GENERALIZED RESIDUATED LATTICES BASED F-TRANSFORM

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Abstract. The aim of the present work is to study the F-transform over a generalized residuated lattice. We discuss the properties that are common with the F-transform over a residuated lattice. We show that the $F^{\uparrow}$-transform can be used in establishing a fuzzy (pre)order on the set of fuzzy sets.

1. Introduction

Fuzzy transform (F-transform in short), firstly proposed by Perfilieva [17] has now been significantly developed and opened a new page in the theory of semi-linear spaces. The main idea of the F-transform is to factorize (or fuzzify) the precise values of independent variables by a closeness relation. It is shown in [17] that this transform encompassed both classical transforms as well as approximation methods based on fuzzy IF-THEN rules studied in fuzzy modeling. The theory of F-transform was further elaborated and extended from real valued to lattice-valued functions (cf., [17, 19]) and from fuzzy sets to parametrized fuzzy sets (cf., [30]). The theory of F-transform is successfully used in signal and image processing [13], compression [18], denoising [25], scheduling [12], trading [33], time series [16], numerical solutions of partial differential equations [11], data analysis [23], and neural network approaches [31].

The concept of fuzziness in the F-transform notion appears in the form of a fuzzy partition, the latter is characterized by a collection of real-valued or lattices-valued fuzzy sets. The lattice is assumed to be residuated. The theory of residuated lattices has been studied and developed by a number of researchers (cf., [2, 6, 7, 8, 9]). The concept of biresiduated multi-adjoint algebra studied in [1, 15] is one of such studies. In such algebras, the conjunctors being neither required to be commutative nor associative, increase the flexibility and shown to be useful in logic programming [15], fuzzy formal concept analysis [14], fuzzy relation equations [5] and fuzzy rough set theory [4, 34]. Among these cited works, recently, in [34], the concept of fuzzy rough set theory based on an integral and complete generalized residuated lattice was studied.

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Let us observe publications that were inspired by the lattice-based F-transforms [17]. The basic idea was first generalized in [27] by considering the so called Q-module transforms [27, 28], where Q stands for a unital quantale[1]. From the algebraic point of view, this structure is a bit more general than that of a residuated lattice. Further generalization, proposed in [27, 28], comes from the usage of a kernel representation of a fuzzy partition instead of the usage of partition elements in [17]. The latter allows the author of [27] to express the lattice-based F-transforms with the help of two residuated homomorphisms between Q-modules.

It is important to make some comments regarding the Q-module transforms and their relationship to the original lattice-based F-transforms, because both consider the similar formalism from dual points of view. The original lattice-based F-transforms as they appeared in [17] follow definitions of classical transforms (Fourier, Laplace, etc.), i.e., they are *results* of certain transformations between two different spaces of functions. Literally, a single lattice-based F-transform can be obtained from the classical one by the formal replacement of arithmetic operations and integration by operations from a chosen residuated lattice and the supremum. On the other hand, the Q-module transform being a homomorphism between Q-modules is the example of an element from the dual space to that of lattice-based F-transforms. The above given remark emphasizes that in dual spaces, similar results can be obtained one from another one after a certain reformulation. This however, does not diminish the importance of such results in both theories.

Another inspired by [17, 19, 32], where the possibilistic interpretation of the fuzzy subsets involved in the partition was proposed. One of the main features of [32] is that it combines two up and down F-transforms and provides an interval-valued estimation of transformed functions.

In the proposed contribution, we continue our study of the theory of lattice-based F-transforms in the classical direction of developing transforms. Specifically, we study the F-transform theory of Perfilieva [17] based on integral and complete generalized residuated lattice and generalized notion of a fuzzy partition. The novelties of this contribution are as follows: a) residuated lattices are considered without commutativity of its monoidal operation, b) fuzzy partitions are taken with minimal requirements and are not restricted to a finite number of elements as it was in the original proposal. We keep the style of the original definition and define the F-transform in terms of partition elements. This gives us a privilege to consider properties of particular F-transform components (Propositions 4.5, 4.10) and assign a structure over them according to the considered problem (Section 6). Another novel application of lattice-based F-transforms is proposed in Section 6, where we show how the direct $F^{\uparrow}$- or $F^{\downarrow}$-transform can be used in determination of non-trivial fuzzy preorders on the set of (L-valued) fuzzy sets. In this application, we were implicitly guided by the idea of Ruspini's implication measures (c.f.[21]).

The structure of the paper is as following. In section 2, the definition of a generalized residuated lattice and some of its properties are given. We also construct an example to show that some of the results of residuated lattices does not hold in

[1]In the literature, quantales are defined as complete residuated po-groupoids.

case of generalized residuated lattices. In section 3, the concept of generalized fuzzy partition has been considered. The direct $F^{\uparrow}(F^{\downarrow})$-transform based on generalized residuated lattices is introduced and studied in section 4. The inverse F-transform is discussed in section 5. The application aimed to establish a fuzzy order on the set of (L-valued) fuzzy sets is given in section 6.

2. Preliminaries

In this section, we recall some concepts related to generalized residuated lattices and lattice-valued fuzzy sets. We begin with the following definition taken from [27, 34].

Definition 2.1. A **generalized residuated lattice** is an algebra $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top)$ such that
(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1;
(ii) $(L, \otimes, \top)$ is a monoid; and
(iii) $\forall a, b, c \in L$;

$$a \otimes b \leq c \text{ if } a \leq b \rightarrow c \text{ if } b \leq a \rightsquigarrow c,$$

$\rightsquigarrow$ and $\rightarrow$ are called the left and right implications (left and right residuals) of $\otimes$, respectively.

Definition 2.2. A generalized residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1, \top)$ is called
(i) **commutative**, if $\otimes$ is commutative;
(ii) **integral**, if $\top = 1$;
(iii) **complete**, if the underlying lattice $(L, \wedge, \vee, 0, 1)$ is complete.

Clearly, if $\otimes$ is commutative, then $\rightarrow = \rightsquigarrow$ holds.

Throughout this paper, we will be working with a fixed integral complete generalized residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$. We simplify its name to a *generalized residuated lattice* and its denotation to L.

The below given Example 2.3 shows that the inequalities $b \otimes (b \rightarrow a) \leq a$ and $(b \rightsquigarrow a) \otimes b \leq a, \forall a, b \in L$, which hold well in a commutative generalized residuated lattice, need not hold in a generic (non-commutative) one.

Example 2.3. Consider the bounded lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given in Figure 1. It is not difficult to verify that $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ is a generalized residuated lattice. Moreover, neither of the two highlighted above inequalities holds, i.e.,

$$c \otimes (c \rightarrow b) = c \otimes d = c \nleq b, \text{ and } (d \rightsquigarrow b) \otimes d = c \otimes d = c \nleq b.$$

The operations '$\otimes$', '$\rightarrow$' and '$\rightsquigarrow$' are defined as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

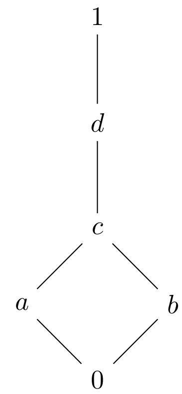

Figure 1. The Hasse Diagram of L

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

and

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	c	c	c	c	1	1
1	0	a	b	c	d	1.

Below, we list the following basic properties of operations on L.

Proposition 2.4. [34] *Let* $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ *be a generalized residuated lattice. Then for all* $a, b, c \in L$,

(i) $1 \rightarrow a = a$ and $1 \rightsquigarrow a = a$;
(ii) $a \leq b \Rightarrow (c \otimes a) \leq (c \otimes b)$ and $(a \otimes c) \leq (b \otimes c)$;
(iii) $b \leq c \Rightarrow (a \rightarrow b) \leq (a \rightarrow c)$ and $(a \rightsquigarrow b) \leq (a \rightsquigarrow c)$;
(iv) $a \leq b \Leftrightarrow a \rightarrow b = 1 \Leftrightarrow a \rightsquigarrow b = 1$;
(v) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$ and $(b \vee c) \otimes a = (b \otimes a) \vee (c \otimes a)$;
(vi) $c \rightarrow (a \wedge b) = (c \rightarrow a) \wedge (c \rightarrow b)$ and $c \rightsquigarrow (a \wedge b) = (c \rightsquigarrow a) \wedge (c \rightsquigarrow b)$;
(vii) $a \leq b \rightarrow (a \otimes b)$ and $b \leq a \rightsquigarrow (a \otimes b)$;
(viii) $(a \rightarrow b) \otimes a \leq b$ and $a \otimes (a \rightsquigarrow b) \leq b$;
(ix) $(a \rightarrow (b \rightsquigarrow c)) = (b \rightsquigarrow (a \rightarrow c))$.

Remark 2.5. Example 2.3 and Property (viii) emphasize that the operation $\rightsquigarrow$ is the left residual, whereas the operation $\rightarrow$ is the right residual of $\otimes$.

Let X be a nonempty set, $(L, \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ be a generalized residuated lattice and $f : X \rightarrow L$ be an L-valued (membership) function identified with an (L-valued)[2] fuzzy set. The set of all fuzzy subsets of X is denoted by L^X. For all

[2]We omit the reference to L, if it is clear from the content.

$a \in L$, $\mathbf{a}(x) = a$ denotes a fuzzy set with a constant membership function. For all $A \in L^X$, the $core(A)$ is a set of all elements $x \in X$, such that $A(x) = 1$. A fuzzy set $A \in L^X$ is normal if $core(A) \neq \emptyset$. Below, we remind the basic relations and operations on L^X.

Definition 2.6. [34] Let X be a nonempty set. Then for $A, B \in L^X$ and for all $x \in X$,
(i) $A \leq B \iff A(x) \leq B(x)$;
(ii) $A = B \iff A(x) = B(x)$;
(iii) $(A \otimes B)(x) = A(x) \otimes B(x)$;
(iv) $(A \to B)(x) = A(x) \to B(x)$;
(v) $(A \rightsquigarrow B)(x) = A(x) \rightsquigarrow B(x)$.

Similarly to fuzzy sets, an L-valued function $R : X \times Y \to L$ is identified with a fuzzy relation on $X \times Y$. If $X = Y$, then the set of fuzzy relations on X is denoted by $L^{X \times X}$.

Definition 2.7. For a nonempty set X, $R \in L^{X \times X}$ is called
(i) **reflexive** if $R(x, x) = 1$, $\forall\, x \in X$, and
(ii) **transitive** if $R(x, y) \otimes R(y, z) \leq R(x, z)$, $\forall\, x, y, z \in X$.

A reflexive and transitive fuzzy relation R is called a **fuzzy preorder**.

Finally, we remind a particular (set-relation) composition $\circ$ between a fuzzy set $A \in L^X$ and a fuzzy relation $R \in L^{X \times X}$:

$$(A \circ R)(x) = \bigvee_{t \in X} (A(t) \otimes R(t, x)).$$

3. Fuzzy Spaces

Let X be a nonempty set, $(L, \wedge, \vee, \otimes, \to, \rightsquigarrow, 0, 1)$ be a generalized residuated lattice and $\{A_\xi : \xi \in \Upsilon\}$ be a collection of fuzzy subsets of X. We say that this collection determines a fuzzy space on X. Every fuzzy set A_ξ can be associated with a certain (vague) property which is shared by elements from X. This space is an example of a possibly weakest structure on X. If we want to compare elements of X according to their properties expressed by $A_\xi, \xi \in \Upsilon$, we must impose additional conditions on these fuzzy sets. Below, we consider a space with fuzzy partition and a fuzzy preordered space.

3.1. Space with Fuzzy Partition.

We begin with the notion of an (L-valued) fuzzy partition. The spaces of this type often appear in various schemes of approximate reasoning where an object is associated with several vague predicates that characterize its properties. These schemes are used in decision-making and expert systems, classification problems, and data analysis in general. A space with a fuzzy partition can be further specified as a fuzzy topology. In our contribution, a fuzzy partition uniquely determines the lattice-based F-transform.

Several definitions of the notion of fuzzy partition can be found in the literature, see, e.g. [10, 17, 22, 26, 30]. Most of them connect this notion with a finite collection of fuzzy sets that are defined on the set of reals $\mathbb{R}$ or its Cartesian product. The extension to the case where the number of fuzzy sets in a partition is infinite was considered in e.g., [22]. Below, we recall the definition from [24] where an (L-valued) fuzzy partition of an arbitrary universe with an arbitrary number of partition elements is introduced.

Definition 3.1. A collection $\mathit{\Pi}$ of normal fuzzy sets $\{A_\xi : \xi \in \mathit{\Upsilon}\}$ in X is an (L-valued) **fuzzy partition** of X, if the corresponding collection of ordinary sets $\{core(A_\xi) : \xi \in \mathit{\Upsilon}\}$ is a partition of X. A pair $(X, \mathit{\Pi})$ where $\mathit{\Pi}$ is an L-valued fuzzy partition of X is called a **space with a fuzzy partition**.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ be a nonempty set and $L = \{0, a, b, c, d, 1\}$ be the generalized residuated lattice of Example 2.3. Then $\{A_1, A_2, A_3, A_4\}$ is a fuzzy partition of X, where

- $A_1 = \dfrac{1}{x_1} + \dfrac{a}{x_2} + \dfrac{b}{x_3} + \dfrac{c}{x_4}$,
- $A_2 = \dfrac{a}{x_1} + \dfrac{1}{x_2} + \dfrac{b}{x_3} + \dfrac{c}{x_4}$,
- $A_3 = \dfrac{b}{x_1} + \dfrac{c}{x_2} + \dfrac{1}{x_3} + \dfrac{d}{x_4}$, and
- $A_4 = \dfrac{c}{x_1} + \dfrac{d}{x_2} + \dfrac{a}{x_3} + \dfrac{1}{x_4}$.

Let $\mathit{\Pi} = \{A_\xi : \xi \in \mathit{\Upsilon}\}$ be a fuzzy partition of X. Then it can be **represented** by the following reflexive fuzzy relation on X

$$R_{\mathit{\Pi}}(x, y) = A_\xi(x), \text{ if } y \in core(A_\xi). \tag{1}$$

Obviously, if for some $\xi \in \mathit{\Upsilon}$, $y, t \in core(A_\xi)$, then for all $x \in X$, $R_{\mathit{\Pi}}(x, y) = R_{\mathit{\Pi}}(x, t)$.

The following statement is an easy consequence from Definition 3.1.

Proposition 3.3. *Fuzzy relation $R_{\mathit{\Pi}}$ on X represents a fuzzy partition of X iff $core(R_{\mathit{\Pi}})$ is an (ordinary) equivalence relation on X.*

3.2. **Fuzzy Preordered Space.** Let $\mathit{\Pi} = \{A_\xi : \xi \in \mathit{\Upsilon}\}$ be a fuzzy partition of X, that is represented by fuzzy relation $R_{\mathit{\Pi}}$. The latter is reflexive, and we look for additional conditions on fuzzy sets in $\mathit{\Pi}$ that make $R_{\mathit{\Pi}}$ transitive. The conditions have been discovered for the case where L is an ordinary complete residuated lattice with a commutative monoidal operation [20]. They are as follows:

$$A_i(x_j) \leq \bigwedge_{x \in X} (A_j(x) \rightarrow A_i(x)), \tag{2}$$

where $i, j \in \mathit{\Upsilon}$. Under these assumptions, fuzzy relation $R_{\mathit{\Pi}}$ is a fuzzy preorder on X. The pair $(X, R_{\mathit{\Pi}})$ is called a fuzzy preordered space. It is easy to see that in the space $(X, R_{\mathit{\Pi}})$, an ordinary preorder on X can be defined as:

$$x \leq_{R_{\mathit{\Pi}}} y \iff R_{\mathit{\Pi}}(x, y) = 1.$$

4. **Lattice-Based F-Transform**

This section is divided into two subsections. The first one is towards the study of direct $F^{\uparrow}$-transform while another one is towards the study of direct $F^{\downarrow}$-transform.

4.1. **Direct $F^{\uparrow}$-transform.** In this subsection, we study the concept of direct $F^{\uparrow}$-transform based on generalized residuated lattices. We show the linearity and monotonicity of direct left (resp. right) $F^{\uparrow}$-transform. We begin with the following.

Definition 4.1. Let f be an L-valued function on X and $\mathit{\Pi} = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X.

(i) The **left $F^{\uparrow}$-transform** of f w.r.t. fuzzy partition $\mathit{\Pi} = \{A_\xi : \xi \in \Upsilon\}$ is a collection of lattice elements (components) $\{F^{\uparrow}_{\xi_l}[f] : \xi \in \Upsilon\}$, where

$$F^{\uparrow}_{\xi_l}[f] = \bigvee_{y \in X} (A_\xi(y) \otimes f(y)),\ \xi \in \Upsilon.$$

(ii) The **right $F^{\uparrow}$-transform** of f w.r.t. fuzzy partition $\mathit{\Pi} = \{A_\xi : \xi \in \Upsilon\}$ is a collection of lattice elements (components) $\{F^{\uparrow}_{\xi_r}[f] : \xi \in \Upsilon\}$, where

$$F^{\uparrow}_{\xi_r}[f] = \bigvee_{y \in X} (f(y) \otimes A_\xi(y)),\ \xi \in \Upsilon.$$

We will omit the reference to f in the denotation of F-transform components, if it is clear from the context.

Example 4.2. In continuation to Example 3.2, let $f : X \to L$ be an L-valued function such that $f = \dfrac{a}{x_1} + \dfrac{b}{x_2} + \dfrac{c}{x_3} + \dfrac{d}{x_4}$. Then the direct left $F^{\uparrow}$-transform of f is the collection $\{c, c, d, d\}$, where $F^{\uparrow}_{1_l} = c$, $F^{\uparrow}_{2_l} = c$, $F^{\uparrow}_{3_l} = d$, $F^{\uparrow}_{4_l} = d$ while the direct right $F^{\uparrow}$-transform of f is the collection $\{a, b, d, d\}$, where $F^{\uparrow}_{1_r} = a$, $F^{\uparrow}_{2_r} = b$, $F^{\uparrow}_{3_r} = d$, $F^{\uparrow}_{4_r} = d$.

The following is towards the linearity of (left/right) $F^{\uparrow}$-transform.

Proposition 4.3. *Let $\alpha, \beta \in L$, f, g be L-valued functions on X and $\mathit{\Pi} = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X. Then for all $\xi \in \Upsilon$,*

(i) $F^{\uparrow}_{\xi_l}[f \otimes \alpha \vee g \otimes \beta] = F^{\uparrow}_{\xi_l}[f] \otimes \alpha \vee F^{\uparrow}_{\xi_l}[g] \otimes \beta$,
(ii) $F^{\uparrow}_{\xi_r}[\alpha \otimes f \vee \beta \otimes g] = \alpha \otimes F^{\uparrow}_{\xi_r}[f] \vee \beta \otimes F^{\uparrow}_{\xi_r}[g]$.

Proof. We only prove (i). For which, for $\xi \in \Upsilon$, let us denote by $[(f \otimes \alpha) \vee (g \otimes \beta)]^{\uparrow}_{\xi_l}, F^{\uparrow}_{\xi_l}, G^{\uparrow}_{\xi_l}$ the ξ-th component of $F^{\uparrow}_{\xi_l}[(f \otimes \alpha) \vee (g \otimes \beta)], F^{\uparrow}_{\xi_l}[f]$ and $F^{\uparrow}_{\xi_l}[g]$ respectively. Then

$$\begin{aligned}
[f\otimes\alpha\vee g\otimes\beta]^{\uparrow}_{\xi_l} &= \bigvee_{y\in X}(A_\xi(y)\otimes(f(y)\otimes\alpha\vee g(y)\otimes\beta))\\
&= \bigvee_{y\in X}(A_\xi(y)\otimes(f(y)\otimes\alpha))\vee(A_\xi(y)\otimes(g(y)\otimes\beta))\\
&= \bigvee_{y\in X}(A_\xi(y)\otimes f(y))\otimes\alpha\vee\bigvee_{y\in X}(A_\xi(y)\otimes g(y))\otimes\beta\\
&= F^{\uparrow}_{\xi_l}\otimes\alpha\vee G^{\uparrow}_{\xi_l}\otimes\beta.
\end{aligned}$$

□

In the following proposition, we give the various properties of direct left (resp. right) $F^{\uparrow}$-transform.

Proposition 4.4. *Let $\alpha,\beta\in L$, f,g be L-valued functions on X and $\Pi=\{A_\xi:\xi\in\Upsilon\}$ be a fuzzy partition of X. Then for all $\xi\in\Upsilon$,*

(i) $F^{\uparrow}_{\xi_l}[\mathbf{a}]=a$, and $F^{\uparrow}_{\xi_r}[\mathbf{a}]=a$,
(ii) $F^{\uparrow}_{\xi_l}[f]\le F^{\uparrow}_{\xi_l}[g]$ and $F^{\uparrow}_{\xi_r}[f]\le F^{\uparrow}_{\xi_r}[g]$, if $f\le g$,
(iii) $f(x_\xi)\le F^{\uparrow}_{\xi_l}[f]$, $f(x_\xi)\le F^{\uparrow}_{\xi_r}[f]$, if $x_\xi\in core(A_\xi)$.

Proof. (i) Follows from Proposition 2.4.
(ii) Follows from Proposition 2.4.
(iii) For $\xi\in\Upsilon$ and $x_\xi\in core(A_\xi)$,

$$\begin{aligned}
F^{\uparrow}_{\xi_l}[f] &= \bigvee_{x\in X}(A_\xi(x)\otimes f(x))\ge(1\otimes f(x_\xi)),\\
F^{\uparrow}_{\xi_l}[f] &\ge f(x_\xi).
\end{aligned}$$

In a similar way we can show that $F^{\uparrow}_{\xi_r}[f]\ge f(x_\xi)$. □

Proposition 4.5. *Let f be an L-valued function on X and $\Pi=\{A_\xi:\xi\in\Upsilon\}$ be a fuzzy partition of X. Then*

(i) the ξ-th component $F^{\uparrow}_{\xi_l}$ of the left $F^{\uparrow}$-transform is the least element of the following set:

$$S=\{a\in L|A_\xi(y)\le f(y)\to a,\ \text{for all } y\in X\},$$

(ii) for $a\in L$, the ξ-th component $F^{\uparrow}_{\xi_l}$ of the left $F^{\uparrow}$-transform is the least solution to the following equation

$$\bigwedge_{y\in X}(f(y)\rightsquigarrow(A_\xi(y)\rightsquigarrow a))=1,$$

(iii) the ξ-th component $F^{\uparrow}_{\xi_r}$ of the right $F^{\uparrow}$-transform is the least element of the following set:

$$S=\{a\in L|A_\xi(y)\le f(y)\rightsquigarrow a,\ \text{for all } y\in X\},$$

(iv) for $a \in L$, the ξ-th component $F^{\uparrow}_{\xi_r}$ of the right $F^{\uparrow}$-transform is the least solution to the following equation

$$\bigwedge_{y\in X}(f(y) \rightarrow (A_\xi(y) \rightarrow a)) = 1.$$

Proof. (i) From Definition 4.1, it is clear that $F^{\uparrow}_{\xi_l} \in S$. We only need to show that $a \in S$ implies $F^{\uparrow}_{\xi_l} \leq a$. Now,

$$\begin{aligned} a \in S &\Rightarrow A_\xi(y) \leq f(y) \rightarrow a \;\forall\; y \in X \\ &\Rightarrow A_\xi(y) \otimes f(y) \leq a \;\forall\; y \in X \\ &\Rightarrow \bigvee_{y\in X}(A_\xi(y) \otimes f(y)) \leq a \\ &\Rightarrow F^{\uparrow}_{\xi_l} \leq a. \end{aligned}$$

Thus $F^{\uparrow}_{\xi_l}$ is the least element of S.

(ii) As from (i), $F^{\uparrow}_{\xi_l}$ is the least element of S, $A_\xi(y) \leq f(y) \rightarrow a, \forall y \in X$, i.e., $A_\xi(y) \otimes f(y) \leq a$, i.e., $f(y) \leq A_\xi(y) \rightsquigarrow a$, or that $1 \leq (f(y) \rightsquigarrow (A_\xi(y) \rightsquigarrow a)), \forall y \in X$, whereby $\bigwedge_{y\in X}(f(y) \rightsquigarrow (A_\xi(y) \rightsquigarrow a)) = 1$. Thus $F^{\uparrow}_{\xi_l}$ is the least solution of the equation.

The proof of (iii) and (iv) are similar to that of (i) and (ii) respectively. □

4.2. **Direct $F^{\downarrow}$-transform.** In this subsection, we study the concept of direct $F^{\downarrow}$-transform based on a generalized residuated lattice. We begin with the following.

Definition 4.6. Let f be an L-valued function on X and $\Pi = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X.

(i) The **left** $F^{\downarrow}$**-transform** of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$ is a collection of lattice elements (components) $\{F^{\downarrow}_{\xi_l}[f] : \xi \in \Upsilon\}$, where

$$F^{\downarrow}_{\xi_l}[f] = \bigwedge_{y\in X}(A_\xi(y) \rightarrow f(y)),\; \xi \in \Upsilon.$$

(ii) The **right** $F^{\downarrow}$**-transform** of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$ is a collection of lattice elements (components) $\{F^{\downarrow}_{\xi_r}[f] : \xi \in \Upsilon\}$, where

$$F^{\downarrow}_{\xi_r}[f] = \bigwedge_{y\in X}(A_\xi(y) \rightsquigarrow f(y)),\; \xi \in \Upsilon.$$

We will omit the reference to f in the denotation of F-transform components, if it is clear from the context.

Example 4.7. In continuation to Example 3.2, let $f : X \rightarrow L$ be an L-valued function such that $f = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} + \frac{d}{x_4}$. Then the direct left $F^{\downarrow}$-transform of f is the collection $\{a, b, c, b\}$, where $F^{\downarrow}_{1_l} = a, F^{\downarrow}_{2_l} = b, F^{\downarrow}_{3_l} = c, F^{\downarrow}_{4_l} = b$, while the direct right $F^{\downarrow}$-transform of f is the collection $\{a, b, c, c\}$, where $F^{\downarrow}_{1_r} = a, F^{\downarrow}_{2_r} = b, F^{\downarrow}_{3_r} = c, F^{\downarrow}_{4_r} = c$.

The following is towards the linearity of (left/right) $F^{\downarrow}$-transform.

Proposition 4.8. *Let $\alpha, \beta \in L$, f, g be L-valued functions on X and $\varPi = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X. Then for all $\xi \in \Upsilon$,*

(i) $F^{\downarrow}_{\xi_l}[(\alpha \rightsquigarrow f) \wedge (\beta \rightsquigarrow g)] = (\alpha \rightsquigarrow F^{\downarrow}_{\xi_l}[f]) \wedge (\beta \rightsquigarrow F^{\downarrow}_{\xi_l}[g])$, and
(ii) $F^{\downarrow}_{\xi_r}[(\alpha \rightarrow f) \wedge (\beta \rightarrow g)] = (\alpha \rightarrow F^{\downarrow}_{\xi_r}[f]) \wedge (\beta \rightarrow F^{\downarrow}_{\xi_r}[g])$.

Proof. We only prove (i). Let us fix some $\xi \in \Upsilon$, and denote by $[(\alpha \rightsquigarrow f) \wedge (\beta \rightsquigarrow g)]^{\downarrow}_{\xi_l}, F^{\downarrow}_{\xi_l}, G^{\downarrow}_{\xi_l}$ the ξ-th components of the respective functions. Then

$$
\begin{aligned}
&[(\alpha \rightsquigarrow f) \wedge (\beta \rightsquigarrow g)]^{\downarrow}_{\xi_l} \\
&\quad = \bigwedge_{y \in X} (A_\xi(y) \rightarrow (\alpha \rightsquigarrow f) \wedge (\beta \rightsquigarrow g)(y)) \\
&\quad = \bigwedge_{y \in X} (A_\xi(y) \rightarrow (\alpha \rightsquigarrow f(y))) \wedge \bigwedge_{y \in X} (A_\xi(y) \rightarrow (\beta \rightsquigarrow g(y))) \\
&\quad = \bigwedge_{y \in X} (\alpha \rightsquigarrow (A_\xi(y) \rightarrow f(y))) \wedge \bigwedge_{y \in X} (\beta \rightsquigarrow (A_\xi(y) \rightarrow g(y))) \\
&\quad = \left(\alpha \rightsquigarrow \bigwedge_{y \in X} (A_\xi(y) \rightarrow f(y))\right) \wedge \left((\beta \rightsquigarrow \bigwedge_{y \in X} (A_\xi(y) \rightarrow g(y))\right) \\
&\quad = (\alpha \rightsquigarrow F^{\downarrow}_{\xi_l}) \wedge (\beta \rightsquigarrow G^{\downarrow}_{\xi_l}).
\end{aligned}
$$

□

Proposition 4.9. *Let $\alpha, \beta \in L$, f, g be L-valued functions on X and $\varPi = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X. Then for all $\xi \in \Upsilon$,*

(i) $F^{\downarrow}_{\xi_l}[\mathbf{a}] = a$ and $F^{\downarrow}_{\xi_r}[\mathbf{a}] = a$,
(ii) $F^{\downarrow}_{\xi_l}[f] \leq F^{\downarrow}_{\xi_l}[g]$ and $F^{\downarrow}_{\xi_r}[f] \leq F^{\downarrow}_{\xi_r}[g]$, if $f \leq g$,
(iii) $f(x_\xi) \geq F^{\downarrow}_{\xi_l}[f]$ and $f(x_\xi) \geq F^{\downarrow}_{\xi_r}[f]$, if $x_\xi \in core(A_\xi)$.

Proof. (i) Follows from Proposition 2.4.
(ii) Follows from Proposition 2.4.
(iii) For $\xi \in \Upsilon$ and $x_\xi \in core(A_\xi)$,

$$
\begin{aligned}
F^{\downarrow}_{\xi_l}[f] &= \bigwedge_{x \in X} (A_\xi(x) \rightarrow f(x)) \leq (1 \rightarrow f(x_\xi)), \\
F^{\downarrow}_{\xi_l}[f] &\leq f(x_\xi).
\end{aligned}
$$

In a similar way we can show that $F^{\downarrow}_{\xi_r}[f] \leq f(x_\xi)$. □

Proposition 4.10. *Let f be an L-valued function on X and $\varPi = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X. Then*

(i) the ξ-th component $F^{\downarrow}_{\xi_l}$ of the left $F^{\downarrow}$-transform is the greatest element of the following set.

$$T = \{a \in L | A_\xi(y) \leq (a \rightsquigarrow f(y)) \text{ for all } y \in X\}, \xi \in \Upsilon,$$

(ii) for $a \in L$, the ξ-th component $F^{\downarrow}_{\xi_l}$ of the left $F^{\downarrow}$-transform is the greatest solution to the following equation

$$\bigwedge_{y \in X} (a \otimes A_\xi(y)) \rightarrow (f(y)) = 1,\ \xi \in \Upsilon,$$

(iii) the ξ-th component $F^{\downarrow}_{\xi_r}$ of the right $F^{\downarrow}$-transform is the greatest element of the following set.

$$T = \{a \in L | A_\xi(y) \leq (a \rightarrow f(y)) \text{ for all } y \in X\}, \xi \in \Upsilon,$$

(iv) for $a \in L$, the ξ-th component $F^{\downarrow}_{\xi_r}$ of the right $F^{\downarrow}$-transform is the greatest solution to the following equation

$$\bigwedge_{y \in X} (A_\xi(y) \otimes a) \rightsquigarrow (f(y)) = 1,\ \xi \in \Upsilon.$$

Proof. (i) From Definition 4.6, it is clear that $F^{\downarrow}_{\xi_l} \in T$. We only need to show that if $a \in T$ then $a \leq F^{\downarrow}_{\xi_l}$. Now,

$$\begin{aligned} a \in T \quad &\Rightarrow \quad A_\xi(y) \leq (a \rightsquigarrow f(y)) \\ &\Rightarrow \quad a \otimes A_\xi(y) \leq f(y) \\ &\Rightarrow \quad a \leq A_\xi(y) \rightarrow f(y)\ \forall y \in X \\ &\Rightarrow \quad a \leq \bigwedge_{y \in X} A_\xi(y) \rightarrow f(y)\ \forall y \in X \\ &\Rightarrow \quad a \leq F^{\downarrow}_{\xi_l}. \end{aligned}$$

Hence (the ξ-th component of left $F^{\downarrow}$-transform)$F^{\downarrow}_{\xi_l}$ is the greatest element of the set T.

(ii) As from (i), $F^{\downarrow}_{\xi_l}$ is the greatest element of the set T, i.e., $A_\xi(y) \leq (a \rightsquigarrow f(y)), \forall y \in X$, i.e., $a \otimes A_\xi(y) \leq f(y)$, or that $a \otimes A_\xi(y) \rightarrow f(y) = 1, \forall y \in X$, whereby $\bigwedge_{y \in X} (a \otimes A_\xi(y)) \rightarrow (f(y)) = 1$. Thus $F^{\downarrow}_{\xi_l}$ is the greatest solution to the equation.

The proof of (iii) and (iv) are similar to that of (i) and (ii) respectively. □

5. Inverse $F^{\uparrow}(F^{\downarrow})$-Transform

In this section, we introduce and study the concept of inverse F-transform based on generalized residuated lattices. This section is divided into two subsections, where the first one is towards the study of inverse $F^{\uparrow}$-transform and the second subsection is towards the study of inverse $F^{\downarrow}$-transform.

5.1. Inverse $F^{\uparrow}$-transform.

We begin with the following concept of an inverse $F^{\uparrow}$-transform.

Definition 5.1. Let f be an L-valued function on X. Then

(i) for left $F^{\uparrow}$-transform $F^{\uparrow}_{\xi_l}[f] = \{F^{\uparrow}_{\xi_l} : \xi \in \Upsilon\}$ of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$, the function

$$\hat{f}^{\uparrow}_l(x) = \bigwedge_{\xi \in \Upsilon} (A_\xi(x) \rightsquigarrow F^{\uparrow}_{\xi_l}),$$

is called the **inverse left $F^{\uparrow}$-transform**.

(ii) For right $F^{\uparrow}$-transform $F^{\uparrow}_{\xi_r}[f] = \{F^{\uparrow}_{\xi_r} : \xi \in \Upsilon\}$ of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$, the function

$$\hat{f}^{\uparrow}_r(x) = \bigwedge_{\xi \in \Upsilon} (A_\xi(x) \rightarrow F^{\uparrow}_{\xi_r}),$$

is called the **inverse right $F^{\uparrow}$-transform**.

Example 5.2. In continuation to Example 4.2, let $f : X \to L$ be an L-valued function such that $f = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} + \frac{d}{x_4}$. Then for the left (right) $F^{\uparrow}$-transform calculated in Example 4.2, the inverse left $F^{\uparrow}$-transform is $\hat{f}^{\uparrow}_l = \frac{c}{x_1} + \frac{c}{x_2} + \frac{d}{x_3} + \frac{d}{x_4}$, while the inverse right $F^{\uparrow}$-transform is $\hat{f}^{\uparrow}_r = \frac{a}{x_1} + \frac{b}{x_2} + \frac{d}{x_3} + \frac{d}{x_4}$.

Now, we have the following.

Theorem 5.3. *Let f be an L-valued function on X. Then for all $y \in X$*

(i) $\hat{f}^{\uparrow}_l(y) \geq f(y)$, and
(ii) $\hat{f}^{\uparrow}_r(y) \geq f(y)$.

Proof. (i) Let $y \in X$. Then from Definition 4.1 and 5.1,

$$\begin{aligned} \hat{f}^{\uparrow}_l(y) &= \bigwedge_{\xi \in \Upsilon} (A_\xi(y) \rightsquigarrow F^{\uparrow}_{\xi_l}) \\ &= \bigwedge_{\xi \in \Upsilon} (A_\xi(y) \rightsquigarrow \bigvee_{y \in X} (A_\xi(y) \otimes f(y)) \\ &\geq \bigwedge_{\xi \in \Upsilon} (A_\xi(y) \rightsquigarrow (A_\xi(y) \otimes f(y)) \\ &\geq f(y). \end{aligned}$$

(ii) Can be proved similarly. □

Theorem 5.4. *Let f be an L-valued function on X, and the left (resp. right) inverse $F^{\uparrow}$-transform $\hat{f}^{\uparrow}_l$ (resp. $\hat{f}^{\uparrow}_r$) of f be computed w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$. Then for all $\xi \in \Upsilon$,*

(i) $F^{\uparrow}_{\xi_l} = \bigvee_{y \in X} A_\xi(y) \otimes \hat{f}^{\uparrow}_l(y)$, and
(ii) $F^{\uparrow}_{\xi_r} = \bigvee_{y \in X} \hat{f}^{\uparrow}_r(y) \otimes A_\xi(y)$.

Proof. We only prove (i). Let $\xi \in \Upsilon$ and $y \in X$. Then, $A_\xi(y) \otimes \hat{f}_l^{\uparrow}(y) = A_\xi(y) \otimes \bigwedge_{\xi \in \Upsilon}(A_\xi(y) \rightsquigarrow F_{\xi_l}^{\uparrow}) \leq A_\xi(y) \otimes (A_\xi(y) \rightsquigarrow F_{\xi_l}^{\uparrow}) \leq F_{\xi_l}^{\uparrow}$. Thus $\bigvee_{y \in X}(A_\xi(y) \otimes \hat{f}_l^{\uparrow}(y)) \leq F_{\xi_l}^{\uparrow}$. It follows that $F_{\xi_l}^{\uparrow} = \bigvee_{y \in X}(A_\xi(y) \otimes f(y)) \leq \bigvee_{y \in X}(A_\xi(y) \otimes \hat{f}_l^{\uparrow}(y))$. Hence $F_{\xi_l}^{\uparrow} = \bigvee_{y \in X} A_\xi(y) \otimes \hat{f}_l^{\uparrow}(y)$. □

5.2. **Inverse $F^{\downarrow}$-transform.** We begin with the following concept of an inverse $F^{\downarrow}$-transform.

Definition 5.5. Let f be an L-valued function on X. Then

(i) for left $F^{\downarrow}$-transform $F_{\xi_l}^{\downarrow}[f] = \{F_{\xi_l}^{\downarrow} : \xi \in \Upsilon\}$ of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$, the function

$$\hat{f}_l^{\downarrow}(x) = \bigvee_{\xi \in \Upsilon}(F_{\xi_l}^{\downarrow} \otimes A_\xi(x)),$$

is called the **inverse left $F^{\downarrow}$-transform**.

(ii) For right $F^{\downarrow}$-transform $F_{\xi_r}^{\downarrow}[f] = \{F_{\xi_r}^{\downarrow} : \xi \in \Upsilon\}$ of f w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$, the function

$$\hat{f}_r^{\downarrow}(x) = \bigvee_{\xi \in \Upsilon}(A_\xi(x) \otimes F_{\xi_r}^{\downarrow}),$$

is called the **inverse right $F^{\downarrow}$-transform**.

Example 5.6. In continuation to Example 4.7, let $f : X \to L$ be an L-valued function such that $f = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} + \frac{d}{x_4}$. Then for the left (right) $F^{\downarrow}$-transform calculated in Example 4.7, the inverse left $F^{\downarrow}$-transform is $\hat{f}_l^{\downarrow} = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} + \frac{c}{x_4}$, while the inverse right $F^{\downarrow}$-transform is $\hat{f}_r^{\downarrow} = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} + \frac{c}{x_4}$.

Theorem 5.7. *Let f be an L-valued function on X. Then for all $y \in X$*

(i) $\hat{f}_l^{\downarrow}(y) \leq f(y)$, and
(ii) $\hat{f}_r^{\downarrow}(y) \leq f(y)$.

Proof. (i) From Definition 4.6 and 5.5, we have

$$\begin{aligned}
\hat{f}_l^{\downarrow}(y) &= \bigvee_{\xi \in \Upsilon}(F_{\xi_l}^{\downarrow} \otimes A_\xi(y)) \\
&= \bigvee_{\xi \in \Upsilon}\left(\bigwedge_{y \in X}(A_\xi(y) \to f(y)) \otimes A_\xi(y)\right) \\
&\leq \bigvee_{\xi \in \Upsilon}(A_\xi(y) \to f(y)) \otimes A_\xi(y)) \\
&\leq f(y).
\end{aligned}$$

(ii) Can be proved similarly. □

Theorem 5.8. *Let f be an L-valued function on X, and the left (resp. right) inverse $F^{\downarrow}$-transform $\hat{f}_l^{\downarrow}(\hat{f}_r^{\downarrow})$ of f be computed w.r.t. fuzzy partition $\Pi = \{A_\xi : \xi \in \Upsilon\}$. Then for all $\xi \in \Upsilon$,*

(i) $F^{\downarrow}_{\xi_l} = \bigwedge_{y\in X} A_\xi(y) \to \hat{f}_l^{\downarrow}(y)$, and
(ii) $F^{\downarrow}_{\xi_r} = \bigwedge_{y\in X} A_\xi(y) \rightsquigarrow \hat{f}_r^{\downarrow}(y)$, $y \in X$.

Proof. We only prove (i). Let us choose $\xi \in \Upsilon$ and $y \in X$. Then, $A_\xi(y) \to \hat{f}_l^{\downarrow}(y) = A_\xi(y) \to \bigvee_{\xi\in\Upsilon}(F^{\downarrow}_{\xi_l} \otimes A_\xi(y)) \geq A_\xi(y) \to (F^{\downarrow}_{\xi_l} \otimes A_\xi(y)) \geq F^{\downarrow}_{\xi_l}$. Thus $\bigwedge_{y\in X}(A_\xi(y) \to \hat{f}_l^{\downarrow}(y)) \geq F^{\downarrow}_{\xi_l}$. It follows that $F^{\downarrow}_{\xi_l} = \bigwedge_{y\in X}(A_\xi(y) \to f(y)) \geq \bigwedge_{y\in X}(A_\xi(y) \to \hat{f}_l^{\uparrow}(y))$. Hence $F^{\downarrow}_{\xi_l} = \bigwedge_{y\in X} A_\xi(y) \to \hat{f}_l^{\downarrow}(y)$. □

6. Lattice-Based F-Transform and Preorder on L^X

In this section, we show how the lattice-based F-transform can be used for establishing various fuzzy preorders on the set L^X of all fuzzy subsets of X. At first, we consider the following general definition of (an L-valued) fuzzy preorder on L^X.

Definition 6.1. A fuzzy preorder on the set L^X of fuzzy subsets of X is an L-valued fuzzy relation denoted by $\lesssim$, which is reflexive and transitive, i.e., for all $A, B, C \in L^X$,

$$A \lesssim A = 1,$$
$$(A \lesssim B) \otimes (B \lesssim C) \leq A \lesssim C.$$

The canonical example of $\lesssim$ is

$$\bigwedge_{y\in X}(A(y) \to B(y)).$$

The meaning of this canonical preorder is that it estimates the measure of inclusion between fuzzy sets A and B. In [26], this measure was extended and used for estimation of closeness between A and B. According to [26], fuzzy sets A and B are close, if there is a fuzzy equivalence, say E, such that A is included into the extension hull of B with respect to E. This idea motivated us to construct two (non-trivial) examples of a fuzzy preorder on L^X. The first one measures the degree of inclusion between fuzzy set A and a certain extension of B, the latter is computed with respect to a fuzzy partition and expressed by the $F^{\uparrow}$-transform of B. The second one measures the degree of inclusion between a certain reduction of fuzzy set A and fuzzy set B, the former is computed with respect to a fuzzy partition and expressed by the $F^{\downarrow}$-transform of A.

Let $\Pi = \{A_\xi : \xi \in \Upsilon\}$ be a fuzzy partition of X, that is represented by the reflexive fuzzy relation R_Π. Let moreover, L be an ordinary complete residuated lattice with a commutative monoidal operation $\otimes$. We claim that the following two fuzzy relations on L^X

$$\bigwedge_{y\in X}(A(y)\to B^{\uparrow}_{\xi^y}),$$

$$\bigwedge_{y\in X}(A^{\downarrow}_{\xi^y}(y)\to B),$$

are fuzzy preorders, where $B^{\uparrow}_{\xi^y}$, $A^{\downarrow}_{\xi^y}$ are the F-transform components $F^{\uparrow}_{\xi^y}[B]$, $F^{\downarrow}_{\xi^y}[A]$, respectively, and $\xi^y\in\Upsilon$ is such that $y\in core(A_{\xi^y})$. We prove the claim for the first fuzzy relation with the $F^{\uparrow}$-transform component and leave the second case as a technical exercise.

We begin with the characterization of the $F^{\uparrow}$-transform components of an L-valued function.

Proposition 6.2. *Let $f\in L^X$ and $\Pi=\{A_\xi:\xi\in\Upsilon\}$ be a fuzzy partition of X represented by the reflexive fuzzy relation R_Π. Then the left and right F-transforms $F^{\uparrow}_{\xi_l}[f]$ and $F^{\uparrow}_{\xi_r}[f]$ coincide, i.e.,*

$$F^{\uparrow}_{\xi_l}[f]=F^{\uparrow}_{\xi_r}[f]=F^{\uparrow}_{\xi}[f],\ \xi\in\Upsilon.$$

Moreover, the F-transform components $\{F^{\uparrow}_{\xi}[f]:\xi\in\Upsilon\}$ of f w.r.t. Π are the only values of the $\circ$ composition between f and R_Π, i.e. for all $\xi\in\Upsilon$, there exists $y_\xi\in X$, such that

$$F^{\uparrow}_{\xi}[f]=(f\circ R_\Pi)(y_\xi)=\bigvee_{x\in X}(f(x)\otimes R_\Pi(x,y_\xi)).\tag{3}$$

Vice-versa, for all $y\in X$, there exists $\xi_y\in\Upsilon$, such that

$$(f\circ R_\Pi)(y)=\bigvee_{x\in X}(f(x)\otimes R_\Pi(x,y))=F^{\uparrow}_{\xi_y}[f].\tag{4}$$

Proof. The first assertion easily follows from commutativity of $\otimes$. We will prove the second one. At first, we choose some $\xi\in\Upsilon$ and select $y_\xi\in core(A_\xi)$. By the construction of R_Π,

$$R_\Pi(x,y_\xi)=A_\xi(x).$$

Then,

$$F^{\uparrow}_{\xi}[f]=\bigvee_{x\in X}(f(x)\otimes A_\xi(x))=\bigvee_{x\in X}(f(x)\otimes R_\Pi(x,y_\xi)).$$

Conversely, let us choose some $y\in X$ and find $\xi_y\in\Upsilon$, such that $y\in core(A_{\xi_y})$. The latter is always possible, because the collection of fuzzy partition cores constitutes an ordinary partition of X. Then,

$$R_\Pi(x,y)=A_{\xi_y}(x),$$

and (4) easily follows. □

Let us additionally assume condition (2) that makes R_Π transitive. We remind that fuzzy sets on X are identified with their L-valued membership functions. Therefore, we can apply the F-transform to fuzzy sets as well. Below, we construct one particular fuzzy preorder $\lesssim$ on L^X which is based on the direct $F^{\uparrow}$-transform.

Proposition 6.3. *Let the assumptions of Proposition 6.2 together with condition (2) on elements of fuzzy partition Π of X be fulfilled, and L^X be a set of fuzzy subsets of X (L-valued functions). Then the following L-valued fuzzy relation $\lesssim$ on L^X*

$$A \lesssim B = \bigwedge_{y \in X} (A(y) \to F^{\uparrow}_{\xi^y}[B]) \tag{5}$$

is a fuzzy preorder, where $\xi^y \in \Upsilon$ is such that $y \in core(A_{\xi^y})$.

Proof. We shall verify that the L-valued fuzzy relation $\lesssim$ is reflexive and transitive. Reflexivity easily follows from Proposition 4.4, (iii) and Proposition 2.4, (iv). To prove transitivity, we rewrite (5) as follows

$$A \lesssim B = \bigwedge_{y \in X} (A(y) \to (B \circ R_\Pi)(y)), \tag{6}$$

where we made use of Proposition 6.2 and equality (4). We shall prove that for all $A, B, C \in L^X$,

$$(A \lesssim B) \otimes (B \lesssim C) \leq (A \lesssim C).$$

Below, we give a sketch of the proof and refer to [21] for technical details. At first, we show that

$$(B \lesssim C) \leq (B \circ R_\Pi \lesssim C \circ R_\Pi).$$

This fact follows from the following two inequalities that are valid in any complete residuated lattice L with a commutative monoidal operation:

$$a \to b \leq a \otimes x \to b \otimes x;$$
$$\bigwedge_{i \in I} (a_i \to b_i) \leq \bigvee_{i \in I} a_i \to \bigvee_{i \in I} b_i.$$

At second, we use the fact that R_Π is a fuzzy preorder on X and therefore, $R_\Pi \circ R_\Pi = R_\Pi$. It follows that

$$(C \circ R_\Pi) \circ R_\Pi = C \circ R_\Pi.$$

Finally,

$$\begin{aligned}
&(A \lesssim B) \otimes (B \lesssim C) \leq (A \lesssim B) \otimes (B \circ R_\Pi \lesssim C \circ R_\Pi) \\
\leq &(A \lesssim B) \otimes \left(\bigwedge_{y \in X} (B \circ R_\Pi)(y) \to ((C \circ R_\Pi) \circ R_\Pi)(y) \right) \\
\leq &(A \lesssim B) \otimes \left(\bigwedge_{y \in X} (B \circ R_\Pi)(y) \to (C \circ R_\Pi)(y) \right) \\
= &\left(\bigwedge_{y \in X} (A(y) \to (B \circ R_\Pi)(y)) \right) \otimes \left(\bigwedge_{y \in X} (B \circ R_\Pi)(y) \to (C \circ R_\Pi)(y) \right) \\
\leq &\bigwedge_{y \in X} (A(y) \to (C \circ R_\Pi)(y)) = (A \lesssim C).
\end{aligned}$$

The last inequality follows from the property of transitivity of the canonical fuzzy preorder on L^X. □

7. Concluding Remarks

In this paper, we have studied the theory of F-transform based on a generalized residuated lattice. This enriches the recent study in [29] and puts one step further towards a similar study in the framework of biresiduated multi-adjoint algebra.

We discussed the notion of a generalized fuzzy partition where the number of elements is not restricted. We showed that a fuzzy partition can be represented by a reflexive fuzzy relation and vice-versa. We defined the F-transform of an L-valued function in terms of partition elements and analyzed the properties that are common with the lattice-based F-transform over a residuated lattice.

We showed that the $F^{\uparrow}$- and $F^{\downarrow}$-transform can be used in determination of non-trivial fuzzy preorders on the set of (L-valued) fuzzy sets. The latter are measures of closeness between fuzzy sets. In this correspondence, the newly defined $F^{\uparrow}$- and $F^{\downarrow}$-transform measures of closeness can be used in various schemes of approximate or similarity-based reasoning, see [3, 21]. This application is a matter of our future research.

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References

[1] A. A. Abdel-Hamid and N. N. Morsi, *Associatively tied implications*, Fuzzy Sets and Systems, **136(3)** (2003), 291-311.

[2] K. Blount and C. Tsinakis, *The structure of residuated lattices*, International Journal of Algebra and Computation, **13(4)** (2003), 437-461.

[3] D. Boixader and J. Jacas, *Extensionality based approximate reasoning*, International Journal of Approximate Reasoning, **19(3-4)** (1998), 221-230.

[4] C. Cornelis, J. Medina, N. Verbiest, *Multi-adjoint fuzzy rough sets: Definition, properties and attribute selection*, International Journal of Approximate Reasoning, **55(1)**(2014), 412-426.

[5] J. C. Diaz and J. Medina, *Multi-adjoint relation equations: Definition, properties and solutions using concept lattices*, Information Sciences, **253** (2013), 100-109.

[6] P. Flondor, G. Georgescu and A. Iorgulescu, *Pseudo-t-norms and pseudo-BL algebras*, Soft Computing, **5(5)** (2001), 355-371.

[7] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated Lattices. An Algebraic Glimpse at Substructural Logics*, Elsevier, 2007.

[8] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Boston, 1998.

[9] P. Hájek, *Observations on non-commutative fuzzy logic*, Soft Computing, **8(1)** (2003), 38-43.

[10] N. T. Hung and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Raton, 1997.

[11] A. Khastan, I. Perfilieva and Z. Alizani, *A new fuzzy approximation method to Cauchy problems by fuzzy transform*, Fuzzy Sets and Systems, **288** (2016), 75-95.

[12] M. Liu, D. Chen, C. Wu and H. Li, *Approximation theorem of the fuzzy transform in fuzzy reasoning and its application to the scheduling problem*, Computers & Mathematics with Applications, **51(3-4)** (2006), 515-526.

[13] F. D. Martino, V. Loia and S. Sessa, *Fuzzy transforms method in prediction data analysis*, Fuzzy Sets and Systems, **180(1)** (2011), 146-163.

[14] J. Medina, M. Ojeda-Aciego and J. Ruiz-Calvino, *Formal concept analysis via multi-adjoint concept lattices*, Fuzzy Sets and Systems, **160(2)** (2009), 130-144.

[15] J. Medina, M. Ojeda-Aciego, A. Valverde and P. Vojtas, *Towards biresiduated multi-adjoint logic programming*, Lecture Notes in Artificial Intelligence, **3040** (2004), 608-617.
[16] V. Novák, I. Perfilieva, M. Holčapek and V. Kreinovich, *Filtering out high frequencies in time series using F-transform*, Information Sciences, **274** (2014), 192-209.
[17] I. Perfilieva, *Fuzzy transforms: Theory and applications*, Fuzzy Sets and Systems, **157(8)** (2006), 993-1023.
[18] I. Perfilieva, *Fuzzy transforms and their applications to image compression*, Lecture Notes in Computer Science, **3849** (2006), 19-31.
[19] I. Perfilieva, *Fuzzy transforms: A challenge to conventional transforms*, Advances in Image and Electron Physics, **147** (2007), 137-196.
[20] I. Perfilieva, *Finitary solvability conditions for systems of fuzzy relation equations*, Information Sciences, **234** (2013), 29-43.
[21] I. Perfilieva, D. Dubois, H. Prade, F. Esteva, L. Godo and P. Hodáková, *interpolation of fuzzy data. analytical approach and overview*, Fuzzy Sets and Systems, **192** (2012), 134-158.
[22] I. Perfilieva, M. Holcapek and V. Kreinovich, *A new reconstruction from the F-transform components*, Fuzzy Sets and Systems, **288** (2016), 3-25.
[23] I. Perfilieva, V. Novák and A. Dvorak, *Fuzzy transforms in the analysis of data*, International Journal of Approximate Reasoning, **48(1)** (2008), 36-46.
[24] I. Perfilieva, A. P. Singh and S. P. Tiwari, *On the relationship among F-transform, fuzzy rough set and fuzzy topology*, Soft Computing, **21(13)** (2017), 3513-3521.
[25] I. Perfilieva and R. Valasek, *Fuzzy transforms in removing noise*, Advances in Soft Computing, **2** (2005), 221-230.
[26] E. H. Ruspini, *On the semantics of fuzzy logic*, International Journal of Approximate Reasoning, **5(1)** (1993), 45-88.
[27] C. Russo, *Quantale Modules with Applications to Logic and Image Processing*, Lambert Academic Publishers, Saarbrucken, 2009.
[28] C. Russo, *Quantale modules and their operators, with applications*, Journal of Logic and Computation, **20(4)** (2010), 917-946.
[29] A. P. Singh and S. P. Tiwari, *F-transform based on generalized residuated lattices*, Decision Making and Soft Computing, **9** (2014), 720-725.
[30] L. Stefanini, *F-transform with parametric generalized fuzzy partitions*, Fuzzy Sets and Systems, **180(1)** (2011), 98-120.
[31] M. Stepnicka and O. Polakovic, *A neural network approach to the fuzzy transform*, Fuzzy Sets and Systems, **160(8)** (2009), 1037-1047.
[32] O. Strauss, *Non-additive interval-valued F-transform*, Fuzzy Sets and Systems, **270** (2014), 1-24.
[33] L. Troiano and P. Kriplani, *Supporting trading strategies by inverse fuzzy transform*, Fuzzy Sets and Systems, **180(1)** (2011), 121-145.
[34] C. Y. Wang and B. Q. Hu, *Fuzzy rough sets based on generalized residuated lattices*, Information Sciences, **248** (2013), 31-49.

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