QUANTALE-VALUED SUP-ALGEBRAS

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Abstract. Based on the notion of Q-sup-lattices (a fuzzy counterpart of complete join-semilattices valuated in a commutative quantale), we present the concept of Q-sup-algebras – Q-sup-lattices endowed with a collection of finitary operations compatible with the fuzzy joins. Similarly to the crisp case investigated in [30], we characterize their subalgebras and quotients, and following [20], we show that the category of Q-sup-algebras is isomorphic to a certain subcategory of a category of Q-modules.

1. **Introduction**

The topic of sets with fuzzy order relations valuated in complete lattices with additional structure has been quite active in the past decade, and a number of papers have been published. Among many others, articles [20, 25, 28] may be used for introduction.

Based on a quantale-valued order relation and subset membership, counterparts to common order-theoretic notions can be defined, like monotone mappings, adjunctions, joins and meets, complete lattices, or join-preserving mappings, and one can consider a category formed from the latter two concepts. An attempt for systematic study of such categories of fuzzy complete lattices with quantale valuation ("Q-sup-lattices") with fuzzy join-preserving mappings has been made by the author in his recent paper [16].

Recent papers [30, 32] brought our attention to general algebras with finitary operations in the categories of posets and sup-lattices ("ordered algebras" and "sup-algebras"), which cover quantales, quantale modules, and quantale algebras as their instances. Having some theory of Q-sup-lattices available, we find it useful to build the concepts of algebraic structures in their category, which might provide a base for further research. In this paper we shall deal with algebras whose operations are compatible with the fuzzy order relation ("Q-ordered algebras") resp. fuzzy joins ("Q-sup-algebras"), building on existing results obtained for algebras based on crisp posets and sup-lattices.

The topic will be presented in relation to standard concepts from the theory of posets, sup-lattices, and sup-algebras. This should convince the reader that our fuzzy structures behave in strong analogy to their crisp counterparts. Throughout the text, we also will try to highlight an important fact: that concepts based on a fuzzy order relation (in the sense of the quantale valuation as studied in this paper)

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should not be treated as generalizations of their crisp variants, but rather as standard crisp concepts of order theory, satisfying certain additional properties. This fact also reduces the work needed to carry out proofs. Thus, even with the additional properties imposed, we shall see that the theory of fuzzy-ordered structures develops consistently with its crisp counterpart.

2. Preliminaries

We just need to recall a few basic concepts related to posets, sup-lattices and quantales. The book [14] and the chapter [7] can be used as a better reference. A concise overview of constructions in the category of sup-lattices can also be found in [6]. The required elements of category theory can be provided by the book [3].

2.1. **Posets and Sup-lattices.** Let $(X, \leq)$ be a poset. A mapping $f\colon X \to X$ is called inflating (deflating) if $x \leq f(x)$ ($f(x) \leq x$) for all $x \in X$. An idempotent monotone inflating (deflating) mapping is called an *order nucleus* (*order conucleus*).

Two monotone mappings $f\colon X \to Y$ and $g\colon Y \to X$ are *adjoint* if $f(x) \leq y \iff x \leq g(y)$ for all $x \in X$, $y \in Y$ (f is a *left adjoint*, g is a *right adjoint*). A right adjoint to f, if it exists, is unique and denoted f^* (likewise, the left adjoint to g is g_*).

By a *sup-lattice* we mean a complete join-semilattice (actually a complete lattice), with its least and greatest element denoted $\perp$ and $\top$. Join-preserving mappings are taken for sup-lattice homomorphisms. Preservation of arbitrary joins implies that $f(\perp) = \perp$, the join of the empty set, for any sup-lattice homomorphism f.

For a join-preserving map $f\colon X \to Y$ between two sup-lattices X and Y, there exists its right adjoint $f_*\colon Y \to X$, which preserves arbitrary meets. The mapping f_* can be defined explicitly as $f_*(y) = \bigvee\{x \in X \mid f(x) \leq y\}$.

The Cartesian product $\prod_{i\in I} X_i$ of posets X_i, $i \in I$, can be equipped with standard product partial order $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ iff $x_i \leq y_i$ for each $i \in I$.

2.2. **Quantales.** A *quantale* is a sup-lattice Q endowed with an associative binary operation (multiplication) $\cdot$ that distributes over arbitrary joins, i.e.

$$a \cdot \bigvee B = \bigvee_{b\in B} (a \cdot b) \qquad \text{and} \qquad \bigvee B \cdot a = \bigvee_{b\in B} (b \cdot a)$$

for any $a \in Q$, $B \subseteq Q$. A quantale is *unital* if there is a unit element for multiplication, denoted by 1. Distributivity of multiplication over arbitrary joins, including that of an empty set, implies that the least element $\perp$ also acts as a multiplicative zero. A quantale is *commutative* if the binary operation is commutative. If all elements of a quantale are idempotent, satisfying $x \cdot x = x$, we call the quantale *idempotent*.

For any element a of a quantale Q, the unary operation $a \cdot - \colon Q \to Q$ is join-preserving, therefore it has a (meet-preserving) right adjoint $a \to - \colon Q \to Q$, characterized by $a \cdot b \leq c \iff b \leq a \to c$. Written explicitly, $a \to c = \bigvee\{b \in Q \mid a \cdot b \leq c\}$.

Similarly, there is a right adjoint $a \leftarrow -\colon Q \to Q$ for $- \cdot a$, characterized by $b \cdot a \le c \iff b \le a \leftarrow c$, and satisfying $a \leftarrow c = \bigvee\{b \in Q \mid b \cdot a \le c\}$. If Q is commutative, the operations $\to$ and $\leftarrow$ clearly coincide, and we will keep denoting them $\to$.

Example 2.1. (1) The real unit interval $[0, 1]$ with standard partial order and multiplication of reals is a commutative quantale.

(2) The set of all open subsets of a topological space with unions as joins and binary meets as multiplication is a commutative idempotent quantale, called a frame.

(3) For any sup-lattice S, the set of its sup-lattice endomorphisms with mapping composition as multiplication and pointwise computed joins is a (generally noncommutative) quantale.

There are a number of (commutative) quantale identities that relate joins and meets with the binary operations, from which we will employ the following:

$$\left(\bigvee_{i\in I} a_i\right) \to b = \bigwedge_{i\in I}(a_i \to b) \qquad \text{and} \qquad 1 \to a = a.$$

3. Q-ordered Sets and Q-sup-lattices

In the standard concept of a fuzzy set due to Zadeh [26], the relation "x is an element of X" is *fuzzified*, that is, replaced by a mapping $X \to [0, 1]$ assigning to each element of X its "membership degree in X". In later generalizations of the concept, the membership degree can be valuated in more general structures than the real unit interval, removing the requirement on linear ordering, or allowing more values – not only the top element – representing 'true'.

There are actually certain requirements the structure used for valuating the fuzzy predicates (assigning truth values) should satisfy. A detailed analysis can be found in [1, Section 2.3] or [2, Section 1.2], which leads to complete residuated lattices. Looking at their internal structure, unital commutative quantales with the top element as the multiplicative unit are precisely complete residuated lattices. The requirement on the top element being the unit can be relaxed as is not applied in calculations, and unital commutative quantales thus provide a sufficiently general framework for valuating fuzzy predicates.

In our paper, the concept of a set remains unchanged, and it will be the partial order relation and the notion of a subset that will be replaced by suitable mappings to a quantale. Instead of considering on X a partial order relation $\le$, we employ a unital commutative quantale Q and mappings $M\colon X \to Q$ and $e\colon X \times X \to Q$, which quantify the truth degree of membership in a subset and of being less or equal. Note that fuzzifying order relations also dates back to Zadeh [27]. However, his condition of antisymmetry is somehow more strict than the one we will employ.

For the rest of this paper, let Q be an arbitrary unital commutative quantale that remains fixed from now on. This structure shall be used for valuating Q-orders and Q-subsets. Note that we do not require the multiplicative unit 1 to be its greatest element $\top$.

Definition 3.1. [5, 29] Let X be a set. A mapping $e\colon X \times X \to Q$ is called a *Q-order* if for any $x, y, z \in X$ the following are satisfied:

(1) $e(x, x) \geq 1$ (reflexivity),
(2) $e(x, y) \cdot e(y, z) \leq e(x, z)$ (transitivity),
(3) if $e(x, y) \geq 1$ and $e(y, x) \geq 1$, then $x = y$ (antisymmetry).

The pair (X, e) is then called a *Q-ordered set*.

It has been shown in [24] that the above definition is equivalent to the notion of fuzzy partial order introduced in [1].

For a Q-order e on X, the relation $\leq_e$ defined as $x \leq_e y \iff e(x, y) \geq 1$ is a partial order in the usual sense. This means that any Q-ordered set can be viewed as an ordinary poset satisfying additional properties.

Vice versa, for a partial order $\leq$ on a set X and any quantale Q we can define a Q-order $e_{\leq}$ by

$$e_{\leq}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Also note that any subset Y of a Q-ordered set (X, e_X) is itself a Q-ordered set with the restricted Q-order $e_Y = e_X|_Y$. Given a Q-order e on a set X, the *dual Q-order* e^{op} can be defined by $e^{\mathrm{op}}(x, y) = e(y, x)$. From the definition of Q-order the following condition can be easily derived:

$$\text{(E)} \qquad e(x, z) = e(y, z) \text{ for all } z \in X \text{ iff } x = y.$$

Definition 3.2. A *Q-subset* of a set X is an element of the set Q^X, i.e., a mapping from X to Q.

For Q-subsets M, N of a set X, we define the *subsethood degree* of M in N as

$$sub_X(M, N) = \bigwedge_{x \in X} (M(x) \to N(x)).$$

Then (Q^X, sub_X) forms a Q-ordered set [1, Theorem 3.12]. In particular, Q itself is a Q-ordered set with $e(x, y) = x \to y$. Note that $sub_X(M, N) \geq 1$ means that $M(x) \to N(x) \geq 1$ for all $x \in X$, which is equivalent to $M(x) \leq N(x)$ for all $x \in X$. We also remark that the definition of Q-subsets corresponds to viewing 'ordinary' subsets of X as mappings from X to the quantale $\mathbf{2} = \{\bot, 1 = \top\}$.

Definition 3.3. Let (X, e_X), (Y, e_Y) be Q-ordered sets, and $f\colon (X, e_X) \to (Y, e_Y)$ and $g\colon (Y, e_Y) \to (X, e_X)$ be mappings.

- The mapping f is called *Q-monotone* if $e_X(x, y) \leq e_Y(f(x), f(y))$ for any $x, y \in X$.
- The mapping f is called *Q-inflating* (resp. *Q-deflating*) if $e_X(x, f(x)) \geq 1$ (resp. $e_X(f(x), x) \geq 1$) for any $x \in X$.
- A Q-monotone, Q-inflating (Q-deflating) mapping is called a *Q-order prenucleus* (*Q-order preconucleus*).
- An idempotent Q-order pre(co)nucleus is called a *Q-order (co)nucleus*.

- Let the mappings f and g be Q-monotone. We say that (f, g) is a *Q-adjunction*, or a (Q-monotone) *Galois connection* [25], if $e_Y(f(x), y) = e_X(x, g(y))$. Then f is called a *left* and g a *right Q-adjoint*.

It can be easily shown that the right and left Q-adjoint to a Q-monotone mapping f are unique if they exist. Then f^* denotes the right Q-adjoint, and f_* denotes the left Q-adjoint to f. From the definition it also easily follows that when $f\colon (X, e_X) \to (Y, e_Y)$ and $g\colon (Y, e_Y) \to (Z, e_Z)$ are left-adjoint Q-monotone mappings, then $(g \circ f)^* = f^* \circ g^*$.

Like with the definition of Q-ordered sets, we should emphasize that:

- Any Q-monotone (Q-inflating, Q-deflating) mapping $(X, e_X) \to (Y, e_Y)$ is an ordinary monotone (inflating, deflating) mapping $(X, \leq_{e_X}) \to (Y, \leq_{e_Y})$.
- Any Q-order-(pre)(co)nucleus on (X, e_X) is an order-(pre)(co)nucleus on $(X, \leq_{e_X})$.
- Any Q-adjunction (f, g) between (X, e_X) and (Y, e_Y) is a poset adjunction between $(X, \leq_{e_X})$ and $(Y, \leq_{e_Y})$.

Let (X_i, e_{X_i}) for $i \in I$ be Q-ordered sets. On the set direct product $\prod_{i \in I} X_i$ we define a valuation

$$e_{\underline{X}}\left((x_i)_{i\in I}, (y_i)_{i\in I}\right) = \bigwedge_{i\in I} e_{X_i}(x_i, y_i),$$

which can be verified that it is a Q-order [25, 16].

3.1. ***Q*-sup-lattices.** Based on the concepts of Q-orders and Q-subsets, notions which are analogous to ordinary joins and meets can be defined, and behave very much like their crisp counterparts. One can even talk about mappings that 'preserve joins', so a category of fuzzy sup-lattices can be introduced and investigated.

Definition 3.4. Let M be a Q-subset of a Q-ordered set (X, e). An element s of X is called a *Q-join* of M, denoted $\bigsqcup M$ if:

(1) $M(x) \leq e(x, s)$ for all $x \in X$, and
(2) for all $y \in X$, $\bigwedge_{x\in X}(M(x) \to e(x, y)) \leq e(s, y)$.

By analogy, an element m of X is called a *Q-meet* of M, denoted $\bigsqcap M$, if $M(x) \leq e(m, x)$ for all $x \in X$, and for all $y \in X$, $\bigwedge_{x\in X}(M(x) \to e(y, x)) \leq e(y, m)$.

Note that also in the Q-fuzzy setting, the concepts of joins and meets are dual – a Q-join of M with respect to the Q-order e is a Q-meet of M with respect to the dual ordering e^{op}. A useful characterization of Q-joins (similarly for Q-meets) is available:

Proposition 3.5. [1, 28] *Let (X, e) be a Q-ordered set, $s, m \in X$ and $M \in Q^X$. Then $s = \bigsqcup M$ iff for all $y \in X$, $e(s, y) = \bigwedge_{x\in X}(M(x) \to e(x, y))$.*

The antisymmetry property of Q-orders implies that if a Q-join (Q-meet) of a Q-subset exists, it is unique. Indeed, Q-ordered sets in which Q-joins and Q-meets of all Q-subsets exist strongly resemble complete lattices:

Proposition 3.6. [25] *For a Q-ordered set (X, e), the following are equivalent:*

(1) *(X, e) is Q-join-complete ($\bigsqcup M$ exists for any Q-subset M of X).*

(2) (X, e) *is Q-meet-complete ($\bigsqcap M$ exists for any Q-subset M of X).*
(3) (X, e) *is Q-complete (both Q-join-complete and Q-meet-complete).*

Since Q-joins and mappings that preserve them will be of our primary interest, we will denote such Q-complete set (X, e_X) by $(X, \bigsqcup_X)$.

Let X and Y be sets, and $f\colon X \to Y$ be a mapping. We define *Zadeh's forward power set operator* [13] that maps Q-subsets of X to Q-subsets of Y as:

$$f_Q^{\rightarrow}(M)(y) = \bigvee_{x \in f^{-1}(y)} M(x).$$

There is also *Zadeh's backward power set operator* $f_Q^{\leftarrow} : Q^Y \to Q^X$ defined for f as $f_Q^{\leftarrow}(Y) = Y \circ f$.

Let (X, e_X) and (Y, e_Y) be Q-ordered sets. We say that a mapping $f\colon X \to Y$ is *Q-join-preserving* if for any Q-subset M of X such that $\bigsqcup M$ exists, $\bigsqcup_Y f_Q^{\rightarrow}(M)$ exists and

$$f\left(\bigsqcup\nolimits_X M\right) = \bigsqcup\nolimits_Y f_Q^{\rightarrow}(M).$$

By analogy, Q-meet-preserving mappings can be defined.

The following proposition provides a useful tool to verify that a mapping f between Q-complete sets is Q-join-preserving. Instead of calculating with joins, one can check that f is Q-monotone and find its right adjoint.

Proposition 3.7. [25] *Let (X, e_X), (Y, e_Y) be Q-ordered sets, and $f\colon X \to Y$, $g\colon Y \to X$ be two mappings. Then the following holds: If (X, e_X) is Q-complete, then f is Q-monotone and has a right adjoint iff $f\left(\bigsqcup M\right) = \bigsqcup f_Q^{\rightarrow}(M)$ for all $M \in Q^X$.*

Any Q-join-preserving mapping $f\colon X \to Y$ between Q-complete sets (X, e_X) and (Y, e_Y) thus has a Q-meet-preserving right adjoint $f^*\colon Y \to X$.

Proposition 3.8. [16] *Let (X, e) be a Q-complete Q-ordered set. Then $(X, \leq_e)$ is a complete poset, and for any $S \subseteq X$ we have $\bigvee S = \bigsqcup \varphi_S$ where*

$$\varphi_S(x) = \begin{cases} 1 & \textit{if } x \in S, \\ \perp & \textit{otherwise.} \end{cases}$$

In what follows, only the Q-joins will be discussed, and we therefore switch to new terms:

Definition 3.9. A *Q-sup-lattice* is a Q-join-complete set. *Homomorphisms of Q-sup-lattices* are Q-join-preserving mappings. By **Q-Sup** we will denote the category whose objects are Q-sup-lattices, and morphisms are Q-sup-lattice homomorphisms.

3.2. **Category Isomorphism to Quantale Modules.** It was already proved by I. Stubbe in [21] on a general level that the category **Q-Sup** is isomorphic to **Q-Mod** (the category of left Q-modules) for a unital quantale Q. This isomorphism was investigated by S. A. Solovyov in [20] with an explicit description of the isomorphism functors:

For every $q \in Q$ and $S \subseteq X$ define a Q-subset of X by

$$M_S^q(x) = \begin{cases} q, & x \in S, \\ \bot & \text{otherwise.} \end{cases}$$

Specifically for $S = \{s\}$ we shall write M_s^q instead.

Proposition 3.10. [20] *There exists a functor $F\colon Q\text{-}\boldsymbol{Mod} \to Q\text{-}\boldsymbol{Sup}$ defined by*

$$F\left(\left(A, \bigvee, *\right) \xrightarrow{\varphi} \left(B, \bigvee, *\right)\right) = \left(A, \bigsqcup\nolimits_A\right) \xrightarrow{\varphi} \left(B, \bigsqcup\nolimits_B\right)$$

where $e_A(a_1, a_2) = a_1 \to_Q a_2$ and $\bigsqcup M = \bigvee_{a \in A}(M(a) * a)$.

Proposition 3.11. [20] *There exists a functor $G\colon Q\text{-}\boldsymbol{Sup} \to Q\text{-}\boldsymbol{Mod}$ defined by*

$$G\left(\left(A, \bigsqcup\nolimits_A\right) \xrightarrow{\varphi} \left(B, \bigsqcup\nolimits_B\right)\right) = \left(A, \bigvee, *\right) \xrightarrow{\varphi} \left(B, \bigvee, *\right)$$

where

(1) *$a_1 \leq a_2$ iff $1 \leq e_A(a_1, a_2)$ for every $a_1, a_2 \in A$,*
(2) *$\bigvee S = \bigsqcup M_S^1$ for every $S \subseteq A$,*
(3) *$q * a = \bigsqcup M_a^q$ for every $q \in Q$ and every $a \in A$.*

Theorem 3.12. [20] *$G \circ F = \mathrm{Id}_{Q\text{-}\boldsymbol{Mod}}$ and $F \circ G = \mathrm{Id}_{Q\text{-}\boldsymbol{Sup}}$, i.e., the two categories Q-**Mod** and Q-**Sup** are isomorphic.*

This category isomorphism might provide a better intuition of Q-joins and Q-meets – they can be viewed as a sort of 'weighted' joins and meets.

3.3. **Quotients and sub-Q-sup-lattices.** Many statements concerning Q-sup-lattices follow directly from the theory of quantale modules [9, 10] (but they can also be proved using the machinery of Q-orders and Q-subsets as in [16, 17]).

As any Q-sup-lattice homomorphism is a left Q-adjoint, we instantly get the following:

Proposition 3.13. *Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and let $f\colon X \to Y$ be a homomorphism. Then $f^* \circ f$ is a Q-order nucleus on X, and $f \circ f^*$ is a Q-order conucleus on Y.*

Proposition 3.14. *Let $(X, \bigsqcup)$ be a Q-sup-lattice, and j be a Q-order-prenucleus on X. Then the subset of fixed points of j, $X_j = \{x \in X \mid j(x) = x\}$ is a Q-sup-lattice.*

Theorem 3.15. *Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and $f\colon X \to Y$ be a surjective homomorphism. Then there exists a Q-order nucleus $j\colon X \to X$ such that $Y \cong X_j$.*

Definition 3.16. Let $(X, \bigsqcup)$ be a Q-sup-lattice, $Y \subseteq X$, and $M \in Q^Y$. We define an extension $M' \in Q^X$ of M as

$$M'(x) = \begin{cases} M(x) & \text{if } x \in Y, \\ \bot & \text{if } x \notin Y. \end{cases}$$

We say that Y is a *sub-Q-sup-lattice* of X if it is closed under Q-joins in the following sense: for any $M \in Q^Y$, the element $\bigsqcup M' \in X$ also belongs to Y.

The definition above has an equivalent statement: A set $Y \subseteq X$ is a sub-Q-sup-lattice of X if for any $M \in Q^X$ such that $M(x) = \bot$ for all $x \in X \setminus Y$, also $\bigsqcup M \in Y$. Since the sub-Q-sup-lattice Y can itself be regarded as a Q-sup-lattice, the inclusion mapping $Y \hookrightarrow X$ is clearly a homomorphism.

Proposition 3.17. *Let $(X, \bigsqcup)$ be a Q-sup-lattice, and $g\colon X \to X$ be a Q-order preconucleus on X. Then $X_g = \{x \in X \mid g(x) = x\}$ is a sub-Q-sup-lattice of X.*

Theorem 3.18. *Let $(X, \bigsqcup_X)$ and $(Y, \bigsqcup_Y)$ be Q-sup-lattices, and $f\colon X \to Y$ be an injective homomorphism. Then there exists a Q-order conucleus $g\colon Y \to Y$ such that $X \cong Y_g$.*

4. Q-sup-algebras

This section deals with Q-sup-lattices endowed with a (possibly infinite) collection of finitary operations. It is based on classical results for quantales, as presented e.g. in [14]. Like with sup-lattices, quantale quotients and subquantales can be characterized by order nuclei and conuclei, which only have to satisfy an additional compatibility condition with respect to multiplication (so-called closed mappings). These results were later extended to sup-lattices with a general algebraic structure in [12] where so-called sup-algebras (actually their many-sorted variant) were introduced and studied. Similar results were also presented for quantale modules in [9, 15] and quantale algebras [19, 18], and have recently been complemented with further results concerning sup-algebras in [30].

A similar path was set out in the fuzzy-order setting for so-called Q-quantales [22, 33, 8], i.e, Q-sup-lattices A endowed with an associative binary operation that preserves Q-joins in both arguments:

$$a \cdot \bigsqcup M = \bigsqcup (a \cdot -)^{\rightarrow}_Q(M) \qquad \text{and} \qquad \bigsqcup M \cdot a = \bigsqcup (- \cdot a)^{\rightarrow}_Q(M)$$

for any $a \in A$ and $M \in Q^A$.

Now we join these directions together as we will discuss Q-ordered structures with arbitrary operations. We shall build on the results presented in the previous section, and combine the concepts of Q-order nuclei and conuclei with so-called subhomomorphisms, which capture the desired property of these mappings concerning the algebraic operations. Since some of the fuzzy concepts will, similarly to what we have seen before, be just special variants of their crisp counterparts, existing results for sup-algebras can be employed. As a general reference on universal algebra, the book [4] can be recommended.

A *type* is a set Ω of *function symbols*. To each $\omega \in \Omega$, a number $n \in \mathbb{N}_0$ is assigned, which is called the *arity* of ω (and ω is called an n-ary function symbol). Then for each $n \in \mathbb{N}_0$, $\Omega_n \subseteq \Omega$ will denote the subset of all n-ary function symbols from Ω.

Definition 4.1. Given a set Ω, an *algebra of type* Ω (shortly, an Ω-*algebra*) is a pair $\mathcal{A} = (A, \Omega)$ where for each $\omega \in \Omega$ with arity n, there is an n-ary operation $f_\omega\colon A^n \to A$.

For better legibility, instead of f_ω we shall denote the respective operation on $\mathcal{A}$ simply by ω_A, or just ω when no confusion is imminent.

In this section we shall follow a similar convention as we have done for using a quantale Q before: let Ω be an arbitrary type, which shall remain fixed from now on.

We are aware that terminology used in this paper might cause some confusion, as different concepts of an algebra (a monoid object in a category of modules) and an Ω-algebra (an object endowed with a collection of well-behaving finitary operations) appear close to each other. However, no collision of them should happen as they are used in different contexts.

4.1. **Po-algebras and Sup-algebras.** We begin with recapitulating the basic concepts and results presented in [12] and [30]. First, to illustrate, possibly with better clarity, what will be investigated in the Q-sup-lattice setting in the next section. And second, as the Q-valued concepts will show to be just special instances of crisp ones, to provide base for our proofs in the next part.

Definition 4.2. (1) A *partially ordered algebra of type* Ω (shortly, a *po-algebra*) is a triple $\mathcal{A} = (A, \leq, \Omega)$ where $(A, \leq)$ is a poset, (A, Ω) is an Ω-algebra, and each operation ω is monotone in any component, that is, $b \leq c$ implies

$$\omega(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \leq \omega(a_1, \ldots, a_{j-1}, c, a_{j+1}, \ldots, a_n)$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, and $a_1, \ldots, a_n, b, c \in A$.

(2) A monotone mapping $\varphi\colon A \to B$ from a po-algebra $(A, \leq_A, \Omega)$ to a po-algebra $(B, \leq_B, \Omega)$ is called a
  (a) *po-algebra subhomomorphism* if
    (i) $\omega_B(\varphi(a_1), \ldots, \varphi(a_n)) \leq_B \varphi(\omega_A(a_1, \ldots, a_n))$ for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and
    (ii) $\omega_B \leq_B \varphi(\omega_A)$ for any $\omega \in \Omega_0$.
  (b) *po-algebra homomorphism* if
    (i) $\omega_B(\varphi(a_1), \ldots, \varphi(a_n)) = \varphi(\omega_A(a_1, \ldots, a_n))$ for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and
    (ii) $\omega_B = \varphi(\omega_A)$ for any $\omega \in \Omega_0$.

(3) A po-algebra $(B, \leq_B, \Omega)$ is called a *sub-po-algebra* of a po-algebra $(A, \leq_A, \Omega)$ if $B \subseteq A$, and the inclusion mapping is a homomorphism.

Posets and poset-based structures, such as partially ordered semigroups and monoids, modules over them (S-posets), or partially ordered groups, provide well-known instances of po-algebras.

Definition 4.3. A subhomomorphism $j\colon A \to A$ on a po-algebra $(A, \leq_A, \Omega)$ is called a *po-algebra (co)nucleus* if it is also an order (co)nucleus on $(A, \leq_A)$.

Definition 4.4. (1) A *sup-algebra of type* Ω (shortly, a *sup-algebra*) is a triple $\mathcal{A} = (A, \bigvee, \Omega)$ where $(A, \bigvee)$ is a sup-lattice, (A, Ω) is an Ω-algebra, and each operation ω is join-preserving in any component, that is,

$$\omega\left(a_1, \ldots, a_{j-1}, \bigvee B, a_{j+1}, \ldots, a_n\right) = \bigvee\{\omega(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \mid b \in B\}$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \dots, n\}$, $a_1, \dots, a_n \in A$, and $B \subseteq A$.

(2) A join-preserving mapping $\varphi\colon A \to B$ from a sup-algebra $(A, \bigvee, \Omega)$ to a sup-algebra $(B, \bigvee, \Omega)$ is called a *sup-algebra homomorphism* if

$$\omega_B(\varphi(a_1), \dots, \varphi(a_n)) = \varphi(\omega_A(a_1, \dots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \dots, a_n \in A$, and

$$\omega_B = \varphi(\omega_A)$$

for any $\omega \in \Omega_0$.

(3) A sup-algebra $(B, \bigvee, \Omega)$ is said to be a *sub-sup-algebra* of a sup-algebra $(A, \bigvee, \Omega)$ if $B \subseteq A$, and the inclusion mapping is a homomorphism.

Since any sup-algebra can also be regarded as a po-algebra, the concept of sub-homomorphism naturally applies to sup-algebras, too.

Instances of sup-algebras one can commonly encounter include (with operation arities evident from context) sup-lattices ($\Omega = \emptyset$), quantales ($\Omega = \{\cdot\}$), unital quantales ($\Omega = \{\cdot, 1\}$), frames ($\Omega = \{\wedge, 1\}$), quantale modules ($\Omega = \{q* \mid q \in Q\}$), or quantale algebras ($\Omega = \{q* \mid q \in Q\} \cup \{\cdot\}$).

Theorem 4.5. (1) *A sup-algebra $\mathcal{B} = (B, \bigvee_B, \Omega)$ is a quotient of a sup-algebra $\mathcal{A} = (A, \bigvee_A, \Omega)$ iff there exists a nucleus j on $\mathcal{A}$ such that $\mathcal{B} \cong \mathcal{A}_j = (A_j, \bigvee_{A_j}, \Omega)$ where $A_j = \{a \in A \mid j(a) = a\}$, $\bigvee_{A_j} S = j\left(\bigvee_A S\right)$ for any $S \subseteq A$, and $\omega_{A_j} = j \circ \omega_A$ for any $\omega \in \Omega$.*

(2) *A sup-algebra $\mathcal{B} = (B, \bigvee_B, \Omega)$ is a sub-sup-algebra of $\mathcal{A} = (A, \bigvee_A, \Omega)$ iff there exists a conucleus g on $\mathcal{A}$ such that $\mathcal{B} = \mathcal{A}_g = (A_g, \bigvee_{A_g}, \Omega)$ where $A_g = \{a \in A \mid g(a) = a\}$, $\bigvee_{A_g} S = \bigvee_A S$ for any $S \subseteq A$, and $\omega_{A_g} = \omega_A|_{A_g}$ for any $\omega \in \Omega$.*

Proof. Follows from [30, Theorem 24 and Theorem 31]. □

4.2. **Q-ordered Algebras.**

Definition 4.6. A *Q-ordered algebra of type* Ω (shortly, a *Q-ordered algebra*) is a triple $\mathcal{A} = (A, e, \Omega)$ where (A, e) is a Q-ordered set, (A, Ω) is an Ω-algebra, and each operation ω of non-zero arity is Q-monotone in any component, that is,

$$e(b, c) \le e(\omega(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n), \omega(a_1, \dots, a_{j-1}, c, a_{j+1}, \dots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \dots, n\}$, and $a_1, \dots, a_n, b, c \in A$.

Example 4.7 (See also [18, Theorem 3.1].)**.** Let $(S, \circ)$ be a groupoid, and define multiplication on Q^S by $(A \circ B)(s) = \bigvee_{a \circ b = s}(A(a) \cdot B(b))$. Let $A, B, C \in Q^S$ be arbitrary. Then for any $A, B, C \in Q^S$ and $q \in Q$ we have

$$\begin{aligned} & q \le sub_S(B, C) \\ & \quad \iff q \le \bigwedge_{s \in S} (B(s) \to C(s)) \\ & \quad \Longrightarrow \forall s \in S\colon q \le B(s) \to C(s) \\ & \quad \iff \forall s \in S\colon q \cdot B(s) \le C(s) \end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \forall s, c, d \in S \text{ s.t. } c \circ d = s \colon q \cdot A(c) \cdot B(d) \leq A(c) \cdot C(d) \\
&\Longrightarrow \forall s, c, d \in S \text{ s.t. } c \circ d = s \colon q \cdot (A(c) \cdot B(d)) \leq \bigvee_{a \circ b = s} (A(a) \cdot C(b)) \\
&\Longrightarrow \forall s \in S \colon q \cdot \bigvee_{a \circ b = s} (A(a) \cdot B(b)) \leq \bigvee_{a \circ b = s} (A(a) \cdot C(b)) \\
&\Longleftrightarrow \forall s \in S \colon q \leq \bigvee_{a \circ b = s} (A(a) \cdot B(b)) \to \bigvee_{a \circ b = s} (A(a) \cdot C(b)) \\
&\Longrightarrow q \leq \bigwedge_{s \in S} \left(\bigvee_{a \circ b = s} (A(a) \cdot B(b)) \to \bigvee_{a \circ b = s} (A(a) \cdot C(b)) \right) \\
&\qquad = \bigwedge_{s \in S} ((A \circ B)(s) \to (A \circ C)(s)) \\
&\qquad = sub_S(A \circ B, A \circ C),
\end{aligned}$$

and the operation $\circ$ is thus Q-monotone on the right-hand side. Verifying Q-monotonicity on the left-hand side of $\circ$ is completely analogous, and $(Q^S, sub_S, \cdot)$ is then a Q-ordered groupoid.

Definition 4.8. Let (A, e_A, Ω) and (B, e_B, Ω) be Q-ordered algebras, and $\varphi \colon A \to B$ be a Q-monotone mapping. Then φ is called:

(1) a *Q-ordered algebra subhomomorphism* if

$$1 \leq e_B(\omega_B(\varphi(a_1), \dots, \varphi(a_n)), \varphi(\omega_A(a_1, \dots, a_n)))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \dots, a_n \in A$, and

$$1 \leq e_B(\omega_B, \varphi(\omega_A))$$

for every $\omega \in \Omega_0$,

(2) a *Q-ordered algebra homomorphism* if

$$\omega_B(\varphi(a_1), \dots, \varphi(a_n)) = \varphi(\omega_A(a_1, \dots, a_n))$$

for any $n \in \mathbb{N}_0$, $\omega \in \Omega_n$, and $a_1, \dots, a_n \in A$.

Any Q-ordered algebra homomorphism is clearly a subhomomorphism.

Definition 4.9. Let (A, e, Ω) be a Q-ordered algebra, and $\varphi \colon A \to A$ be a subhomomorphism. Then φ is called a *Q-ordered algebra (pre)(co)nucleus* if it is a Q-order (pre)(co)nucleus on (A, e).

Again, we remind that any Q-ordered algebra is a po-algebra, subhomomorphisms of Q-ordered algebras are subhomomorphisms of the respective po-algebras, and all the concepts introduced in the definition above are just special cases of their crisp variants. Moreover, any po-algebra whose operations are coordinatewise Q-monotone with respect to the Q-order $e_{\leq}$ is a Q-ordered algebra.

Because a subhomomorphism between Q-ordered algebras is actually a 'crisp concept' (defined using inequality '≥ 1', hence actually making a statement concerning only the induced order $\leq_e$), one can see, using [30, Section 2], that the composition of subhomomorphisms is a subhomomorphism, too. For homomorphisms, which are defined by equality, such statement is evident.

4.3. **Q-sup-algebras.** Now we are able to combine the concepts presented above, and present a sup-algebra analogue with the ordering relation valuated in Q.

Definition 4.10. A *Q-sup-algebra of type* Ω (shortly, a *Q-sup-algebra*) is a triple $\mathcal{A} = (A, \bigsqcup, \Omega)$ where $(A, \bigsqcup)$ is a Q-sup-lattice, (A, Ω) is an Ω-algebra, and each operation ω is Q-join-preserving in any component, that is,

$$\omega\left(a_1, \dots, a_{j-1}, \bigsqcup M, a_{j+1}, \dots, a_n\right)$$
$$= \bigsqcup \omega(a_1, \dots, a_{j-1}, -, a_{j+1}, \dots, a_n)^{\rightarrow}_{Q}(M)$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \dots, n\}$, $a_1, \dots, a_n \in A$, and $M \in Q^A$.

Example 4.11. As instances of Q-sup-algebras, we may typically encounter the Q-valued counterparts of the examples of sup-algebras:

(1) Q-sup-lattices ($\Omega = \emptyset$),
(2) Q-quantales ($\Omega = \{\cdot\}$),
(3) unital Q-quantales ($\Omega = \{\cdot, 1\}$),
(4) modules over a Q-quantale A, ($\Omega = \{a* \mid a \in A\}$),
(5) Q-quantale algebras over a Q-quantale A ($\Omega = \{a* \mid a \in A\} \cup \{\cdot\}$).

Example 4.12. As shown e.g. in [17, Theorem 2.2.43], Q^X is the free Q-sup-lattice over a set X. By the procedure analogous to Example 4.7, any Ω-algebra A gives rise to a Q-sup-algebra Q^A with operations defined by

$$\omega_{Q^A}(A_1, \dots, A_n)(a) = \bigvee_{\omega_A(a_1, \dots, a_n) = a} A_1(a_1) \cdot \cdots \cdot A_n(a_n),$$

given $n \in \mathbb{N}$, $\omega \in \Omega_n$, $A_1, \dots, A_n \in Q^A$, $a_1, \dots, a_n \in A$.

Definition 4.13. Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup algebras, and $\varphi\colon A \to B$ be a Q-join-preserving mapping. Then φ is called a *Q-sup-algebra homomorphism* if

$$\omega_B(\varphi(a_1), \dots, \varphi(a_n)) = \varphi(\omega_A(a_1, \dots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega$, and $a_1, \dots, a_n \in A$, and

$$\omega_B = \varphi(\omega_A)$$

for any $\omega \in \Omega_0$.

Due to Proposition 3.7, as preserving Q-joins implies Q-monotonicity, any Q-sup-algebra is a Q-ordered algebra, and any homomorphism of Q-sup-algebras is also a homomorphism of Q-ordered algebras.

By Proposition 3.8, any Q-sup-algebra is a sup-algebra, and the same holds for homomorphisms. Conversely, any sup-algebra that is Q-complete, and its operations are coordinatewise Q-monotone, is a Q-sup-algebra.

Proposition 4.14. *Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup algebras, and $f\colon A \to B$ be a homomorphism. Then $f^*\colon B \to A$ is a subhomomorphism.*

Proof. By Proposition 3.7, f^* preserves Q-meets, so it is Q-monotone.

Because f is also a homomorphism between the induced sup-algebras $(A, \bigvee_A, \Omega)$ and $(B, \bigvee_B, \Omega)$, f^* is a sup-algebra subhomomorphism by [30, Proposition 9]. □

Proposition 4.15. *Let $\mathcal{A} = (A, \bigsqcup_A, \Omega)$ and $\mathcal{B} = (B, \bigsqcup_B, \Omega)$ be Q-sup algebras, and $f\colon A \to B$ be a homomorphism. Then $f^* \circ f$ is a nucleus on $\mathcal{A}$, and $f \circ f^*$ is a conucleus on $\mathcal{B}$.*

Proof. We already know that from Proposition 3.13 that $j = f^* \circ f$ is a Q-order nucleus, and $g = f \circ f^*$ is a Q-order conucleus. By the previous proposition, f^* is a subhomomorphism. Then both j and g, being compositions of two subhomomorphisms, are subhomomorphisms, too. □

Proposition 4.16. *If j is a nucleus on a Q-sup-algebra $(A, \bigsqcup, \Omega)$, then*

$$j(\omega(a_1, \dots, a_n)) = j(\omega(a_1, \dots, a_{i-1}, j(a_i), a_{i+1}, \dots, a_n))$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \dots, n\}$, $a_1, \dots, a_n \in A$. Consequently,

$$j(\omega(a_1, \dots, a_n)) = j(\omega(j(a_1), \dots, j(a_n))).$$

Proof. Follows from [12, Lemma 2.2.6] because any Q-sup-algebra nucleus is also a sup-algebra nucleus. □

For any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \dots, n\}$, and $a_1, \dots, a_n \in A$, the mapping $\overline{\omega}\colon A \to A$ given as $\overline{\omega} = \omega(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)$ preserves Q-joins, and it therefore has a right adjoint $\overline{\omega}^*\colon A \to A$, which then satisfies for any $a \in A$,

$$\begin{aligned} e\left(\overline{\omega}(\overline{\omega}^*(a)), a\right) &\geq 1, \\ e\left(a, \overline{\omega}^*(\overline{\omega}(a))\right) &\geq 1. \end{aligned}$$

4.4. **Quotients of Q-sup-algebras.** A *quantic nucleus* on a quantale Q is an order nucleus j satisfying $j(x) \cdot j(y) \leq j(x \cdot y)$ for any $x, y \in Q$. The set of fixed points of j, Q_j, is a quantale with $x \cdot_j y = j(x \cdot y)$. Given a surjective quantale homomorphism $f\colon Q \to R$, the mapping $f^* \circ f$ is a quantic nucleus on Q, and R is isomorphic to Q_j for $j = f^* f$. We will demonstrate an analogous claim for Q-sup-algebras.

Proposition 4.17. *Let $\mathcal{A} = (A, \bigsqcup_A, \Omega)$ be a Q-sup-algebra, and $j\colon A \to A$ be a nucleus on $\mathcal{A}$. Put $A_j = \{a \in A \mid j(a) = a\}$, $\bigsqcup_{A_j} M = j\left(\bigsqcup_A M'\right)$ for every $M \in Q^{A_j}$, $\omega_{A_j}(a_1, \dots, a_n) = j(\omega_A(a_1, \dots, a_n))$ for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \dots, a_n \in A_j$, and $\omega_{A_j} = j(\omega_A)$ for any $\omega \in \Omega_0$. Then $(A_j, \bigsqcup_{A_j}, \Omega)$ is a Q-sup-algebra, and j is a Q-sup-algebra homomorphism from A to A_j.*

Proof. (1) By Proposition 3.14, A_j is a Q-sup-lattice, and j is a Q-sup-lattice homomorphism from A to A_j. It is also an Ω-algebra homomorphism since $j \circ \omega_A = \omega_{A_j}$ for any $\omega \in \Omega$.

(2) We show that every operation ω_{A_j} distributes over Q-joins: take $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, $\omega \in \Omega_n$, $a_1, \dots, a_n \in A_j$, and $M \in Q^{A_j}$. Then

$$\begin{aligned}
&\omega_{A_j}\left(a_1, \dots, a_{i-1}, \bigsqcup\nolimits_{A_j} M, a_{i+1}, \dots, a_n\right)\\
&\quad= j\left(\omega_A\left(a_1, \dots, a_{i-1}, \bigsqcup\nolimits_{A_j} M, a_{i+1}, \dots, a_n\right)\right)\\
&\quad= j\left(\omega_A\left(a_1, \dots, a_{i-1}, j\left(\bigsqcup\nolimits_{A} M'\right), a_{i+1}, \dots, a_n\right)\right)\\
&\quad= j\left(\omega_A\left(a_1, \dots, a_{i-1}, \bigsqcup\nolimits_{A} M', a_{i+1}, \dots, a_n\right)\right)\\
&\quad= j\left(\bigsqcup\nolimits_{A} \omega_A(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)_Q^{\rightarrow}(M')\right)\\
&\quad= j\left(\bigsqcup\nolimits_{A} (\omega_A(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)|_{A_j})_Q^{\rightarrow}(M)\right)\\
&\quad= \bigsqcup\nolimits_{A_j} j_Q^{\rightarrow}\left((\omega_A(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)|_{A_j})_Q^{\rightarrow}(M)\right)\\
&\quad= \bigsqcup\nolimits_{A_j} \left(j \circ (\omega_A(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)|_{A_j})_Q^{\rightarrow}\right)(M)\\
&\quad= \bigsqcup\nolimits_{A_j} \omega_{A_j}(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)_Q^{\rightarrow}(M).
\end{aligned}$$

□

Theorem 4.18. *Let $f\colon A \to B$ be a surjective homomorphism of Q-sup-algebras. Then there exists a nucleus j on A such that $B \cong A_j$.*

Proof. By Theorem 3.15, $B \cong A_j$ as Q-sup-lattices for $j = f^* \circ f$. By Proposition 4.15, j is also a Q-ordered algebra nucleus. □

4.5. **Subalgebras of Q-sup-algebras.** On a quantale Q, a *quantic conucleus* is an order conucleus g satisfying $g(x) \cdot g(y) \leq g(x \cdot y)$ for any $x, y \in Q$. The set Q_g is a subquantale of Q, and for any subquantale R of Q there exists a conucleus g on Q such that $R \cong Q_g$. Again, we show that an analogous characterization holds for Q-sup-algebras.

Definition 4.19. A Q-sup-algebra $(B, \bigsqcup_B, \Omega)$ is a *sub-Q-sup-algebra* of a Q-sup-algebra $(A, \bigsqcup_A, \Omega)$ if $B \subseteq A$, and the inclusion mapping is a Q-sup-algebra homomorphism.

Example 4.20. Let $(A, \bigsqcup_A, \cdot)$ be a Q-quantale, and consider its subset $L = \{x \in A \mid e_A(a \cdot x, x) \geq 1 \text{ for all } a \in A\}$, the analogue of left-sided elements of a quantale, with the inherited Q-order $e_L = e_A|_{L \times L}$. Then for any $x, y \in L$ and $a \in A$ we have $e_A(a \cdot (x \cdot y), (x \cdot y)) = e_A((a \cdot x) \cdot y, x \cdot y) \geq e_A(a \cdot x, x) \geq 1$, so L is closed under multiplication.

Given $M \in Q^L$ and $a \in A$, we calculate:

$$e_A\left(a \cdot \bigsqcup_A M', \bigsqcup_A M'\right) = e_A\left(\bigsqcup_A (a \cdot -)_Q^{\rightarrow}(M'), \bigsqcup_A M'\right)$$

$$= \bigwedge_{x\in A}\left((a\cdot -)^{\to}_{Q}(M')(x) \to e_A\left(x, \bigsqcup_A M'\right)\right)$$
$$= \bigwedge_{x\in A}\left(\bigvee_{\substack{y\in A\\ a\cdot y=x}}(M'(y)) \to e_A\left(x, \bigsqcup_A M'\right)\right)$$
$$= \bigwedge_{x\in A}\bigwedge_{\substack{y\in A\\ a\cdot y=x}}\left(M'(y) \to e_A\left(x, \bigsqcup_A M'\right)\right)$$
$$= \bigwedge_{y\in A}\left(M'(y) \to e_A\left(a\cdot y, \bigsqcup_A M'\right)\right).$$

Since $e_A(y, \bigsqcup_A M') \geq e_A(y, a\cdot y)\cdot e_A(a\cdot y, \bigsqcup_A M') \geq 1\cdot e_A(a\cdot y, \bigsqcup_A M') = e_A(a\cdot y, \bigsqcup_A M')$ for any $y \in L$, and the operation $\to$ is monotone in its right argument, we finish with

$$\bigwedge_{y\in A}\left(M'(y) \to e_A\left(a\cdot y, \bigsqcup_A M'\right)\right) \geq \bigwedge_{y\in A}\left(M'(y) \to e_A\left(y, \bigsqcup_A M'\right)\right)$$
$$= e_A\left(\bigsqcup_A M', \bigsqcup_A M'\right) \geq 1.$$

L is thus closed under Q-joins in A, and for $M \in Q^L$ we can define $\bigsqcup_L M = \bigsqcup_A M'$. Then $(L, \bigsqcup_L, \cdot)$ is a sub-Q-quantale of $(A, \bigsqcup_A, \cdot)$.

Proposition 4.21. *Let $\mathcal{A} = (A, \bigsqcup_A, \Omega)$ be a Q-sup-algebra, and $g\colon A \to A$ be a conucleus on $\mathcal{A}$. Then $(A_g, \bigsqcup_A|_{A_g}, \Omega)$ where $A_g = \{a \in A \mid g(a) = a\}$ is a sub-Q-sup-algebra of $\mathcal{A}$.*

Proof. (1) From Proposition 3.17 we know that $(A_g, \bigsqcup_A|_{A_g})$ is a sub-Q-sup-lattice of $(A, \bigsqcup_A)$.

(2) Since g is also a conucleus on the induced sup-algebra $(A, \bigvee, \Omega)$, we have $\omega_A(a_1, \ldots, a_n) = g(\omega_A(a_1, \ldots, a_n))$ for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A_g$ by [30, Lemma 29].

□

Theorem 4.22. *Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup-algebras, and let $f\colon A \to B$ be an injective homomorphism. Then there exists a conucleus $g\colon B \to B$ such that $A \cong B_g$.*

Proof. By Theorem 3.18, $A \cong B_g$ as Q-sup-lattices for $g = f \circ f^*$. By Proposition 4.15, g is also a Q-ordered-algebra conucleus. As f already is a homomorphism, the statement follows. □

As a special case of the results presented in this section, considering Q-quantales, we get those of [33]. We also note that a many-sorted variant of the above could be considered, like it has been done for sup-algebras in the recent article [32]. For the sake of conciseness, we will not pursue that direction in this paper.

4.6. **Category Isomorphism to Algebras Among Quantale Modules.** We now return back to the category isomorphism theorem for Q-sup-lattices and Q-modules due to S. A. Solovyov [20]. In the same paper, its extension to Q-quantales (semigroups in Q-**Sup**) and Q-algebras (semigroups in Q-**Mod**) was presented, which was also proved independently in [23].

One can immediately notice that the transition between Q-quantales and quantale algebras only takes place between the Q-order and Q-module part of the structure, leaving multiplication intact. A similar isomorphism thus exists when any operation is considered, and we can therefore add one for the category of Q-sup-algebras.

By *Q-module-algebra of type* Ω (shortly, *Q-module-algebra*) we will denote the structure $\mathcal{A} = (A, \bigvee, *, \Omega)$ where $(A, \bigvee, *)$ is a Q-module, (A, Ω) is an Ω-algebra, and each operation ω is a Q-module homomorphism in any component, that is,

$$\begin{aligned}
&\omega\left(a_1, \dots, a_{j-1}, \bigvee B, a_{j+1}, \dots, a_n\right) \\
&\qquad = \bigvee \{\omega(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \mid b \in B\}, \\
&\omega\left(a_1, \dots, a_{j-1}, q * b, a_{j+1}, \dots, a_n\right) \\
&\qquad = q * \omega(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n)
\end{aligned}$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \dots, n\}$, $a_1, \dots, a_n, b \in A$, and $B \subseteq A$.

A mapping $\varphi\colon A \to B$ from a Q-module-algebra $(A, \bigvee, *, \Omega)$ to a Q-module-algebra $(B, \bigvee, *, \Omega)$ is called a *Q-module-algebra homomorphism* if it is both a Q-module homomorphism and an Ω-algebra homomorphism.

For a given quantale Q and a type Ω, let Q-**Sup**-Ω-**Alg** denote the category of Q-sup-algebras of type Ω with Q-sup-algebra homomorphisms, and Q-**Mod**-Ω-**Alg** the category of Q-module-algebras of type Ω with Q-module-algebra homomorphisms.

Proposition 4.23. *There exists a functor*

$$F\colon Q\text{-}\boldsymbol{Mod}\text{-}\Omega\text{-}\boldsymbol{Alg} \to Q\text{-}\boldsymbol{Sup}\text{-}\Omega\text{-}\boldsymbol{Alg}$$

defined by

$$F\left(\left(A, \bigvee, *, \Omega\right) \xrightarrow{\varphi} \left(B, \bigvee, *, \Omega\right)\right) = \left(A, \bigsqcup\nolimits_A, \Omega\right) \xrightarrow{\varphi} \left(B, \bigsqcup\nolimits_B, \Omega\right)$$

where the mappings e and $\bigsqcup$ are obtained as in Proposition 3.10.

Proof. By Proposition 3.10, F acts as a functor from Q-**Mod** to Q-**Sup**. Similarly to [20, Proposition 33], we only need to show that F acts correctly with respect to the operations. Let $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \dots, n\}$ and $a_1, \dots, a_n \in A$ be arbitrary. Then, by Proposition 3.5, the definition of the forward power set operator, and straightforward calculation,

$$e\left(\bigsqcup \omega(a_1, \dots, a_{i-1}, -, a_{i+1}, \dots, a_n)^{\rightarrow}_{Q}(M), b\right)$$

$$
\begin{aligned}
&= \bigwedge_{c\in A} \left(\omega(a_1,\ldots,a_{i-1},-,a_{i+1},\ldots,a_n)^{\rightarrow}_{Q}(M)(c) \to e(c,b)\right)\\
&= \bigwedge_{c\in A} \left(\left(\bigvee_{\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)=c} M(d)\right) \to (c \to_Q b)\right)\\
&= \bigwedge_{c\in A} \bigwedge_{\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)=c} (M(d) \to (c \to_Q b))\\
&= \bigwedge_{c\in A} \bigwedge_{\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)=c} (M(d) \to \\
&\qquad (\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n) \to_Q b))\\
&= \bigwedge_{d\in A} (M(d) \to (\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n) \to_Q b))\\
&= \bigwedge_{d\in A} ((M(d) * \omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)) \to_Q b)\\
&= \bigwedge_{d\in A} (\omega(a_1,\ldots,a_{i-1},M(d)*d,a_{i+1},\ldots,a_n) \to_Q b)\\
&= \left(\bigvee_{d\in A} \omega(a_1,\ldots,a_{i-1},M(d)*d,a_{i+1},\ldots,a_n)\right) \to_Q b\\
&= \omega\left(a_1,\ldots,a_{i-1},\bigvee_{d\in A}(M(d)*d),a_{i+1},\ldots,a_n\right) \to_Q b\\
&= \omega\left(a_1,\ldots,a_{i-1},\bigsqcup M,a_{i+1},\ldots,a_n\right) \to_Q b\\
&= e\left(\omega\left(a_1,\ldots,a_{i-1},\bigsqcup M,a_{i+1},\ldots,a_n\right),b\right),
\end{aligned}
$$

which implies that all the operations are Q-sup-lattice homomorphisms in any coordinate. □

Proposition 4.24. *There exists a functor*

$$G\colon Q\textbf{-}\boldsymbol{Sup}\textbf{-}\Omega\textbf{-}\boldsymbol{Alg} \to Q\textbf{-}\boldsymbol{Mod}\textbf{-}\Omega\textbf{-}\boldsymbol{Alg},$$

$$G\left(\left(A,\bigsqcup\nolimits_A,\Omega\right) \xrightarrow{\varphi} \left(B,\bigsqcup\nolimits_B,\Omega\right)\right) = \left(A,\bigvee,\Omega,*\right) \xrightarrow{\varphi} \left(B,\bigvee,\Omega,*\right)$$

where $\leq$, $\bigvee$ *and* $*$ *are obtained as in Proposition 3.11.*

Proof. By Proposition 3.11, G acts as a functor from Q**-Sup** to Q**-Mod**. Again, like in [20, Proposition 34], we only have to check that G is correct for the operations.

First, let us recall the notation used in the previous section: for every $q \in Q$ and $S \subseteq A$ define a Q-subset of A by

$$M_S^q(a) = \begin{cases} q, & x \in S,\\ \bot & \text{otherwise.}\end{cases}$$

Let $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $i \in \{1, \ldots, n\}$.

First we show that operations preserve joins in any coordinate. Notice that for $S \subseteq A$ and $a_1, \ldots, a_n \in A$,

$$\begin{aligned}
&\omega(a_1, \ldots, a_{i-1}, \bigvee S, a_{i+1}, \ldots, a_n) \\
&\quad = \omega(a_1, \ldots, a_{i-1}, \bigsqcup M_S^1, a_{i+1}, \ldots, a_n) \\
&\quad = \bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)_Q^{\rightarrow}(M_S^1).
\end{aligned}$$

We also have $\bigvee_{s \in S} \omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n) = \bigsqcup M_T^1$, where T stands for the set $\{\omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n) \mid s \in S\}$.

For every $b \in A$ we can calculate

$$\begin{aligned}
&e\left(\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)_Q^{\rightarrow}(M_S^1), b\right) \\
&\quad = \bigwedge_{c \in A} \big(\omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)_Q^{\rightarrow}(M_S^1)(c) \to e(c, b)\big) \\
&\quad = \bigwedge_{c \in A} \left(\left(\bigvee_{\omega(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n) = c} M_S^1(d)\right) \to e(c, b)\right) \\
&\quad = \bigwedge_{c \in A} \bigwedge_{\omega(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n) = c} \big(M_S^1(d) \to e(c, b)\big) \\
&\quad = \bigwedge_{d \in A} \big(M_S^1(d) \to e(\omega(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n), b)\big) \\
&\quad = \bigwedge_{s \in S} (1 \to e(\omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n), b)) \\
&\quad = \bigwedge_{s \in S} e(\omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n), b).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
e\left(\bigsqcup M_T^1, b\right) &= \bigwedge_{c \in A} \big(M_T^1(c) \to e(c, b)\big) \\
&= \bigwedge_{s \in S} (1 \to e(\omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n), b)) \\
&= \bigwedge_{s \in S} e(\omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n), b).
\end{aligned}$$

Combining the above calculations, we get for any $b \in A$

$$e\left(\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)_Q^{\rightarrow}(M_S^1), b\right) = e\left(\bigsqcup M_T^1, b\right),$$

therefore, using condition (E), $\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)_Q^{\rightarrow}(M_S^1) = \bigsqcup M_T^1$, which yields

$$\omega(a_1, \ldots, a_{i-1}, \bigvee S, a_{i+1}, \ldots, a_n) = \bigvee_{s \in S} \omega(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n).$$

Now we verify that the operations preserve the quantale action. Notice that given $q \in Q$ and $a_1, \ldots, a_n \in A$,

$$q * \omega(a_1, \ldots, a_n) = \bigsqcup M^q_{\omega(a_1,\ldots,a_n)},$$

and for any $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
&\omega(a_1, \ldots, a_{i-1}, q * a_i, a_{i+1}, \ldots, a_n) \\
&\quad = \omega(a_1, \ldots, a_{i-1}, \bigsqcup M^q_{a_i}, a_{i+1}, \ldots, a_n) \\
&\quad = \bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)^{\rightarrow}_Q(M^q_{a_i}).
\end{aligned}$$

Then for every $b \in A$ we have

$$\begin{aligned}
&e\left(\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)^{\rightarrow}_Q(M^q_{a_i}), b\right) \\
&\quad = \bigwedge_{c \in A} \big(\omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)^{\rightarrow}_Q(M^q_{a_i})(c) \to e(c, b)\big) \\
&\quad = \bigwedge_{c \in A} \left(\left(\bigvee_{\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)=c} M^q_{a_i}(d)\right) \to e(c, b)\right) \\
&\quad = \bigwedge_{c \in A} \bigwedge_{\omega(a_1,\ldots,a_{i-1},d,a_{i+1},\ldots,a_n)=c} \big(M^q_{a_i}(d) \to e(c, b)\big) \\
&\quad = \bigwedge_{d \in A} \big(M^q_{a_i}(d) \to e(\omega(a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n), b)\big) \\
&\quad = q \to e(\omega(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n), b).
\end{aligned}$$

On the other hand, we get that

$$\begin{aligned}
e\left(\bigsqcup M^q_{\omega(a_1,\ldots,a_n)}, b\right) &= \bigwedge_{c \in A} \left(M^q_{\omega(a_1,\ldots,a_n)}(c) \to e(c, b)\right) \\
&= q \to e\left(\omega(a_1, \ldots, a_n), b\right).
\end{aligned}$$

From the above it follows that, for all $b \in A$,

$$e\left(\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)^{\rightarrow}_Q(M^q_{a_i}), b\right) = e\left(\bigsqcup M^q_{\omega(a_1,\ldots,a_n)}, b\right),$$

which gives us

$$\bigsqcup \omega(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)^{\rightarrow}_Q(M^q_{a_i}) = \bigsqcup M^q_{\omega(a_1,\ldots,a_n)}$$

by condition (E), and this yields

$$\omega(a_1, \ldots, a_{i-1}, q * a_i, a_{i+1}, \ldots, a_n) = q * \omega(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n).$$

□

Theorem 4.25. $G \circ F = \mathrm{Id}_{Q\text{-}\boldsymbol{Mod}\text{-}\Omega\text{-}\boldsymbol{Alg}}$, *and* $F \circ G = \mathrm{Id}_{Q\text{-}\boldsymbol{Sup}\text{-}\Omega\text{-}\boldsymbol{Alg}}$, *i.e., the two categories* Q-***Mod***-Ω-***Alg*** *and* Q-***Sup***-Ω-***Alg*** *are isomorphic.*

Proof. Follows from Theorem 3.12 and Propositions 4.23, 4.24. □

5. **Conclusion**

Following existing research of algebraic structures based on sup-lattices enabled us to introduce general algebras in the category Q-**Sup** ("Q-sup-algebras") for which a few basic results have been obtained. Q-sup-algebras appear to provide a perspective area for future work as it should be possible to follow current research on sup-algebras, and to investigate concepts like representation [30], general forms of projectivity [11], injectivity with respect to subhomomorphisms [31], or the many-sorted variants [12, 32].

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