TESTING STATISTICAL HYPOTHESES UNDER FUZZY DATA AND BASED ON A NEW SIGNED DISTANCE

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Abstract. This paper deals with the problem of testing statistical hypotheses when the available data are fuzzy. In this approach, we first obtain a fuzzy test statistic based on fuzzy data, and then, based on a new signed distance between fuzzy numbers, we introduce a new decision rule to accept/reject the hypothesis of interest. The proposed approach is investigated for two cases: the case without nuisance parameters and the case with nuisance parameters. This method is employed to test the hypotheses for the mean of a normal distribution with known/unknown variance, the variance of a normal distribution, the difference of means of two normal distributions with known/unknown variances, and the ratio of variances of two normal distributions.

1. Introduction

An important problem in the statistical inference is testing statistical hypothesis. In classical environments, testing statistical hypothesis depends on precise data and exact hypothesis. However, in practical situations, different kinds of uncertainty are present, especially in situations with imprecise data. As an example, the available data of lifetime of a system are usually reported as imprecise data such as: about 10 (h), approximately 18 (h), about between 5 (h) and 12 (h), essentially less than 20 (h), and so on. In such situations, the classical approaches for testing statistical hypotheses are not appropriate for dealing with such imprecise data. Therefore, we need to use some new approaches in non-exact environments based on soft computing methods [9, 26, 30] and/or using the methods of computing with words [25, 44]. In recent decades, some attentions have been attempted to analyze the problem of testing hypotheses for such situations using fuzzy sets theory, introduced by Zadeh [45]. In the following, we review some works on testing statistical hypotheses in fuzzy environments.

Testing hypotheses based on fuzzy data are investigated by Casals and Gil [11], Casals et al. [12], Grzegorzewski [20], and Kahraman et al. [21]. Arnold [6, 7], Delgado et al. [16], Taheri and Arefi [32], and Taheri and Behboodian [33, 34] presented the problem of testing statistical hypotheses when the hypotheses are formulated as vague (fuzzy) sets. Arefi and Taheri [4, 5], Grzegorzewski [19], Taheri and Behboodian [35], and Torabi et al. [38] studied the topic of testing statistical

Received: November 2016; Revised: April 2017; Accepted: September 2017

Key words and phrases: Fuzzy data, Fuzzy test statistic, Signed distance, Statistical hypothesis, Testing hypothesis.

hypotheses when both hypotheses and available data are fuzzy (imprecise). The p-value approach to test statistical hypotheses in fuzzy environment has been studied by Couso and Sánchez [15], Filzmoser and Viertl [18] and Parchami et al. [27, 28]. In addition, some hierarchical soft methods were developed for testing statistical hypotheses in imprecise (fuzzy/vague) environments by Akbari and Rezaei [1], Arefi and Taheri [3], Buckley [10], Chachi et al. [13], Taheri and Hesamian [36, 37], and Yosefi et al. [43]. For more studies about statistical methods with fuzzy data, see, e.g. Taheri [31], and Viertl [40, 41].

This article is organized as follows. After reviewing some necessary concepts and notations in the present section, we define a new signed distance between fuzzy numbers in Section 2. In Section 3, a method for testing statistical hypotheses under fuzzy data, when the underlying model has no any nuisance parameter, is investigated. In this section, we apply the proposed method to test the mean of a normal distribution (with known variance), the variance of a normal distribution, the difference of means of two normal distributions (with known variances), and finally the ratio of variances of two normal distributions. In Section 4, we focus on the case in which there exists a nuisance parameter in the underlying model. In this section, we apply the proposed method to test the mean of a normal distribution with unknown variance and also to test the difference between means of two normal distributions with unknown (but equal) variances. Finally, in Section 5 a brief conclusion is provided.

First, let us recall some preliminary concepts and notations about fuzzy numbers and interval arithmetic. For details, the reader is referred to standard texts, e.g. Klir and Yuan [22] and Kruse and Meyer [23].

A fuzzy set $\widetilde{A}$ on the universal set X is defined by a membership function $\widetilde{A}: X \to [0,1]$. The α-cut of $\widetilde{A}$ is given as $\widetilde{A}[\alpha] = \{x | \widetilde{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$.

Definition 1.1. The fuzzy set $\widetilde{A}$ of X is called a fuzzy number if

i) $\widetilde{A}(x) = 1$ for some x,

ii) $\widetilde{A}[\alpha]$ is a closed bounded interval for $0 < \alpha \leq 1$.

The set of all fuzzy numbers is denoted by $FN(R)$.

Remark 1.2. A fuzzy number is a generalization of a real number. A real number refers to a single value, but a fuzzy number refers to a set of possible values, where each value has its own weight between 0 and 1 based on a membership function. Note that the fuzzy numbers depict the physical world more realistically than real numbers (for more detial, see Dijkman et al. [17]). For example, suppose that we are driving along a way where the speed limit is 90 kilometer per hour (kh). We try to hold our speed at exactly 90 kh, but we cannot control it at exactly 90 kh, and it varies from moment to moment. So, our speed is approximately 90 kh. We can model it based on a fuzzy number with a suitable membership function.

Definition 1.3. A fuzzy number $\tilde{A}$ is called a LR fuzzy number, if the membership function is defined as follows

:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{s_1}\right) & x \leq m, \\ R\left(\frac{x-m}{s_2}\right) & m < x, \\ 0 & o.w., \end{cases}$$

where, $L(.)$ and $R(.)$ are strictly decreasing functions from R^+ to $[0,1]$, and $L(0) = R(0) = 1$. $L(.)$ and $R(.)$ are called the reference functions. A LR fuzzy number is denoted by $\tilde{A} = (m; s_1, s_2)_{LR}$.

Some special cases of LR fuzzy numbers are given as follows:

I1): If $L(x) = R(x) = \max\{0, 1-x\}$ for all $x \in [0,1]$, then it is called a triangular fuzzy number, and is denoted by $\tilde{A} = (m; s_1, s_2)_T$. Also, if $s_1 = s_2 = s$, then it is called a symmetric triangular fuzzy number as $\tilde{A} = (m; s)_T$.

I2): If $L(x) = R(x) = \exp\{-x^2\}$ for all $x \in R$, then it is called a normal fuzzy number, and is denoted by $\tilde{A} = (m; s_1, s_2)_N$, and if $s_1 = s_2 = s$, then it is a symmetric normal fuzzy number as $\tilde{A} = (m; s)_N$.

Let $I = [a, b]$ and $J = [c, d]$ be two closed intervals. Then, based on the interval arithmetic, we have

$$\begin{aligned} I + J &= [a + c, b + d], \\ I - J &= [a - d, b - c], \\ I.J &= [\alpha_1, \beta_1],\ \alpha_1 = \min\{ac, ad, bc, bd\},\ \beta_1 = \max\{ac, ad, bc, bd\}, \\ I \div J &= [\alpha_2, \beta_2],\ \alpha_2 = \min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\},\ \beta_2 = \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \end{aligned}$$

where, zero does not belong to $J = [c, d]$ in the last case.

2. A New Signed Distance Between Fuzzy Numbers

In this section, we define a new signed distance between fuzzy numbers. This distance is an extended signed version of the distance introduced by Allahviranloo et al. [2] and Yao and Wu [42]. It should be mentioned that the distance between fuzzy sets introduced by some other authors, for instance, see [8, 14, 24, 39, 42].

Definition 2.1. [2] Let $\tilde{A}$ be a fuzzy number with the α-cut $\tilde{A}[\alpha] = [L^A_\alpha, R^A_\alpha]$. The values of average and width of $\tilde{A}$, respectively, are defined as follows

$$\begin{aligned} I(\tilde{A}) &= \frac{1}{2}\int_0^1 (L^A_\alpha + R^A_\alpha) d\alpha, \\ D(\tilde{A}) &= \int_0^1 (R^A_\alpha - L^A_\alpha) f(\alpha) d\alpha, \end{aligned}$$

where, $f(\alpha)$ is a nonnegative and increasing function on [0,1] with $f(0) = 0$, $f(1) = 1$, and $\int_0^1 f(\alpha) d\alpha = \frac{1}{2}$. In this paper, we assume that $f(\alpha) = \alpha$.

Definition 2.2. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. Then, we define a new signed distance between $\tilde{A}$ and $\tilde{B}$ as follows

$$
\begin{aligned}
T(\tilde{A},\tilde{B}) &= \begin{cases} I(\tilde{A}) - I(\tilde{B}) + |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) \geq I(\tilde{B}) \\ I(\tilde{A}) - I(\tilde{B}) - |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) < I(\tilde{B}) \end{cases} \\
&= \begin{cases} |I(\tilde{A}) - I(\tilde{B})| + |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) \geq I(\tilde{B}) \\ -|I(\tilde{A}) - I(\tilde{B})| - |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) < I(\tilde{B}) \end{cases} \\
&= \begin{cases} d(\tilde{A},\tilde{B}) & I(\tilde{A}) \geq I(\tilde{B}), \\ -d(\tilde{A},\tilde{B}) & I(\tilde{A}) < I(\tilde{B}), \end{cases}
\end{aligned}
$$

where, $d(\tilde{A},\tilde{B}) = |I(\tilde{A}) - I(\tilde{B})| + |D(\tilde{A}) - D(\tilde{B})|$ satisfies the following properties.

Remark 2.3. Let $\tilde{A},\tilde{B},\tilde{C} \in FN(R)$. Then, $d(.,.)$ given in Definition 2.2 satisfies the following properties:

A1): $d(\tilde{A},\tilde{A}) = 0$,
A2): $d(\tilde{A},\tilde{B}) \geq 0$,
A3): $d(\tilde{A},\tilde{B}) = d(\tilde{B},\tilde{A})$,
A4): $d(\tilde{A},\tilde{B}) + d(\tilde{B},\tilde{C}) \geq d(\tilde{A},\tilde{C})$.

Remark 2.4. In some special cases, the signed distance given in Definition 2.2 is reduced as follows:

i: If $\tilde{A}$ is a fuzzy number and b is a crisp real number, then the signed distance between $\tilde{A}$ and b is reduced as

$$
T(\tilde{A},b) = \begin{cases} I(\tilde{A}) + |D(\tilde{A})| - b & I(\tilde{A}) \geq b, \\ I(\tilde{A}) - |D(\tilde{A})| - b & I(\tilde{A}) < b. \end{cases}
$$

ii: If a and b are two crisp real number, then the signed distance between a and b is reduced as

$$
T(a,b) = a - b.
$$

Definition 2.5. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. Then, the ranking of $\tilde{A}$ and $\tilde{B}$ is expressed as

$$
\begin{aligned}
T(\tilde{A},\tilde{B}) > 0 &\quad \Leftrightarrow \quad \tilde{A} \succ \tilde{B}, \\
T(\tilde{A},\tilde{B}) < 0 &\quad \Leftrightarrow \quad \tilde{A} \prec \tilde{B}, \\
T(\tilde{A},\tilde{B}) = 0 &\quad \Leftrightarrow \quad \tilde{A} \approx \tilde{B}.
\end{aligned}
$$

Lemma 2.6. *Let $\tilde{A},\tilde{B},\tilde{C} \in FN(R)$. Then, the signed distance introduced in Definition 2.2 satisfies the following properties:*

(i): *For $I(\tilde{A}) \neq I(\tilde{B})$, then $T(\tilde{A},\tilde{B}) = -T(\tilde{B},\tilde{A})$, and for $I(\tilde{A}) = I(\tilde{B})$, then $T(\tilde{A},\tilde{B}) = T(\tilde{B},\tilde{A})$.*
(ii): *We order $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ such that $I(\tilde{A}) \geq I(\tilde{B}) \geq I(\tilde{C})$, then $T(\tilde{A},\tilde{B}) + T(\tilde{B},\tilde{C}) \geq T(\tilde{A},\tilde{C})$.*
(iii): $\tilde{A} \approx \tilde{B} \Leftrightarrow \tilde{B} \approx \tilde{A}$.

Proof. (i) For $I(\tilde{A}) \neq I(\tilde{B})$, we have

$$\begin{aligned} T(\tilde{A},\tilde{B}) &= \begin{cases} I(\tilde{A}) - I(\tilde{B}) + |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) > I(\tilde{B}) \\ I(\tilde{A}) - I(\tilde{B}) - |D(\tilde{A}) - D(\tilde{B})| & I(\tilde{A}) < I(\tilde{B}) \end{cases} \\ &= \begin{cases} -\Big[I(\tilde{B}) - I(\tilde{A}) - |D(\tilde{B}) - D(\tilde{A})|\Big] & I(\tilde{A}) > I(\tilde{B}) \\ -\Big[I(\tilde{B}) - I(\tilde{A}) + |D(\tilde{B}) - D(\tilde{A})|\Big] & I(\tilde{A}) < I(\tilde{B}) \end{cases} \\ &= \begin{cases} -\Big[I(\tilde{B}) - I(\tilde{A}) + |D(\tilde{B}) - D(\tilde{A})|\Big] & I(\tilde{B}) > I(\tilde{A}) \\ -\Big[I(\tilde{B}) - I(\tilde{A}) - |D(\tilde{B}) - D(\tilde{A})|\Big] & I(\tilde{B}) < I(\tilde{A}) \end{cases} \\ &= -T(\tilde{B},\tilde{A}), \end{aligned}$$

and for $I(\tilde{A}) = I(\tilde{B})$, we have

$$T(\tilde{A},\tilde{B}) = |D(\tilde{A}) - D(\tilde{B})| = |D(\tilde{B}) - D(\tilde{A})| = T(\tilde{B},\tilde{A}).$$

(ii) If $I(\tilde{A}) \geq I(\tilde{B}) \geq I(\tilde{C})$, we have

$$\begin{aligned} T(\tilde{A},\tilde{C}) &= I(\tilde{A}) - I(\tilde{C}) + |D(\tilde{A}) - D(\tilde{C})| \\ &= I(\tilde{A}) - I(\tilde{B}) + I(\tilde{B}) - I(\tilde{C}) \\ &\quad + |D(\tilde{A}) - D(\tilde{B}) + D(\tilde{B}) - D(\tilde{C})| \\ &\leq I(\tilde{A}) - I(\tilde{B}) + I(\tilde{B}) - I(\tilde{C}) \\ &\quad + |D(\tilde{A}) - D(\tilde{B})| + |D(\tilde{B}) - D(\tilde{C})| \\ &= I(\tilde{A}) - I(\tilde{B}) + |D(\tilde{A}) - D(\tilde{B})| \\ &\quad + I(\tilde{B}) - I(\tilde{C}) + |D(\tilde{B}) - D(\tilde{C})| \\ &= T(\tilde{A},\tilde{B}) + T(\tilde{B},\tilde{C}). \end{aligned}$$

(iii) From item (i) and Definition 2.5, we have

$$\tilde{A} \approx \tilde{B} \;\Leftrightarrow\; T(\tilde{A},\tilde{B}) = 0 = -T(\tilde{B},\tilde{A}),$$

hence, $T(\tilde{B},\tilde{A}) = 0$, and $\tilde{B} \approx \tilde{A}$. □

Remark 2.7. A signed distance between fuzzy numbers $\tilde{A}$ and $\tilde{B}$ with $\tilde{A}[\alpha] = [L_\alpha^A, R_\alpha^A]$ and $\tilde{B}[\alpha] = [L_\alpha^B, R_\alpha^B]$ also is introduced by Yao and Wu [42] as follows

$$\begin{aligned} d(\tilde{A},\tilde{B}) &= \int_0^1 (M_\alpha(\tilde{A}) - M_\alpha(\tilde{B}))d\alpha \\ &= \int_0^1 (\frac{L_\alpha^A + R_\alpha^A}{2} - \frac{L_\alpha^B + R_\alpha^B}{2})d\alpha \\ &= I(\tilde{A}) - I(\tilde{B}). \end{aligned}$$

Note that this signed distance has a disadvantage such that it is reduced to a crisp signed distance $d(a,b) = a - b$, when $\tilde{A}$ and $\tilde{B}$ are the symmetric fuzzy numbers with the centers (mids) a and b.

3. Testing Statistical Hypotheses Based On Fuzzy Data (One Parameter Case)

3.1. **Statement of the Method.** Let $X_1, X_2, ..., X_n$ be a random sample of a probability density (mass) function $f(x;\theta)$, where the parameter θ is unknown. Suppose the available data of the random sample are as the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$ rather than the crisp data $x_1, x_2, ..., x_n$. Based on the method introduced by Arefi and Taheri [4], a fuzzy point estimation for θ is obtained as follows:

Definition 3.1. Let $\theta^* = u(x_1, x_2, ..., x_n)$ be a point estimation for θ. By substituting the α-cuts of the fuzzy numbers $\widetilde{X}_i,\ i = 1, ..., n$ for $x_i,\ i = 1, ..., n$ into θ^*, the α-cut of the fuzzy point estimation $\widetilde{\theta}^*$ is obtained as follows

$$\widetilde{\theta}^*[\alpha] := \left\{u(x_1, x_2, ..., x_n);\ x_i \in \widetilde{X}_i[\alpha],\ i = 1, 2, ..., n\right\}.$$

In the following, we introduce a new procedure for testing the one-sided and two-sided hypotheses, respectively.

3.1.1. **Testing Simple Hypothesis Against Two-sided Hypothesis.** Suppose that we want to test the following hypotheses

$$\begin{cases} H_0:\ \theta = \theta_0, \\ H_1:\ \theta \neq \theta_0. \end{cases}$$

In the crisp case, the decision rule for testing a null hypothesis $H_0 : \theta = \theta_0$ against an alternative $H_0 : \theta \neq \theta_0$, at the significance level δ, is as

$$\begin{cases} Q_0 \leq Q_{\delta/2} \ \textit{or}\ Q_0 \geq Q_{1-\delta/2} & \Rightarrow \quad RH_0\ (\textit{Reject}\ H_0), \\ Q_{\delta/2} < Q_0 < Q_{1-\delta/2} & \Rightarrow \quad AH_0\ (\textit{Accept}\ H_0), \end{cases}$$

where, $Q_0 = Q_0(\theta^*, \theta_0)$ is the value of the crisp test statistic (under H_0), and Q_δ is the δ-quantile of the crisp test statistic. Suppose that the signed distance $d(x, y) = x - y$ is available, hence the above decision rule can rewrite as follows

$$\begin{cases} d(Q_0, Q_{\delta/2}) \leq 0 \ \textit{or}\ d(Q_0, Q_{1-\delta/2}) \geq 0 & \Rightarrow \quad RH_0, \\ d(Q_0, Q_{\delta/2}) > 0 \ \textit{and}\ d(Q_0, Q_{1-\delta/2}) < 0 & \Rightarrow \quad AH_0. \end{cases}$$

Now, we extend this decision rule based on fuzzy data. First, we obtain the fuzzy test statistic under fuzzy data based on the method introduced by Arefi and Taheri [4] as follows:

- **i:** We obtain a fuzzy point estimation $\widetilde{\theta}^*$ using Definition 3.1.
- **ii:** By substituting the α-cuts of the fuzzy point estimation $\widetilde{\theta}^*$ for the point estimation θ^* in the crisp test statistic (Q_0) and using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$ as follows:

$$\widetilde{Z}[\alpha] := \left\{Q_0(\theta^*, \theta_0);\ \theta^* \in \widetilde{\theta}^*[\alpha]\right\}. \tag{1}$$

Then, a new decision rule for testing two-sided hypotheses based on the signed distance (Definition 2.2) is introduced as follows:

Definition 3.2. New decision rule: Let $T(\widetilde{Z}, Q_{\delta/2})$ and $T(\widetilde{Z}, Q_{1-\delta/2})$ be the signed distances between the fuzzy test statistic $\widetilde{Z}$ and the quantiles $Q_{\delta/2}$ and $Q_{1-\delta/2}$, respectively. Then, the decision rule for testing null hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ is introduced as follows

$$\begin{cases} T(\widetilde{Z}, Q_{\delta/2}) \leq 0 \text{ or } T(\widetilde{Z}, Q_{1-\delta/2}) \geq 0 & \Rightarrow \quad RH_0, \\ T(\widetilde{Z}, Q_{\delta/2}) > 0 \text{ and } T(\widetilde{Z}, Q_{1-\delta/2}) < 0 & \Rightarrow \quad AH_0. \end{cases}$$

3.1.2. **Testing Simple Hypothesis Against One-sided Hypothesis.** Suppose that we wish to test the following hypotheses

$$\begin{cases} H_0 : \theta = \theta_0, \\ H_1 : \theta > \theta_0. \end{cases}$$

The decision rule for testing such hypotheses in a precise environment, at the significance level δ, is of the form

$$\begin{cases} Q_0 \geq Q_{1-\delta} \;\Rightarrow\; RH_0 \; (Reject\ H_0), \\ Q_0 < Q_{1-\delta} \;\Rightarrow\; AH_0 \; (Accept\ H_0), \end{cases}$$

where, Q_0 is the value of the crisp test statistic (under H_0), and $Q_{1-\delta}$ is the $(1-\delta)$-quantile of the distribution of the crisp test statistic. Based on the signed distance $d(x, y) = x - y$, the above decision rule can rewrite as follows:

$$\begin{cases} d(Q_0, Q_{1-\delta}) \geq 0 & \Rightarrow \quad RH_0, \\ d(Q_0, Q_{1-\delta}) < 0 & \Rightarrow \quad AH_0. \end{cases}$$

Now, let the available data be as fuzzy (imprecise). We extend the above decision rule based on the signed distance between fuzzy numbers. First, we obtain a fuzzy test statistic $\widetilde{Z}$ based on relation (1). Then, the new decision rule for testing one-sided hypotheses based on the signed distance is introduced as follows:

Definition 3.3. New decision rule: Let $T(\widetilde{Z}, Q_{1-\delta})$ be the signed distance between $\widetilde{Z}$ and $Q_{1-\delta}$. Then, the decision rule for testing simple hypothesis $H_0 : \theta = \theta_0$ against one-sided hypothesis $H_1 : \theta > \theta_0$ is introduced as follows:

$$\begin{cases} T(\widetilde{Z}, Q_{1-\delta}) \geq 0 & \Rightarrow \quad RH_0, \\ T(\widetilde{Z}, Q_{1-\delta}) < 0 & \Rightarrow \quad AH_0. \end{cases}$$

Remark 3.4. In a similar way, we can also introduce the above decision rule for testing simple hypothesis $H_0 : \theta = \theta_0$ against one-sided hypothesis $H_1 : \theta < \theta_0$ as

$$\begin{cases} T(\widetilde{Z}, Q_{\delta}) \leq 0 & \Rightarrow \quad RH_0, \\ T(\widetilde{Z}, Q_{\delta}) > 0 & \Rightarrow \quad AH_0. \end{cases}$$

Remark 3.5. In testing statistical hypotheses, if we calculate the crisp test statistic based on the centers of fuzzy data, it is similar to the α-cut of fuzzy test statistic for $\alpha = 1$, i.e. $\widetilde{Z}[\alpha = 1] = Q_0(\theta^*, \theta_0)$. Hence, the decision rule for accepting/rejecting the hypothesis H_0 is reduced to original case with precise data.

Remark 3.6. In testing statistical hypotheses, we may encounter with the following cases:

i): The value of crisp test statistic is close to the related quantile.

ii): The available/observed data are as imprecise values.

In case i, the decision to accept or reject the null hypothesis is very sensitive, and in case ii, the classical methods can not support such data. In such cases, we first model the data based on fuzzy sets introduced by Zadeh [45], and then, we use the method given in this paper (see Example 3.7). Also, for solving this problem, we can use the approach presented by Arefi and Taheri [3]. They introduced a decision rule for evaluating statistical hypotheses by two indices: the degree of consistency DC that measure the consistency of data with the hypothesis H_0, and degree of inconsistency DI that measure the inconsistency of data with the hypothesis H_0 (the consistency of data with the hypothesis H_1).

Example 3.7. Assume that, based on a random sample of size $n = 25$ from a population $N(\theta, 9)$, we observe the symmetric triangular fuzzy data in Table 1. Now, we want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

Based on the centers of fuzzy data x_i, the crisp test statistic is obtained as $z_0 = \frac{\overline{x}-\theta_0}{\sigma/\sqrt{n}} = \frac{2.9868-2}{3/5} = 1.6447$. Since $z_0 < z_{1-\delta} = 1.6449$, we accept the hypothesis H_0. But, in this case, the value of crisp test statistic is close to the related quantile, and the decision to accept the null hypothesis H_0 is very sensitive (note that based on the signed distance $d(x, y) = x - y$, we have $d(z_0, z_{1-\delta}) = -0.0002 < 0$). If we use the fuzzy data, then the α-cuts of fuzzy point estimation $\widetilde{\overline{X}}$ are obtained as ($\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U]$)

$$\widetilde{\overline{X}}[\alpha] = \frac{1}{n}\sum_{i=1}^{n}[\widetilde{X}_i^L, \widetilde{X}_i^U] = [2.6872 + 0.2996\alpha, 3.2864 - 0.2996\alpha]$$

Also, based on Subsection 3.1.2, the α-cuts of the fuzzy test statistic are obtained as

$$\widetilde{Z}[\alpha] = \frac{\widetilde{\overline{X}}[\alpha] - \theta_0}{\sigma/\sqrt{n}} = [1.1454 + 0.4993\alpha, 2.1440 - 0.4993\alpha]\,.$$

Hence, based on the new signed distance, we have

$$\begin{aligned} T(\widetilde{Z}, z_{1-\delta}) &= I(\tilde{Z}) - |D(\tilde{Z})| - z_{0.95} \\ &= 1.6447 - 0.4993 - 1.6449 \\ &= -0.4995, \end{aligned}$$

where, $I(\tilde{Z}) = 1.6447$ and $D(\tilde{Z}) = 0.4993$. Since, $T(\widetilde{Z}, z_{1-0.05}) = -0.4995 < 0$, we surely accept the hypothesis H_0.

No.	$\widetilde{X}_i = (x_i; r_i)_T$	No.	$\widetilde{X}_i = (x_i; r_i)_T$	No.	$\widetilde{X}_i = (x_i; r_i)_T$	No.	$\widetilde{X}_i = (x_i; r_i)_T$
1	$(2.66; 0.27)_T$	8	$(2.59; 0.26)_T$	15	$(3.35; 0.34)_T$	22	$(3.22; 0.32)_T$
2	$(2.98; 0.30)_T$	9	$(3.23; 0.32)_T$	16	$(2.83; 0.28)_T$	23	$(2.85; 0.29)_T$
3	$(2.58; 0.26)_T$	10	$(2.90; 0.29)_T$	17	$(3.39; 0.34)_T$	24	$(3.26; 0.33)_T$
4	$(2.77; 0.28)_T$	11	$(3.19; 0.32)_T$	18	$(2.68; 0.27)_T$	25	$(2.87; 0.29)_T$
5	$(2.61; 0.26)_T$	12	$(3.08; 0.31)_T$	19	$(3.05; 0.30)_T$		
6	$(3.09; 0.31)_T$	13	$(3.17; 0.32)_T$	20	$(3.20; 0.32)_T$		
7	$(3.24; 0.32)_T$	14	$(2.78; 0.28)_T$	21	$(3.10; 0.31)_T$		

TABLE 1. Fuzzy Data from a Normal Population in Example 3.7

3.2. **Testing Statistical Hypotheses in the Normal Distribution.**

3.2.1. **Testing Hypotheses for the Mean.** Suppose that we have taken a random sample of size n from a $N(\theta, \sigma^2)$ (σ^2 known) and observed the fuzzy numbers $\widetilde{X}_1, ..., \widetilde{X}_n$ rather than the crisp data $x_1, ..., x_n$. The usual point estimation for θ is $\theta^* = \overline{x}$. By substituting the α-cuts of $\widetilde{X}_i,\ i = 1, ..., n$, ($\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U]$) for x_i in the point estimation, the α-cuts of the fuzzy point estimation $\widetilde{\overline{X}}$ is obtained as

$$\widetilde{\overline{X}}[\alpha] = \frac{1}{n}\sum_{i=1}^{n}[\widetilde{X}_i^L, \widetilde{X}_i^U] = \left[\frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i^L, \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i^U\right] = [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U].$$

Under $H_0 : \theta = \theta_0$, the value of the crisp test statistic is $z_0 = \frac{\theta^* - \theta_0}{\sigma/\sqrt{n}}$. By substituting the α-cuts of the fuzzy point estimation $\widetilde{\overline{X}}$ for θ^* in z_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as follows

$$\tilde{Z}[\alpha] = \frac{\widetilde{\overline{X}}[\alpha] - \theta_0}{\sigma/\sqrt{n}} = \left[\frac{\widetilde{\overline{X}}^L - \theta_0}{\sigma/\sqrt{n}}, \frac{\widetilde{\overline{X}}^U - \theta_0}{\sigma/\sqrt{n}}\right].$$

Hence, the decision rule for testing the null hypothesis $H_0 : \theta = \theta_0$ against the hypothesis H_1 is as follows:

- **i):** for $H_1 : \theta \neq \theta_0$, we reject H_0, if $T(\widetilde{Z}, z_{\delta/2}) \leq 0$ or $T(\widetilde{Z}, z_{1-\delta/2}) \geq 0$,
- **ii):** for $H_1 : \theta > \theta_0$, we reject H_0, if $T(\widetilde{Z}, z_{1-\delta}) \geq 0$,
- **iii):** for $H_1 : \theta < \theta_0$, we reject H_0, if $T(\widetilde{Z}, z_\delta) \leq 0$'

where, z_δ is δ-quantile of the standard normal distribution.

For example, suppose that the available data are as the symmetric LR fuzzy numbers $\widetilde{X}_i = (x_i; r_i)_{LR}$, $i = 1, ..., n$. Then, the fuzzy point estimation is given by $\widetilde{\overline{X}} = (\overline{x}; \overline{r})_{LR}$. Hence, the α-cuts of the fuzzy test statistic are obtained as

$$\begin{aligned} \widetilde{Z}[\alpha] &= \left[\frac{\widetilde{\overline{X}}^L - \theta_0}{\sigma/\sqrt{n}}, \frac{\widetilde{\overline{X}}^U - \theta_0}{\sigma/\sqrt{n}}\right] \\ &= \left[z_0 - L^{-1}(\alpha)\frac{\overline{r}\ \sqrt{n}}{\sigma}, z_0 + R^{-1}(\alpha)\frac{\overline{r}\ \sqrt{n}}{\sigma}\right]. \end{aligned} \quad (2)$$

It is obvious that the fuzzy test statistic is as a symmetric LR fuzzy number $\widetilde{Z} = (z_0; \frac{\overline{r}\ \sqrt{n}}{\sigma}, \frac{\overline{r}\ \sqrt{n}}{\sigma})_{LR}$. Now, we can apply the decision rule for rejecting/accepting the hypothesis H_0.

$(x_i;r_i)_T$	$(x_i;r_i)_T$	$(x_i;r_i)_T$	$(x_i;r_i)_T$	$(x_i;r_i)_T$
$(1.8;0.2)_T$	$(2.4;0.1)_T$	$(0.3;0.1)_T$	$(6.0;1.2)_T$	$(6.2;1.1)_T$
$(2.9;0.4)_T$	$(1.9;0.3)_T$	$(1.6;0.2)_T$	$(1.3;0.3)_T$	$(5.8;1.1)_T$
$(0.9;0.2)_T$	$(6.0;0.8)_T$	$(6.9;1.0)_T$	$(3.4;0.4)_T$	$(-0.4;0.2)_T$
$(3.7;0.7)_T$	$(3.1;0.4)_T$	$(1.8;0.2)_T$	$(5.8;1.0)_T$	$(0.9;0.2)_T$
$(1.2;0.2)_T$	$(7.3;1.5)_T$	$(4.9;1.0)_T$	$(-2.1;0.2)_T$	$(-1.2;0.2)_T$
$(1.7;0.3)_T$	$(5.1;1.0)_T$	$(2.8;0.3)_T$	$(1.0;0.2)_T$	$(-2.8;0.4)_T$
$(0.5;0.1)_T$	$(1.3;0.2)_T$	$(-4.6;0.3)_T$	$(0.8;0.1)_T$	$(1.6;0.3)_T$
$(3.4;0.5)_T$	$(2.1;0.2)_T$	$(6.8;1.4)_T$	$(5.0;1.0)_T$	$(0.9;0.2)_T$
$(-1.3;0.2)_T$	$(1.4;0.1)_T$	$(3.8;0.4)_T$	$(3.0;0.6)_T$	$(5.7;1.0)_T$
$(-3.4;0.4)_T$	$(3.0;0.6)_T$	$(4.4;0.4)_T$	$(6.9;1.2)_T$	$(-0.7;0.1)_T$

TABLE 2. Fuzzy Data from a Normal Population in Example 3.8

Example 3.8. [4] Assume that, based on a random sample of size $n = 50$ from $N(\theta, \sigma^2 = 9)$, we observe the triangular fuzzy data listed in Table 2. Now, suppose that we want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

Based on the fuzzy data $\widetilde{X}_i,\ i = 1, ..., n$, the α-cuts of the fuzzy point estimation $\widetilde{\overline{X}}$ is obtained as

$$\begin{aligned} \widetilde{\overline{X}}[\alpha] &= [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U] \\ &= \left[\frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i^L, \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i^U\right] \\ &= [\overline{x} - (1-\alpha)\overline{r}, \overline{x} + (1-\alpha)\overline{r}] \\ &= [1.924 + 0.492\alpha, 2.908 - 0.492\alpha], \end{aligned}$$

where $\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U] = [x_i - (1-\alpha)r_i, x_i + (1-\alpha)r_i]$. Using the relation (2), the α-cuts of fuzzy test statistic are obtained as

$$\begin{aligned} \widetilde{Z}[\alpha] &= \left[\frac{\widetilde{\overline{X}}^L - \theta_0}{\sigma/\sqrt{n}}, \frac{\widetilde{\overline{X}}^U - \theta_0}{\sigma/\sqrt{n}}\right] \\ &= \left[\frac{1.924 + 0.492\alpha - 2}{3/\sqrt{50}}, \frac{2.908 - 0.492\alpha - 2}{3/\sqrt{50}}\right] \\ &= [-0.1791 + 1.1597\alpha, 2.1402 - 1.1597\alpha]. \end{aligned}$$

So, the fuzzy test statistic is a triangular fuzzy number as $\widetilde{Z} = (0.9805; 1.1597)_T$. Hence, we have

$$\begin{aligned} T(\widetilde{Z}, z_{1-\delta}) &= T(\widetilde{Z}, z_{0.95}) \\ &= I(\tilde{Z}) - |D(\tilde{Z})| - z_{0.95} \\ &= 0.9805 - 0.3866 - 1.6449 \\ &= -1.0510, \end{aligned}$$

where, $I(\tilde{Z}) = 0.9805$ and $D(\tilde{Z}) = 0.3866$. Since, $T(\widetilde{Z}, z_{1-0.05}) = T(\widetilde{Z}, 1.6449) = -1.0510 < 0$, we accept H_0.

In this example, suppose that we want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta \neq 2. \end{cases}$$

Based on the above fuzzy test statistics, we have

$$\begin{aligned} T(\widetilde{Z}, z_{\delta/2}) &= T(\widetilde{Z}, z_{0.025}) \\ &= I(\tilde{Z}) + |D(\tilde{Z})| - z_{0.025} \\ &= 0.9805 + 0.3866 + 1.9600 \\ &= 3.3271, \\ T(\widetilde{Z}, z_{1-\delta/2}) &= T(\widetilde{Z}, z_{0.975}) \\ &= I(\tilde{Z}) - |D(\tilde{Z})| - z_{0.975} \\ &= 0.9805 - 0.3866 - 1.9600 \\ &= -1.3661. \end{aligned}$$

Since $T(\widetilde{Z}, z_{\delta/2}) = 3.3271 > 0$ and $T(\widetilde{Z}, z_{1-\delta/2}) = -1.3661 < 0$. Hence, based on the decision rule the hypothesis H_0 is accepted, at the significance level $\delta = 0.05$.

Based on Remark 3.5 for $\alpha = 1$, the fuzzy test statistic reduces to the crisp test statistic as $\widetilde{Z}[\alpha = 1] = z_0 = 0.9805$. Since $|z_0| < z_{1-\delta/2} = 1.96$, the hypothesis H_0 is accepted, at the significance level $\delta = 0.05$ based on the centers of fuzzy data.

3.2.2. **Testing Hypotheses for the Variance.** Suppose that we have taken a random sample of size n from a population $N(\mu, \theta)$ (μ is unknown) and observed the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$. The usual point estimation for θ is $\theta^* = s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2$. By substituting the α-cuts of $\widetilde{X}_i$, $i = 1, ..., n$, ($\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U]$) for x_i in the point estimation θ^*, and using the interval arithmetic, the α-cuts of the fuzzy point estimation $\widetilde{S^2}$ are obtained as follows (see also Arefi and Taheri [4])

$$\widetilde{S^2}[\alpha] = \frac{1}{n-1}\sum_{i=1}^{n}\left([\widetilde{X}_i^L, \widetilde{X}_i^U] - [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U]\right) \cdot \left([\widetilde{X}_i^L, \widetilde{X}_i^U] - [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U]\right) = \left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha]\right],$$

where,

$$\widetilde{\overline{X}}[\alpha] = [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U] = \left[\tfrac{1}{n}\textstyle\sum_{i=1}^n \widetilde{X}_i^L, \tfrac{1}{n}\sum_{i=1}^n \widetilde{X}_i^U\right],$$

$$\widetilde{S^2}^L[\alpha] = \max\Big[0, \min\Big[\tfrac{1}{n-1}\textstyle\sum_{i=1}^n(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)^2, \tfrac{1}{n-1}\sum_{i=1}^n(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)^2,$$

$$\tfrac{1}{n-1}\textstyle\sum_{i=1}^n(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)\Big]\Big],$$

$$\widetilde{S^2}^U[\alpha] = \max\Big[\tfrac{1}{n-1}\textstyle\sum_{i=1}^n(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)^2, \tfrac{1}{n-1}\sum_{i=1}^n(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)^2,$$

$$\tfrac{1}{n-1}\textstyle\sum_{i=1}^n(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)\Big].$$

Under the null hypothesis $H_0 : \theta = \theta_0$, the crisp test statistic is $Q_0 = \frac{(n-1)s^2}{\theta_0}$ (distributed according to χ^2_{n-1}).

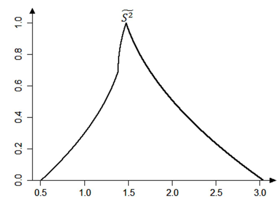

FIGURE 1. Fuzzy Point Estimation in Example 3.9

No.	$\widetilde{X}_i = (x_i; s_i)_T$	No.	$\widetilde{X}_i = (x_i; s_i)_T$	No.	$\widetilde{X}_i = (x_i; s_i)_T$	No.	$\widetilde{X}_i = (x_i; s_i)_T$
1	$(1.29; 0.26)_T$	6	$(2.72; 0.54)_T$	11	$(1.62; 0.32)_T$	16	$(2.60; 0.52)_T$
2	$(0.23; 0.05)_T$	7	$(2.11; 0.42)_T$	12	$(3.91; 0.78)_T$	17	$(2.25; 0.45)_T$
3	$(0.70; 0.14)_T$	8	$(3.29; 0.66)_T$	13	$(3.53; 0.71)_T$	18	$(0.54; 0.11)_T$
4	$(3.54; 0.71)_T$	9	$(2.63; 0.53)_T$	14	$(3.08; 0.62)_T$	19	$(1.61; 0.32)_T$
5	$(1.51; 0.30)_T$	10	$(1.37; 0.27)_T$	15	$(3.05; 0.61)_T$	20	$(4.85; 0.97)_T$

TABLE 3. Fuzzy Data from a Normal Population in Example 3.9

By substituting the α-cuts of the fuzzy point estimation $\widetilde{S^2}$ for s^2 in Q_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as

$$\tilde{Z}[\alpha] = \frac{(n-1)}{\theta_0}\widetilde{S^2}[\alpha] = \left[\frac{(n-1)\widetilde{S^2}^L[\alpha]}{\theta_0}, \frac{(n-1)\widetilde{S^2}^U[\alpha]}{\theta_0}\right].$$

Now, based on the above fuzzy test statistic, we can test the null hypothesis $H_0 : \theta = \theta_0$ against the hypothesis H_1 as follows:

i): for $H_1 : \theta \neq \theta_0$, we reject H_0, if $T(\widetilde{Z}, \chi^2_{n-1,\delta/2}) \leq 0$ or $T(\widetilde{Z}, \chi^2_{n-1,1-\delta/2}) \geq 0$,

ii): for $H_1 : \theta > \theta_0$, we reject H_0, if $T(\widetilde{Z}, \chi^2_{n-1,1-\delta}) \geq 0$,

iii): for $H_1 : \theta < \theta_0$, we reject H_0, if $T(\widetilde{Z}, \chi^2_{n-1,\delta}) \leq 0$.

Example 3.9. [4] Suppose that, based on a random sample of size $n = 20$ from $N(\mu, \theta)$, we observe the symmetric triangular fuzzy numbers given in Table 3. The α-cuts of the fuzzy point estimation for θ are obtained as (see Figure 1)

$$\widetilde{S^2}[\alpha] = \left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha]\right],$$

where,

$$\widetilde{S^2}^L[\alpha] = \begin{cases} -0.9677\alpha^2 + 1.9354\alpha + 0.5108 & 0 \leq \alpha \leq 0.6942, \\ 0.9677\alpha^2 - 1.3435\alpha + 1.8543 & 0.6942 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S^2}^U[\alpha] = 0.9677\alpha^2 - 2.5273\alpha + 3.0381, \ 0 \leq \alpha \leq 1.$$

Now, we test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

The α-cuts of the fuzzy test statistic are calculated as follows

$$\widetilde{Z}[\alpha] = \left[\frac{(n-1)\widetilde{S^2}^L[\alpha]}{\theta_0}, \frac{(n-1)\widetilde{S^2}^U[\alpha]}{\theta_0}\right] = \left[9.5\widetilde{S^2}^L[\alpha], 9.5\widetilde{S^2}^U[\alpha]\right] = 9.5\widetilde{S^2}[\alpha].$$

So, we have

$$\begin{aligned} I(\widetilde{Z}) &= \frac{1}{2}\int_0^1 \left(9.5\widetilde{S^2}^U[\alpha] + 9.5\widetilde{S^2}^L[\alpha]\right) d\alpha = 15.4077, \\ D(\widetilde{Z}) &= \int_0^1 \left(9.5\widetilde{S^2}^U[\alpha] - 9.5\widetilde{S^2}^L[\alpha]\right) \alpha \; d\alpha = 2.5436. \end{aligned}$$

Since $I(\widetilde{Z}) = 15.4077 < \chi^2_{19,0.95} = 30.1435$, we obtain

$$\begin{aligned} T(\widetilde{Z}, \chi^2_{n-1,1-\delta}) &= I(\widetilde{Z}) - |D(\widetilde{Z})| - \chi^2_{19,0.95} \\ &= 15.4077 - 2.5436 - 30.1435 \\ &= -17.2794. \end{aligned}$$

Hence, $T(\widetilde{Z}, \chi^2_{n-1,1-\delta}) < 0$, and we accept the null hypothesis, at the significance level $\delta = 0.05$.

Based on Remark 3.5, the α-cut of fuzzy test statistic for $\alpha = 1$ is reduced to the crisp test statistic as

$$\widetilde{Z}[\alpha = 1] = Q_0 = \frac{\sum_{i=1}^{20}(x_i - \overline{x})^2}{\theta_0} = 1.4785,$$

where $x_i,\ i = 1, 2, ..., 20$ are the centers of the fuzzy data in Table 3. Since $Q_0 < \chi^2_{19,0.95} = 30.1435$, we accept the null hypothesis based on the centers of fuzzy data.

3.3. **Testing Hypothesis for Difference Between Means of two Normal Populations with Known Variances.** Suppose that we have taken two independent random samples of sizes n_1 and n_2 from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively (with σ_1^2 and σ_2^2 known), and observed the fuzzy numbers $\widetilde{X}_1, ...\widetilde{X}_{n_1}$ and $\widetilde{Y}_1, ..., \widetilde{Y}_{n_2}$ instead of the crisp numbers $x_1, ...x_{n_1}$ and $y_1, ..., y_{n_2}$. The usual point estimation for $\theta = \mu_1 - \mu_2$ is $\theta^* = \overline{x} - \overline{y}$. Under the fuzzy data, the α-cuts of the fuzzy point estimation $\widetilde{\theta^*}$ is given by

$$\begin{aligned} \widetilde{\theta^*}[\alpha] &= [\widetilde{\theta^*}^L, \widetilde{\theta^*}^U] \\ &= \overline{\widetilde{X}}[\alpha] - \overline{\widetilde{Y}}[\alpha] \\ &= \left[\tfrac{1}{n_1}\textstyle\sum_{i=1}^{n_1} \widetilde{X}_i^L, \tfrac{1}{n_1}\sum_{i=1}^{n_1} \widetilde{X}_i^U\right] - \left[\tfrac{1}{n_2}\sum_{i=1}^{n_2} \widetilde{Y}_i^L, \tfrac{1}{n_2}\sum_{i=1}^{n_2} \widetilde{Y}_i^U\right] \\ &= [\overline{\widetilde{X}}^L, \overline{\widetilde{X}}^U] - [\overline{\widetilde{Y}}^L, \overline{\widetilde{Y}}^U] \\ &= [\overline{\widetilde{X}}^L - \overline{\widetilde{Y}}^U, \overline{\widetilde{X}}^U - \overline{\widetilde{Y}}^L]. \end{aligned}$$

Under the null hypothesis $H_0 : \mu_1 - \mu_2 = \theta_0$, the value of the crisp test statistic is $z_0 = \frac{(\overline{x}-\overline{y})-\theta_0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = \frac{\theta^*-\theta_0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ (distributed according to standard normal $N(0, 1)$).

$\widetilde{X}_i = (x_i; r_i)_N$	$\widetilde{X}_i = (x_i; r_i)_N$	$\widetilde{X}_i = (x_i; r_i)_N$	$\widetilde{Y}_i = (y_i; s_i)_N$	$\widetilde{Y}_i = (y_i; s_i)_N$
$(8.10; 0.81)_N$	$(10.13; 1.01)_N$	$(10.36; 1.04)_N$	$(4.59; 0.46)_N$	$(7.24; 0.72)_N$
$(2.17; 0.22)_N$	$(5.43; 0.54)_N$	$(6.89; 0.69)_N$	$(8.10; 0.81)_N$	$(2.44; 0.24)_N$
$(10.55; 1.06)_N$	$(3.23; 0.32)_N$	$(8.91; 0.89)_N$	$(5.73; 0.57)_N$	$(4.99; 0.50)_N$
$(-0.80; 0.08)_N$	$(6.84; 0.68)_N$	$(9.38; 0.94)_N$	$(2.63; 0.26)_N$	$(6.35; 0.64)_N$
$(6.49; 0.65)_N$	$(2.38; 0.24)_N$	$(8.83; 0.88)_N$	$(4.67; 0.47)_N$	$(4.41; 0.44)_N$
$(5.53; 0.55)_N$	$(8.22; 0.82)_N$	$(5.28; 0.53)_N$	$(5.46; 0.55)_N$	$(4.26; 0.43)_N$
$(9.10; 0.91)_N$	$(4.86; 0.49)_N$	$(2.72; 0.27)_N$	$(5.05; 0.51)_N$	$(1.99; 0.20)_N$
$(12.61; 1.26)_N$	$(12.30; 1.23)_N$		$(3.85; 0.39)_N$	
$(7.99; 0.80)_N$	$(3.05; 0.30)_N$		$(2.17; 0.22)_N$	

Table 4. Fuzzy Data of Two Normal Populations in Example 3.10

By substituting the α-cuts of the fuzzy point estimation $\widetilde{\theta^*}$ for θ^*, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as follows

$$\tilde{Z}[\alpha] = \frac{\widetilde{\theta^*}[\alpha] - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \left[\frac{\widetilde{\theta^*}^L - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}, \frac{\widetilde{\theta^*}^U - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right].$$

Hence, the decision rule based on the new signed distance for testing the null hypothesis $H_0 : \mu_1 - \mu_2 = \theta_0$ against the hypothesis H_1 is introduced as follows:

i) for $H_1 : \mu_1 - \mu_2 \neq \theta_0$, we reject H_0 if $T(\widetilde{Z}, z_{\delta/2}) \leq 0$ or $T(\widetilde{Z}, z_{1-\delta/2}) \geq 0$,

ii) for $H_1 : \mu_1 - \mu_2 > \theta_0$, we reject H_0 if $T(\widetilde{Z}, z_{1-\delta}) \geq 0$,

iii) for $H_1 : \mu_1 - \mu_2 < \theta_0$, we reject H_0 if $T(\widetilde{Z}, z_{\delta}) \leq 0$.

Example 3.10. Assume that based on two independent random samples of sizes $n_1 = 25$ and $n_2 = 16$ from $N(\mu_1, 9)$ and $N(\mu_2, 4)$, we obtain the symmetric normal fuzzy numbers given in Table 4. Now, assume that we wish to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \mu_1 - \mu_2 = 2, \\ H_1 : \mu_1 - \mu_2 > 2. \end{cases}$$

Based on fuzzy data given in Table 4, we calculate $\widetilde{\overline{X}}[\alpha]$ and $\widetilde{\overline{Y}}[\alpha]$ as follows

$$\begin{aligned} \widetilde{\overline{X}}[\alpha] &= \left[\frac{1}{25} \sum_{i=1}^{25} (x_i - r_i\sqrt{-\ln(\alpha)}), \frac{1}{25} \sum_{i=1}^{25} (x_i + r_i\sqrt{-\ln(\alpha)}) \right] \\ &= \left[6.8220 - 0.6884\sqrt{-\ln(\alpha)}, 6.8220 + 0.6884\sqrt{-\ln(\alpha)} \right]. \\ \widetilde{\overline{Y}}[\alpha] &= \left[\frac{1}{16} \sum_{i=1}^{16} (y_i - s_i\sqrt{-\ln(\alpha)}), \frac{1}{16} \sum_{i=1}^{16} (y_i + s_i\sqrt{-\ln(\alpha)}) \right] \\ &= \left[4.6206 - 0.4631\sqrt{-\ln(\alpha)}, 4.6206 + 0.4631\sqrt{-\ln(\alpha)} \right]. \end{aligned}$$

So, the α-cuts of fuzzy point estimation are obtained as

$$\widetilde{\theta^*}[\alpha] = \left[\widetilde{\theta^*}^L, \widetilde{\theta^*}^U \right] = \left[2.2014 - 1.1515\sqrt{-\ln(\alpha)}, 2.2014 + 1.1515\sqrt{-\ln(\alpha)} \right].$$

Hence, the α-cuts of fuzzy test statistic are obtained as

$$\begin{aligned}
\tilde{Z}[\alpha] &= \left[Z_\alpha^L, Z_\alpha^U\right] \\
&= \left[\frac{\widetilde{\theta^*}^L - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}, \frac{\widetilde{\theta^*}^U - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right] \\
&= \left[\frac{2.2014 - 1.1515\sqrt{-\ln(\alpha)} - 2}{\sqrt{\frac{9}{25} + \frac{4}{16}}}, \frac{2.2014 + 1.1515\sqrt{-\ln(\alpha)} - 2}{\sqrt{\frac{9}{25} + \frac{4}{16}}}\right] \\
&= \left[0.2578 - 1.4744\sqrt{-\ln(\alpha)}, 0.2578 + 1.4744\sqrt{-\ln(\alpha)}\right].
\end{aligned}$$

Using Maple software [29], $I(\widetilde{Z})$ and $D(\widetilde{Z})$ are calculated as

$$\begin{aligned}
I(\widetilde{Z}) &= \frac{1}{2}\int_0^1 (Z_\alpha^U + Z_\alpha^L) d\alpha = 0.2578, \\
D(\widetilde{Z}) &= \int_0^1 (Z_\alpha^U - Z_\alpha^L)\alpha \, d\alpha = \int_0^1 2.9488\alpha\sqrt{-\ln(\alpha)} \, d\alpha = 0.9239.
\end{aligned}$$

Since $I(\widetilde{Z}) = 0.2578 < z_{0.95} = 1.6449$, we obtain

$$\begin{aligned}
T(\widetilde{Z}, z_{1-\delta}) &= I(\widetilde{Z}) - |D(\widetilde{Z})| - z_{0.95} \\
&= 0.2578 - 0.9239 - 1.6449 \\
&= -2.3110.
\end{aligned}$$

Hence, $T(\widetilde{Z}, z_{1-\delta}) < 0$, and we accept the null hypothesis, at the significance level $\delta = 0.05$.

3.4. **Testing Hypothesis for $\frac{\sigma_1^2}{\sigma_2^2}$ of Two Normal Populations.** Suppose that we have taken two independent random samples of sizes n_1 and n_2 from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, and observed the fuzzy numbers $\widetilde{X}_1, ...\widetilde{X}_{n_1}$ and $\widetilde{Y}_1, ..., \widetilde{Y}_{n_2}$, respectively. The usual point estimation for $\theta = \frac{\sigma_1^2}{\sigma_2^2}$ is $\theta^* = \frac{s_1^2}{s_2^2} = \frac{\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\overline{x})^2}{\frac{1}{n_2-1}\sum_{j=1}^{n_2}(y_i-\overline{y})^2}$. Under the fuzzy data, the α-cuts of the fuzzy point estimation $\widetilde{\theta^*}$ are obtained as

$$\widetilde{\theta^*}[\alpha] = \frac{\widetilde{S_1^2}[\alpha]}{\widetilde{S_2^2}[\alpha]} = \frac{\left[\widetilde{S_1^2}^L, \widetilde{S_1^2}^U\right]}{\left[\widetilde{S_2^2}^L, \widetilde{S_2^2}^U\right]} = \left[\frac{\widetilde{S_1^2}^L}{\widetilde{S_2^2}^U}, \frac{\widetilde{S_1^2}^U}{\widetilde{S_2^2}^L}\right],$$

where, $\widetilde{S_1^2}[\alpha]$ and $\widetilde{S_2^2}[\alpha]$ are calculated based on the method introduced in Subsection 3.2.2. Under the hypothesis $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = \theta_0$, the crisp test statistic is distributed as $\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} = \frac{S_2^2}{S_1^2} \cdot \theta_0 \sim F_{n_2-1, n_1-1}$. Substituting the bounds of $\widetilde{\theta^*}[\alpha]$ instead of θ^* in the test statistic $Q_0 = \frac{\theta_0}{\theta^*}$, and using the interval arithmetic, we obtain the α-cuts of the fuzzy test statistic as

$$\tilde{Z}[\alpha] = \frac{\theta_0}{\widetilde{\theta^*}[\alpha]} = \left[\frac{\theta_0\widetilde{S_2^2}^L}{\widetilde{S_1^2}^U}, \frac{\theta_0\widetilde{S_2^2}^U}{\widetilde{S_1^2}^L}\right].$$

Hence, the decision rule for testing the null hypothesis $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = \theta_0$ against the different cases of hypothesis H_1 is derived as follows:

i): for $H_1 : \frac{\sigma_1^2}{\sigma_2^2} \neq \theta_0$, the hypothesis H_0 is rejected, if

$$T(\widetilde{Z}, F_{n_2-1,n_1-1,\delta/2}) \leq 0 \textit{ or } T(\widetilde{Z}, F_{n_2-1,n_1-1,1-\delta/2}) \geq 0,$$

ii): for $H_1 : \frac{\sigma_1^2}{\sigma_2^2} > \theta_0$, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, F_{n_2-1,n_1-1,1-\delta}) \geq 0$,

iii): for $H_1 : \frac{\sigma_1^2}{\sigma_2^2} < \theta_0$, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, F_{n_2-1,n_1-1,\delta}) \leq 0$.

Example 3.11. Consider the data in Table 4. Assume that these data are two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ with unknown variances. We want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 3, \\ H_1 : \frac{\sigma_1^2}{\sigma_2^2} < 3. \end{cases}$$

Based on Subsection 3.2.2, the α-cuts of $\widetilde{S_1^2}$ and $\widetilde{S_2^2}$ are obtained as

$$\widetilde{S_1^2}[\alpha] = \left[\widetilde{S_1^2}^L, \widetilde{S_1^2}^U\right], \ \widetilde{S_2^2}[\alpha] = \left[\widetilde{S_2^2}^L, \widetilde{S_2^2}^U\right],$$

where

$$\widetilde{S_1^2}^L = \begin{cases} 0 & 0 < \alpha \leq 0.00374 \\ 11.6382 + 2.0820\ln(\alpha) & 0.00374 < \alpha \leq 0.7510, \\ 11.6382 - 2.0820\ln(\alpha) - 2.2280\sqrt{-\ln(\alpha)} & 0.7510 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S_1^2}^U = 11.6382 - 2.0820\ln(\alpha) + 2.2280\sqrt{-\ln(\alpha)},$$

$$\widetilde{S_2^2}^L = \begin{cases} 0 & 0 < \alpha \leq 0.03743 \\ 3.1086 + 0.9462\ln(\alpha) & 0.03743 < \alpha \leq 0.8977, \\ 3.1086 - 0.9462\ln(\alpha) - 0.6218\sqrt{-\ln(\alpha)} & 0.8977 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S_2^2}^U = 3.1086 - 0.9462\ln(\alpha) + 0.6218\sqrt{-\ln(\alpha)}.$$

Hence, the α-cuts of the fuzzy test statistic are obtained as follows (see Figure 2)

$$\tilde{Z}[\alpha] = \left[\frac{\theta_0 \widetilde{S_2^2}^L}{\widetilde{S_1^2}^U}, \frac{\theta_0 \widetilde{S_2^2}^U}{\widetilde{S_1^2}^L}\right] = \left[\tilde{Z}^L, \tilde{Z}^U\right],$$

where

$$\tilde{Z}^L = \begin{cases} 0 & 0 < \alpha \leq 0.03743 \\ \frac{9.3258+2.8386\ln(\alpha)}{11.6382-2.0820\ln(\alpha)+2.2280\sqrt{-\ln(\alpha)}} & 0.03743 < \alpha \leq 0.8977, \\ \frac{9.3258-2.8386\ln(\alpha)-1.8654\sqrt{-\ln(\alpha)}}{11.6382-2.0820\ln(\alpha)+2.2280\sqrt{-\ln(\alpha)}} & 0.8977 < \alpha \leq 1, \end{cases}$$

$$\tilde{Z}^U = \begin{cases} 1.6436 & 0 < \alpha \leq 0.00374 \\ \frac{9.3258-2.8386\ln(\alpha)+1.8654\sqrt{-\ln(\alpha)}}{11.6382-2.0820\ln(\alpha)-2.2280\sqrt{-\ln(\alpha)}} & 0.00374 < \alpha \leq 0.7510, \\ \frac{9.3258-2.8386\ln(\alpha)+1.8654\sqrt{-\ln(\alpha)}}{11.6382+2.0820\ln(\alpha)} & 0.7510 < \alpha \leq 1. \end{cases}$$

Using Maple software [29], $I(\widetilde{Z})$ and $D(\widetilde{Z})$ are calculated as

$$\begin{aligned} I(\widetilde{Z}) &= \frac{1}{2}\int_0^1 (Z_\alpha^U + Z_\alpha^L)d\alpha = 0.8084, \\ D(\widetilde{Z}) &= \int_0^1 (Z_\alpha^U - Z_\alpha^L)\alpha \ d\alpha = 0.2333. \end{aligned}$$

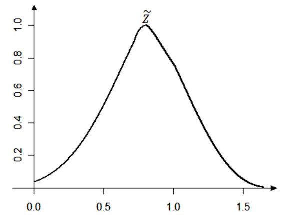

FIGURE 2. Fuzzy Test Statistic in Example 3.11

Since $I(\widetilde{Z}) = 0.8084 > F_{15,24,0.05} = 0.4371$, we obtain

$$\begin{aligned} T(\widetilde{Z}, F_{15,24,0.05}) &= I(\widetilde{Z}) + |D(\widetilde{Z})| - F_{15,24,0.05} \\ &= 0.8085 + 0.2333 - 0.4371 \\ &= 0.6047. \end{aligned}$$

Therefore, $T(\widetilde{Z}, F_{15,24,0.05}) = 0.6047 > 0$, the null hypothesis H_0 is accepted, at the significance level $\delta = 0.05$.

4. Testing Statistical Hypotheses Based on Fuzzy Data in Models with Nuisance Parameter

In statistical inference, we sometimes encounter with some parameters of statistical model which is not of immediate interest but may be accounted for analysis of other parameters which are of interest. The parameters of interest are called the "original parameters" and the parameters which is not of interest, are called the "nuisance parameters". For example, the variance σ^2 of a normal distribution $N(\mu, \sigma^2)$ is a nuisance parameter, when the mean μ is the original parameter of interest.

In this section, we test hypotheses of interest when some nuisance parameters are given in statistical models.

4.1. **Statement of the Method.** Suppose that we have taken a random sample from a population with probability density (mass) function $f(x; \theta, \nu)$, where θ is an unknown original parameter and ν is an unknown nuisance parameter, and we have observed the fuzzy data $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$ rather than the crisp data $x_1, x_2, ..., x_n$. Based on Arefi and Taheri's method [4], the fuzzy point estimations for the parameters of model under fuzzy data are obtained as follows:

Definition 4.1. [4] Let $\theta^* = u_1(x_1, x_2, ..., x_n)$ and $\nu^* = u_2(x_1, x_2, ..., x_n)$ be the point estimations for θ and ν, respectively. By substituting the α-cuts of the fuzzy numbers $\widetilde{X}_i,\ i = 1, ..., n$ for $x_i,\ i = 1, ..., n$ into θ^* and ν^*, the α-cuts of the fuzzy point estimations $\widetilde{\theta}^*$ and $\widetilde{\nu}^*$ are obtained as follows

$$\begin{aligned} \widetilde{\theta}^*[\alpha] &:= \left\{ u_1(x_1, x_2, ..., x_n);\ x_i \in \widetilde{X}_i[\alpha],\ i = 1, 2, ..., n \right\}, \\ \widetilde{\nu}^*[\alpha] &:= \left\{ u_2(x_1, x_2, ..., x_n);\ x_i \in \widetilde{X}_i[\alpha],\ i = 1, 2, ..., n \right\}. \end{aligned}$$

Now, consider the null hypothesis $H_0 : \theta = \theta_0$, where, θ_0 is a known constant. By substituting the α-cuts of the fuzzy point estimations $\widetilde{\theta}^*$ and $\widetilde{\nu}^*$ instead of the point estimations θ^* and ν^* in the crisp test statistic ($Q_0 = Q_0(\theta^*, \nu^*; \theta_0)$) and by using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$ as follows

$$\widetilde{Z}[\alpha] := \left\{ Q_0(\theta^*, \nu^*; \theta_0);\ \theta^* \in \widetilde{\theta}^*[\alpha],\ \nu^* \in \widetilde{\nu}^*[\alpha] \right\}.$$

Based on fuzzy test statistic $\widetilde{Z}$ and the signed distance, the **new decision rule** for testing simple hypothesis $H_0 : \theta = \theta_0$ against the hypothesis H_1 is introduced as follows:

i): for $H_1 : \theta \neq \theta_0$, we reject H_0, if $T(\widetilde{Z}, Q_{\delta/2}) \leq 0$ or $T(\widetilde{Z}, Q_{1-\delta/2}) \geq 0$,
ii): for $H_1 : \theta > \theta_0$, we reject H_0, if $T(\widetilde{Z}, Q_{1-\delta}) \geq 0$,
iii): for $H_1 : \theta < \theta_0$, we reject H_0, if $T(\widetilde{Z}, Q_{\delta}) \leq 0$,

where, Q_δ is the δ-quantile of the distribution of crisp test statistic.

4.2. **Testing Hypothesis for Mean of a Normal Distribution with Unknown Variance.** Suppose that, based on a random sample of size n from $N(\mu, \sigma^2)$ (σ^2 unknown), we have observed the fuzzy numbers $\widetilde{X}_1, ..., \widetilde{X}_n$. The α-cuts of the fuzzy point estimations for the original parameter μ and the nuisance parameter σ^2 are obtained in Subsections 3.2.1 and 3.2.2 as $\widetilde{\overline{X}}[\alpha] = [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U]$ and $\widetilde{S^2}[\alpha] = \left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha]\right]$, respectively. Now, we test the following hypotheses, at the significance level δ

$$\begin{cases} H_0 : \mu = \mu_0, \\ H_1 : \mu \neq \mu_0. \end{cases}$$

Under H_0, the crisp test statistic is distributed as $\frac{\overline{X}-\mu_0}{S/\sqrt{n}} \sim t_{n-1}$. By substituting the bounds of $\widetilde{\overline{X}}[\alpha]$ and $\widetilde{S^2}[\alpha]$ instead of $\widehat{\mu} = \overline{x}$ and $\widehat{\sigma^2} = s^2$ in $t_0 = \frac{\overline{x}-\mu_0}{s/\sqrt{n}}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as

$$\tilde{Z}[\alpha] = \frac{\widetilde{\overline{X}}[\alpha] - \mu_0}{\sqrt{\widetilde{S^2}[\alpha]/n}} = \left[\frac{\widetilde{\overline{X}}^L - \mu_0}{\sqrt{\widetilde{S^2}^U[\alpha]/n}}, \frac{\widetilde{\overline{X}}^U - \mu_0}{\sqrt{\widetilde{S^2}^L[\alpha]/n}} \right].$$

Hence, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, z_{\delta/2}) \leq 0$ or $T(\widetilde{Z}, z_{1-\delta/2}) \geq 0$.

Example 4.2. Consider the fuzzy data from a normal population $N(\mu, \sigma^2)$ in Example 3.9. Assume that we want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \mu = 2.5, \\ H_1 : \mu \neq 2.5. \end{cases}$$

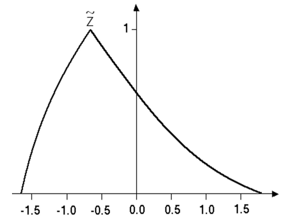

FIGURE 3. Fuzzy Test Statistic in Example 4.2

The α-cuts of the fuzzy point estimations for μ and σ^2 are obtained in Example 3.9 as

$$\begin{aligned}\widetilde{\widetilde{X}}[\alpha] &= [\widetilde{\widetilde{X}}^L, \widetilde{\widetilde{X}}^U] = [1.8570 + 0.4645\alpha, 2.786 - 0.4645\alpha],\\ \widetilde{\widetilde{S^2}}[\alpha] &= \left[\widetilde{\widetilde{S^2}}^L[\alpha], \widetilde{\widetilde{S^2}}^U[\alpha]\right],\end{aligned}$$

where

$$\begin{aligned}\widetilde{\widetilde{S^2}}^L[\alpha] &= \begin{cases} -0.9677\alpha^2 + 1.9354\alpha + 0.5108 & 0 \le \alpha \le 0.6942, \\ 0.9677\alpha^2 - 1.3435\alpha + 1.8543 & 0.6942 < \alpha \le 1, \end{cases}\\ \widetilde{\widetilde{S^2}}^U[\alpha] &= 0.9677\alpha^2 - 2.5273\alpha + 3.0381.\end{aligned}$$

Under the null hypothesis H_0, the α-cuts of fuzzy test statistic (Figure 3), which shows a fuzzy number "approximately -0.6566", are obtained as $\widetilde{Z}[\alpha] = \left[\widetilde{Z}^L, \widetilde{Z}^U\right]$, where

$$\begin{aligned}\widetilde{Z}^L &= \frac{-0.6430 + 0.4645\alpha}{\sqrt{0.0739\alpha^2 + 0.1774\alpha(1-\alpha) + 0.1519(1-\alpha)^2}},\\ \widetilde{Z}^U &= \begin{cases} \frac{0.286 - 0.4645\alpha}{\sqrt{0.0739\alpha^2 + 0.1478\alpha(1-\alpha) + 0.0255(1-\alpha)^2}} & 0 \le \alpha \le 0.6942, \\ \frac{0.286 - 0.4645\alpha}{\sqrt{0.0739\alpha^2 + 0.1183\alpha(1-\alpha) + 0.0927(1-\alpha)^2}} & 0.6942 < \alpha \le 1. \end{cases}\end{aligned}$$

Using Maple software [29], $I(\widetilde{Z})$ and $D(\widetilde{Z})$ are calculated as

$$\begin{aligned}I(\widetilde{Z}) &= \frac{1}{2}\int_0^1 (Z_\alpha^U + Z_\alpha^L) d\alpha = -0.4680,\\ D(\widetilde{Z}) &= \int_0^1 (Z_\alpha^U - Z_\alpha^L)\alpha \, d\alpha = 0.5114.\end{aligned}$$

Since $I(\widetilde{Z}) = -0.4680 > t_{19,0.025} = -2.0930$ and $I(\widetilde{Z}) = -0.4680 < t_{19,0.975} = 2.0930$, we obtain

$$\begin{aligned}T(\widetilde{Z}, t_{19,0.025}) &= I(\widetilde{Z}) + |D(\widetilde{Z})| - t_{19,0.025} = -0.4680 + 0.5114 - (-2.0930) = 2.1364,\\ T(\widetilde{Z}, t_{19,0.975}) &= I(\widetilde{Z}) - |D(\widetilde{Z})| - t_{19,0.975} = -0.4680 - 0.5114 - 2.0930 = -3.0724,\end{aligned}$$

Therefore, $T(\widetilde{Z}, t_{19,0.025}) > 0$ and $T(\widetilde{Z}, t_{19,0.975}) < 0$. Based on the decision rule, the null hypothesis H_0 is accepted, at the significance level $\delta = 0.05$.

4.3. **Testing Hypothesis for Difference Between Means of Two Normal Distributions with Equal Unknown Variances.** Assume that we have taken two independent random samples of sizes n and m from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ (where μ_1, μ_2 and σ^2 are unknown), and observed the fuzzy data $\widetilde{X}_1, ..., \widetilde{X}_n$ and $\widetilde{Y}_1, ..., \widetilde{Y}_m$, respectively. Here, $\theta = \mu_1 - \mu_2$ is the parameter of interest and $\nu = \sigma^2$ is the nuisance parameter. The usual point estimations for θ and ν are $\widehat{\theta} = \overline{x} - \overline{y}$ and $\widehat{\nu} = s_p^2 = \frac{(n-1)s_x^2+(m-1)s_y^2}{n+m-2} = \frac{\sum_{i=1}^n (x_i-\overline{x})^2 + \sum_{j=1}^m (y_i - \overline{y})^2}{n+m-2}$, respectively. Under the null hypothesis $H_0 : \mu_1 - \mu_2 = \theta_0$, the crisp test statistic is distributed as $Q_0 = \frac{(\overline{X}-\overline{Y})-\theta_0}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim t_{n+m-2}$. Based on Subsections 3.2.1 and 3.2.2, the α-cuts of the fuzzy point estimations for the parameters θ and ν are obtained as

$$\widehat{\widetilde{\theta}}_1[\alpha] = [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U] - [\widetilde{\overline{Y}}^L, \widetilde{\overline{Y}}^U] = [\widetilde{\overline{X}}^L - \widetilde{\overline{Y}}^U, \widetilde{\overline{X}}^U - \widetilde{\overline{Y}}^L] = [\widehat{\widetilde{\theta}}_1^L, \widehat{\widetilde{\theta}}_1^U],$$

and

$$\begin{aligned}
\widetilde{S_p^2}[\alpha] &= \frac{(n-1)\widetilde{S_x^2}[\alpha]+(m-1)\widetilde{S_y^2}[\alpha]}{n+m-2} \\
&= \frac{(n-1)\left[\widetilde{S_x^2}^L, \widetilde{S_x^2}^U\right]+(m-1)\left[\widetilde{S_y^2}^L, \widetilde{S_y^2}^U\right]}{n+m-2} \\
&= \left[\frac{(n-1)\widetilde{S_x^2}^L+(m-1)\widetilde{S_y^2}^L}{n+m-2}, \frac{(n-1)\widetilde{S_x^2}^U+(m-1)\widetilde{S_y^2}^U}{n+m-2}\right] \\
&= \left[\widetilde{S_p^2}^L, \widetilde{S_p^2}^U\right].
\end{aligned}$$

Hence, using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as

$$\tilde{Z}[\alpha] = \frac{\widehat{\widetilde{\theta}}_1[\alpha] - \theta_0}{\sqrt{\widetilde{S_p^2}[\alpha](\frac{1}{n}+\frac{1}{m})}} = \left[\frac{\widehat{\widetilde{\theta}}_1^L - \theta_0}{\sqrt{\widetilde{S_p^2}^U(\frac{1}{n}+\frac{1}{m})}}, \frac{\widehat{\widetilde{\theta}}_1^U - \theta_0}{\sqrt{\widetilde{S_p^2}^L(\frac{1}{n}+\frac{1}{m})}}\right].$$

Based on fuzzy test statistic $\widetilde{Z}$ and the signed distance, the decision rule for testing simple hypothesis $H_0 : \mu_1 - \mu_2 = \theta_0$ against the hypothesis H_1 is given as follows:

- **i):** for $H_1 : \mu_1 - \mu_2 \neq \theta_0$, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, t_{n+m-2,\delta/2}) \leq 0$ or $T(\widetilde{Z}, t_{n+m-2,1-\delta/2}) \geq 0$,
- **ii):** for $H_1 : \mu_1 - \mu_2 > \theta_0$, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, t_{n+m-2,1-\delta}) \geq 0$,
- **iii):** for $H_1 : \mu_1 - \mu_2 < \theta_0$, the hypothesis H_0 is rejected, if $T(\widetilde{Z}, t_{n+m-2,\delta}) \leq 0$.

Example 4.3. Suppose that we have taken two independent random samples of sizes $n = 20$ and $m = 30$ from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively, and observed the symmetric triangular fuzzy data given in Table 5.

Based on these data, we have

$$\begin{aligned}
\widetilde{\overline{X}}[\alpha] &= [9.477 + 0.498\alpha, 10.473 - 0.498\alpha], \\
\widetilde{\overline{Y}}[\alpha] &= [11.286 + 0.594\alpha, 12.474 - 0.594\alpha], \\
\widetilde{S_x^2}[\alpha] &= \left[\widetilde{S_x^2}^L, \widetilde{S_x^2}^U\right], \\
\widetilde{S_y^2}[\alpha] &= \left[\widetilde{S_y^2}^L, \widetilde{S_y^2}^U\right],
\end{aligned}$$

$\widetilde{X}_i = (x_i, s_i)_T$	$\widetilde{X}_i = (x_i, s_i)_T$	$\widetilde{Y}_j = (y_j, r_j)_T$	$\widetilde{Y}_j = (y_j, r_j)_T$	$\widetilde{Y}_j = (y_j, r_j)_T$
$(9.51; 0.48)_T$	$(10.37; 0.52)_T$	$(12.01; 0.60)_T$	$(11.29; 0.56)_T$	$(12.96; 0.65)_T$
$(8.27; 0.41)_T$	$(9.07; 0.45)_T$	$(13.29; 0.66)_T$	$(9.97; 0.50)_T$	$(13.20; 0.66)_T$
$(8.62; 0.43)_T$	$(10.37; 0.52)_T$	$(14.06; 0.70)_T$	$(10.83; 0.54)_T$	$(10.24; 0.51)_T$
$(11.54; 0.58)_T$	$(6.92; 0.35)_T$	$(10.48; 0.52)_T$	$(12.22; 0.61)_T$	$(12.12; 0.61)_T$
$(9.69; 0.48)_T$	$(10.09; 0.50)_T$	$(11.65; 0.58)_T$	$(11.38; 0.57)_T$	$(12.31; 0.62)_T$
$(10.69; 0.53)_T$	$(11.87; 0.59)_T$	$(10.08; 0.50)_T$	$(12.51; 0.63)_T$	$(11.04; 0.55)_T$
$(12.06; 0.60)_T$	$(9.36; 0.47)_T$	$(11.71; 0.59)_T$	$(11.54; 0.58)_T$	$(13.09; 0.65)_T$
$(11.12; 0.56)_T$	$(9.05; 0.45)_T$	$(7.94; 0.40)_T$	$(13.83; 0.69)_T$	$(11.11; 0.56)_T$
$(13.18; 0.66)_T$	$(6.95; 0.35)_T$	$(12.88; 0.64)_T$	$(12.20; 0.61)_T$	$(13.04; 0.65)_T$
$(8.49; 0.42)_T$	$(12.28; 0.61)_T$	$(12.22; 0.61)_T$	$(13.31; 0.67)_T$	$(11.89; 0.60)_T$

TABLE 5. Fuzzy Data from Normal Populations in Example 4.3

where

$$\widetilde{S_x^2}^L = \begin{cases} 2.9422 - 1.0515(1-\alpha)^2 & 0 < \alpha \leq 0.8608, \\ 2.9422 - 0.2926(1-\alpha) + 1.0515(1-\alpha)^2 & 0.8608 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S_x^2}^U = 2.9422 + 0.2926(1-\alpha) + 1.0515(1-\alpha)^2,$$

$$\widetilde{S_y^2}^L = \begin{cases} 1.7432 - 1.4643(1-\alpha)^2 & 0 < \alpha \leq 0.9408, \\ 1.7432 - 0.1735(1-\alpha) + 1.4643(1-\alpha)^2 & 0.9408 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S_y^2}^U = 1.7432 + 0.1735(1-\alpha) + 1.4643(1-\alpha)^2.$$

Hence, the fuzzy point estimations for $\theta = \mu_1 - \mu_2$ and σ^2 are obtained as follows

$$\widetilde{\widehat{\theta}}_1[\alpha] = \widetilde{\overline{X}}[\alpha] - \widetilde{\overline{Y}}[\alpha] = [-2.9970 + 1.0920\alpha, -0.8130 - 1.0920\alpha],$$

and

$$\widetilde{S_p^2}[\alpha] = \left[\widetilde{S_p^2}^L, \widetilde{S_p^2}^U\right],$$

where

$$\widetilde{S_p^2}^L = \frac{(n-1)\widetilde{S_x^2}^L + (m-1)\widetilde{S_y^2}^L}{n+m-2} = \begin{cases} 2.2178 - 1.3009(1-\alpha)^2 & 0 < \alpha \leq 0.8608, \\ 2.2178 - 0.1158(1-\alpha) - 0.4685(1-\alpha)^2 & 0.8608 < \alpha \leq 0.9408, \\ 2.2178 - 0.2206(1-\alpha) + 1.3009(1-\alpha)^2 & 0.9408 < \alpha \leq 1, \end{cases}$$

$$\widetilde{S_p^2}^U = \frac{(n-1)\widetilde{S_x^2}^U + (m-1)\widetilde{S_y^2}^U}{n+m-2} = 2.2178 + 0.2206(1-\alpha) + 1.3009(1-\alpha)^2.$$

Now, we want to test the following hypotheses, at the significance level $\delta = 0.05$

$$\begin{cases} H_0 : \mu_1 - \mu_2 = -2, \\ H_1 : \mu_1 - \mu_2 > -2. \end{cases}$$

The α-cuts of fuzzy test statistic are obtained as follows (see Figure 4)

$$\tilde{Z}[\alpha] = \left[\frac{\widetilde{\widehat{\theta}_1}^L - \theta_0}{\sqrt{\widetilde{S_p^2}^U\left(\frac{1}{n} + \frac{1}{m}\right)}}, \frac{\widetilde{\widehat{\theta}_1}^U - \theta_0}{\sqrt{\widetilde{S_p^2}^L\left(\frac{1}{n} + \frac{1}{m}\right)}}\right] = [\tilde{Z}^L, \tilde{Z}^U],$$

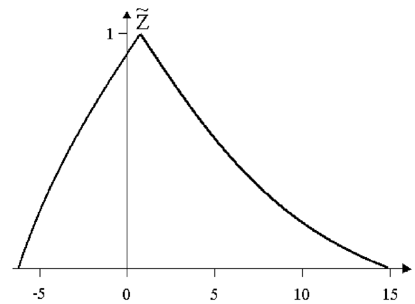

FIGURE 4. Fuzzy Test Statistic in Example 4.3

where

$$\tilde{Z}^L = \frac{-0.9970 + 1.0920\alpha}{\sqrt{0.1848 + 0.0184(1-\alpha) + 0.1084(1-\alpha)^2}},$$

and

$$\tilde{Z}^U = \begin{cases} \frac{1.1870-1.0920\alpha}{\sqrt{0.1848-0.1084(1-\alpha)^2}} & 0 < \alpha \leq 0.8608, \\ \frac{1.1870-1.0920\alpha}{\sqrt{0.1848-0.0097(1-\alpha)-0.0390(1-\alpha)^2}} & 0.8608 < \alpha \leq 0.9408, \\ \frac{1.1870-1.0920\alpha}{\sqrt{0.1848-0.0184(1-\alpha)+0.1084(1-\alpha)^2}} & 0.9408 < \alpha \leq 1. \end{cases}$$

Using Maple software [29], $I(\widetilde{Z})$ and $D(\widetilde{Z})$ are calculated as

$$I(\widetilde{Z}) = \frac{1}{2}\int_0^1 (Z_\alpha^U + Z_\alpha^L) d\alpha = 0.4500,$$

$$D(\widetilde{Z}) = \int_0^1 (Z_\alpha^U - Z_\alpha^L)\alpha \, d\alpha = 0.8694.$$

Since $I(\widetilde{Z}) = 0.4500 < t_{48,0.95} = 1.6772$, we obtain

$$\begin{aligned} T(\widetilde{Z}, t_{48,0.95}) &= I(\widetilde{Z}) - |D(\widetilde{Z})| - t_{48,0.95} \\ &= 0.4500 - 0.8694 - 1.6772 \\ &= -2.0966. \end{aligned}$$

Therefore, $T(\widetilde{Z}, t_{19,0.025}) < 0$ and based on the decision rule, the null hypothesis H_0 is accepted, at the significance level $\delta = 0.05$.

5. Conclusion

An approach to the problem of testing statistical hypotheses when the available data are fuzzy, is presented. In the proposed approach, we have provided a new decision rule for testing statistical hypotheses based on a new signed distance. The advantage of proposed approach is that it is a natural analogue of the usual approach to the problem of testing statistical hypotheses, since the decision rule is essentially based on comparing between the so called fuzzy test statistic and the quantile of interest of distribution.

The p-value approach for testing hypotheses in fuzzy environment based on the proposed new signed distance can investigate in the future research.

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