathcal{E} \mid A \leq B\}$ probably be empty, it means the right hand side of the following equation:

$$co(A) = \bigwedge\{B \in \mathcal{E} \mid A \leq B\}$$

could be equal to χ_X. Note that χ_X is not an L-convex set in general. It follows that $co(A)$ (the least element in $\mathcal{E}$ containing A) may not exist in $(X, \mathcal{E})$. In this case, the next definition is needed.

Definition 3.4. Let $(X, \mathcal{E})$ be an L-convex system. An L-subset $A \in L^X$ is called *admissible*, provided it is included in some partially L-convex set, and the collection of all admissible L-subsets is denoted by $ads(X, \mathcal{E})$, that is,

$$ads(X, \mathcal{E}) = \{A \in \mathbf{2}^X \mid \exists B \in \mathcal{E} \text{ such that } A \leq B\}.$$

Definition 3.5. Let $(X, \mathcal{E})$ be an L-convex system. For each $A \in ads(X, \mathcal{E})$, we define

$$co(A) = \bigwedge\{B \in \mathcal{E} \mid A \leq B\},$$

that is, $co(A)$ is the least element of $\mathcal{E}$ that contains A, called the *L-convex hull* of A. The operator co is called the *partially L-convex hull operator* on $(X, \mathcal{E})$.

Remark 3.6. (1) For each L-convex structure $(X, \mathcal{C})$, $ads(X, \mathcal{C}) = L^X$.

(2) $\mathcal{E} \subseteq ads(X, \mathcal{E})$, and hence co can be regarded as a mapping on $ads(X, \mathcal{E})$.

Proposition 3.7. *Let $(X, \mathcal{E})$ be an L-convex system and let co be the partially L-convex hull operator on $(X, \mathcal{E})$. Then for any $A, B \in ads(X, \mathcal{C})$ and each directed family $\{A_i \mid i \in I\} \subseteq ads(X, \mathcal{C})$, the operator co satisfies the following properties:*

(PH1) *Normalization Law: $co(\chi_\emptyset) = \chi_\emptyset$;*

(PH2) *Extensive Law: $A \leq co(A)$;*

(PH3) *Idempotent Law: $co\,(co\,(A)) = co(A)$;*

(PH4) *Directed-preserving Law: $co\left(\bigvee^{\uparrow}_{i\in I} A_i\right) = \bigvee^{\uparrow}_{i\in I} co(A_i)$.*

Moreover, (PH4) *implies*

(PH5) *Monotone Law: $A \leq B$, then $co(A) \leq co(B)$;*

Proof. The verifications of (PH1) and (PH2) are straightforward.

(PH3) By (PC2) and $A \in ads(X, \mathcal{C})$, we know $co(A) = \bigwedge\{B \in \mathcal{E} \mid A \leq B\} \in \mathcal{E}$. Then we obtain $co(co(A)) = \bigwedge\left\{B \in L^X \mid co(A) \leq B \in \mathcal{E}\right\} \leq co(A)$. The inverse inequality $co(A) \leq co(co(A))$ holds obviously.

(PH4) It is trivial that co is monotone, and therefore $\{co(A_i) \mid i \in I\} \subseteq \mathcal{E}$ is directed. Further, by (PC3), we have $\bigvee^{\uparrow}_{i\in I} co(A_i) \in \mathcal{E}$. Since $\bigvee^{\uparrow}_{i\in I} A_i \leq \bigvee^{\uparrow}_{i\in I} co(A_i)$, it follows that

$$co\left(\bigvee_{i\in I}^{\uparrow} A_i\right) = \bigwedge\left\{B \in L^X \,\middle|\, \bigvee_{i\in I}^{\uparrow} A_i \leq B \in \mathcal{E}\right\} \leq \bigvee_{i\in I}^{\uparrow} co(A_i).$$

The inverse inequality $\bigvee^{\uparrow}_{i\in I} co(A_i) \leq co\left(\bigvee^{\uparrow}_{i\in I} A_i\right)$ holds obviously. □

Proposition 3.8. *Let $(X, \mathcal{E})$ be an L-convex system. Then $ads(X, \mathcal{E})$ is a non-empty Scott closed set on L^X with the pointwise order, that is, it satisfies the following conditions:*

(SC1) *$ads(X, \mathcal{E})$ is non-empty;*
(SC2) *$ads(X, \mathcal{E})$ is a lower set, that is, if $A \leq B$ and $B \in ads(X, \mathcal{E})$, then $A \in ads(X, \mathcal{E})$;*
(SC3) *if $\{D_i \mid i \in I\} \subseteq ads(X, \mathcal{E})$ is directed, then $\bigvee^{\uparrow}_{i\in I} D_i \in ads(X, \mathcal{E})$.*

Proof. The verifications of (SC1) and (SC2) are straightforward. We verify (SC3). Take any directed family $\{D_i \mid i \in I\} \subseteq ads(X, \mathcal{E})$. It follows that $\{co(D_i) \mid i \in I\} \subseteq \mathcal{E}$ is directed. Then $\bigvee^{\uparrow}_{i\in I} D_i \leq \bigvee^{\uparrow}_{i\in I} co(D_i) \in \mathcal{E}$. Hence $\bigvee^{\uparrow}_{i\in I} D_i \in ads(X, \mathcal{E})$. □

Motivated by [23], we propose the notion of homomorphisms between L-convex systems.

Definition 3.9. A mapping $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ between two L-convex systems is called *partial L-convexity preserving* (*PCP*, for short) if it satisfies the following conditions:

(PCP1) for each $A \in ads(X, \mathcal{E}_X)$, $f_L^{\rightarrow}(A) \in ads(Y, \mathcal{E}_Y)$;
(PCP2) if $B \in \mathcal{E}_Y$, then for each $A \in ads(X, \mathcal{E}_X)$ such that $A \leq f_L^{\leftarrow}(B)$, we have $co_X(A) \leq f_L^{\leftarrow}(B)$.

Next, we will show some equivalent conditions of PCP mappings which are relatively simple in form. Before this, the following lemma is needed.

Lemma 3.10. *Let $(X, \mathcal{E})$ be an L-convex system and let $A \in ads(X, \mathcal{E})$. Then $co(A) = \bigvee^{\uparrow}_{F\in \mathbf{2}^X_{fin}} co(\chi_F \wedge A)$.*

Proof. By the directness of $\{co(\chi_F \wedge A) \mid F \in \mathbf{2}^X_{fin}\}$, we have $A \leq \bigvee^{\uparrow}_{F\in \mathbf{2}^X_{fin}} co(\chi_F \wedge A) \in \mathcal{E}$. It follows that $co(A) \leq \bigvee^{\uparrow}_{F\in \mathbf{2}^X_{fin}} co(\chi_F \wedge A)$. The inverse inequality holds obviously by the monotonicity of co. □

Proposition 3.11. *Let $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ be a mapping between two L-convex systems. Then the following conditions are equivalent:*

(1) *f is PCP;*
(2) *for each $A \in ads(X, \mathcal{E})$, we have $f_L^{\rightarrow}(A) \in ads(Y, \mathcal{E}_Y)$ and $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A))$;*
(3) *for each finite L-subset $F \in ads(X, \mathcal{E})$, we have $f_L^{\rightarrow}(F) \in ads(Y, \mathcal{E}_Y)$ and $f_L^{\rightarrow}(co_X(F)) \leq co_Y(f_L^{\rightarrow}(F))$.*

Proof. The implication (2)⇒(3) is clear.

(1)⇒(2). It's trivial that $f_L^{\rightarrow}(A) \in ads(Y, \mathcal{E}_Y)$. Furthermore, by (PCP2) and the following inequality

$$A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \leq f_L^{\leftarrow}(co_Y(f_L^{\rightarrow}(A))),$$

we obtain $co_X(A) \leq f_L^{\leftarrow}(co_Y(f_L^{\rightarrow}(A)))$. Therefore $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A))$.

(3)⇒(1). It suffices to verify (PCP1) and (PCP2). In fact,

(PCP1) Suppose $A \in ads(X, \mathcal{E}_X)$. Then by Lemma 3.10, we have

$$\begin{aligned} f_L^{\rightarrow}(A) &\leq f_L^{\rightarrow}(co_X(A)) \\ &= f_L^{\rightarrow}\left(\bigvee\nolimits^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge A)\right) \\ &= \bigvee\nolimits^{\uparrow}_{F \in \mathbf{2}^X_{fin}} f_L^{\rightarrow}(co_X(\chi_F \wedge A)) \\ &\leq \bigvee\nolimits^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_Y(f_L^{\rightarrow}(\chi_F \wedge A)). \end{aligned}$$

Since the family $\left\{co_Y(f_L^{\rightarrow}(\chi_F \wedge A)) \;\middle|\; F \in \mathbf{2}^X_{fin}\right\} \subseteq \mathcal{E}_Y$ is directed. By(PC3), we obtain $\bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_Y(f_L^{\rightarrow}(\chi_F \wedge A)) \in \mathcal{E}_Y$ and $f_L^{\rightarrow}(A) \leq \bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_Y(f_L^{\rightarrow}(\chi_F \wedge A))$. This shows $f_L^{\rightarrow}(A) \in ads(Y, \mathcal{E}_Y)$, as desired.

(PCP2) Suppose $B \in \mathcal{E}_Y$, $A \in ads(X, \mathcal{E}_X)$ and $A \leq f_L^{\leftarrow}(B)$. Then we obtain $f_L^{\rightarrow}(\chi_F \wedge A) \leq f_L^{\rightarrow}(A) \leq B$ for all $F \in \mathbf{2}^X_{fin}$. Further, since $ads(X, \mathcal{E}_X)$ is a lower set and $\chi_F \wedge A \leq A$, we have $\chi_F \wedge A \in ads(X, \mathcal{E}_X)$. It follows from $B \in \mathcal{E}_Y$ that $co_Y(f_L^{\rightarrow}(\chi_F \wedge A)) \leq B$. From the proof in (PCP1), we have

$$f_L^{\rightarrow}(co_X(A)) \leq \bigvee\nolimits^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_Y(f_L^{\rightarrow}(\chi_F \wedge A)) \leq B.$$

This means $co_X(A) \leq f_L^{\leftarrow}(B)$. □

In order to construct a category of L-convex systems, the following trivial proposition is necessary.

Proposition 3.12. *Suppose both mappings $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ and $g : (Y, \mathcal{E}_Y) \longrightarrow (Z, \mathcal{E}_Z)$ between L-convex systems are PCP. Then the composite mapping $g \circ f$ is also PCP.*

The category whose objects are L-convex systems and whose morphisms are PCP mappings will be denoted by L-**PCS**.

Next, we will propose a new type of mappings between L-convex systems like the CP mappings in Definition 2.5, and investigate the relations with PCP mappings.

Definition 3.13. A mapping $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ between two L-convex systems is called *strongly partial L-convexity preserving* (*SPCP*, for short) if it satisfies the following conditions:

(PCP1) for each $A \in ads(X, \mathcal{E}_X)$, $f_L^{\rightarrow}(A) \in ads(Y, \mathcal{E}_Y)$;

(PCP2)* for each $B \in \mathcal{E}_Y$, $f_L^{\leftarrow}(B) \in \mathcal{E}_X$.

Proposition 3.14. *Let $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ be a mapping between two L-convex systems. Then the following conditions are equivalent:*

(1) *f is SPCP;*

(2) *f is PCP, and satisfies the following condition:*
 (PCP3) *for each $B \in ads(Y, \mathcal{E}_Y)$, $f_L^{\leftarrow}(B) \in ads(X, \mathcal{E}_X)$.*

Proof. (1)⇒(2). Condition (PCP1) is clear, and by an easy induction one can get (PCP2). We verify (PCP3). Take any $B \in ads(Y, \mathcal{E}_Y)$. Then $co_Y(B) = \bigwedge\{C \in$

$\mathcal{E}_Y \mid B \leq C\} \in \mathcal{E}_Y$. It follows from (PCP2)* that $f_L^{\leftarrow}(co_Y(B)) \in \mathcal{E}_X$. Since $f_L^{\leftarrow}(B) \leq f_L^{\leftarrow}(co_Y(B))$, it follows $f_L^{\leftarrow}(B) \in ads(X, \mathcal{E}_X)$.

(2)⇒(1). It suffices to prove (PCP2)*. Take any $B \in \mathcal{E}_Y$. Then $B \in ads(Y, \mathcal{E}_Y)$. By (PCP3) we have $f_L^{\leftarrow}(B) \in ads(X, \mathcal{E}_X)$. Since $\chi_F \wedge f_L^{\leftarrow}(B) \leq f_L^{\leftarrow}(B)$ for each $F \in \mathbf{2}^X_{fin}$, it follows from (SC2) that $\chi_F \wedge f_L^{\leftarrow}(B) \in ads(X, \mathcal{E}_X)$. By the directness of $\left\{co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \;\middle|\; F \in \mathbf{2}^X_{fin}\right\}$ and (PC3), we have $\bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \in \mathcal{E}_X$. Furthermore, by the following inequality

$$f_L^{\leftarrow}(B) = \bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} \chi_F \wedge f_L^{\leftarrow}(B) \leq \bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \in \mathcal{E},$$

we obtain $co_X(f_L^{\leftarrow}(B)) \leq \bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge f_L^{\leftarrow}(B))$. Thus we have $co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(B)$. This shows that $\bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(B)$. Hence $f_L^{\leftarrow}(B) = \bigvee^{\uparrow}_{F \in \mathbf{2}^X_{fin}} co_X(\chi_F \wedge f_L^{\leftarrow}(B)) \in \mathcal{E}_X$. □

The following example is a PCP mapping but not an SPCP mapping.

Example 3.15. Let $X = \{x, y\}$, let $\mathcal{E}_1 = \{\emptyset, \{x\}, \{y\}\}$ and let $\mathcal{E}_2 = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Then it is easy to check that the identity mapping id_X on X is a PCP mapping from the convex system $(X, \mathcal{E}_1)$ to the convex system $(X, \mathcal{E}_2)$. However, id_X is not an SPCP mapping since $\{x, y\} \in \mathcal{E}_2$, but $\mathrm{id}_X^{-1}(\{x, y\}) = \{x, y\} \notin \mathcal{E}_1$.

By making use of the SPCP mappings as morphisms, we can naturally define another category of L-convex systems . Before this, the following lemma is needed.

Lemma 3.16. *Suppose both mappings $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ and $g : (Y, \mathcal{E}_Y) \longrightarrow (Z, \mathcal{E}_Z)$ between L-convex systems are SPCP. Then the composite mapping $g \circ f$ is also SPCP.*

The category whose objects are L-convex systems and whose morphisms are SPCP mappings will be denoted by L-**SPCS**.

Next, we will show the relations among L-**SPCS** , L-**PCS** and L-**CS**. Before this, we will give a relevant result, which shows the connection between L-convex structures and L-convex systems. The proof is trivial and will be omitted.

Proposition 3.17. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between two L-convex structures. Then the following conditions are equivalent:*

(1) *f is CP;*
(2) *f is PCP;*
(3) *f is SPCP.*

By Proposition 3.17, we obtain a functor $\mathbb{I}_{\mathbb{S}}$:L-**CS**⟶ L-**PCS** defined as follows:

$$\mathbb{I}_{\mathbb{S}} : \begin{cases} L\text{-}\mathbf{CS} & \longrightarrow & L\text{-}\mathbf{PCS}, \\ (X, \mathcal{E}) & \longmapsto & (X, \mathcal{E}), \\ f & \longmapsto & f. \end{cases}$$

The functor $\mathbb{I}_{\mathbb{S}}$, called inclusion functor, also can be considered as a functor from L-**CS** to L-**SPCS**. From Lemma 3.16, Proposition 3.17 and the inclusion functor $\mathbb{I}_{\mathbb{S}}$, the following theorem is straightforward.

Theorem 3.18. *L-**CS** is a full subcategory of both L-**SPCS** and L-**PCS**.*

4. Scott-hull Operators

In the classical setting, there is a bijective relationship between algebra closure operators and convex structures [23]. Rosa extends this relation to fuzzy case [16]. For a convex system, the universal set does not need to be convex, which restricts the domain of the convex hull operator from the powerset to the collection of all admissible sets. By axiomatizing the properties that admissible sets fulfill, we present the notion of Scott-hull operators, and then construct the categorical isomorphism to L-convex systems.

In Proposition 3.8, we show that the collection of all admissible sets actually is a non-empty Scott closed set on L^X with the pointwise order. Based on this fact, we propose the following definition.

Definition 4.1. Let $\mathscr{B} \subseteq L^X$ be a non-empty Scott closed set on L^X, that is, it satisfies (SC1)–(SC3) in Propositon 3.8, and let co be a mapping on $\mathscr{B}$ satisfying (PH1)–(PH4) in Proposition 3.7. Then we call the triple $(X, \mathscr{B}, co)$ a *Scott-hull space*, and co a *Scott-hull operator* on $(X, \mathscr{B})$.

Definition 4.2. Let $(X, \mathscr{B}, co_X)$ and $(Y, \mathscr{D}, co_Y)$ be two Scott-hull spaces. A mapping $f : X \longrightarrow Y$ is called *Scott-hull preserving* (SHP, for short) provided it satisfies the following conditions:

(NSC1) for each $A \in \mathscr{B}$, $f_L^{\rightarrow}(A) \in \mathscr{D}$;
(NSC2) for each $A \in \mathscr{B}$, $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A))$.

Proposition 4.3. *Let $f : (X, \mathscr{B}, co_X) \longrightarrow (Y, \mathscr{D}, co_Y)$ and $g : (Y, \mathscr{D}, co_Y) \longrightarrow (Z, \mathscr{F}, co_Z)$ be SHP mappings between Scott-hull spaces. Then the composite mapping $g \circ f$ is also SHP.*

The category whose objects are Scott-hull spaces and whose morphisms are SHP mappings will be denoted by L-**SHS**.

Proposition 4.4. *Let $(X, \mathscr{B}, co)$ be a Scott-hull space and let*

$$\mathcal{E}^{(\mathscr{B}, co)} = \{A \in \mathscr{B} \mid co(A) = A\}.$$

Then $\mathcal{E}^{(\mathscr{B}, co)}$ is a partially L-convex structure on X, and we call $\left(X, \mathcal{E}^{(\mathscr{B}, co)}\right)$ the L-convex system induced by $(X, \mathscr{B}, co)$.

Proof. It suffices to verify that $\mathcal{E}^{(\mathscr{B}, co)}$ satisfies (PC1)–(PC3).

(PC1) It is straightforward by (PH1).

(PC2) Take any $\{A_i \mid i \in I\} \subseteq \mathcal{E}^{(\mathscr{B}, co)}$. Then by (SC2), $co(A_i) = A_i$ for each $i \in I$ and $\bigwedge_{i \in I} A_i \in \mathscr{B}$. Since co is monotone, we have

$$co\left(\bigwedge_{i \in I} A_i\right) \leq \bigwedge_{i \in I} co(A_i) = \bigwedge_{i \in I} A_i.$$

By (PH3), the inverse inequality holds obviously. Thus $co\left(\bigwedge_{i\in I} A_i\right) = \bigwedge_{i\in I} A_i$. This proves $\bigwedge_{i\in I} A_i \in \mathcal{E}^{(\mathscr{B},co)}$, as desired.

(PC3) Take any directed family $\{D_i \mid i \in I\} \subseteq \mathcal{E}^{(\mathscr{B},co)}$. Then $co(D_i) = D_i$ for each $i \in I$. By (PH5), we obtain

$$co\left(\bigvee_{i\in I}^{\uparrow} D_i\right) = \bigvee_{i\in I}^{\uparrow} co(D_i) = \bigvee_{i\in I}^{\uparrow} D_i.$$

This proves $\bigvee_{i\in I}^{\uparrow} D_i \in \mathcal{E}^{(\mathscr{B},co)}$, as desired. □

Proposition 4.5. *Let $(X, \mathscr{B}, co)$ be a Scott-hull space and let $\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$ be the partially L-convex system induced by $(X, \mathscr{B}, co)$. Then*
(1) $ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right) = \mathscr{B}$;
(2) for each $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$, $co(A) = \bigwedge\{B \in \mathcal{E}^{(\mathscr{B},co)} \mid A \leq B\}$.

Proof. (1) Take any $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$. Then there exists $B \in \mathcal{E}^{(\mathscr{B},co)}$ such that $A \leq B$. As $B \in \mathscr{B}$, then by (SC2), $A \in \mathscr{B}$. Conversely, for each $A \in \mathscr{B}$, by(PH2) and (PH3), we obtain $A \leq co(A) = co(co(A))$. It follows that $co(A) \in \mathcal{E}^{(\mathscr{B},co)}$. Hence $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$.

(2) By(PH2), we have $A \leq co(A) \in \mathcal{E}^{(\mathscr{B},co)}$. It follows that

$$\bigwedge\left\{B \in \mathcal{E}^{(\mathscr{B},co)} \,\middle|\, A \leq B\right\} \leq co(A).$$

Conversely, suppose $B \in \mathcal{E}^{(\mathscr{B},co)}$ and $A \leq B$. Then $co(A) \leq co(B) = B$. It follows $co(A) \leq \bigwedge\left\{B \in \mathcal{E}^{(\mathscr{B},co)} \mid A \leq B\right\}$. This proves $co(A) = \bigwedge\left\{B \in \mathcal{E}^{(\mathscr{B},co)} \mid A \leq B\right\}$, as desired. □

Proposition 4.6. *Let $f : (X, \mathscr{B}, co_X) \longrightarrow (Y, \mathscr{D}, co_Y)$ be an SHP mapping between two Scott-hull spaces. Then $f : \left(X, \mathcal{E}^{(\mathscr{B},co_X)}\right) \longrightarrow \left(Y, \mathcal{E}^{(\mathscr{D},co_Y)}\right)$ is PCP.*

Proof. It suffices to verify that f satisfies (PCP1) and (PCP2).

(PCP1) It is straightforward by Proposition 4.5 that $ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right) = \mathscr{B}$.

(PCP2) Assume $B \in \mathcal{E}^{(\mathscr{D},co_Y)}$, $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co_X)}\right) = \mathscr{B}$ and $A \leq f_L^{\leftarrow}(B)$. Then $f_L^{\rightarrow}(A) \leq B$. By (PH2), we obtain $co_Y(f_L^{\rightarrow}(A)) \leq co_Y(B) = B$. Since f is an SHP mapping, it follows $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A)) \leq B$. Therefore $co_X(A) \leq f_L^{\leftarrow}(B)$. □

By Propositions 4.4 and 4.6, we obtain a functor $\mathbb{H}_{\mathbb{S}} : L\text{-}\mathbf{SHS} \longrightarrow L\text{-}\mathbf{PCS}$ defined as follows:

$$\mathbb{H}_{\mathbb{S}} : \begin{cases} L\text{-}\mathbf{SHS} & \longrightarrow & L\text{-}\mathbf{PCS}, \\ (X, \mathscr{B}, co) & \longmapsto & \left(X, \mathcal{E}^{(\mathscr{B},co)}\right), \\ f & \longmapsto & f. \end{cases}$$

Proposition 4.7. *Let $(X, \mathcal{E})$ be an L-convex system and let $co^{\mathcal{E}}$ be the partially L-convex hull operator, that is,*

$$\forall A \in ads(X, \mathcal{E}),\ co^{\mathcal{E}}(A) = \bigwedge\left\{B \in \mathcal{E} \mid A \leq B\right\}.$$

Then $\left(X, ads(X, \mathcal{E}), co^{\mathcal{E}}\right)$ is a Scott-hull space.

Proof. It is straightforward by Propositions 3.7 and 3.8. □

Proposition 4.8. *If a mapping* $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ *between two* L*-convex systems is PCP, then* $f : \left(X, ads(X, \mathcal{E}_X), co^{\mathcal{E}_X}\right) \longrightarrow \left(Y, ads(Y, \mathcal{E}_Y), co^{\mathcal{E}_Y}\right)$ *is SHP.*

Proof. The verification of (NSC1) is straightforward by (PCP1). It suffices to prove (NSC2). Take any $A \in ads(X, \mathcal{E}_X)$. Since $co^{\mathcal{E}_Y}\,(f_L^{\rightarrow}(A)) \in \mathcal{E}_Y$ and

$$A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \leq f_L^{\leftarrow}\left(co^{\mathcal{E}_Y}(f_L^{\rightarrow}(A))\right),$$

it follows $co^{\mathcal{E}_X}(A) \leq f_L^{\leftarrow}\left(co^{\mathcal{E}_Y}(f_L^{\rightarrow}(A))\right)$. This proves $f_L^{\rightarrow}\left(co^{\mathcal{E}_X}(A)\right) \leq co^{\mathcal{E}_Y}(f_L^{\rightarrow}(A))$. □

By Propositions 4.7 and 4.8, we obtain a functor $\mathbb{S}_{\mathbb{H}} : L$-**PCS**$\longrightarrow L$-**SHS** defined as follows:

$$\mathbb{S}_{\mathbb{H}} : \begin{cases} L\text{-}\mathbf{PCS} & \longrightarrow & L\text{-}\mathbf{SHS}, \\ (X, \mathcal{E}) & \longmapsto & \left(X, ads(X, \mathcal{E}), co^{\mathcal{E}}\right), \\ f & \longmapsto & f. \end{cases}$$

Theorem 4.9. L*-**PCS** is isomorphic to* L*-**SHS**.*

Proof. It suffices to verify that $\mathbb{S}_{\mathbb{H}} \circ \mathbb{H}_{\mathbb{S}} = \mathbb{I}_{L\text{-}\mathbf{SHS}}$ and $\mathbb{H}_{\mathbb{S}} \circ \mathbb{S}_{\mathbb{H}} = \mathbb{I}_{L\text{-}\mathbf{PCS}}$. That is to say, we only need to verify (1) $\mathcal{E}^{\left(ads(X,\mathcal{E}),co^{\mathcal{E}}\right)} = \mathcal{E}$, (2) $ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right) = \mathscr{B}$ and (3) $co^{\mathcal{E}^{(\mathscr{B},co)}} = co$.

(1) It's trivial since $\mathcal{E}^{\left(ads(X,\mathcal{E}),co^{\mathcal{E}}\right)} = \{A \in L^X \mid co^{\mathcal{E}}(A) = A\} = \mathcal{E}$.

(2) If $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$, then there exists a $B \in \mathcal{E}^{(\mathscr{B},co)}$ such that $A \leq B$. Since $\mathcal{E}^{(\mathscr{B},co)} = \{C \in \mathscr{B} \mid co(C) = C\}$, we obtain $B = co(B) \in \mathscr{B}$. It follows from (SC2) that $A \in \mathscr{B}$. Conversely, if $A \in \mathscr{B}$, then by (PH3), $co(A) = co(co(A))$. It follows that $co(A) \in \mathcal{E}^{(\mathscr{B},co)}$. By (PH2), that is, $A \leq co(A)$, we obtain $A \in ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right)$. This proves $ads\left(X, \mathcal{E}^{(\mathscr{B},co)}\right) = \mathscr{B}$.

(3) Take any $A \in L^X$. Then

$$A \in \mathcal{E}^{(\mathscr{B},co)} \Longleftrightarrow A \in \mathscr{B} \text{ and } coA = A.$$

Further, we have

$$\begin{aligned} co^{\mathcal{E}^{(\mathscr{B},co)}}(A) &= \bigwedge\left\{B \in L^X \mid A \leq B \in \mathcal{E}^{(\mathscr{B},co)}\right\} \\ &= \bigwedge\{B \in L^X \mid A \leq B \in \mathscr{B} \text{ and } co(B) = B\}. \end{aligned}$$

Since $A \leq co(A) \in \mathscr{B}$ and $co(co(A)) = co(A)$, it follows $co^{\mathcal{E}^{(\mathscr{B},co)}}(A) \leq co(A)$. If $A \leq B = co(B) \in \mathscr{B}$, then by (PH2), $co(A) \leq co(B) = B$. This means that $co(A) \leq co^{\mathcal{E}^{(\mathscr{B},co)}}(A)$. Thus, $co^{\mathcal{E}^{(\mathscr{B},co)}}(A) = co(A)$. □

5. L-convex Structures Induced by L-convex Systems

As we know, convex systems are closely connected with convex structures. Many concepts and results in convexity can actually be formulated in terms of convex systems. From Theorem 3.18, we know L-**CS** is a full subcategory of L-**PCS**. In this section, our main purpose is to show a deeper relation between them.

Definition 5.1. Let $(X,\mathcal{E})$ be an L-convex system. An L-subset $C \in L^X$ is called an *L-convex set induced by* $\mathcal{E}$, or an *induced L-convex set* provided $C \wedge A \in \mathcal{E}$ for all $A \in \mathcal{E}$. The collection of all induced L-convex sets is denoted by $\mathcal{E}^{ind}$.

Proposition 5.2. *Let $(X,\mathcal{E})$ be an L-convex system. Then $A \in \mathcal{E}^{ind}$ iff for each $F \in \mathbf{2}^X_{fin}$ such that $\chi_F \wedge A \in ads(X,\mathcal{E})$, we have $co(\chi_F \wedge A) \leq A$.*

Proof. **Necessity**. Suppose $A \in \mathcal{E}^{ind}$. If $F \in \mathbf{2}^X_{fin}$ and $\chi_F \wedge A \in ads(X,\mathcal{E})$, it follows $co(\chi_F \wedge A) \in \mathcal{E}$ and $co(\chi_F \wedge A) \wedge A \in \mathcal{E}$. Further, since $\chi_F \wedge A \leq co(\chi_F \wedge A) \wedge A \in \mathcal{E}$, it follows $co(\chi_F \wedge A) \leq co(\chi_F \wedge A) \wedge A \leq A$.

Sufficiency. Take any $B \in \mathcal{E}$. Since $\chi_F \wedge A \wedge B \leq B \in \mathcal{E}$ for each $F \in \mathbf{2}^X_{fin}$, it follows $\chi_F \wedge A \wedge B \in ads(X,\mathcal{E})$. By assumption, $co(\chi_F \wedge A \wedge B) \leq A \wedge B$, which means $\bigvee^{\uparrow}_{F\in\mathbf{2}^X_{fin}} co(\chi_F \wedge A \wedge B) \leq A \wedge B$. The inverse inequality holds obviously. It follows that $A \wedge B = \bigvee^{\uparrow}_{F\in\mathbf{2}^X_{fin}} co(\chi_F \wedge A \wedge B)$. Further, by the directness of $\left\{co\left(\chi_F \wedge A \wedge B\right) \,\middle|\, F \in \mathbf{2}^X_{fin}\right\} \subseteq \mathcal{E}$, we obtain

$$A \wedge B = \bigvee^{\uparrow}_{F\in\mathbf{2}^X_{fin}} co(\chi_F \wedge A \wedge B) \in \mathcal{E}.$$

Hence $A \in \mathcal{E}^{ind}$. □

The following theorem shows that the collection of all L-convex sets induced by an L-convex system forms an L-convexity. The proof of this theorem is not difficult, so will not be reviewed.

Theorem 5.3. *Let $(X,\mathcal{E})$ be an L-convex system. Then $\mathcal{E}^{ind}$ is an L-convexity.*

Lemma 5.4. *If $f : X \longrightarrow Y$ is a mapping, $B \in L^Y$ and $F \in \mathbf{2}^X_{fin}$, then $f^{\rightarrow}_L(\chi_F \wedge f^{\leftarrow}_L(B)) = \chi_{f(F)} \wedge B \leq B$.*

Proof. For each $y \in Y$, we have

$$\begin{aligned} f^{\rightarrow}_L(\chi_F \wedge f^{\leftarrow}_L(B))(y) &= \textstyle\bigvee^{\uparrow}_{x\in X,\ f(x)=y} (\chi_F \wedge f^{\leftarrow}_L(B))(x) \\ &= \textstyle\bigvee^{\uparrow}_{x\in F,\ f(x)=y} f^{\leftarrow}_L(B)(x) = \bigvee^{\uparrow}_{x\in F,\ f(x)=y} B(f(x)) \\ &= \textstyle\bigvee^{\uparrow}_{y\in f(F)} B(y) = (\chi_{f(F)} \wedge B)(y) \leq B(y). \end{aligned}$$

The proof is completed. □

Theorem 5.5. *Let $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ be two L-convex systems. A mapping $f : X \longrightarrow Y$ is PCP iff it satisfies* (PCP1) *and $f : \left(X,\mathcal{E}^{ind}_X\right) \longrightarrow \left(Y,\mathcal{E}^{ind}_Y\right)$ is CP.*

Proof. **Necessity**. It suffices to prove $f : \left(X,\mathcal{E}^{ind}_X\right) \longrightarrow \left(Y,\mathcal{E}^{ind}_Y\right)$ is CP, that is, $f^{\leftarrow}_L(B) \in \mathcal{E}^{ind}_X$ for each $B \in \mathcal{E}^{ind}_Y$. By Lemma 5.4 that $f^{\rightarrow}_L(\chi_F \wedge f^{\leftarrow}_L(B)) = \chi_{f(F)} \wedge B \leq B \in \mathcal{E}^{ind}_Y$, we obtain

$$co_Y(f^{\rightarrow}_L(\chi_F \wedge f^{\leftarrow}_L(B))) = co_Y(\chi_{f(F)} \wedge B) \leq co_Y(B) = B.$$

It is clear from Proposition 3.11 that $f^{\rightarrow}_L(co_X(\chi_F \wedge f^{\leftarrow}_L(B))) \leq co_Y(f^{\rightarrow}_L(\chi_F \wedge f^{\leftarrow}_L(B))) \leq B$. This means $co_X(\chi_F \wedge f^{\leftarrow}_L(B)) \leq f^{\leftarrow}_L(B)$.

Sufficiency. By Propositon 3.11, it suffices to prove $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A))$ for all $A \in ads(X, \mathcal{E}_X)$. Since $co_Y(f_L^{\rightarrow}(A)) \in \mathcal{E}_Y \subseteq \mathcal{E}_Y^{ind}$, it follows

$$A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \leq f_L^{\leftarrow}(co_Y(f_L^{\rightarrow}(A))) \in \mathcal{E}_X^{ind}.$$

Further, from $A \in ads(X, \mathcal{E}_X)$, we have $co_X(A) \in \mathcal{E}_X$. It follows that

$$co_X(A) \leq co_X(A) \wedge f_L^{\leftarrow}(co_Y(f_L^{\rightarrow}(A))) \in \mathcal{E}_X.$$

Then $co_X(A) \leq f_L^{\leftarrow}(co_Y(f_L^{\rightarrow}(A)))$. This proves $f_L^{\rightarrow}(co_X(A)) \leq co_Y(f_L^{\rightarrow}(A))$. □

From Theorems 5.3 and 5.5, we obtain a functor $\mathbb{S}_{\mathbb{I}}$:L-**PCS**$\longrightarrow$ L-**CS** defined as follows:

$$\mathbb{S}_{\mathbb{I}}: \begin{cases} L\text{-}\mathbf{PCS} & \longrightarrow & L\text{-}\mathbf{CS}, \\ (X,\mathcal{E}) & \longmapsto & \left(X, \mathcal{E}^{ind}\right), \\ f & \longmapsto & f. \end{cases}$$

Next, we would like to construct the relationship between L-**PCS** and L-**CS**. Before this, the following several useful lemmas are needed.

Lemma 5.6. *Let $(X, \mathcal{E})$ be an L-convex system. Then the identity mapping* $\mathrm{id}_X : (X, \mathcal{E}) \longrightarrow \left(X, \mathcal{E}^{ind}\right)$ *is PCP.*

Proof. The verification of (PCP1) is trivial since $ads\left(X, \mathcal{E}^{ind}\right) = L^X$. We verify (PCP2). Take any $B \in \mathcal{E}^{ind}$. If $A \in ads(X, \mathcal{E})$ such that $A \leq (\mathrm{id}_X)_L^{\leftarrow}(B) = B$, then $co(A) \in \mathcal{E}$ and $A \leq co(A) \wedge B \in \mathcal{E}$. This means $co(A) \leq co(A) \wedge B$. Hence $co(A) \leq B$. □

Lemma 5.7. *Let $(X, \mathcal{E})$ be an L-convex system and let $(Y, \mathcal{C})$ be an L-convex structure. If a mapping $f : (X, \mathcal{E}) \longrightarrow (Y, \mathcal{C})$ is PCP, then $f : \left(X, \mathcal{E}^{ind}\right) \longrightarrow (Y, \mathcal{C})$ is CP, and hence PCP.*

Proof. Take any $B \in \mathcal{C}$. For each $F \in \mathbf{2}_{fin}^X$ such that $\chi_F \wedge f_L^{\leftarrow}(B) \in ads(X, \mathcal{E})$, since $f : (X, \mathcal{E}) \longrightarrow (Y, \mathcal{C})$ is PCP, we obtain $co(\chi_F \wedge f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(B)$. Then by Proposition 5.2, $f_L^{\leftarrow}(B) \in \mathcal{E}^{ind}$. This shows that $f : \left(X, \mathcal{E}^{ind}\right) \longrightarrow (Y, \mathcal{C})$ is CP. Further, since both $\left(X, \mathcal{E}^{ind}\right)$ and $(Y, \mathcal{C})$ are L-convex structures, it follows from Proposition 3.17 that $f : \left(X, \mathcal{E}^{ind}\right) \longrightarrow (Y, \mathcal{C})$ is also PCP. □

Proposition 5.8. [1] *Suppose $\mathbb{F} : \mathcal{A} \longrightarrow \mathcal{B}$ and $\mathbb{G} : \mathcal{B} \longrightarrow \mathcal{A}$ is a pair of functors. Then the following conditions are equivalent:*

(1) *$\mathbb{F}$ is the left adjoint of $\mathbb{G}$;*
(2) *There exists a natural transformations η :$\mathrm{id}_{\mathcal{A}} \longrightarrow \mathbb{G} \circ \mathbb{F}$, such that for each $A \in ob\mathcal{A}$, $\eta_A : A \longrightarrow \mathbb{G}(\mathbb{F}(A))$ satisfying the universal property: for any $\mathcal{A}$-morphism $f : A \longrightarrow \mathbb{G}(B)$ from A to some $\mathcal{B}$-object B, there exists a unique $\mathcal{B}$-morphism $g : \mathbb{F}(A) \longrightarrow B$ such that $f = \mathbb{G}(g) \circ \eta_A$.*

Theorem 5.9. *The functor $\mathbb{S}_{\mathbb{I}}$ is the left adjoint of $\mathbb{I}_{\mathbb{S}}$.*

Proof. We prove the conclusion in two steps.

Step1: We firstly define a natural transformation η from $\mathrm{id}_{L\text{-}\mathbf{PCS}}$ to $\mathbb{I}_{\mathbb{S}} \circ \mathbb{S}_{\mathbb{I}}$ as follows: For each L-convex system $(X, \mathcal{E})$, $\eta_{(X,\mathcal{E})} = \mathrm{id}_X : (X, \mathcal{E}) \longrightarrow \mathbb{I}_{\mathbb{S}}\left(\mathbb{S}_{\mathbb{I}}((X, \mathcal{E}))\right)$, that is, $\eta_{(X,\mathcal{E})} : (X,\mathcal{E}) \longrightarrow \left(X,\mathcal{E}^{ind}\right)$, $\eta_{(X,\mathcal{E})}(x) = x$ for each $x \in X$. By Lemma 5.6, $\eta_{(X,\mathcal{E})}$ is PCP which is one of the morphisms of L-**PCS**.

Step2: Take any L-convex system $(X,\mathcal{E})$, L-convex structure $(Y,\mathcal{C})$ and PCP mapping $f : (X,\mathcal{E}) \longrightarrow \mathbb{I}_{\mathbb{S}}((Y,\mathcal{C})) = (Y,\mathcal{C})$. We define $g = f : \mathbb{S}_{\mathbb{I}}((X,\mathcal{E})) = \left(X,\mathcal{E}^{ind}\right) \longrightarrow (Y,\mathcal{C})$. Then the uniqueness of g is trivial, and by Lemma 5.7, g is CP, that is, g is a morphism of L-**CS**. Since $\mathbb{I}_{\mathbb{S}}\left(\mathbb{S}_{\mathbb{I}}((X,\mathcal{E}))\right) = \mathbb{S}_{\mathbb{I}}((X,\mathcal{E}))$ and $\mathbb{I}_{\mathbb{S}}((Y,\mathcal{C})) = (Y,\mathcal{C})$, it follows $\mathbb{I}_{\mathbb{S}}(g) = g : \mathbb{I}_{\mathbb{S}}\left(\mathbb{S}_{\mathbb{I}}((X,\mathcal{E}))\right) \longrightarrow \mathbb{I}_{\mathbb{S}}((Y,\mathcal{C}))$ is PCP and satisfies

$$\mathbb{I}_{\mathbb{S}}(g) \circ \eta_{(X,\mathcal{E})} = g \circ \mathrm{id}_X = f \circ \mathrm{id}_X = f.$$

The proof is completed. □

From Theorems 3.18 and 5.9, we know L-**CS** is a full subcategory of L-**PCS** and the inclusion functor $\mathbb{I}_{\mathbb{S}}$ has a left adjuction. Also, we note that for each L-convex system $(X,\mathcal{E})$, the morphism $\eta_{(X,\mathcal{E})} : (X,\mathcal{E}) \longrightarrow \mathbb{I}_{\mathbb{S}}\left(\mathbb{S}_{\mathbb{I}}((X,\mathcal{E}))\right)$ in the proof of Theorem 5.9 is the identity mapping on X, and hence a bimorphism in the sense of [1]. Then we obtain the following result.

Theorem 5.10. *L-**CS** is the bireflective subcategory of L-**PCS**.*

Next we will show that L-**CS** cannot be the coreflective subcategory of L-**PCS**. Before this, the following lemma is necessary, and can be proved by an easy induction.

Lemma 5.11. *For a mapping $f : X \longrightarrow Y$, $f_L^{\rightarrow}(\chi_X) = \chi_{f(X)}$.*

Proposition 5.12. *Let $(X,\mathcal{C})$ be an L-convex structure, let $(X,\mathcal{E})$ be an L-convex system and let $f : X \longrightarrow Y$ be a PCP mapping. Then $\chi_{f(X)} \in ads(X,\mathcal{E})$, or equivalently*

(P) *there exists some $B \in \mathcal{E}$ satisfying $\chi_{f(X)} \leq B$.*

Proof. Note that $ads(X,\mathcal{C}) = L^X$. It follows $\chi_X \in ads(X,\mathcal{C})$. Then by (PCP1), $f_L^{\rightarrow}(\chi_X) = \chi_{f(X)} \in ads(X,\mathcal{E})$, which is equivalent to the statement (P). □

Remark 5.13. (1) In fact, not all L-convex systems satisfy (P). To see this, let $E = \{x,y\}$ and let $L = [0,1]$. We define $\mathcal{E}_E = \{\chi_\emptyset, A, B, C\} \subseteq L^E$ as follows:

	$\chi_\emptyset$	A	B	C
x	0	0	0.5	0.5
y	0	0.5	0	0.5

Figure 1. The L-convex System $\mathcal{E}_E$

It's trivial to verify $(E,\mathcal{E}_E)$ is an L-convex system which doesn't satisfy (P).

(2) By Propositin 5.12, we know for each L-convex structure $(X,\mathcal{C})$, there doesn't exist a PCP mapping from $(X,\mathcal{C})$ to the system $(E,\mathcal{E}_E)$ in the case of (1) above, which shows that L-**CS** cannot be the coreflective subcategory of L-**PCS** in the sense of [1].

6. **Quotient Systems**

In this section, we will introduce the concept of quotient systems, which further shows a close connection between L-convex systems and L-convex structures.

Definition 6.1. Let R be an equivalence relation on X and let $(X,\mathcal{E})$ be an L-convex system with a corresponding quotient function $q : X \longrightarrow X/R$. An L-subset $\widetilde{A} \in L^{X/R}$ is called a *partially L-convex set* on X/R, if $\widetilde{A} \in \mathcal{E}^{ind}/R$ and satisfies that for each $\widetilde{F} \in \mathbf{2}^{X/R}_{fin}$, there exists some $G^{\widetilde{F}} \in ads(X,\mathcal{E})$ such that $\chi_{\widetilde{F}} \wedge \widetilde{A} \leq co_I\left(q_L^{\rightarrow}\left(G^{\widetilde{F}}\right)\right)$, where $\mathcal{E}^{ind}/R$ is the quotient space of $\left(X,\mathcal{E}^{ind}\right)$ (refer to Definition 2.6), and the operator co_I the L-convex hull on $\left(X/R,\mathcal{E}^{ind}/R\right)$.

Theorem 6.2. *We denote the collection of all partially L-convex sets on X/R by $\mathcal{E}/R$. Then $(X/R,\mathcal{E}/R)$ is an L-convex system, called the quotient system of $(X,\mathcal{E})$.*

Proof. The verifications of (PC1) and (PC2) are straightforward. It suffices to prove (PC3), that is, $\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i \in \mathcal{E}/R$ for all directed family $\left\{\widetilde{D}_i \;\middle|\; i\in I\right\} \subseteq \mathcal{E}/R$. We prove this conclusion in two steps.

Step 1: $\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i \in \mathcal{E}^{ind}/R$. Take any $C\in\mathcal{E}$. Since $\widetilde{D}_i \in \mathcal{E}/R \subseteq \mathcal{E}^{ind}/R$, that is, $q_L^{\leftarrow}\left(\widetilde{D}_i\right) \in \mathcal{E}^{ind}$, it follows $C\wedge q_L^{\leftarrow}\left(\widetilde{D}_i\right)\in\mathcal{E}$. Note that $\left\{C\wedge q_L^{\leftarrow}\left(\widetilde{D}_i\right) \;\middle|\; i\in I\right\}$ is directed, and hence by (PC3), we obtain $\bigvee^{\uparrow}_{i\in I} C\wedge q_L^{\leftarrow}\left(\widetilde{D}_i\right)\in\mathcal{E}$. By the equality

$$C\wedge q_L^{\leftarrow}\left(\bigvee_{i\in I}^{\uparrow}\widetilde{D}_i\right) = C\wedge\bigvee_{i\in I}^{\uparrow} q_L^{\leftarrow}\left(\widetilde{D}_i\right) = \bigvee_{i\in I}^{\uparrow} C\wedge q_L^{\leftarrow}\left(\widetilde{D}_i\right),$$

we have $C\wedge q_L^{\leftarrow}\left(\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i\right)\in\mathcal{E}$. Thus $q_L^{\leftarrow}\left(\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i\right)\in\mathcal{E}^{ind}$, which means $\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i \in \mathcal{E}^{ind}/R$.

Step 2: $\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i \in \mathcal{E}/R$. Since $\widetilde{D}_i\in\mathcal{E}/R$, it follows for each $i\in I$ and each $\widetilde{F}\in \mathbf{2}^{X/R}_{fin}$, that there exists an $G_i^{\widetilde{F}}\in ads(X,\mathcal{E})$ such that $\chi_{\widetilde{F}}\wedge\widetilde{D}_i \leq co_I\left(q_L^{\rightarrow}\left(G_i^{\widetilde{F}}\right)\right)$. We might just as well let

$$G_i^{\widetilde{F}} = \bigwedge\left\{G\in ads(X,\mathcal{E}) \;\middle|\; \chi_{\widetilde{F}}\wedge\widetilde{D}_i \leq co_I(q_L^{\rightarrow}(G))\right\}$$

for each $i\in I$. Then by (SC2), $G_i^{\widetilde{F}}\in ads(X,\mathcal{E})$. Note that $\widetilde{D}_i\leq\widetilde{D}_j$ implies $G_i^{\widetilde{F}}\leq G_j^{\widetilde{F}}$. This means that $\left\{G_i^{\widetilde{F}} : i\in I\right\}$ is directed. It is trivial from (SC3) that $\bigvee^{\uparrow}_{i\in I} G_i^{\widetilde{F}}\in ads(X,\mathcal{E})$. Further,

$$\chi_{\widetilde{F}}\wedge\bigvee_{i\in I}^{\uparrow}\widetilde{D}_i = \bigvee_{i\in I}^{\uparrow}\chi_{\widetilde{F}}\wedge\widetilde{D}_i \leq co_I\left(\pi_L^{\rightarrow}\left(\bigvee_{i\in I}^{\uparrow} G_i^{\widetilde{F}}\right)\right).$$

This proves $\bigvee^{\uparrow}_{i\in I}\widetilde{D}_i\in\mathcal{E}/R$. □

In the following, for each $\widetilde{A} \in ads(X/R, \mathcal{E}/R)$, we denote the partially L-convex hull of $\tilde{A}$ by $co_R\left(\widetilde{A}\right)$. Then it is trivial

$$co_I\left(\widetilde{A}\right) \leq co_R\left(\widetilde{A}\right) \quad \text{and} \quad co_R\left(\widetilde{A}\right) \in \mathcal{E}_{ind}/R.$$

Lemma 6.3. *Let R be an equivalence relation on X with the corresponding quotient mapping q. Then*

$$q_L^{\rightarrow} \circ q_L^{\leftarrow} = \mathrm{id}_{X/R}.$$

Proof. Take any $\widetilde{A} \in L^{X/R}$ and $\widetilde{y} \in X/R$. Then

$$q_L^{\rightarrow} \circ q_L^{\leftarrow}(\widetilde{A})(\widetilde{y}) = q_L^{\rightarrow}(\widetilde{A} \circ q)(\widetilde{y}) = \bigvee_{q(x)=\widetilde{y}} \widetilde{A} \circ q(x) = \widetilde{A}(\widetilde{y}).$$

Hence $q_L^{\rightarrow} \circ q_L^{\leftarrow} = \mathrm{id}_{X/R}$. □

Theorem 6.4. *Let $(X, \mathcal{E})$ be an L-convex system and let R be an equivalence relation on X with the corresponding quotient mapping q. Then $q : (X, \mathcal{E}) \longrightarrow (X/R, \mathcal{E}/R)$ is PCP.*

Proof. (PCP1) Take any $G \in ads(X, \mathcal{E})$. Then we obtain

$$\chi_{\widetilde{F}} \wedge co_I(q_L^{\rightarrow}(G)) \leq co_I(q_L^{\rightarrow}(G)) \in \mathcal{E}^{ind}/R$$

for each $\widetilde{F} \in \mathbf{2}^X_{fin}$. It means that $co_I(q_L^{\rightarrow}(G)) \in \mathcal{E}/R$, and hence $q_L^{\rightarrow}(G) \in ads(X/R, \mathcal{E}/R)$.

(PCP2) Take any finite L-set $F \in ads(X, \mathcal{E})$. Then $q_L^{\rightarrow}(F) \in ads(X/R, \mathcal{E}/R)$ and $co_R(q_L^{\rightarrow}(F)) \in \mathcal{E}/R \subseteq \mathcal{E}^{ind}/R$. This means $q_L^{\leftarrow}(co_R(q_L^{\rightarrow}(F))) \in \mathcal{E}^{ind}$. Since $F \leq q_L^{\leftarrow}(co_R(q_L^{\rightarrow}(F))) \in \mathcal{E}^{ind}$, it follows from Proposition 5.2 that $co(F) \leq q_L^{\leftarrow}(co_R(q_L^{\rightarrow}(F)))$. This proves $q_L^{\rightarrow}(co(F)) \leq co_R(q_L^{\rightarrow}(F))$. Hence by Proposition 3.11, q is PCP. □

Theorem 6.5. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two L-convex systems and let R be an equivalence relation on X with the corresponding quotient mapping q. Suppose q satisfies* (PCP3)*, that is, q is SPCP. Then a mapping $f : (X/R, \mathcal{E}_X/R) \longrightarrow (Y, \mathcal{E}_Y)$ is SPCP iff the composite mapping $f \circ q : (X, \mathcal{E}) \longrightarrow (Y, \mathcal{E}_Y)$ is SPCP.*

Proof. It suffices to prove sufficiency, that is, f satisfies (PCP1) and (PCP2)*.

(PCP1) Take any $\widetilde{A} \in ads\,(X/R, \mathcal{E}_X/R)$. Since q satisfies (PCP3), it follows $q_L^{\leftarrow}\left(\widetilde{A}\right) \in ads(X, \mathcal{E}_X)$. Further, by assumption that $f \circ q$ is PCP, we obtain $(f \circ q)_L^{\rightarrow}\left(q_L^{\leftarrow}\left(\widetilde{A}\right)\right) = f_L^{\rightarrow}\left(\widetilde{A}\right) \in ads(Y, \mathcal{E}_Y)$.

(PCP2)* Take any $B \in \mathcal{E}_Y$. Then $(f \circ q)_L^{\leftarrow}(B) = q_L^{\leftarrow}(f_L^{\leftarrow}(B)) \in \mathcal{E}_X \subseteq \mathcal{E}_X^{ind}$. This means $f_L^{\leftarrow}(B) \in \mathcal{E}_X^{ind}$. For each $\widetilde{F} \in \mathbf{2}^{X/R}_{fin}$, let $G^{\widetilde{F}} = (f \circ q)_L^{\leftarrow}(B)$. Then $G^{\widetilde{F}} \in ads(X, \mathcal{E}_X)$ and

$$\chi_{\widetilde{F}} \wedge f_L^{\leftarrow}(B) \leq co_I(f_L^{\leftarrow}(B)) = co_I(q_L^{\rightarrow}((f \circ q)_L^{\leftarrow}(B))) = co_I\left(q_L^{\rightarrow}\left(G^{\widetilde{F}}\right)\right).$$

It follows that $f_L^{\leftarrow}(B) \in \mathcal{E}_X/R$. □

Now we give the main result in this section, which shows a close connection between L-convex systems and L-convex structures.

Theorem 6.6. *Let $(X, \mathcal{E})$ be an L-convex system and let R be an equivalence relation on X with the corresponding quotient mapping q. Then $\mathcal{E}^{ind}/R = (\mathcal{E}/R)^{ind}$.*

Proof. We prove this conclusion in two steps.

Step 1: $\mathcal{E}^{ind}/R \subseteq (\mathcal{E}/R)^{ind}$. Take any $\widetilde{D} \in \mathcal{E}^{ind}/R$ and $\widetilde{C} \in \mathcal{E}/R$. Since $\mathcal{E}/R \subseteq \mathcal{E}^{ind}/R$, it follows $\widetilde{C} \wedge \widetilde{D} \in \mathcal{E}^{ind}/R$. Furthermore, for each $\widetilde{F} \in \mathbf{2}^{X/R}_{fin}$, there exists an L-subset $G^{\widetilde{F}} \in ads(X, \mathcal{E})$ such that

$$\chi_{\widetilde{F}} \wedge \left(\widetilde{C} \wedge \widetilde{D}\right) \leq \chi_{\widetilde{F}} \wedge \widetilde{C} \leq co_I \left(q_L^{\rightarrow}(G)\right).$$

This means that $\widetilde{C} \wedge \widetilde{D} \in \mathcal{E}/R$. Thus $\widetilde{D} \in (\mathcal{E}/R)^{ind}$.

Step 2: $(\mathcal{E}/R)^{ind} \subseteq \mathcal{E}^{ind}/R$. Take any $\widetilde{B} \in (\mathcal{E}/R)^{ind}$ and take any $F \in \mathbf{2}^X_{fin}$ such that $\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right) \in ads(X, \mathcal{E})$. Since q is PCP, it follows $q_L^{\rightarrow}\left(\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right)\right) \in ads(X/R, \mathcal{E}/R)$. Further,

$$q_L^{\rightarrow}\left(\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right)\right) = \chi_{q(F)} \wedge \widetilde{B} \leq \widetilde{B} \in (\mathcal{E}/R)^{ind}.$$

Then by Theorem 6.4 and Proposition 3.11, we obtain

$$q_L^{\rightarrow}\left(co\left(\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right)\right)\right) \leq co_R\left(q_L^{\rightarrow}\left(\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right)\right)\right) = co_R\left(\chi_{q(F)} \wedge \widetilde{B}\right) \leq \widetilde{B}.$$

This means $co\left(\left(\chi_F \wedge q_L^{\leftarrow}\left(\widetilde{B}\right)\right)\right) \leq q_L^{\leftarrow}\left(\widetilde{B}\right)$. It follows $q_L^{\leftarrow}\left(\widetilde{B}\right) \in \mathcal{E}^{ind}$. Hence $\widetilde{B} \in \mathcal{E}^{ind}/R$. □

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Chong Shen, School of Mathematics and statistics, Beijing Institute of Technology, Beijing 100081, P.R. China

E-mail address: shenchong0520@163.com

Fu-Gui Shi*, School of Mathematics and statistics, Beijing Institute of Technology, Beijing 100081, P.R. China

E-mail address: fuguishi@bit.edu.cn

*Corresponding author