POWERSET OPERATORS OF EXTENSIONAL FUZZY SETS

J. MOČKOŘ

Abstract. Powerset structures of extensional fuzzy sets in sets with similarity relations are investigated. It is proved that extensional fuzzy sets have powerset structures which are powerset theories in the category of sets with similarity relations, and some of these powerset theories are defined also by algebraic theories (monads). Between Zadeh's fuzzy powerset theory and the classical powerset theory there exists a strong relation, which can be represented as a homomorphism. Analogical results are also proved for new powerset theories of extensional fuzzy sets.

1. Introduction

The notion of the so-called powersets in classical set theory is one of the most useful and exploited tools in various branches of mathematics. Recall that given a set X, there exists the set $P(X) = \{S : Y \subseteq X\}$, called the powerset of X and such that every map $f : X \to Y$ can be extended to the powerset operators $f^{\to} : P(X) \to P(Y)$ and $f^{\leftarrow} : P(Y) \to P(X)$, such that

$$f^{\to}(S) = f(S), \quad f^{\leftarrow}(T) = f^{-1}(T).$$

The powerset structures are widely used in many branches of mathematics, for illustrative examples of possible applications in algebra, logic and topology see, e.g., the introductory part of the paper of [21], for applications of powerset objects in abstract interpretation see, e.g.[2]. A classical set theory can be considered to be a special part of fuzzy set theory, introduced by [23]. A fuzzy set in a set A with values in the interval $I = [0, 1]$ is defined as a map $A \to I$ and it is then natural that an investigation of powerset objects I^X of fuzzy sets was of interest. The first approach was done again by [23], who defined I^X as a new powerset object instead of $P(X)$ and introduced new powerset operators $f_Z^{\to} : I^X \to I^Y$ and $f_Z^{\leftarrow} : I^Y \to I^X$, such that for $s \in I^X, t \in I^Y, y \in Y$,

$$f_Z^{\to}(s)(y) = \bigvee_{x, f(x)=y} s(x), \quad f_Z^{\leftarrow}(t) = t.f.$$

A lot of papers were published about Zadeh's extension and its generalizations, see, e.g. [6, 15, 22]. Zadeh's extension (which could be considered as an extension of a forward powerset operator $f^{\to}$) was intensively studied in [17], especially the relation between $f^{\to}$ and $f_Z^{\to}$. This paper was in fact the first real attempt to uniquely derive the powerset operators $f_Z^{\leftarrow}, f_Z^{\to}$ from $f^{\to}$ and $f^{\leftarrow}$ and not only

Received: December 2016; Revised: May 2017; Accepted: August 2017
Key words and phrases: Extensional fuzzy sets, Powerset operators, Monads in categories.

explicitly stipulate them. Other generalizations of Zadeh powerset operators determined not only by functions but also by binary many-valued relations taking values in residuated lattices were investigated in [3, 5]. Generalizations of Zadeh powerset operators which are based on lattice-theoretic approach only were investigated in [1, 4].

It should be noted that the development of the investigation of powerset objects properties in fuzzy sets was initially realized in two relatively independent branches. The first type of investigation dealt with explicit properties of powerset operators and their relationships, mostly by using classical tools from set theory. That approach was accepted in standard fuzzy set community for its simplicity and only basic scope of knowledge of category theory.

On the contrary to such approach, the other type of investigation was based on an attempt to establish a general theory of powerset objects and operators, mostly with the use of advanced category theory. The principal idea of the categorical approach was an observation that classical powerset objects constitute the so-called *algebraic theory* (or *monad*), introduced by [9]. Roughly speaking, there exists an algebraic theory (in clone form) $\mathbf{P} = (P, \eta, \Diamond)$ in the category *Set*, where $P(X) = 2^X$ (see [18]), such that the operator $f_P^{\rightarrow} : P(X) \to P(Y)$ induced from $\mathbf{P}$ by the Kleisli composition $\Diamond$ is the same as $f^{\rightarrow}$. Analogously can be derived the powerset operator $f_P^{\leftarrow} = f^{\leftarrow}$. In this sense, an algebraic theory $\mathbf{P}$ generates the traditional powerset theory [18]. In the same manner as in the case of an algebraic theory $\mathbf{P} = (P, \eta, \Diamond)$, which generates powerset operators in the category *Set*, we can formally proceed in defining algebraic theory $\mathbf{Z} = (Z, \mu, \Box)$ which defines powerset operators for Q-valued fuzzy sets, where Q is an appropriate lattice. In that case, $Z(X) = Q^X$, and for any $f : X \to Q^Y$, $g : Y \to Q^Z$, the Kleisli composition $g \Box f : X \to Q^Z$ is defined by

$$[(g \Box f)(x)](z) = \bigvee_{y \in Y} f(x)(y) \otimes g(y)(z).$$

Rodabaugh [19] then proved that $\mathbf{Z}$ is an algebraic theory if and only if Q is a unital quantale. Moreover, he also proved that in that case, for each morphism $f : X \to Y$, the lifting of f equals to the classical Zadeh's powerset operator $f_Z^{\rightarrow} : Z(X) \to Z(Y)$. Because a significant part of the theory of Q-valued fuzzy sets is built on some variants of residuated lattices which fulfill that condition, it is possible even in such cases to use algebraic theory for the construction of powerset operators.

Manes [10] introduced also a new structure $\mathbf{T} = (T, e, (-)^*)$, called *fuzzy theory*, such that T assigns to each set X a set $T(X)$, e assigns to each set X a map $e_X : X \to T(X)$, and $(-)^*$ assigns to each function $f : X \to T(Y)$ a function $f^* : T(X) \to T(Y)$, satisfying some additional conditions. Fuzzy theory is, in fact, identical with the algebraic theory $(T, \Diamond, e)$ in the category *Set*, if we set $g \Diamond f = g^*.f$.

Instead of *algebraic theory* (in clone form) introduced by [9], more explicit *powerset theory* was introduced by Rodabaugh [18] as a special structure describing powerset objects. A slight modification of that structure defined in a category

K, is represented by a system $\mathbf{P} = (P, \rightarrow, V, \eta)$, where $P : |\mathbf{K}| \rightarrow CSLAT$ is a powerset generator (where $CSLAT$ is the category of complete $\bigvee$-semilattices with $\bigvee$-preserving maps), $\rightarrow$ is a forward powerset operator, such that for each $f : X \rightarrow Y$ in **K**, $f^{\rightarrow}_{\mathbf{P}} : P(X) \rightarrow P(Y)$ is a morphism in $CSLAT$, $V : \mathbf{K} \rightarrow Set$ is a concrete functor and $\eta_X : V(X) \rightarrow P(X)$ is a map for each object X. He then proved that some powerset theories can be generated by algebraic theories.

Since the original Zadeh's paper was published, the notion of "fuzzy set" has been changed significantly and it is now more general. The first important modification concerns the value set: instead of real number interval $I = [0, 1]$, more general lattice structures Q are considered. Among these lattice structures, complete residuated lattices play important role, (see e.g. [16]), in some terminology *unital and commutative quantale*, (see [20]). A well known example of that structure is the Łukasiewicz algebra.

Fuzzy sets (or even fuzzy sets with values in residuated lattice Q) were originally defined on sets. But any set A can be considered as a couple $(A, =)$, where $=$ is a standard equality relation defined on A. It is then natural instead of the crisp equality relation $=$, to consider some more "fuzzy" equality relation defined on A, which is called a *similarity relation*. Hence, instead of a classical set A as a basic set and a fuzzy set $s : A \rightarrow Q$, we can use a set with Q-valued similarity relation (A, δ) (called a Q-set) and a map $s : A \rightarrow Q$, satisfying some additional properties with respect to δ. In the paper we use fuzzy sets s, which are *extensional* with respect to δ. Such a map then represents a new "fuzzy object" in (A, δ).

Extensional maps can be characterized as special morphisms in a category of sets with similarity relations. In fact, let Set(Q) be the category with Q-sets as objects and morphisms $f : (A, \delta) \rightarrow (B, \gamma)$ which are maps $f : A \rightarrow B$ such that $\gamma(f(x), f(y)) \geq \delta(x, y)$ for all $x, y \in A$. Then a fuzzy sets in A, extensional with respect to δ are morphisms $(A, \delta) \rightarrow (Q, \leftrightarrow)$, where $\leftrightarrow$ is the similarity relation defined by the biresiduation operation in Q. In that case, a powerset object $(Q, \leftrightarrow)^{(A,\delta)}$ of fuzzy sets extensional with respect to δ generalizes Zadeh's powerset object Q^A.

Any Q-valued fuzzy set s in a set A can be defined equivalently by a system of level sets X_α, $\alpha \in Q$, where $X_\alpha = \{a \in A : X(a) \geq \alpha\}$. In our previous papers [14], [13], we proved that also any morphisms $(A, \delta) \rightarrow (Q, \leftrightarrow)$ in the category Set(Q) can be equivalently defined by a system of special subsets of A, indexed by Q, called **f-cuts**. In that case, f-cuts represent another tool for investigation of fuzzy sets, which are extensional with respect to similarity relations. By using f-cuts, instead of a powerset object $(Q, \leftrightarrow)^{(A,\delta)}$ a system of all f-cuts in (A, δ) can be equivalently used.

In the paper we prove that these new powerset objects presented above have similar properties to those of Zadeh's powerset objects. We show that all these new fuzzy objects (i.e., extensional fuzzy sets and f-cuts) have powerset structures which are powerset theories in corresponding categories, in the sense of Rodabaugh and some of these powerset theories are defined by algebraic theories. For classical powerset theories **Z** and **P** there exists a strong relation between these two theories, which can be represented as some homomorphism $\mathbf{P} \rightarrow \mathbf{Z}$. We show that for these new fuzzy theories $\mathbf{F}$ there also exist "new classical" powerset theories $\mathbf{R}$ and homomorphisms $\mathbf{R} \to \mathbf{F}$.

2. Preliminaries

To be more self-contained we repeat several definitions and notations which will be used in the paper, and we also recall several results from previous papers which can be useful for full understanding and notation of our results. For some basic notions from the category theory, see, e.g., [21].

In the paper we use a notion of a complete residuated lattice (see e.g. [16]), in some terminology *unital and commutative quantale*, (see [20]), i.e. a structure $Q = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that $(L, \wedge, \vee)$ is a complete lattice, $(L, \otimes, 1)$ is a commutative monoid with operation $\otimes$ isotone in both arguments and $\to$ is a binary operation which is adjoint with respect to $\otimes$, i.e.

$$\alpha \otimes \beta \le \gamma \text{ iff } \alpha \le \beta \to \gamma.$$

A well known example is the Łukasiewicz algebra $Ł = ([0,1], \vee, \wedge, \otimes, \to_Ł, 0, 1)$, where

$$a \otimes b = 0 \vee (a + b - 1)$$
$$a \to_Ł b = 1 \wedge (1 - a + b).$$

Definition 2.1 (E.G.Manes [9])**.** $\mathbf{T} = (T, \eta, \Diamond)$ is an **algebraic theory** (in clone form) in a category $\mathbf{K}$, if

1. $T : |\mathbf{K}| \to |\mathbf{K}|$ is an object function,
2. η is a system of $\mathbf{K}$-morphisms $\eta_A : A \to T(A)$, for any object A,
3. for each pair of $\mathbf{K}$-morphisms $f : A \to T(B)$, $g : B \to T(C)$, there exists a composition $g \Diamond f : A \to T(C)$, which is associative,
4. for every $\mathbf{K}$-morphism $f : A \to T(B)$, $\eta_B \Diamond f = f$,
5. $\Diamond$ is compatible with composition $\circ$ of morphisms of $\mathbf{K}$, i.e., for each $\mathbf{K}$-morphisms $f : A \to B$, $g : B \to T(C)$, we have $g \Diamond (\eta_B.f) = g.f$.

In that case any morphism $f : A \to B$ in $\mathbf{K}$ lifts to the morphism $f_T^{\to} = T(f) : T(A) \to T(B)$, such that $T(f) = (\eta_B.f) \Diamond 1_{T(A)}$. Hence, T can be extended to a functor $\mathbf{K} \to \mathbf{K}$. Classical powersets then form an algebraic theory $\mathbf{P} = (P, \eta, \Diamond)$ in the category Set, where $P : |Set| \to |Set|$ is the classical powerset function, $\eta_X : X \to P(X)$ is defined by $\eta_X(x) = \{x\}$, and for every $f : X \to P(Y)$, $g : Y \to P(Z)$, $g \Diamond f : X \to P(Z)$ is defined by

$$(g \Diamond f)(x) = \bigcup_{y \in f(x)} g(y).$$

Moreover, the classical powerset operator $f^{\to}$ equals to $P(f) = (\eta_B.f) \Diamond 1_{P(A)}$.

Recall that if $f : L \to M, g : M \to L$ are isotone maps between presets, then $f \vdash g$, provided that for any $a \in L, b \in M$, $a \le g(b) \Leftrightarrow f(a) \le b$. It is clear that $f \vdash g$ iff (f, g) is an isotonic Galois connection, i.e. $f.g \le 1_M$, $g.f \ge 1_L$. From a general lattice theory the following Adjoint Functor Theorem is well known (see, e.g. [7],[17]).

Theorem 2.2. *Let L, M be partially ordered sets such that L has arbitrary $\bigvee$ and let $f^{\rightarrow} : L \to M$ be a map which preserves arbitrary $\bigvee$. Then for any $y \in M$, the map $f^{\leftarrow} : M \to L$ defined by*

$$f^{\leftarrow}(y) = \bigvee_{\{x \in L, f^{\rightarrow}(x) \leq y\}} x,$$

is the unique map $M \to L$ such that

(1) *$(f^{\rightarrow}, f^{\leftarrow})$ is an isotonic Galois connection,*
(2) *$f^{\leftarrow}$ preserves all meets in M.*

Let $CSLAT$ be the category of all complete $\bigvee$-semilattices with $\bigvee$-preserving maps as morphisms. The following definition of Rodabaugh then introduces a general notion of $CSLAT$-powerset theory.

Definition 2.3 (Rodabaugh [18])**.** Let $\mathbf{K}$ be a category. Then $\mathbf{P} = (P, \to, V, \eta)$ is called **$CSLAT$-powerset theory in $\mathbf{K}$**, if

(1) $P : |\mathbf{K}| \to |CSLAT|$ is an object-mapping,
(2) for each $f : A \to B$ in $\mathbf{K}$, there exists $f_{\mathbf{P}}^{\rightarrow} : P(A) \to P(B)$ in $CSLAT$,
(3) $V : \mathbf{K} \to Set$ is a concrete functor, such that for each $A \in \mathbf{K}$, η determines in Set a mapping $\eta_A : V(A) \to P(A)$,
(4) For each $f : A \to B$ in $\mathbf{K}$, $f_{\mathbf{P}}^{\rightarrow}.\eta_A = \eta_B.V(f)$.

For simplicity, instead of $CSLAT$-powerset theory we will speak only about powerset theory. It is clear that an algebraic theory (in clone form) $\mathbf{P} = (P, \eta, \Diamond)$ is also a powerset theory, where the forward powerset operator $f_{\mathbf{P}}^{\rightarrow}$ is derived from the Kleisli composition $\Diamond$, i.e., $f_{\mathbf{P}}^{\rightarrow} = (\eta_B.f)\Diamond 1_{P(A)}$. In this case we say that the *algebraic theory generates a powerset theory.*

We will frequently deal with the following situation. Let $\mathbf{K}$ be a category and let $P : \mathbf{K} \to CSLAT$ be a covariant functor. It follows that for any morphism $f : A \to B$, $P(f)$ is a map preserving all sup. Instead of $P(f)$, we use $f_P^{\rightarrow}$.

By using Theorem 2.2, for any morphism $f : A \to B$ in $\mathbf{K}$, there exists the map $f_P^{\leftarrow} : P(B) \to P(A)$ defined by

$$f_P^{\leftarrow}(Y) = \bigvee_{\{X \in P(A) : f_P^{\rightarrow}(X) \leq Y\}} X, \tag{1}$$

for any $Y \in P(B)$. It is then clear that $f_P^{\leftarrow} : P(B) \to P(A)$ preserves all existing meets and $(f_P^{\rightarrow}, f_P^{\leftarrow})$ is an isotonic Galois connection. *If a functor P is given, then we will reffer by $f_P^{\leftarrow}$ always to the map defined by (1).*

We need the following simple modification of a natural transformation. Let $H, L : \mathbf{K}^{op} \to CSLAT$ be contravariant functors. We say that $\eta : H \to L$ is a *lex natural transformation*, if for any morphism $f : A \to B$, we have $\eta_A.H(f) \leq L(f).\eta_B$.

Proposition 2.4. *Let P and F be covariant functors $\mathbf{K} \to CSLAT$ and let $\eta : P \to F$ be a natural transformation.*

(1) *There exist contravariant functors $F^{op}, P^{op} : \mathbf{K} \to CSLAT$, such that*

(a) $F^{op}(A) = F(A)$, $P^{op}(A) = P(A)$ *for all* $A \in |\mathbf{K}|$,
(b) $F^{op}(f) = f_F^{\leftarrow}$, $P^{op}(f) = f_P^{\leftarrow}$ *for any morphism* $f : A \to B$.

(2) $\eta : P^{op} \to F^{op}$ *is a lex natural transformation.*

Proof. (1) We need to prove $F^{op}(g.f) = F^{op}(f).F^{op}(g)$, for all morphisms $A \to^f B \to^g C$ in $\mathbf{K}$. For any $t \in F^{op}(C)$, we set

$$S_1 = \{s \in F(A) : f_F^{\rightarrow}(s) \leq g_F^{\leftarrow}(t)\},$$
$$S_2 = \{s \in F(A) : g_F^{\rightarrow}(f_F^{\rightarrow}(s)) \leq t\}.$$

Since F is a functor, it is clear then that

$$F^{op}(g.f)(t) = (g.f)_F^{\leftarrow}(t) = \bigvee_{s \in S_2} s, \quad F^{op}(f).F^{op}(g)(t) = f_F^{\leftarrow}.g_F^{\leftarrow}(t) = \bigvee_{s \in S_1} s.$$

Let $s \in S_1$. Since $g_F^{\rightarrow}$ preserves sup, we have

$$g_F^{\rightarrow}(f_F^{\rightarrow}(s)) \leq g_F^{\rightarrow}(g_F^{\leftarrow}(t)) = g_F^{\rightarrow}(\bigvee_{u, g_F^{\rightarrow}(u) \leq t} u) = \bigvee_{u, g_F^{\rightarrow}(u) \leq t} g_F^{\rightarrow}(u) \leq t.$$

Hence, $s \in S_2$. Conversely, let $s \in S_2$, i.e. $g_F^{\rightarrow}(f_F^{\rightarrow}(s)) \leq t$. We set $u := f_F^{\rightarrow}(s)$. Then $g_F^{\rightarrow}(u) \leq t$ and we have

$$g_F^{\leftarrow}(t) = \bigvee_{v, g_F^{\rightarrow}(v) \leq t} v \geq u = f_F^{\rightarrow}(s).$$

Hence, $s \in S_1$ and it follows that F^{op} is a contravariant functor. Similarly it can be proved for P.

(2) Let $f : A \to B$ be a morphism in $\mathbf{K}$ and let us consider the following diagram.

$$\begin{array}{ccc} P(A) & \xleftarrow{f_P^{\leftarrow}} & P(B) \\ \eta_A \downarrow & & \downarrow \eta_B \\ F(A) & \xleftarrow{f_F^{\leftarrow}} & F(B). \end{array}$$

Since $\eta : P \to F$ is a natural transformation, we have $f_F^{\rightarrow}.\eta_A = \eta_B.f_P^{\rightarrow}$. Since $(f_F^{\rightarrow}, f_F^{\leftarrow})$ and $(f_P^{\rightarrow}, f_P^{\leftarrow})$ are isotonic Galois connections, we have $f_F^{\leftarrow}.f_F^{\rightarrow} \geq id_{F(A)}$, $f_P^{\rightarrow}.f_P^{\leftarrow} \leq id_{P(A)}$. It follows that

$$\eta_A \leq f_F^{\leftarrow}.f_F^{\rightarrow}.\eta_A = f_F^{\leftarrow}.\eta_B.f_P^{\rightarrow},$$

and it follows that

$$\eta_A.f_P^{\leftarrow} \leq f_F^{\leftarrow}.\eta_B.f_P^{\rightarrow}.f_P^{\leftarrow} \leq f_F^{\leftarrow}.\eta_B.$$

□

For some natural transformations $\eta : P \to F$, $\eta : P^{op} \to F^{op}$ is also a natural transformation. In fact, let us consider the classical powerset functor $\mathbf{P} : Set \to CSLAT$ and Zadeh's functor $Z : Set \to CSLAT$ of Q-valued fuzzy sets, and a natural transformation $\eta : \mathbf{P} \to Z$, such that $\eta_X = \chi_X$ is the characteristic map of X. Then $\eta : \mathbf{P}^{op} \to Z^{op}$ is a natural transformation, as it can be proved by a simple computation.

On the other hand, in the next section we show an example of powerset objects functors P, F and their natural transformation $\eta : P \to F$, such that $\eta : P^{op} \to F^{op}$ is not a natural transformation.

We need to define a morphism between two powerset theories.

Definition 2.5. Let $\mathbf{T} = (T, \rightarrow, V, \eta)$ and $\mathbf{R} = (R, \Rightarrow, W, \tau)$ be powerset theories in a category $\mathbf{K}$. Then $(\Phi, \Psi) : \mathbf{T} \to \mathbf{R}$ is a morphism, if Φ and Ψ assign to each $X \in |\mathbf{K}|$, a morphism $\Phi_X : T(X) \to R(X)$ and a morphism $\Psi_X : V(X) \to W(X)$, respectively, such that

(1) for each morphism $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} T(X) & \xrightarrow{f_T^{\rightarrow}} & T(Y) \\ {\scriptstyle \Phi_X}\downarrow & & {\scriptstyle \Phi_Y}\downarrow \\ R(X) & \xrightarrow{f_R^{\Rightarrow}} & R(Y), \end{array}$$

(2) for each object X, the following diagram commutes:

$$\begin{array}{ccc} V(X) & \xrightarrow{\Psi_X} & W(X) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \tau_X} \\ T(X) & \xrightarrow[\Phi_X]{} & R(X). \end{array}$$

If $\mathbf{T} = (T, \rightarrow, V, \eta)$ and $\mathbf{R} = (R, \Rightarrow, W, \tau)$ are powerset theories in a category $\mathbf{K}$ and T, R and V, W are functors, then (Φ, Ψ) is a morphism $\mathbf{T} \to \mathbf{R}$, iff $\Phi : T \to R$ and $\Psi : V \to W$ are natural transformations such that the diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\Psi} & W \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \tau} \\ T & \xrightarrow{\Phi} & R. \end{array}$$

Recall some facts about sets with similarity relations. A set with similarity relation (or Q-set) is a couple (A, δ), where $\delta : A \times A \to Q$ is a map such that

(a) $(\forall x \in A) \quad \delta(x, x) = 1$,
(b) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,
(c) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$ (generalized transitivity).

As we mentioned in Introduction, we use the category $\mathrm{Set}(Q)$, with Q-sets as objects. Within morphisms of $\mathrm{Set}(Q)$ we are interested in morphism $(A, \delta) \to (Q, \leftrightarrow)$, which are identical to extensional fuzzy sets. Let

$$F(A, \delta) = (Q, \leftrightarrow)^{(A,\delta)}$$

be the powerset object of all maps $A \to Q$, which are extensional with respect to δ. For simplicity, elements of $F(A, \delta)$ are called *fuzzy objects*.

In many papers (see, e.g.,[13, 12, 8]) the following extensional hull

$$\widehat{\ } : Q^A \to F(A, \delta)$$

was introduced, such that for any map $s : A \to Q$, $\widehat{s}(a) = \bigvee_{x\in A} \delta(a,x) \otimes s(x)$, for any $a \in A$.

As mentioned in Introduction, extensional fuzzy sets in the category Set(Q) can be equivalently represented by special cut systems, which are called *f-cuts* and defined as follows.

Definition 2.6. Let (A,δ) be a Q-set. Then a system $\mathbf{C} = (C_\alpha)_\alpha$ of subsets of A is called an **f-cut** in (A,δ) in the category Set(Q) if

(a) $\forall a, b \in A, \quad a \in C_\alpha \Rightarrow b \in C_{\alpha\otimes\delta(a,b)}$,
(b) $\forall a \in A, \forall \alpha \in Q, \quad \bigvee_{\{\beta : a\in C_\beta\}} \beta \geq \alpha \Rightarrow a \in C_\alpha$.

Let

$$C(A,\delta) \subseteq (2^A)^Q$$

be set of all f-cuts in (A,δ). Recall (see [14],[13]), that any system of subsets indexed by elements of Q can be extended to f-cut system. In fact, for any Q-set (A,δ) there exist closure maps

$$c_{(A,\delta)} : (2^A)^Q \to C(A,\delta),$$

such that $c_{(A,\delta)}(\mathbf{C}) = \overline{\mathbf{C}} = (\overline{C_\alpha})_\alpha$, where

$$\overline{C_\alpha} = \{a \in A : \bigvee_{\{(x,\beta) : x\in C_\beta\}} \beta \otimes \delta(a,x) \geq \alpha\}, \text{ for any } (C_\alpha)_\alpha, C_\alpha \subseteq A.$$

According to [13], Theorem 4.2, for any Q-set (A,δ), there exists a bijection $\phi_{(A,\delta)} : C(A,\delta) \to F(A,\delta)$, such that for each f-cut $\mathbf{X} \in C(A,\delta)$ and each $s \in F(A,\delta)$, we have

$$\phi_{(A,\delta)}(\mathbf{X})(a) = \bigvee_{\beta, a\in X_\beta} \beta, \quad \phi^{-1}_{(A,\delta)}(s) = (s_\alpha)_\alpha, \tag{2}$$

where $s_\alpha = \{a \in A : s(a) \geq \alpha\}$.

3. Powerset Objects in Set(Q)

In this section we investigate powerset objects $F(A,\delta) = (Q,\leftrightarrow)^{(A,\delta)}$ of extensional fuzzy sets, which are defined in the category Set(Q). We show that these powerset objects are *powerset theories* in the sense of Definition 2.3, and that these powerset theories are algebraic, i.e., defined by an *algebraic theory*, in the sense of Definition 2.1.

3.1. Extensional Fuzzy Sets as Fuzzy Objects.

In the paper [11] we proved, that $F(A,\delta) = (Q,\leftrightarrow)^{(A,\delta)}$ represents an object function of a functor $F : \mathrm{Set}(Q) \to CSLAT$ of extensional fuzzy sets in Set(Q), where $CSLAT$ is the category of complete $\bigvee$-semilattices with $\bigvee$-preserving maps as morphisms. For any morphism $f : (A,\delta) \to (B,\gamma)$ in Set(Q), we have

$$f^{\rightarrow}_F(s)(b) = F(f)(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(f(x), b),$$

where ordering on $F(A,\delta)$ is defined point-wise. It is clear that F is a generalization of the classical Zadeh's powerset functor $Z : Set \to CSLAT$. In fact, if $(A,=),(B,=)$ are Q-sets with equalities as similarity relations, then a morphism $f : (A,=) \to (B,=)$ in Set(Q) is a classical map $f : A \to B$, $F(A,=)$ is the set of all maps $A \to Q$ and $F(f) = Z(f)$.

We show that, analogously as the Zadeh's functor Z is a generalization of a classical powerset functor $\mathbf{P}$ in the category Set of sets, the functor F is a generalization of a powerset functor T in the category Set(Q), where T is a powerset functor $T : \text{Set}(Q) \to CSLAT$, defined by

$$T(A,\delta) = (2^A, \subseteq), \quad X \in 2^X, f_T^{\to}(X) = f(X).$$

The following propositions are extensions of some results from Rodabaugh [17, 18, 19], where we use functor F of extensional fuzzy sets with respect to similarity relation instead of classical Zadeh's functor Z and powerset functor P in the category Set.

Proposition 3.1. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in Set(Q).*

(1) *There exists a natural transformation $\eta : T \to F$.*
(2) $(\forall t \in F(B,\gamma))(\forall a \in A) \quad f_F^{\leftarrow}(t)(a) = t.f(a)$,
(3) $(\forall Y \subseteq B) \quad f_T^{\leftarrow}(Y) = f^{-1}(Y)$.

Proof. (1) We define a natural transformation $\eta : T \to F$, by

$$(\forall (A,\delta))(\forall X \subseteq A) \quad \eta_{(A,\delta)}(X) = \widehat{\chi_A(X)},$$

where $\chi_A(X)$ is a characteristic map of X in A and $\widehat{\chi_A(X)}$ is an extensional hull of $\chi_A(X)$ (see Section 2). It can be proved easily that $\eta_{(A,\delta)}$ is a morphism in $CSLAT$. Then for any morphism $f : (A,\delta) \to (B,\gamma)$, the following diagram commutes:

$$\begin{array}{ccc} T(A,\delta) & \xrightarrow{f_T^{\to}} & T(B,\gamma) \\ {\scriptstyle \eta_{(A,\delta)}}\downarrow & & \downarrow{\scriptstyle \eta_{(B,\gamma)}} \\ F(A,\delta) & \xrightarrow{f_F^{\to}} & F(B,\gamma). \end{array}$$

In fact, let $X \subseteq A, b \in B$. Then we have

$$\eta_{(B,\gamma)}.f_T^{\to}(X)(b) = \chi_B\widehat{(f_T^{\to}(X))}(b) = \bigvee_{y\in B} \chi_B(f_T^{\to}(X))(y) \otimes \gamma(b,y) =$$

$$\bigvee_{y\in f_T^{\to}(X)} \gamma(b,y) = (*),$$

and

$$f_F^{\to}.\eta_{(A,\delta)}(X)(b) = f_F^{\to}.\widehat{\chi_A(X)}(b) = \bigvee_{x\in A} \widehat{\chi_A(X)}(x) \otimes \gamma(f(x),b) =$$

$$\bigvee_{x,z\in A} \chi_A(X)(z) \otimes \delta(z,x) \otimes \gamma(f(x),b) = (**).$$

Since f is a morphism in Set(Q), we obtain

$$(*) = \bigvee_{z\in X} \gamma(f(z), b) \leq (**) \leq \bigvee_{x\in A, z\in X} \gamma(f(z), f(x)) \otimes \gamma(f(x), b) \leq \bigvee_{z\in X} \gamma(f(z), b) = (*),$$

and the diagram commutes.

(2) Let $s \in F(A,\delta)$ be such that $f_F^{\rightarrow}(s) \leq t$, i.e.,

$$(\forall b \in B) \quad f_F^{\rightarrow}(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(f(x), b) \leq t(b).$$

For $b = f(a)$ and for any such s, we have

$$s(a) \leq f_F^{\rightarrow}(s)(f(a)) \leq t(f(a)),$$

and it follows that

$$f_F^{\leftarrow}(t)(a) = \bigvee_{s\in F(A,\delta), f_F^{\rightarrow}(s)\leq t} s(a) \leq t(f(a)).$$

On the other hand, since $t : (B,\gamma) \to (Q,\leftrightarrow)$ is a morphism in Set(Q), for $s := t.f$ we have

$$f_F^{\rightarrow}(t.f)(b) = \bigvee_{x\in A} t(f(x)) \otimes \gamma(f(x), b) \leq t(b),$$

and we receive

$$t.f(a) \leq \bigvee_{s\in F(A,\delta), f_F^{\rightarrow}(s)\leq t} s(a) \leq t(f(a)).$$

(3) The proof for $f_T^{\leftarrow}$ it trivial. □

Let (A,δ) be a Q-set and let $\{-\}_{(A,\delta)} : A \to T(A,\delta)$, $\widehat{\{-\}}_{(A,\delta)} : A \to F(A,\delta)$ be maps defined by $\{-\}_{(A,\delta)}(a) := \{a\}$, $\widehat{\{-\}}_{(A,\delta)}(a) := \widehat{\chi_A(\{x\})}(a)$. Hence, $\widehat{\{-\}}_{(A,\delta)}(a) = \delta(a,x)$. If V : Set(Q) $\to$ *Set* is a forgetful functor, i.e. $V(A,\delta) = A$, then $\{-\} : V \to T$ and $\widehat{\{-\}} : V \to F$ are corresponding natural transformations. Rodabaugh [18] proved that both classical functors $\mathbf{P}$ and Z generates a powerset theories and that these theories are algebraic. In the next theorem we prove that the same is true for our functors T and F.

Theorem 3.2. *The following statements hold.*

(1) $\mathbf{T} = (T, \to, V, \{-\})$ *and* $\mathbf{F} = (F, \to, V, \widehat{\{-\}})$ *are* $CSLAT$*-powerset theories.*
(2) *There exists a morphism* $\mathbf{T} \to \mathbf{F}$.
(3) $\mathbf{F}$ *is an algebraic theory, i.e. it is defined by monad.*

Proof. (1) For a functor T, for any morphism $f : (A,\delta) \to (B,\gamma)$ in Set(Q), we set $f_T^{\rightarrow} = T(f)$. Since $\{-\} : V \to T$ is a natural transformation, $\mathbf{T}$ is a powerset theory in Set(Q). Analogously, for a functor F, we put $f_F^{\rightarrow} = F(f)$. Since $\widehat{\{-\}} : V \to F$ is a natural transformation, $\mathbf{F}$ is a powerset theory.

(2) It is clear that the following diagram of natural transformations commutes,

$$\begin{array}{ccc} T & \xrightarrow{\eta} & F \\ {\scriptstyle \{-\}}\big\uparrow & & \big\uparrow{\scriptstyle \widehat{\{-\}}} \\ V & \xrightarrow{1_V} & V, \end{array}$$

where η is a natural transformation from Prop. 3.1. Part (2) then follows from directly the Definition 2.5 and Prop. 3.1.

(3) Let $\widetilde{F} : |\mathrm{Set}(Q)| \to |\mathrm{Set}(Q)|$ be the object function defined by

$$\widetilde{F}(A, \delta) = (F(A, \delta), \sigma_{(A,\delta)}),$$

where the similarity relation $\sigma_{(A,\delta)}$ is defined by $\sigma_{(A,\delta)}(s,t) = \bigwedge_{x\in A} s(x) \leftrightarrow t(x)$. For each (A, δ), we define

$$\psi_{(A,\delta)} : (A, \delta) \to \widetilde{F}(A, \delta),$$
$$\psi_{(A,\delta)}(x) = \widehat{\chi_A(\{x\}})}.$$

Since

$$\sigma_{(A,\delta)}(\psi_{(A,\delta)}(x), \psi_{(A,\delta)}(y)) = \bigwedge_{z\in A} \delta(z, x) \leftrightarrow \delta(z, y) \geq \delta(x, y),$$

$\psi_{(A,\delta)}$ is a morphism in $\mathrm{Set}(Q)$.

Now, let $f : (A, \delta) \to \widetilde{F}(B, \gamma)$ and $g : (B, \gamma) \to \widetilde{F}(C, \omega)$ be morphisms in $\mathrm{Set}(Q)$. Then $g\Diamond f : (A, \delta) \to \widetilde{F}(C, \omega)$ is defined by

$$(g\Diamond f)(a)(c) = \bigvee_{b\in B} f(a)(b) \otimes g(b)(c).$$

We show that $(g\Diamond f)(a) \in F(C, \omega)$. In fact, we have

$$(g\Diamond f)(a)(c) \otimes \omega(c, c') = \bigvee_{b\in B} f(a)(b) \otimes g(b)(c) \otimes \omega(c, c') \leq$$
$$\bigvee_{b\in B} f(a)(b) \otimes g(b)(c') = (g\Diamond f)(a)(c').$$

Further, $g\Diamond f$ is a morphism in $\mathrm{Set}(Q)$. In fact, we have

$$\sigma_{(C,\omega)}((g\Diamond f)(a), (g\Diamond f)(a')) = \bigwedge_{c\in C} (g\Diamond f)(a)(c) \leftrightarrow (g\Diamond f)(a')(c) =$$
$$\bigwedge_{c\in C} (\bigvee_{b\in B} f(a)(b) \otimes g(b)(c)) \leftrightarrow (\bigvee_{b'\in B} f(a')(b') \otimes g(b')(c)) \geq$$
$$\bigwedge_{c\in C} (\bigwedge_{b\in B} (f(a)(b) \otimes g(b)(c) \leftrightarrow f(a')(b) \otimes g(b)(c))) \geq$$
$$\bigwedge_{b\in B} f(a)(b) \leftrightarrow f(a')(b) \geq \delta(a, a').$$

Then $\widetilde{\mathbf{F}} = (\widetilde{F}, \psi, \Diamond)$ is an algebraic theory (in clone form) in Set(Q). Let $f : (A,\delta) \to \widetilde{F}(B,\gamma)$ be a morphism in Set(Q). Then $\psi_{(B,\gamma)} \Diamond f = f$. In fact, let $a \in A, b \in B$.

$$\psi_{(B,\gamma)} \Diamond f(a)(b) = \bigvee_{y \in B} f(a)(y) \otimes \psi_{(B,\gamma)}(y)(b) =$$
$$\bigvee_{y \in B} f(a)(y) \otimes \gamma(y,b) = f(a)(b).$$

Further, $\Diamond$ is compatible with composition in Set(Q). In fact, let $f : (A,\delta) \to (B,\gamma)$ and $g : (B,\gamma) \to \widetilde{F}(C,\omega)$ be morphisms, then $g \Diamond (\psi_{(B,\gamma)}.f) = g.f$. Indeed, for $a \in A, c \in C$, we have

$$g \Diamond (\psi_{(B,\gamma)}.f)(a)(c) = \bigvee_{b \in B} (\psi_{(B,\gamma)}.f)(a)(b) \otimes g(b)(c) =$$
$$\bigvee_{b \in B} \gamma(b, f(a)) \otimes g(b)(c) = (*).$$

Since g is a morphism in Set(Q), we have $\gamma(f(a), b) \le \sigma_{(C,\omega)}(gf(a), g(b))$, and it follows that

$$(*) \le \bigvee_{b \in B} \sigma_{(C,\omega)}(gf(a), g(b)) \otimes g(b)(c) \le \bigvee_{b \in B} (g(b)(c) \to gf(a)(c)) \otimes g(b)(c)$$
$$\le g(f(a))(c).$$

On the other hand, $(*) \ge \gamma(f(a), f(a)) \otimes g(f(a))(c) = g(f(a))(c)$.

Finally, let $f : (A,\delta) \to \widetilde{F}(B,\gamma)$, $g : (B,\gamma) \to \widetilde{F}(C,\omega)$, $h : (C,\omega) \to \widetilde{F}(D,\rho)$ be morphisms in Set(Q). Then we have $h \Diamond (g \Diamond f) = (h \Diamond g) \Diamond f$. The proof can be done similarly as the proof of [18], Lemma 5.1. Hence, $\widetilde{\mathbf{F}}$ is a monad (in clone form). To prove that $\widetilde{\mathbf{F}}$ algebraically generates powerset theory $\mathbf{F}$ in the category Set(Q), we need to prove that for each morphism $f : (A,\delta) \to (B,\gamma)$ in Set(Q), $f_F^{\rightarrow} = (\psi_{(B,\gamma)}.f) \Diamond 1_{\widetilde{F}(A,\delta)}$. For each $s \in F(A,\delta)$ and $b \in B$, we have

$$(\psi_{(B,\gamma)}.f) \Diamond 1_{\widetilde{F}(A,\delta)}(s)(b) = \bigvee_{x \in A} s(x) \otimes \chi_B \widehat{(\{f(x)\})}(b) =$$
$$\bigvee_{x \in A} \bigvee_{y \in B} s(x) \otimes \chi_B(\{f(x)\})(y) \otimes \gamma(y,b) = \bigvee_{x \in A} s(x) \otimes \gamma(f(x), b) = f_F^{\rightarrow}(b).$$

□

Example 3.3. We show that for the natural transformation $\eta : T \to F$, η, in general, is not a natural transformation $T^{op} \to F^{op}$. In fact, let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and let δ be defined by $\delta(a_1,a_2) = \delta(a_2,a_1) = 0_Q, \delta(a_1,a_1) = \delta(a_2,a_2) = 1_Q$, and analogously, $\gamma(b_1,b_2) = \gamma(b_2,b_1) = \alpha$, $\gamma(b_1,b_1) = \gamma(b_2,b_2) = 1_Q$, where $\alpha \in Q, \alpha > 0_Q$. Then $(A,\delta), (B,\gamma) \in$ Set(Q) and the map $f : A \to B$, defined by $f(a_i) = b_i$, $i = 1,2$, is a morphism in Set(Q). Let us consider the following diagram:

$$\begin{array}{ccc} T(A,\delta) & \xleftarrow{f_T^{\leftarrow}} & T(B,\gamma) \\ {\scriptstyle \eta_{(A,\delta)}}\downarrow & & \downarrow{\scriptstyle \eta_{(B,\gamma)}} \\ F(A,\delta) & \xleftarrow{f_F^{\leftarrow}} & F(B,\gamma). \end{array}$$

Let $Y = \{b_1\} \in T(B,\gamma)$, then $f_T^{\leftarrow}(Y) = \{a_1\}$, and

$$\eta_{(A,\delta)}.f_T^{\leftarrow}(Y)(a_2) = \chi_A\widehat{(f_T^{\leftarrow}(Y))}(a_2) = \bigvee_{x\in A} \chi_A(\{a_1\})(x) \otimes \delta(a_2, x) =$$

$$\delta(a_2, a_1) = 0_Q < \alpha = \gamma(b_2, b_1) = \bigvee_{y\in Y} \gamma(f(a_2), y) = \bigvee_{y\in B} \chi_B(Y)(y) \otimes \gamma(y, f(a_2))$$

$$= \widehat{\chi_B(Y)}(f(a_2)) = f_F^{\leftarrow}.\eta_B(Y)(a_2).$$

Hence, the diagram does not commute and η is only a lex natural transformation.

In the category Set(Q), the extension $f_F^{\rightarrow}$ is connected with the extension $f_T^{\rightarrow}$. We can show that $f_F^{\rightarrow}$ can be uniquely derived also as the unique extension of the original morphism $f : (A,\delta) \to (B,\gamma)$. Moreover, we also show that $f_F^{\rightarrow}$ is the unique $\bigvee$-preserving map, which preserves special fuzzy singletons. Recall that an extensional fuzzy set $s \in F(A,\delta)$ is a *fuzzy singleton*, if $s(x) \otimes s(y) \le \delta(x, y)$, for $x, y \in A$. In the proof we will use decomposition methods generalizing results from [17].

By $\underline{\alpha} = \underline{\alpha}_A$ we denote a map $(A,\delta) \to (Q,\leftrightarrow)$ with the constant value $\alpha \in Q$. It is clear that $\underline{\alpha} \in F(A,\delta)$. In the following lemma we show that any extensional fuzzy set can be represented as a supremum of special fuzzy singletons.

Lemma 3.4. *Let $s \in F(A,\delta)$. Then*

1. $s = \bigvee_{x\in A} \underline{s(x)} \otimes \widehat{\chi_A(\{x\})}$,
2. $s = \bigvee_{\alpha\in Q} \underline{\alpha} \otimes \widehat{\chi_A(s_\alpha)}$, *where* $s_\alpha = \{x \in A : s(x) \ge \alpha\}$.

Proof. (1) Let $a \in A$, then

$$(\bigvee_{x\in A} \underline{s(x)} \otimes \widehat{\chi_A(\{x\})})(a) = \bigvee_{x\in A} s(x) \otimes \widehat{\chi_A(\{x\})}(a) = \bigvee_{x\in A} s(x) \otimes \delta(a, x) = (*).$$

Since s is a morphism in Set(Q), we have $(*) \le s(a)$ and if we put $x := a$ in (*), we obtain $(*) \ge s(a)$.

(2) Let $a \in A$. Then we have

$$\bigvee_{\alpha\in Q} \underline{\alpha} \otimes \widehat{\chi_A(s_\alpha)})(a) = \bigvee_{\alpha\in Q} \alpha \otimes \widehat{\chi_A(s_\alpha)}(a) =$$

$$\bigvee_{\alpha\in Q} \alpha \otimes (\bigvee_{x\in s_\alpha} \delta(a, x)) = \bigvee_{\alpha\in Q}\bigvee_{x\in s_\alpha} \alpha \otimes \delta(a, x) = (*).$$

Since s is a morphism in Set(Q), for every $\alpha \in Q, x \in s_\alpha$, we have $s(a) \ge s(x) \otimes \delta(a, x) \ge \alpha \otimes \delta(a, x)$, and it follows that $s(a) \ge (*)$. On the other hand, if we set $\alpha = s(a), x = a$, we obtain $(*) \ge s(a) \otimes \delta(a, a) = s(a)$. □

The following theorem states that $f_F^{\rightarrow}$ can be characterized as a sup-preserving map which also preserves some fuzzy singletons. It generalizes Theorem 6.8 in [17].

Theorem 3.5. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in Set(Q) and let $h : F(A,\delta) \to F(B,\gamma)$ be a map. Then the following statements are equivalent.*

1. $h = f_F^{\rightarrow}$.

(2) *The following hold:*
  (i) *h preserves all $\bigvee$,*
  (ii) *h preserves fuzzy singletons, i.e.* $(\forall \alpha > 0, x \in A)$

$$h(\underline{\alpha}_A \otimes \widehat{\chi_A(\{x\})}) = \underline{\alpha}_B \otimes \chi_B\widehat{(\{f(x)\}}).$$

Proof. Let h satisfies (i) and (ii). Then according to Lemma 3.4, for $s \in F(A, \delta)$, we have $s = \bigvee_{x \in A} \underline{s(x)}_A \otimes \widehat{\chi_A(\{x\}})$. It follows that for any $b \in B$,

$$h(s)(b) = h(\bigvee_{x \in A} \underline{s(x)}_A \otimes \widehat{\chi_A(\{x\}}))(b) = \bigvee_{x \in A} h(\underline{s(x)}_A \otimes \delta(x, -))(b) =$$
$$\bigvee_{x \in A} \underline{s(x)}_A \otimes \chi_B\widehat{(\{f(x)\}})(b) = \bigvee_{x \in A} s(x) \otimes \gamma(f(x), b) = f_F^{\rightarrow}(s)(b).$$

Conversely, if $h = f_F^{\rightarrow}$, then for any $b \in B$, we have

$$f_F^{\rightarrow}(\underline{\alpha}_A \otimes \widehat{\chi_A(\{x\}}))(b) = \bigvee_{z \in A} \alpha \otimes \delta(x, z) \otimes \gamma(f(z), b) =$$
$$\alpha \otimes \bigvee_{z \in A} \delta(x, z) \otimes \gamma(f(z), b) = (*).$$

Since f is a morphism in Set(Q), we have

$$(*) \le \alpha \otimes \bigvee_{z \in A} \gamma(f(x), f(z)) \otimes \gamma(f(z), b) \le$$
$$\alpha \otimes \gamma(f(x), b) = (\underline{\alpha}_B \otimes \chi_B\widehat{\{f(x)}\})(b) \le (*)$$

□

In the following theorem we show another characterization of $f_F^{\rightarrow}$ as a $\bigvee$-preserving map which commutes with some fuzzy singletons. It generalizes [17], Theorem 6.13.

Theorem 3.6. *Let $f : (A, \delta) \to (B, \gamma)$ be a morphism in Set(Q). For $\alpha \in Q$, let $\eta_\alpha^{(A,\delta)} : (A, \delta) \to F(A, \delta)$ be defined by*

$$(\forall x \in A) \quad \eta_\alpha^{(A,\delta)}(x) = \underline{\alpha}_A \otimes \widehat{\chi_A(\{x\}}).$$

Let $g : F(A, \delta) \to F(B, \gamma)$ be a map. Then the following statements are equivalent.

(i) $g = f_F^{\rightarrow}$.
(ii) *g preserves arbitrary sup and for every $\alpha \in Q$, the following local diagram commutes:*

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{f} & (B,\gamma) \\ {\scriptstyle \eta_\alpha^{(A,\delta)}}\downarrow & & \downarrow{\scriptstyle \eta_\alpha^{(B,\gamma)}} \\ F(A,\delta) & \xrightarrow{g} & F(B,\gamma). \end{array}$$

Proof. (1) Let $g = f_F^{\rightarrow}$, and let $\alpha \in Q$, $a \in A$, $b \in B$. Then we have

$$f_F^{\rightarrow}.\eta_\alpha^{(A,\delta)}(a)(b) = f_F^{\rightarrow}(\eta_\alpha^{(A,\delta)}(a))(b) = \bigvee_{x\in A} \eta_\alpha^{(A,\delta)}(a)(x) \otimes \gamma(f(x),b) =$$
$$\bigvee_{x\in A} (\underline{\alpha}_A \otimes \widehat{\chi_A(\{a\}})(x) \otimes \gamma(f(x),b) = \bigvee_{x\in A} (\alpha \otimes \widehat{\chi_A(\{a\}})(x)) \otimes \gamma(f(x),b) =$$
$$\bigvee_{x\in A} (\alpha \otimes \delta(a,x)) \otimes \gamma(f(x),b) = (*).$$

On the other hand, we have

$$\eta_\alpha^{(B,\gamma)}.f(a)(b) = \eta_\alpha^{(B,\gamma)}(f(a))(b) = (\underline{\alpha}_B \otimes \chi_B\widehat{(\{f(a)\}})(b) =$$
$$\alpha \otimes \chi_B\widehat{(\{f(a)\}})(b) = \alpha \otimes \gamma(f(a),b) = (**).$$

Then we obtain

$$(*) \leq \bigvee_{x\in A} (\alpha \otimes \gamma(f(a),f(x)) \otimes \gamma(f(x),b) \leq \bigvee_{x\in A} (\alpha \otimes \gamma(f(a),b) \leq (**),$$

and, if we put $x := a$ in $(*)$, we obtain $(*) \geq (**)$. Since the operation $\otimes$ is distributive with respect to the operation $\bigvee$ in Q, $f_F^{\rightarrow}$ preserves arbitrary sup.

(2) Let g satisfies conditions from (ii). Then for any $s \in F(A,\delta), b \in B$, according to Lemma 3.4, we have

$$g(s)(b) = g(\bigvee_{x\in A} \underline{s(x)}_A \otimes \widehat{\chi_A(\{x\}})(b) = \bigvee_{x\in A} g(\underline{s(x)}_A \otimes \widehat{\chi_A(\{x\}})(b) =$$
$$\bigvee_{x\in A} g.\eta_{s(x)}^{(A,\delta)}(x)(b) = \bigvee_{x\in A} \eta_{s(x)}^{(B,\gamma)}.f(x)(b) = \bigvee_{x\in A} \eta_{s(x)}^{(B,\gamma)}(f(x))(b) =$$
$$\bigvee_{x\in A} \underline{s(x)}_B \otimes \chi_B\widehat{(\{f(x)\}})(b) = \bigvee_{x\in A} s(x) \otimes \gamma(f(x),b) = f_F^{\rightarrow}(s)(b).$$

□

It is clear that $f_F^{\rightarrow}$ is an extension of a morphism f. In fact, if $\eta_{(A,\delta)} : (A,\delta) \to F(A,\delta)$ is an embedding in Set(Q), defined by $\eta_{(A,\delta)}(x) = \delta(x,-)$, then the following diagram commutes:

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{f} & (B,\gamma) \\ \eta_{(A,\delta)}\downarrow & & \downarrow\eta_{(B,\gamma)} \\ F(A,\delta) & \xrightarrow{f_F^{\rightarrow}} & F(B,\gamma). \end{array}$$

On the other hand, if $h : F(A,\delta) \to F(B,\gamma)$ is another map such that the above diagram commutes, it is not true, in general, that $h = f_F^{\rightarrow}$. Let us consider the following example.

Example 3.7. Let (A,δ) and (B,γ) be the same as in Example 3.3 and let $f : A \to B$ be defined such that $f(a_1) = f(a_2) = b_1$. Then $f : (A,\delta) \to (B,\gamma)$ is a morphism in Set(Q). Let a function $h : F(A,\delta) \to F(B,\gamma)$ be defined by

$$(\forall \underline{\alpha}_A \in F(A,\delta)) \quad h(\underline{\alpha}_A) = \underline{\alpha}_B, \text{ where } \alpha \in Q,$$
$$(\forall s \in F(A,\delta))(\forall \alpha \in Q) \quad s \neq \underline{\alpha}_A \Rightarrow h(s) = f_F^{\rightarrow}(s).$$

Then we have $h \neq f_F^{\rightarrow}$. In fact, let $\beta > \alpha$, then we have

$$f_F^{\rightarrow}(\underline{\beta}_A)(b_2) = \beta \otimes \bigvee_{x\in A} \gamma(f(x), b_2) = \beta \otimes (\alpha \vee \alpha) = \beta \otimes \alpha \leq \beta \wedge \alpha = \alpha < \beta,$$
$$h(\underline{\beta}_A)(b_2) = \underline{\beta}_B(b_2) = \beta.$$

On the other hand, the following diagram commutes:

$$\begin{array}{ccc} (A,\delta) & \xrightarrow{f} & (B,\gamma) \\ {\scriptstyle \eta_{(A,\delta)}}\downarrow & & \downarrow{\scriptstyle \eta_{(B,\gamma)}} \\ F(A,\delta) & \xrightarrow{h} & F(B,\gamma). \end{array}$$

In fact, let $a \in A$, then we have $\eta_{(A,\delta)}(a) = \delta(a,-) \neq \underline{\alpha}_A$, for all $\alpha \in Q$ and $h(\eta_{(A,\delta)}(a)) = f_F^{\rightarrow}(\eta_{(A,\delta)}(a)) = \eta_{(B,\gamma)}.f(a)$.

3.2. ***f*-Cut Systems as Fuzzy Objects.** As we mentioned in Introduction, any fuzzy set extensional with respect to a similarity relation, can be expressed equivalently as an f-cut in a Q-set (see equation (2) in Section 2). This relationship can be expressed in a categorical form as a natural isomorphisms between the functor F and the functor C : Set$(Q) \to CSLAT$, such that for any morphism $f : (A,\delta) \to (B,\gamma)$,

$$C(A,\delta) = (\{\mathbf{E} | \mathbf{E} = (E_\alpha)_\alpha \text{ is an f-cut in } (A,\delta)\}, \leq)$$
$$f_C^{\rightarrow} : C(A,\delta) \to C(B,\gamma), \quad f_C^{\rightarrow}(\mathbf{E}) = (\overline{f(E_\alpha)})_\alpha.$$

where $(C_\alpha)_\alpha \leq (D_\alpha)_\alpha$, iff $C_\alpha \subseteq D_\alpha$, for any $\alpha \in Q$. Recall that

$$\overline{f(E_\alpha)} = \{b \in B : \bigvee_{(y,\beta):y\in f(E_\beta)} \beta \otimes \gamma(b,y) \geq \alpha\},$$

(see [13]). The natural isomorphism $\phi : C \to F$ is defined by equations (2) in Section 2. Using that natural isomorphism, we can translate results from previous Section 3.1 to results about f-cuts and powerset objects of f-cuts. Some results can be obtained by a simple application of this natural isomorphism, and these proofs will be omitted, some other results require additional proofs.

The natural isomorphism ϕ was defined in [13] for functors F, C : Set$(Q) \to Set$. To be able to use isomorphism ϕ for our purposes, we need to prove that $C(A,\delta)$ is a complete $\bigvee$-semilattice and $C(f)$ is a $\bigvee$-preserving map.

Lemma 3.8. *C is a functor Set$(Q) \to CSLAT$.*

Proof. In [13]; Th. 4.1, we proved that C is a functor Set$(Q) \to Set$. Let $\mathbf{E}_i = (E_{i\alpha})_\alpha, i \in I$, be f-cuts in (A,δ). Then a join operation in $C(A,\delta)$ is defined by $\bigvee_i \mathbf{E}_i = \bigvee_i (E_{i,\alpha})_\alpha = (\overline{\bigcup_i E_{i,\alpha}})_\alpha$, as it can be verified simply. To prove that C is a functor $\mathrm{Set}(Q) \to CSLAT$, we have to prove only that $f_C^{\to}$ preserves all joins, i.e. $f_C^{\to}(\bigvee_i \mathbf{E}_i) = \bigvee_i f_C^{\to}(\mathbf{E}_i)$. We have

$$f_C^{\to}(\bigvee_i \mathbf{E}_i) = f_C^{\to}((\overline{\bigcup_i E_{i,\alpha}})_\alpha) = (\overline{f(\overline{\bigcup_i E_{i,\alpha}})})_\alpha,$$
$$\bigvee_i f_C^{\to}(\mathbf{E}_i) = \bigvee_i (\overline{f(E_{i,\alpha})})_\alpha = (\overline{\bigcup_i \overline{f(E_{i,\alpha})}})_\alpha.$$

Since $f(E_{i,\alpha}) \subseteq f(\overline{\bigcup_i E_{i,\alpha}})$, we obtain

$$\overline{f(E_{i,\alpha})} \subseteq \overline{f(\overline{\bigcup_i E_{i,\alpha}})}$$

and it follows that $f_C^{\to}(\bigvee_i \mathbf{E}_i) \geq \bigvee_i f_C^{\to}(\mathbf{E}_i)$.

On the other hand, according to [13]; Prop. 4.3, we have $f(\overline{E_{i,\alpha}}) \subseteq \overline{f(E_{i,\alpha})}$, and we obtain

$$f(\overline{\bigcup_i E_{i,\alpha}}) \subseteq \overline{f(\bigcup_i E_{i,\alpha})} \subseteq \overline{\bigcup_i f(E_{i,\alpha})} \subseteq \overline{\bigcup_i \overline{f(E_{i,\alpha})}},$$

and the opposite inequality also holds. Hence, C is the functor $\mathrm{Set}(Q) \to CSLAT$. □

We show firstly that the powerset object functor C is also a generalization of the powerset functor T from Section 3.1.

Proposition 3.9. *Let $f : (A, \delta) \to (B, \gamma)$ be a morphism in Set(Q).*

(1) *There exists a natural transformation $\zeta : T \to C$.*
(2) $(\forall \mathbf{D} \in C(B, \gamma)) \quad f_C^{\leftarrow}(\mathbf{D}) = (f^{-1}(D_\alpha))_\alpha$.

Proof. (1) Let $\zeta_{(A,\delta)} : T(A, \delta) \to C(A, \delta)$ be defined by

$$(\forall X \subseteq A) \quad \zeta_{(A,\delta)}(X) = (\overline{X_\alpha})_\alpha, \quad X_\alpha = X, \quad (\forall \alpha \in Q).$$

Hence, $\overline{X_\alpha} = \{a \in A : \bigvee_{x \in X} \delta(a, x) \geq \alpha\}$. It can be proved easily that $\zeta_{(A,\delta)}$ is a morphism in the category $CSLAT$. Then for any morphism $f : (A, \delta) \to (B, \gamma)$, the diagram commutes:

$$\begin{array}{ccc} T(A,\delta) & \xrightarrow{f_T^{\to}} & T(B,\gamma) \\ \zeta_{(A,\delta)} \downarrow & & \downarrow \zeta_{(B,\gamma)} \\ C(A,\delta) & \xrightarrow{f_C^{\to}} & C(B,\gamma). \end{array}$$

In fact, for $X \in T(A, \delta)$, we have $f_C^{\to}.\zeta_{(A,\delta)}(X) = (\overline{f(\overline{X_\alpha})})_\alpha$, and analogously, $\zeta_{(B,\gamma)}.f_T^{\to}(X) = (\overline{f(X)_\alpha})_\alpha$. Since $f(\overline{X_\alpha}) \subseteq \overline{f(X_\alpha)}$, we obtain that the diagram commutes and $\zeta : T \to C$ is a natural transformation.

(2) Let $\mathbf{D} \in C(B, \gamma)$. Then according to (1), we have

$$f_C^{\leftarrow}(\mathbf{D}) = \bigvee_{\mathbf{E} \in C(A,\delta), f_C^{\to}(\mathbf{E}) \leq \mathbf{D}} \mathbf{E}.$$

Let $\mathbf{E} \in C(A,\delta)$ be such that $f_C^{\rightarrow}(\mathbf{E}) \leq \mathbf{D}$. It follows that $\overline{f(E_\alpha)} \subseteq D_\alpha$, for all $\alpha \in Q$. Then $f(E_\alpha) \subseteq \overline{f(E_\alpha)} \subseteq D_\alpha$, and it follows that $E_\alpha \subseteq f^{-1}(D_\alpha)$. Hence, $f_C^{\leftarrow}(\mathbf{D}) \leq (f^{-1}(D_\alpha))_\alpha$. On the other hand, we set $\mathbf{E} = (f^{-1}(D_\alpha))_\alpha$. Then we have $f_C^{\rightarrow}(\mathbf{E}) \leq \mathbf{D}$ and we received the opposite inequality. □

We want to show that the fuzzy object functor C also generates a powerset theory which is algebraic. Let (A,δ) be a Q-set and let $\overline{\{-\}}_{(A,\delta)} : A \to C(A,\delta)$ be defined by

$$\overline{\{-\}}_{(A,\delta)}(a) = (\overline{X_\alpha})_\alpha, \quad X_\alpha = \{a\}, \forall \alpha \in Q.$$

Then an analogy of Theorem 3.2 holds for the functor $C_{\mathrm{Set}(Q)}$.

Theorem 3.10. *The following statements hold.*

(1) $\mathbf{C} = (C, \rightarrow, V, \overline{\{-\}})$ *is a CSLAT-powerset theory.*
(2) *There exists a morphism* $\mathbf{T} \to \mathbf{C}$, *where* $\mathbf{T}$ *is the powerset theory from Section 3.1.*
(3) $\mathbf{C}$ *is an algebraic theory, i.e., it is defined by a monad.*

Proof. (1) We show firstly that $\overline{\{-\}} : V \to C$ is a natural transformation. Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in $\mathrm{Set}(Q)$. Then for $a \in A$, we have $f_C^{\rightarrow}.\overline{\{-\}}_{(A,\delta)}(a) = (\overline{f(\overline{X}_\alpha)})_\alpha$, where $X_\alpha = \{a\}$, for each α. On the other hand, $\overline{\{-\}}_{(A,\delta)}.f(a) = \overline{\{-\}}_{(B,\gamma)}(f(a)) = (\overline{Y}_\alpha)_\alpha$, where $\overline{Y}_\alpha = \overline{\{f(a)\}}_\alpha = \{b \in B : \gamma(f(a),b) \geq \alpha\}$. Let $b \in \overline{Y}_\alpha$. Then we have $\bigvee_{(x,\beta):\delta(a,x)\geq\beta} \beta \otimes \gamma(b,f(x)) \geq 1_Q \otimes \gamma(b,f(a)) \geq \alpha$, and it follows that $b \in \overline{f(\overline{X}_\alpha)}$. On the other hand, if $b \in \overline{f(\overline{X}_\alpha)}$, then since f is a morphism, we obtain

$$\alpha \leq \bigvee_{(x,\beta):\delta(a,x)\geq\beta} \beta \otimes \gamma(b,f(x)) \leq \bigvee_{(x,\beta):\delta(a,x)\geq\beta} \delta(a,x) \otimes \gamma(b,f(x)) \leq \bigvee_{(x,\beta):\delta(a,x)\geq\beta} \gamma(f(x),f(a)) \otimes \gamma(b,f(x)) \leq \gamma(b,f(a)),$$

and $b \in \overline{Y}_\alpha$. Hence, $\overline{\{-\}}$ is a natural transformation. The rest follows directly from Proposition 3.9.

(2) It follows directly from Definition 2.5, and Proposition 3.9.

(3) For the construction of the monad $\widetilde{\mathbf{C}}$ we use the existence of a natural isomorphism ϕ and results from Theorem 3.2.

The object function $\widetilde{C} : |\mathrm{Set}(Q)| \to |\mathrm{Set}(Q)|$ of the monad is defined by

$$\widetilde{C}(A,\delta) = (C(A,\delta), \tau_{(A,\delta)}), \text{ where}$$
$$\tau_{(A,\delta)}(\mathbf{X},\mathbf{Y}) = \sigma_{(A,\delta)}(\phi_{(A,\delta)}(\mathbf{X}), \phi_{(A,\delta)}(\mathbf{Y})).$$

For each (A,δ), we define

$$\rho_{(A,\delta)} : (A,\delta) \to \widetilde{C}(A,\delta),$$
$$\rho_{(A,\delta)}(x) = (\overline{\{x\}}_\alpha)_\alpha = \phi^{-1}_{(A,\delta)}(\widehat{\chi_A(\{x\})}).$$

Then for each $x, y \in A$, we have

$$\tau_{(A,\delta)}(\rho_{(A,\delta)}(x), \rho_{(A,\delta)}(y)) = \sigma_{(A,\delta)}(\widehat{\chi_A(\{x\})}, \widehat{\chi_A(\{y\})}) \geq \delta(x,y),$$

as follows from Theorem 3.2, and $\rho_{(A,\delta)}$ is a morphism in $\mathrm{Set}(Q)$.

Now, let $f : (A,\delta) \to \widetilde{C}(B,\gamma)$ and $g : (B,\gamma) \to \widetilde{C}(D,\omega)$ be morphisms in $\mathrm{Set}(Q)$. Then $g\Box f : (A,\delta) \to \widetilde{C}(D,\omega)$ is defined by

$$g\Box f := \phi^{-1}_{(D,\omega)}.((\phi_{(D,\omega)}.g)\Diamond(\phi_{(B,\gamma)}.f)).$$

Then from Theorem 3.2, it follows that $\widetilde{\mathbf{C}} = (\widetilde{C}, \rho, \Box)$ is an algebraic theory (in clone form) in $\mathrm{Set}(Q)$. □

Similarly as for extensional fuzzy sets from $F(A,\delta)$, we can define a singleton f-cut.

Definition 3.11. An f-cut $\mathbf{C} = (C_\alpha)_\alpha \in C(A,\delta)$ is called a **singleton f-cut**, if for every $x, y \in A$, the following holds:

$$\bigvee_{\{(\alpha,\beta):x\in C_\alpha, y\in C_\beta\}} \alpha \otimes \beta \leq \delta(x,y).$$

The following example represents example of a singleton f-cut, which can be used for decomposition of any f-cut, as we can see from the next proposition.

Example 3.12. Let $\alpha, \beta \in Q, a \in A$. We set $S^{a,\beta}_\alpha = S^{a,\beta}_{\alpha,(A,\delta)} = \{x \in A : \beta \otimes \delta(a,x) \geq \alpha\}$. Then $\mathbf{S}^{a,\beta} = \mathbf{S}^{a,\beta}_{(A,\delta)} = (S^{a,\beta}_{\alpha,(A,\delta)})_\alpha$ is a singleton f-cut.

Lemma 3.13. *Let (A,δ) be a Q-set, then the following statements are equivalent.*

(1) *$s : (A,\delta) \to (Q, \leftrightarrow)$ is a singleton,*
(2) *$\phi^{-1}_{(A,\delta)}(s) = (\{x \in A : s(x) \geq \alpha\})_\alpha$ is a singleton f-cut.*

The singleton f-cuts $(S^{a,\beta}_{\alpha,(A,\delta)})_\alpha$ can be used for a decomposition of any f-cut. In fact, the following proposition holds.

Proposition 3.14. *Let $\mathbf{C} = (C_\alpha)_\alpha \in C(A,\delta)$ be an f-cut. Then we have*

$$\mathbf{C} = \bigvee_{\{(\beta,x):x\in C_\beta\}} \mathbf{S}^{x,\beta}.$$

Proof. We show firstly that $\bigcup_{\{(\beta,x):x\in C_\beta\}} S^{x,\beta}_\alpha \subseteq C_\alpha$. In fact, let for some $x \in C_\beta$, we have $a \in S^{x,\beta}_\alpha$. Then $\beta \otimes \delta(x,a) \geq \alpha$. Since $\mathbf{C}$ is an f-cut, from $x \in C_\beta$ it follows that $a \in C_{\beta\otimes\delta(a,x)} \subseteq C_\alpha$ and the inclusion holds. Hence, we have $\overline{\bigcup_{\{(\beta,x):x\in C_\beta\}} S^{x,\beta}_\alpha} \subseteq \overline{C_\alpha} = C_\alpha$ and $\bigvee_{x\in C_\beta} \mathbf{S}^{x,\beta} \leq \mathbf{C}$. On the other hand, for any $a \in C_\alpha$, we have $a \in S^{a,\alpha}_\alpha \subseteq \bigcup_{\{(\beta,x):x\in C_\beta\}} S^{x,\beta}_\alpha$, and the opposite inequality holds. □

The following theorem is an f-cut variant of Theorem 3.5.

Theorem 3.15. *Let $f : (A,\delta) \to (B,\gamma)$ be a morphism in Set(Q) and let $h : C(A,\delta) \to C(B,\gamma)$ be a map. Then the following statements are equivalent.*

(1) $h = f^{\rightarrow}_{C}$.
(2) *The following hold:*
   (i) *h preserves all* $\bigvee$,
   (ii) *h preserves singletons f-cuts, i.e.*

$$(\forall \beta > 0, x \in A) \quad h(\mathbf{S}^{x,\beta}_{(A,\delta)}) = \mathbf{S}^{f(x),\beta}_{(B,\gamma)}.$$

Proof. The proof follows directly from the ϕ-translation of Theorem 3.5., and from the equality

$$\phi_{(A,\delta)}(\mathbf{S}^{x,\beta}_{(A,\delta)}) = \underline{\alpha}_A \otimes \widehat{\chi_A(\{x\})}.$$

□

4. Conclusions

The fundamental importance of powerset operators for fuzzy sets was first recognized by L.A. Zadeh in his principal paper ([23]). In that paper he describes an extension principle which enables to extend any mapping between two sets A, B to a mapping between sets of fuzzy sets $Z(A), Z(B)$ defined over corresponding sets. To explain Zadeh's extension principle, Rodabaugh used *algebraic theories* (or monads), introduced by E.G. Manes. Namely he proved that there exists an algebraic theory $\mathbf{Z} = (Z, \mu, \Box)$ defined in the category Set, such that $\mathbf{Z}$ induces *powerset theory* $(Z, \rightarrow, V, \chi)$ (introduced also by Rodabaugh). In that case the Zadeh's extension $f^{\rightarrow}_{Z} : Z(A) \rightarrow Z(B)$ can be derived as the lifting $(\mu_A \circ f)\Box 1_{Z(A)}$.

Zadeh's work was limited to classically defined fuzzy set, i.e. to maps from the underlying set to real interval $[0, 1]$. Since then, many other generalizations of fuzzy sets were defined. These generalizations are related to another structures Q of value sets of fuzzy sets (e.g. lattices with some additional properties) or instead of fuzzy sets defined as maps $A \rightarrow Q$, another structures are considered. In that way, new fuzzy sets can be, e.g., morphisms in some categories, or some systems of level sets, analogical to α-level sets in classical fuzzy sets.

In the paper we were interested in fuzzy sets extensional with respect to a similarity relation defined in the category Set(Q), and in powerset objects of these extensional fuzzy sets. We proved that the powerset objects and powerset operators of these new extensional fuzzy sets have similar properties to those of classical fuzzy sets $s : X \rightarrow Q$. We proved that all these extensional fuzzy sets have powerset structures which are powerset theories in the category Set(Q), in the sense of Rodabaugh [18] and some of these powerset theories are defined also by algebraic theories (monads). Between Zadeh's fuzzy powerset theory $\mathbf{Z}$ and the classical powerset theory $\mathbf{P}$ there exists a relationship, which can be represented as a homomorphism from $\mathbf{P}$ to $\mathbf{Z}$. We proved that also for these new powerset theories $\mathbf{F}$ there exists a "new classical" powerset theory $\mathbf{R}$ and a homomorphism $\mathbf{R} \rightarrow \mathbf{F}$.

All these results extend results of Rodabaugh [17, 18, 19] and explain the powerset structures of frequently used extensional fuzzy sets.

Acknowledgements. This work was supported by the Centre of Excellence project LQ1602).

References

[1] C. De Mitri and C. Guido, *Some remarks on fuzzy powerset operators*, Fuzzy Sets and Systems **126** (2002), 241-251.

[2] G. Filé, and F. Ranzato, *Improving Abstract Interpretations by Systematic Lifting to the Powerset*, Proceedings of the 1994 International Symposium on Logic programming, MIT Press Cambridge, MA, USA (1994), 655–669.

[3] A. Frascella and C. Guido, *Transporting many-valued sets along many-valued relations*, Fuzzy Sets and Systems **159**(1) (2008), 1-22.

[4] A. Frascella and C. Guido, *Structured lattices and ground categories of L-sets*, Int. J. Math. and Math. Sci. **17** (2005), 2783-2803.

[5] G. Georgescu and A. Popescu, *Non-dual fuzzy connections*, Archive Math. Logic **43** (2004), 1009-1039.

[6] G. Gerla and L. Scarpati, *Extension principles for fuzzy set theory*, Journal of Information Sciences **106** (1998), 49–69.

[7] H. Herrlich and C. G. Strecker, *Category Theory*, Sigma Ser. Pure Math. Vol. 1, Heldermann, Berlin, 1979.

[8] U. Höhle, *Fuzzy sets and sheaves. Part I, Basic concepts*, Fuzzy Sets and Systems **158** (2007), 1143–1174.

[9] E. G. Manes, *Algebraic Theories*, Springer-Verlag, Berlin, New York, 1976.

[10] E. G. Manes, *A class of fuzzy theories*, Journal of mathematical Analysis and Applications, (85) (1982), 409–451.

[11] J. Močkoř, *Extensional subobjects in categories of Ω-fuzzy sets*, Czech.Math.J. **57(132)** (2007), 631–645.

[12] J. Močkoř, *Morphisms in categories of sets with similarity relations*, Proceedings of IFSA Congress/EUSFLAT Conference. Lisabon (2009), 560–568.

[13] J. Močkoř, *Cut systems in sets with similarity relations*, Fuzzy Sets and Systems, (161) (2010), 3127–3140.

[14] J. Močkoř, *Fuzzy sets and cut systems in a category of sets with similarity relations*, Soft Computing **16** (2012), 101–107.

[15] H. T. Nguyen, *A note on the extension principle for Fuzzy sets*, J. Math. Anal. Appl., **64** (1978), 369–380.

[16] V. Novák and I. Perfilijeva and J. Močkoř, *Mathematical Principles of Fuzzy Logic* , Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.

[17] S. E. Rodabaugh, *Powerset operator based foundation for point-set lattice theoretic (poslat) fuzzy set theories and topologies*, *Quaestiones Mathematicae* **20(3)** (1997), 463–530.

[18] S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, International Journal of Mathermatics and Mathematical Sciences, (2007), 1–71.

[19] S. E. Rodabaugh, *Relationship of algebraic theories to powersets over objects in Set and* $Set \times C$, Fuzzy Sets and Systems, **161(3)** (2010), 453–470.

[20] K. I. Rosenthal, *Quantales and Their Applications* , Pittman Res. Notes in Math. 234, Longman, Burnt Mill, Harlow, 1990.

[21] S. A. Solovyov, *Powerset operator foundations for catalc fuzzy set theories*, Iranian Journal of Fuzzy Systems **8(2)** (2011), 1–46.

[22] R. R. Yager, *A characterization of the extension principle*, Fuzzy Sets and Systems, **18** (1996), 205–217.

[23] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.

J. Močkoř, University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

E-mail address: `jiri.mockor@osu.cz`