ON CONTROLLABILITY AND OBSERVABILITY OF FUZZY CONTROL SYSTEMS

R. MASTIANI AND S. EFFATI

Abstract. In order to more effectively cope with the real world problems of vagueness, imprecise and subjectivity, fuzzy event systems were proposed recently. In this paper, we investigate the controllability and the observability property of two systems that one of them has fuzzy variables and the other one has fuzzy coefficients and fuzzy variables (fully fuzzy system). Also, sufficient conditions for the controllability and the observability of such systems are established. Some examples are given to substantiate the results obtained.

1. Introduction

Mathematical models describing real systems will usually require knowing exactly parameter model values. However, in practice, exact values are not available and models usually exhibit a certain degree of uncertainty. These lead to use models with uncertain parameters and sometimes also uncertain initial conditions. In order to consider the whole spectrum of possible results, such uncertainty should be taken into account. One possible way to handle this uncertainty in parameters and initial conditions is to use fuzzy numbers and fuzzy arithmetic operations for the model simulation. Thus, for a long time, there remained a need for effective means to describe and quantify information in terms of its associated vagueness and imprecision. This need had been answered by the emergence of fuzzy theory in 1965.

The controllability and the observability of any system (linear and non-linear) are very important concepts in the control design process. The controllability conditions ensure that the control action exists to drive the states from the initial condition to a target point. Hence, both metrics provide a tool to determine if the system under investigation, can achieve the control design requirements. Parallel to controllability, observability of systems is also a metric in the design of observers. Thus, paralleling a similar approach presented herein, observability conditions can be also easily obtained. While the controllability of linear systems is easy to determine, unfortunately a similar simple criterion, in general, for non-linear controllers does not exist [17].

The controllability and the observability in the fuzzy sense, as the foundation of fuzzy dynamical system theory, are one of the most important issues that need to be explored in fuzzy dynamical system theory [14]. In the classical control theory,

Received: April 2016; Revised: June 2017; Accepted: February 2018

Key words and phrases: Fuzzy dynamical systems, Controllability, Observability, Fuzzy number, Fuzzy state.

the controllability is the characteristic of the system to transfer a crisp initial state to any desired crisp state in a finite time interval by applying an appropriate control input. The observability is the characteristic of the system to estimate the crisp initial state according to the knowledge of the input and the output in a finite time interval. However, when the coefficients and variables of fuzzy dynamical system are fuzzy, not deterministic, the concepts of the controllability and the observability in the classical control theory cannot be applied for the fuzzy dynamical system. As a result, it is considered important to research the controllability and the observability of fuzzy dynamical system. The controllability of fuzzy systems has been explored by many authors, such as, Cai and Tang [3], Ding and Kandel [4, 5], Farinwata et al. [7], Feng et el. [8], and Gupta et al. [11]. Recently, Biglarbegian et al. [2] have studied the accessability and the controllability properties of T-S fuzzy logic control systems by using differential geometric and Lie-algebraic techniques. The literature [6] investigated the observability of the fuzzy dynamical system with the fuzzy initial state, and pointed out that the observability was the characteristic of the system to estimate the range of the fuzzy initial state according to the knowledge of the fuzzy input and the fuzzy output in a finite time interval. Also, Gabr [10] presented a new approach for the modelling, analysis, and design of automatic control systems in fully fuzzy environment based on the normalized fuzzy matrices. The approach was also suitable for determining the propagation of fuzziness in automatic control and dynamical systems where all system coefficients are expressed as fuzzy parameters.

In this paper, we consider linear time-invariant systems with fuzzy condition and establish results on the controllability and the observability properties of the system. In fact, we study the controllability and the observability of two different systems. That one of them has fuzzy variables and another one has fuzzy coefficients and fuzzy variables (fully fuzzy system). Furthermore, we give some sufficient conditions for the controllability and the observability about such systems.

The organization of the paper is as follows. In Section 2, we state some preliminaries from fuzzy set theory and control systems. Section 3, introduce the fuzzy linear system (FLS). We describe the evolution of the solutions of systems with fuzzy variables in Section 4. In fact, in this section the controllability and the observability of systems with fuzzy variables (FLCS) are discussed. In Section 5, we investigate the controllability and the observability system with fuzzy coefficients and variables. Section 6 gives an interesting application of FLCS (i. e., the fuzzy inverted pendulum). Some examples are given to demonstrate the results obtained in Section 7. Finally, we conclude the paper in Section 8.

2. **Preliminaries**

In this section, some basic definitions of fuzzy numbers are given. Let $\mathbb{R}$ be the set of all real numbers. $\mathbb{R}^+$ denotes the set of all non-negative real numbers. $\mathbb{N}$ is the set of all natural numbers. For $n \in \mathbb{N}$, E^n is the set of all n-dimensional vectors of fuzzy numbers on $\mathbb{R}$. $F(X)$ denotes the set of all fuzzy sets defined in a set X.

Definition 2.1. [1, 18, 13] If X is a collection of objects denoted by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A(x)$ is called the membership function or grade of membership of x in A. The range of membership function is a subset of non-negative real numbers whose supremum is finite.

Definition 2.2. [1, 18, 13] By a fuzzy number on $\mathbb{R}$, we mean a mapping $\mu : \mathbb{R} \longrightarrow [0, 1]$ with the following properties:

(1) μ is fuzzy convex, that is, $\mu(\alpha x+(1-\alpha)y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$.
(2) Closure of the support of μ is compact, that is, $cl(x \in \mathbb{R} : \mu(x) > 0)$ is compact in $\mathbb{R}$.

Definition 2.3. [1, 18, 13] Let A be a fuzzy number or a fuzzy set defined on the universe X which, in general, could be a subset of $\mathbb{R}^n$. The α-cut or α-level set of A is denoted by A^α or $[A]_\alpha$ and is defined as $A^\alpha = \{x \in X, \mu(x) \geq \alpha\}$ for $\alpha \in (0, 1]$. For $\alpha = 0$, the α-cut of A is defined as the closure of union of all non-zero α-cut of A. That is, $A^0 = \overline{\bigcup_{\alpha \in (0,1]} A^\alpha}$.

It is well known that for every $A \in E^1$, the α-level sets of A are closed and bounded intervals defined by $[A]_\alpha = [\underline{A^\alpha}, \overline{A^\alpha}]$, where $\underline{A^\alpha}$ and $\overline{A^\alpha}$ are called the lower α-cut and the upper α-cut of A, respectively. Every fuzzy set can be uniquely represented in terms of its α-cut. The following decomposition theorem of fuzzy sets depicts this fact.

Definition 2.4. [1, 18] A fuzzy number $\tilde{A}$ is LR-type if there exit L (for left), R (for right), and scalars $\alpha, \beta > 0$ with the following representations:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), & x \leq a \\ R(\frac{x-a}{\beta}), & x \geq a, \end{cases}$$

where L and R are strictly decreasing functions defined on $[0, 1]$ and satisfy the following conditions:

$$\begin{aligned} &L(0) = R(0) = 1, \\ &L(1) = R(1) = 0, \\ &0 < L(x) < 1, \quad 0 < R(x) < 1, \quad x \neq 0. \end{aligned}$$

The mean value of $\tilde{A}$, (m) is a real number, and α, β are called the left and the right spreads, respectively. $\tilde{A}$ is denoted by $(\alpha, m, \beta)_{LR}$.

Definition 2.5. [1, 18, 13] A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix if for all i and j, $\tilde{a}_{ij} \in F(\mathbb{R})$. $\tilde{A}$ will be positive (negative) and denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$), if for all i and j, $\tilde{a}_{ij} > 0$ ($\tilde{a}_{ij} < 0$). Clearly, $\tilde{N} = (a, b, c)$ is positive (negative), if and only if, $a > 0$ ($c < 0$). Non-negative and non-positive fuzzy matrices will be defined similarly.

Next theorem describes the fuzzy arithmetic.

Theorem 2.6. [1, 12, 18] *Let $\tilde{M} = (\alpha, m, \beta)$ and $\tilde{N} = (\gamma, n, \delta)$ are two arbitrary triangular fuzzy numbers and $\lambda > 0$ is a real number. Then,*

(1) $\tilde{M} \oplus \tilde{N} = (\alpha + \gamma, m + n, \beta + \delta)$,

(2) $-\tilde{M} = (\beta, -m, \alpha)$,
(3) $\tilde{M} \ominus \tilde{N} = (\alpha + \delta, m - n, \beta + \gamma)$,
(4) *Let* $\tilde{M} = (a, b, c)$ *be any triangular fuzzy number and* $\tilde{N} = (x, y, z)$ *be a non-negative triangular fuzzy number, then*

$$\tilde{M} \otimes \tilde{N} = \begin{cases} (ax, by, cz), & a \geq 0 \\ (az, by, cz), & a < 0, \quad c \geq 0, \\ (az, by, cx), & c < 0. \end{cases}$$

The rest of this section gives some preliminaries from control system. Consider the following linear system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \tag{1}$$

where $A = (a_{ij})$ is an $n \times n$ real matrix, $B = (b_{ij})$ is an $n \times m$ real matrix, and $C = (c_{ij})$ is a $r \times n$ real matrix.

Definition 2.7 (Hukuhara difference [16, 13])**.** Given $\tilde{u}, \tilde{v} \in E^1$, the H-difference is defined by:

$$\tilde{u} \ominus_H \tilde{v} = \tilde{w} \Longleftrightarrow \tilde{u} = \tilde{v} + \tilde{w}.$$

If $\tilde{u} \ominus_H \tilde{v}$ exists, it is unique and its α-cuts are $(\tilde{u} \ominus_H \tilde{v})[\alpha] = [\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)]$, $\alpha \in [0, 1]$. Clearly, $\tilde{u} \ominus_H \tilde{u} = 0$. Also, when $\tilde{u}$ and $\tilde{v}$ are fuzzy vectors, this conclusion is true if it is satisfied for the elements of $\tilde{u}$ and $\tilde{v}$.

Remark 2.8. There exists an $n \times n$ non-singular matrix P such that,

$$P^{-1} AP = \Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\},$$

where λ_i for $i = 1, 2, \ldots, n$ is an eigenvalue of A. In fact, Λ is a diagonal matrix.

Now, substituting $x = Pz$ in (1), yields:

$$\begin{cases} \dot{z} = \Lambda z + \beta u \\ y = \gamma z, \end{cases} \tag{2}$$

where $\beta = P^{-1} B$ and $\gamma = CP$. The system (2) is controllable if β has no zero row. Also, the system (2) is observable if γ has no zero column.

3. **Fuzzy Linear System**

In this section, the fuzzy linear system (FLS) is stated. In fact, some new results about the FLS are given.

Consider the following $n \times n$ linear system of equations:

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n = \tilde{y}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n = \tilde{y}_2, \\ \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n = \tilde{y}_n. \end{cases}$$

The above system can be written as the matrix form as follow:

$$A\tilde{x} = \tilde{y}, \tag{3}$$

where the coefficient matrix $A = (a_{ij})$ is a crisp $n \times n$ matrix and $\tilde{x}_i, \tilde{y}_i \in E^1$ for $1 \le i \le n$. This system is called fuzzy linear system (FLS).

Definition 3.1. [9] A fuzzy number vector $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)^T$ given by

$$\tilde{x}_j = (\underline{x}_j(\alpha), \overline{x}_j(\alpha)), \quad j = 1, 2, \ldots, n, \quad 0 \le \alpha \le 1,$$

is called the solution of the FLS (3) if,

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j(\alpha)} = \sum_{j=1}^{n} \underline{a_{ij} x_j(\alpha)} = \underline{y}_i(\alpha), \quad i = 1, 2, \cdots, n, \quad 0 \le \alpha \le 1, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j(\alpha)} = \sum_{j=1}^{n} \overline{a_{ij} x_j(\alpha)} = \overline{y}_i(\alpha), \quad i = 1, 2, \cdots, n, \quad 0 \le \alpha \le 1. \end{cases}$$

If for a particular index i, $a_{ij} > 0$ for all $j = 1, 2, \ldots, n$, then we simply get:

$$\begin{cases} \sum_{j=1}^{n} a_{ij} \underline{x_j(\alpha)} = \underline{y}_i(\alpha), \quad i = 1, 2, \cdots, n \\ \sum_{j=1}^{n} a_{ij} \overline{x_j(\alpha)} = \overline{y}_i(\alpha), \quad i = 1, 2, \cdots, n. \end{cases}$$

In order to solve the given system $A\tilde{x} = \tilde{y}$, one must solve a crisp $2n \times 2n$ linear system where the right hand side column is the following vector function:

$$(\underline{y}_1(\alpha), \underline{y}_2(\alpha), \cdots, \underline{y}_n(\alpha), -\overline{y}_1(\alpha), -\overline{y}_2(\alpha), \cdots, -\overline{y}_n(\alpha))^T.$$

In fact, the following $2n \times 2n$ linear system is derived:

$$\begin{cases} s_{11}\underline{x}_1(r) + \cdots + s_{1n}\underline{x}_n(r) + s_{1,n+1}(-\overline{x}_1(r)) + \cdots + s_{1,2n}(-\overline{x}_n(r)) = \underline{y}_1(r), \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ s_{n1}\underline{x}_1(r) + \cdots + s_{nn}\underline{x}_n(r) + s_{n,n+1}(-\overline{x}_1(r)) + \cdots + s_{n,2n}(-\overline{x}_n(r)) = \underline{y}_n(r), \\ s_{n+1,1}\underline{x}_1(r) + \cdots + s_{n+1,n}\underline{x}_n(r) + s_{n+1,n+1}(-\overline{x}_1(r)) + \cdots + s_{n+1,2n}(-\overline{x}_n(r)) = -\overline{y}_1(r), \\ \vdots \\ s_{2n,1}\underline{x}_1(r) + \cdots + s_{2n,n}\underline{x}_n(r) + s_{2n,n+1}(-\overline{x}_1(r)) + \cdots + s_{2n,2n}(-\overline{x}_n(r)) = -\overline{y}_n(r), \end{cases} \tag{4}$$

in which,

$$\begin{aligned} a_{ij} \ge 0 &\Longrightarrow s_{ij} = s_{i+n,j+n} = a_{ij}, \qquad & s_{i+n,j} = s_{i,j+n} = 0, \\ a_{ij} < 0 &\Longrightarrow s_{i+n,j} = s_{i,j+n} = -a_{ij}, \qquad & s_{ij} = s_{i+n,j+n} = 0. \end{aligned}$$

Applying the matrix form, equations (4) can be expressed as follows:

$$SX = Y,$$

where,

$$\begin{aligned} X &= (\underline{x}_1(\alpha), \underline{x}_2(\alpha), \cdots, \underline{x}_n(\alpha), -\overline{x}_1(\alpha), -\overline{x}_2(\alpha), \cdots, -\overline{x}_n(\alpha))^T, \\ Y &= (\underline{y}_1(\alpha), \underline{y}_2(\alpha), \cdots, \underline{y}_n(\alpha), -\overline{y}_1(\alpha), -\overline{y}_2(\alpha), \cdots, -\overline{y}_n(\alpha))^T. \end{aligned}$$

The structure of S implies that, $S = (s_{ij}) \geq 0$ for $1 \leq i, j \leq 2n$, i. e,:

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}, \tag{5}$$

where S_1 contains the positive entries of A, and S_2 contains the absolute values of the negative entries of A. In fact, $A = S_1 - S_2$.

The following theorems are used in the rest of the paper.

Theorem 3.2. *The matrix S is non-singular, if and only if, the matrices $A = S_1 - S_2$ and $S_1 + S_2$ are non-singular.*

Proof. See Theorem 1 in [9]. □

Theorem 3.3. *If S^{-1} exists, then it must have the same structure as S, i.e.,*

$$S^{-1} = \begin{bmatrix} E & F \\ F & E \end{bmatrix},$$

where E contains the positive entries of S^{-1}, and F contains the absolute values of the negative entries of S^{-1} (i.e., $S^{-1} = E - F$). Also,

$$E = \frac{1}{2}[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}], \quad F = \frac{1}{2}[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}].$$

Moreover, if S is non-singular then, $X = S^{-1}Y$.

Proof. See Theorem 2 in [9]. □

4. Controllability and Observability of Systems with Fuzzy Variables

Here, the controllability and the observability of a system with fuzzy variables are investigated. In fact, we study the following fuzzy linear control system (FLCS):

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \\ \tilde{y} = C\tilde{x} \end{cases} \tag{6}$$

where $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ are real $n \times n$, $n \times m$, and $r \times n$ matrices, respectively. Assume that in (6) $\tilde{x} = P\tilde{z}$, where $P = (p_{ij})$ is an $n \times n$ real matrix. According to Definition 3.1, the system $\tilde{x} = P\tilde{z}$ can be written as equivalence form $X = SZ$ where,

$$X = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \dots & \underline{x}_n & -\overline{x}_1 & \dots & -\overline{x}_n \end{bmatrix}^T, \; Z = \begin{bmatrix} \underline{z}_1 & \underline{z}_2 & \dots & \underline{z}_n & -\overline{z}_1 & \dots & -\overline{z}_n \end{bmatrix}^T.$$

Also, the matrix S is similar to (5). Therefore, the FLCS (6) is reformulated as follows:

$$\begin{cases} P\dot{\tilde{z}} = AP\tilde{z} + B\tilde{u} \\ \tilde{y} = CP\tilde{z}, \end{cases}$$

By using the equivalence form $X = SZ$ we get,

$$\begin{cases} \dot{Z} = S'A'SZ + S'B'U \\ Y = C'SZ, \end{cases} \tag{7}$$

where matrices A', B', and C' and vectors $\dot{Z}$, U, and Y are as follow:

$$A' = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}, \qquad B' = \begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix}, \qquad C' = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix},$$
$$\dot{Z} = \begin{bmatrix} \dot{\underline{z}}_1 \\ \dot{\underline{z}}_2 \\ \vdots \\ \dot{\underline{z}}_n \\ -\dot{\overline{z}}_1 \\ \vdots \\ -\dot{\overline{z}}_n \end{bmatrix} = \begin{bmatrix} \dot{\underline{z}} \\ -\dot{\overline{z}} \end{bmatrix}, \qquad U = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_n \\ -\overline{u}_1 \\ \vdots \\ -\overline{u}_n \end{bmatrix} = \begin{bmatrix} \underline{u} \\ -\overline{u} \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix} = \begin{bmatrix} \underline{y} \\ -\overline{y} \end{bmatrix}. \tag{8}$$

Also, A_1, B_1, C_1, and A_2, B_2, C_2 contain the positive entries and the absolute values of the negative entries of A, B, C, respectively. Moreover, we know that:

$$A = A_1 - A_2, \quad B = B_1 - B_2, \quad C = C_1 - C_2.$$

Now, the system (7) can be stated in the following form:

$$\begin{cases} S\dot{Z} = A'SZ + B'U, \\ Y = C'SZ, \end{cases}$$

where, $S' = \begin{bmatrix} S_1' & S_2' \\ S_2' & S_1' \end{bmatrix}$, and S_1', S_2' contain the positive entries and the absolute values of the negative entries of P^{-1}, respectively. Also, $P^{-1} = S_1' - S_2'$.
The following theorem states the connection between the defined matrices.

Theorem 4.1. *Let* $S'A'S = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$, $S'B' = \begin{bmatrix} F & G \\ G & F \end{bmatrix}$, *and* $C'S = \begin{bmatrix} H & I \\ I & H \end{bmatrix}$. *Then* $D - E = P^{-1}AP$, $F - G = P^{-1}B$, *and* $H - I = CP$.

Proof. According to the definitions of matrices we have:

$$\begin{aligned} S'A'S &= \begin{bmatrix} S_1' & S_2' \\ S_2' & S_1' \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \\ &= \begin{bmatrix} S_1'A_1S_1 + S_2'A_2S_1 + S_1'A_2S_2 + S_2'A_1S_2 & S_1'A_1S_2 + S_2'A_2S_2 + S_1'A_2S_1 + S_2'A_1S_1 \\ S_1'A_1S_2 + S_2'A_2S_2 + S_1'A_2S_1 + S_2'A_1S_1 & S_1'A_1S_1 + S_2'A_2S_1 + S_1'A_2S_2 + S_2'A_1S_2 \end{bmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} &D - E \\ &= \underbrace{S_1'A_1S_1 + S_2'A_2S_1 + S_1'A_2S_2 + S_2'A_1S_2}_{D} - (\underbrace{S_1'A_1S_2 + S_2'A_2S_2 + S_1'A_2S_1 + S_2'A_1S_1}_{E}) \\ &= S_1'A_1S_1 + S_2'A_2S_1 + S_1'A_2S_2 + S_2'A_1S_2 - S_1'A_1S_2 - S_2'A_2S_2 - S_1'A_2S_1 - S_2'A_1S_1 \\ &= S_1'A_1(S_1 - S_2) + S_2'A_2(S_1 - S_2) - S_1'A_2(S_1 - S_2) - S_2'A_1(S_1 - S_2) \\ &= S_1'A_1P + S_2'A_2P - S_1'A_2P - S_2'A_1P \\ &= S_1'(A_1 - A_2)P - S_2'(A_1 - A_2)P \\ &= S_1'AP - S_2'AP \\ &= (S_1' - S_2')AP \\ &= P^{-1}AP. \end{aligned}$$

Moreover,

$$S'B' = \begin{bmatrix} S_1' & S_2' \\ S_2' & S_1' \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix} = \begin{bmatrix} S_1'B_1 + S_2'B_2 & S_1'B_2 + S_2'B_1 \\ S_1'B_2 + S_2'B_1 & S_1'B_1 + S_2'B_2 \end{bmatrix}.$$

Therefore, $F - G$ is constructed as below:

$$\begin{aligned} F - G &= \underbrace{S_1'B_1 + S_2'B_2}_{F} - (\underbrace{S_1'B_2 + S_2'B_1}_{G}) \\ &= S_1'(B_1 - B_2) - S_2'(B_1 - B_2) \\ &= (S_1' - S_2')B \\ &= P^{-1}B. \end{aligned}$$

Finally,

$$C'S = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} = \begin{bmatrix} C_1S_1 + C_2S_2 & C_1S_2 + C_2S_1 \\ C_1S_2 + C_2S_1 & C_1S_1 + C_2S_2 \end{bmatrix}$$

Thus, $H - I$ is become,

$$\begin{aligned} H - I &= \underbrace{C_1S_1 + C_2S_2}_{H} - (\underbrace{C_1S_2 + C_2S_1}_{I}) \\ &= C_1(S_1 - S_2) - C_2(S_1 - S_2) \\ &= (C_1 - C_2)S \\ &= CP. \end{aligned}$$

So, the results follow. □

Corollary 4.2. *Based on Remark 2.8 the system (7) is controllable if $F - G$ has no zero row. Also, the system (7) is observable if $H - I$ has no zero column.*

Theorem 4.3. *The sufficient condition for the controllability of the system (7) is,*

$$\operatorname{rank}(B', A'B', A'^2B', ..., A'^{n-1}B') = n.$$

Proof. According to Remark 2.8 the sufficient condition for the controllability of the system (7) is that the matrix

$$\left[S'B', S'A'SS'B', S'A'^2SS'B', ..., S'A'^{n-1}SS'B'\right]$$

be non-singular. Furthermore, we know that:

$$\left[S'B', S'A'SS'B', S'A'^2SS'B', ..., S'A'^{n-1}SS'B'\right] = S'[B', A'B', A'^2B', ..., A'^{n-1}B'].$$

Since S' is non-singular so $\operatorname{rank}(B', A'B', A'^2B', \dots, A'^{n-1}B') = n$. Therefore, the proof is completed. □

Theorem 4.4. *The sufficient condition for the observability of the system (7) is:*

$$\operatorname{rank}(\begin{bmatrix} C' & C'A' & C'A'^2 & \dots & C'A'^{n-1} \end{bmatrix}^T) = n.$$

Proof. The proof is similar to the former theorem therefore, we omit here. □

From the aforementioned discussion, the process and algorithm for deducing the controllability and the observability of systems with fuzzy variables are summarized as follows.

Algorithm 4.5. The algorithm for investigating the controllability and the observability of systems with fuzzy variables.

1: Identify the matrices A, B, and C.
2: Construct the matrices P and P^{-1} from matrix A.
3: According to Equations (8), matrices A, B, C, and vectors x, u, y are changed into matrices A', B', C', and vectors X, U, Y, respectively.
4: From (5) and the fact that, $P^{-1} = S_1' - S_2'$ the matrices S and S' are obtained.
5: Based on Theorem 4.1 the matrices $D - E$, $F - G$, and $H - I$ are derived.
6: Applying the matrices $F-G$, $H-I$, and Corollary (4.2) the controllability and the observability of the system (7) is stated.
7: The sufficient condition for the controllability and the observability of the system (7) can be obtained from Theorems 4.3 and 4.4.

5. Controllability and Observability of System with Fuzzy Coefficients and Variables

In this section, the controllability and the observability of system with fuzzy coefficients and fuzzy variables are studied. As a matter of fact, we consider the following FLCS:

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} = \tilde{C}\tilde{x} \end{cases} \tag{9}$$

where fuzzy matrices and fuzzy variables are LR-type, that are:

$$\tilde{A} = (\tilde{a}_{ij}) = (a'_{ij}, a_{ij}, a''_{ij})_{LR}, \quad \tilde{B} = (\tilde{b}_{ij}) = (b'_{ij}, b_{ij}, b''_{ij})_{LR}, \quad \tilde{C} = (\tilde{c}_{ij}) = (c'_{ij}, c_{ij}, c''_{ij})_{LR},$$
$$\tilde{x} = (\tilde{x}_i) = (x'_i, x_i, x''_i)_{LR}, \qquad \tilde{u} = (\tilde{u}_i) = (u'_i, u_i, u''_i)_{LR}, \qquad \tilde{y} = (\tilde{y}_i) = (y'_i, y_i, y''_i)_{LR},$$

where $1 \leq i, j \leq n$. For convenience, we define the following matrices:

$$\begin{aligned} &A_1 = (a'_{ij}),\ A_2 = (a_{ij}),\ A_3 = (a''_{ij}),\ B_1 = (b'_{ij}),\ B_2 = (b_{ij}),\ B_3 = (b''_{ij}), \\ &C_1 = (c'_{ij}),\ C_2 = (c_{ij}),\ \ C_3 = (c''_{ij}),\ X_1 = (x'_i),\ X_2 = (x_i),\ \ X_3 = (x''_i), \\ &U_1 = (u'_i),\ \ U_2 = (u_i),\ \ U_3 = (u''_i),\ \ Y_1 = (y'_i),\ \ Y_2 = (y_i),\ \ \ Y_3 = (y''_i). \end{aligned} \tag{10}$$

We break up the matrix A_1 into two $n \times n$ matrices such that their addition is A_1. Let $A_1^+ = ({a'_{ij}}^+)$ and $A_1^- = ({a'_{ij}}^-)$ where,

$${a'_{ij}}^+ = \begin{cases} a'_{ij} & a'_{ij} \geq 0 \\ 0 & a'_{ij} < 0 \end{cases}, \qquad {a'_{ij}}^- = \begin{cases} 0 & a'_{ij} \geq 0 \\ a'_{ij} & a'_{ij} < 0. \end{cases}$$

Then $A_1^+ + A_1^- = A_1$. We also break up the matrix A_3 into two matrices, similarly. Let $A_3^+ = ({a''_{ij}}^+)$ and $A_3^- = ({a''_{ij}}^-)$, where,

$${a''_{ij}}^+ = \begin{cases} a''_{ij} & a''_{ij} \geq 0 \\ 0 & a''_{ij} < 0 \end{cases}, \qquad {a''_{ij}}^- = \begin{cases} 0 & a''_{ij} \geq 0 \\ a''_{ij} & a''_{ij} < 0. \end{cases}$$

Also we break up matrices B_1 and B_3 and C_1 and C_3 like above.

Theorem 5.1. *The system $\dot{\tilde{x}} = \tilde{A} \otimes \tilde{x} + \tilde{B} \otimes \tilde{u}$ with the multiplication defined in Theorem 2.6 is equivalent to the following system:*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} A_1^+ & 0 & A_1^- \\ 0 & A_2 & 0 \\ A_3^- & 0 & A_3^+ \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_1^+ & 0 & B_1^- \\ 0 & B_2 & 0 \\ B_3^- & 0 & B_3^+ \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}. \tag{11}$$

Proof. According to Theorem 2.6 we have:

$$\begin{aligned}
\dot{\tilde{x}}_i &= (\dot{x}_i', \dot{x}_i, \dot{x}_i'')^T \\
&= (\tilde{A} \otimes \tilde{x})_i + (\tilde{B} \otimes \tilde{u})_i \\
&= \sum_{j=1}^{n} \{(\tilde{a}_{ij} \otimes \tilde{x}_j) + (\tilde{b}_{ij} \otimes \tilde{u}_j)\} \\
&= \sum_{j=1}^{n} (((a_{ij}', a_{ij}, a_{ij}'') \otimes (x_j', x_j, x_j'')) + ((b_{ij}', b_{ij}, b_{ij}'') \otimes (u_j', u_j, u_j''))) \\
&= \begin{cases} \sum_{j=1}^{n} ((a_{ij}' x_j', a_{ij} x_j, a_{ij}'' x_j'') + (b_{ij}' u_j', b_{ij} u_j, b_{ij}'' u_j'')), & a_{ij}' \geq 0, b_{ij}' \geq 0 \\ \sum_{j=1}^{n} ((a_{ij}' x_j'', a_{ij} x_j, a_{ij}'' x_j'') + (b_{ij}' u_j'', b_{ij} u_j, b_{ij}'' u_j'')), & a_{ij}' < 0, b_{ij}' < 0,\ a_{ij}'' \geq 0,\ b_{ij}'' \geq 0 \\ \sum_{j=1}^{n} ((a_{ij}' x_j'', a_{ij} x_j, a_{ij}'' x_j') + (b_{ij}' u_j'', b_{ij} u_j, b_{ij}'' u_j')), & a_{ij}'' < 0, b_{ij}'' < 0 \end{cases} \\
&= \begin{cases} \sum_{j=1}^{n} (a_{ij}' x_j' + b_{ij}' u_j', a_{ij} x_j + b_{ij} u_j, a_{ij}'' x_j'' + b_{ij}'' u_j''), & a_{ij}' \geq 0, b_{ij}' \geq 0 \\ \sum_{j=1}^{n} (a_{ij}' x_j'' + b_{ij}' u_j'', a_{ij} x_j + b_{ij} u_j, a_{ij}'' x_j'' + b_{ij}'' u_j''), & a_{ij}' < 0, b_{ij}' < 0,\ a_{ij}'' \geq 0,\ b_{ij}'' \geq 0 \\ \sum_{j=1}^{n} (a_{ij}' x_j'' + b_{ij}' u_j'', a_{ij} x_j + b_{ij} u_j, a_{ij}'' x_j' + b_{ij}'' u_j'), & a_{ij}'' < 0, b_{ij}'' < 0 \end{cases}
\end{aligned}$$

By adding the first component and the third component together we get:

$$\begin{bmatrix} \dot{x}_i' \\ \dot{x}_i \\ \dot{x}_i'' \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \underbrace{((a_{ij}' x_j' + b_{ij}' u_j')}_{a_{ij}'^{+}, b_{ij}'^{+}} + \underbrace{(a_{ij}' x_j'' + b_{ij}' u_j''))}_{a_{ij}'^{-}, b_{ij}'^{-}} \\ \\ \sum_{j=1}^{n} (a_{ij} x_j + b_{ij} u_j) \\ \\ \sum_{j=1}^{n} \underbrace{((a_{ij}'' x_j'' + b_{ij}'' u_j'')}_{a_{ij}''^{+}, b_{ij}''^{+}} + \underbrace{(a_{ij}'' x_j' + b_{ij}'' u_j'))}_{a_{ij}''^{-}, b_{ij}''^{-}} \end{bmatrix}.$$

According to the notations (10) the above system can be stated as the following form:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} (A_1^+ X_1 + B_1^+ U_1) + (A_1^- X_3 + B_1^- U_3) \\ \\ A_2 X_2 + B_2 U_2 \\ \\ (A_3^- X_1 + B_3^- U_1) + (A_3^+ X_3 + B_3^+ U_3) \end{bmatrix}.$$

Or it can be replaced as follows:

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\end{bmatrix}=\underbrace{\begin{bmatrix}A_1^+ & 0 & A_1^-\\ 0 & A_2 & 0\\ A_3^- & 0 & A_3^+\end{bmatrix}}_{A}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}+\underbrace{\begin{bmatrix}B_1^+ & 0 & B_1^-\\ 0 & B_2 & 0\\ B_3^- & 0 & B_3^+\end{bmatrix}}_{B}\begin{bmatrix}U_1\\ U_2\\ U_3\end{bmatrix}.$$

Thus, the result follows. □

Theorem 5.2. *The system $\tilde{y}=\tilde{C}\otimes\tilde{x}$ with the multiplication defined in Theorem 2.6 is equivalent to the following system:*

$$\begin{bmatrix}Y_1\\ Y_2\\ Y_3\end{bmatrix}=\underbrace{\begin{bmatrix}C_1^+ & 0 & C_1^-\\ 0 & C_2 & 0\\ C_3^- & 0 & C_3^+\end{bmatrix}}_{C}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}. \tag{12}$$

Proof. The proof is similar to that of the former theorem and so, we omit here. □

Remark 5.3. According to Theorem 5.1, instead of studying the controllability of the system (9) one can study the system (11). As you can see, the system (11) is a crisp system and for the controllability of the system (11) we can use the method described in Remark 2.8. In other words, the system (11) is controllable whenever all rows corresponding to u_i', u_i, u_i'' in term $P^{-1}B$ are not zero. Also, for discussing the observability of the system (9) one can check the system (12). In other words, the system (12) is observable whenever all columns corresponding to y_i', y_i, y_i'' in term CP are not zero.

Theorem 5.4. *The sufficient condition for the controllability and the observability of the system (9) is:*

$$\operatorname{rank}[B, AB, A^2B, ..., A^{n-1}B]=n, \qquad \operatorname{rank}(\begin{bmatrix}C & CA & CA^2 & \dots & CA^{n-1}\end{bmatrix}^T)=n,$$

respectively.

Proof. The proof is clear, so omit here. □

From the aforementioned discussion, the process and algorithm for deducing the controllability and the observability of systems with fuzzy coefficients and variables are summarized as follows.

Algorithm 5.5. The algorithm for investigating the controllability and the observability of systems with fuzzy coefficients and variables.

1: Identify the LR-type matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$.
2: According to equations (10), the system (9) is become to the equations (11) and (12).
3: The controllability and the observability of the system (9) are obtained from Remark 5.3.
4: The sufficient condition for the controllability and the observability of the system (9) can be obtained from Theorem 5.4.

6. Application: The Inverted Pendulum

In this section, the application of the FLS is given. In fact, we consider a crisp Inverted Pendulum problem as a fuzzy Inverted Pendulum with assuming the fuzzy coefficient matrix and fuzzy variables.

The single inverted pendulum is a classical problem in the field of non-linear control theory; it also offers a good example for control engineers to verify a modern control theory. The inverted pendulum is a highly non-linear and open-loop unstable system. The characteristics of the inverted pendulum make identification and control more challenging. Inverted pendulum can be considered as a popular system that is used to approximate highly complex models such as rockets during liftoff, bipedal walking, cranes, robots, and etc. After ignoring the air resistance and a variety of friction, the linear inverted pendulum can be abstracted into a cart and a homogeneous rod, shown in Figure 1.

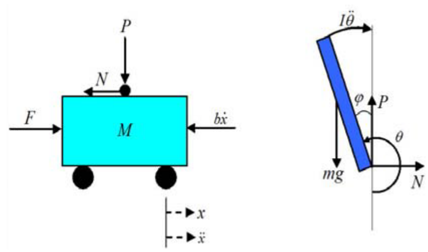

FIGURE 1. Pendulum System Force Analysis

From the sum of force in both horizontal and vertical directions and the sum of moments around the centroid of the pendulum, the system can be described by the following two nonlinear differential equations [15]:

$$\begin{aligned} &(M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u, \\ &(I+ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x}, \end{aligned} \tag{13}$$

where the parameters in equations (13) are describe in Table 1.

Symbol	Meaning
M	Cart mass
m	Rod mass
b	Friction coefficient of the cart
l	Distance from the rod axis rotation center to the rod mass center
I	Rod inertia
F	Force acting on the cart
x	Cart position
ϕ	The angle of the pendulum with the vertical upward direction
θ	The angle of the pendulum and the vertical downward direction

TABLE 1. Description of the Parameters in equations (13)

A set of state variables sufficient to describe this system are chosen as the position and velocity of the cart, the angular position and change of angular position of the pendulum. Therefore, the equations that describe the behaviour of the inverted pendulum are as the following:

$$\begin{aligned}
\dot{x} &= \dot{x} \\
\ddot{x} &= \frac{-(I+ml^2)b}{I(M+m)+Mml^2}\dot{x} + \frac{m^2gl^2}{I(M+m)Mml^2}\phi + \frac{(I+ml^2)}{I(M+m)+Mml^2}u \\
\dot{\phi} &= \dot{\phi} \\
\ddot{\phi} &= \frac{-mlb}{I(M+m)+Mml^2}\dot{x} + \frac{mgl(M+m)}{I(M+m)Mml^2}\phi + \frac{ml}{I(M+m)+Mml^2}u.
\end{aligned}$$

Now, in the above equations consider the fuzzy parameters $\tilde{x}$, $\tilde{u}$, $\tilde{\phi}$, and $\tilde{\theta}$ therefore, the equations that describe the behaviour of the fuzzy inverted pendulum can be as the following:

$$\begin{aligned}
(M+m)\ddot{x} + b\dot{x} \ominus_H ml\ddot{\phi} &= u, \\
(I+ml^2)\ddot{\phi} \ominus_H mgl\phi &= ml\ddot{x},
\end{aligned}$$

where the $\ominus_H$ denotes the Hukuhara difference (see Definition 2.7). Or

$$\begin{aligned}
\dot{\tilde{x}} &= \dot{\tilde{x}} \\
\ddot{\tilde{x}} &= \frac{-(I+ml^2)b}{I(M+m)+Mml^2}\dot{\tilde{x}} + \frac{m^2gl^2}{I(M+m)Mml^2}\tilde{\phi} + \frac{(I+ml^2)}{I(M+m)+Mml^2}\tilde{u} \\
\dot{\tilde{\phi}} &= \dot{\tilde{\phi}} \\
\ddot{\tilde{\phi}} &= \frac{-mlb}{I(M+m)+Mml^2}\dot{\tilde{x}} + \frac{mgl(M+m)}{I(M+m)Mml^2}\tilde{\phi} + \frac{ml}{I(M+m)+Mml^2}\tilde{u}
\end{aligned}$$

The fuzzy system state-space equation can be written as follow:

$$\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\
\dot{\tilde{y}} &= C\tilde{x} + D\tilde{u}.
\end{aligned}$$

Or it can be:

$$\begin{bmatrix} \dot{\tilde{x}} \\ \ddot{\tilde{x}} \\ \dot{\tilde{\phi}} \\ \ddot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)Mml^2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \\ \tilde{\phi} \\ \dot{\tilde{\phi}} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} \tilde{u},$$

$$\tilde{y} = \begin{bmatrix} \tilde{x} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \\ \tilde{\phi} \\ \dot{\tilde{\phi}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tilde{u}.$$

7. Numerical Examples

In order to demonstrate the effectiveness and efficiency of the proposed method, we solve some examples.

Example 7.1. Consider the following FLCS:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tilde{u}, \\ \tilde{y} = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \tilde{x}. \end{cases}$$

The eigenvalues of matrix A are $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$, so the matrices P and P^{-1} can be as follow:

$$P = \begin{bmatrix} 6 & -3 & 4 \\ 5 & -4 & 6 \\ 1 & -1 & 2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 1 & -2 & 4 \\ 0.25 & -0.75 & 2.25 \end{bmatrix}.$$

Now, we obtain the following matrices,

$$S = \begin{bmatrix} 6 & 0 & 4 & 0 & 3 & 0 \\ 5 & 0 & 6 & 0 & 4 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 6 & 0 & 4 \\ 0 & 4 & 0 & 5 & 0 & 6 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}, \qquad S' = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0.5 & 0 \\ 1 & 0 & 4 & 0 & 2 & 0 \\ 0.25 & 0 & 2.25 & 0 & 0.75 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0.5 \\ 0 & 2 & 0 & 1 & 0 & 4 \\ 0 & 0.75 & 0 & 0.25 & 0 & 2.25 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & 0 & 0 & 11 \\ 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 11 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 1 & 0 \end{bmatrix}, \qquad B' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

Therefore,

$$S'A'S = \begin{bmatrix} 8 & 7.5 & 14 & 9 & 7.5 & 14 \\ 42 & 36 & 68 & 42 & 38 & 68 \\ 19 & 17.25 & 30 & 19 & 17.25 & 33 \\ 9 & 7.5 & 14 & 8 & 7.5 & 14 \\ 42 & 38 & 68 & 42 & 36 & 68 \\ 19 & 17.25 & 33 & 19 & 17.25 & 30 \end{bmatrix}, S'B' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C'S = \begin{bmatrix} 17 & 1 & 14 & 1 & 10 & 2 \\ 1 & 10 & 2 & 17 & 1 & 14 \end{bmatrix}.$$

As one can see,

$$D = \begin{bmatrix} 8 & 7.5 & 14 \\ 42 & 36 & 68 \\ 19 & 17.25 & 30 \end{bmatrix}, \qquad E = \begin{bmatrix} 9 & 7.5 & 14 \\ 42 & 38 & 68 \\ 19 & 17.25 & 33 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 17 & 1 & 14 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 10 & 2 \end{bmatrix}.$$

And also,

$$D-E=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix},\quad F-G=\begin{bmatrix}1\\1\\0\end{bmatrix},\quad H-I=\begin{bmatrix}16 & -9 & 12\end{bmatrix}.$$

The third row of the matrix $F-G$ or $P^{-1}B$ is zero so the above system is not controllable. But the above system is the observable because the matrix $H-I$ has no zero columns. In other way,

$\mathrm{rank}(B', A'B', A'^2B', A'^3B', A'^4B', A'^5B') = 5$, $\mathrm{rank}([C'\ C'A'\ C'A'^2\ C'A'^3\ C'A'^4\ C'A'^5]^T) = 6$.

Therefore, the above system is not the controllable but it is the observable.

Example 7.2. Consider the following FLCS:

$$\begin{cases}\dot{\tilde{x}}=\begin{bmatrix}3 & -1\\ 2 & 0\end{bmatrix}\tilde{x}+\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\tilde{u},\\ \tilde{y}=\begin{bmatrix}2 & 0\\ 1 & -1\end{bmatrix}\tilde{x}.\end{cases}$$

The eigenvalues of matrix A are $\lambda_1 = 1, \lambda_2 = 2$, so the matrices P and P^{-1} can be as follow:

$$P=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix},\quad P^{-1}=\begin{bmatrix}-1 & 1\\ -2 & 1\end{bmatrix}$$

Now, we obtain the following matrices,

$$S=\begin{bmatrix}1&0&0&1\\2&0&0&1\\0&1&1&0\\0&1&2&0\end{bmatrix},\qquad S'=\begin{bmatrix}0&1&1&0\\0&1&2&0\\1&0&0&1\\2&0&0&1\end{bmatrix}$$

$$A'=\begin{bmatrix}3&0&0&1\\2&0&0&0\\0&1&3&0\\0&0&2&0\end{bmatrix},\qquad B'=\begin{bmatrix}0&2&1&0\\1&0&0&1\\1&0&0&2\\0&1&1&0\end{bmatrix},\quad C'=\begin{bmatrix}2&0&0&0\\1&0&0&1\\0&0&2&0\\0&1&1&0\end{bmatrix}$$

Therefore,

$$S'A'S=\begin{bmatrix}4&3&3&3\\6&6&6&4\\3&3&4&3\\6&4&6&6\end{bmatrix},\qquad S'B'=\begin{bmatrix}2&0&0&2\\3&0&0&5\\0&3&2&0\\0&5&3&0\end{bmatrix},\qquad C'S=\begin{bmatrix}2&0&0&2\\1&1&2&1\\0&2&2&0\\2&1&1&1\end{bmatrix},$$

$$D-E=\begin{bmatrix}1&0\\0&2\end{bmatrix},\qquad F-G=\begin{bmatrix}2&-3\\3&-5\end{bmatrix},\qquad H-I=\begin{bmatrix}2&-2\\-1&0\end{bmatrix}.$$

As a result, none of the rows of the matrix $F-G$ or $P^{-1}B$ and none of the columns of the matrix $H-I$ or CP are not zero thus, the system is both the controllable and the observable. Also, in other way, since

$$\mathrm{rank}(B', A'B', A'^2B', A'^3B') = 4,\qquad \mathrm{rank}(\begin{bmatrix}C' & C'A' & C'A'^2 & C'A'^3\end{bmatrix}^T) = 4,$$

therefore, the above system is both the controllable and the observable.

Example 7.3. Consider the following FLCS:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} (1,-1,1) & (2,2,3) \\ (2,-3,3) & (1,4,1) \end{bmatrix} \tilde{x} + \begin{bmatrix} (-0.7,4,0) & (0,4,0.7) \\ (0.7,6,0) & (0,6,0.7) \end{bmatrix} \tilde{u}, \\ \tilde{y} = \begin{bmatrix} (-2,-1,0) & (1,2,3) \\ (-4,-3,-2) & (3,4,5) \end{bmatrix} \tilde{x}. \end{cases}$$

Now, for solving this example assume that:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2),$$

where,

$$\begin{aligned} &\tilde{x}_1 = (x_1', x_1, x_1''), && \tilde{x}_2 = (x_2', x_2, x_2''), && \tilde{u}_1 = (u_1', u_1, u_1''), \\ &\tilde{u}_2 = (u_2', u_2, u_2''), && \tilde{y}_1 = (y_1', y_1, y_1''), && \tilde{y}_2 = (y_2', y_2, y_2''). \end{aligned}$$

Also, suppose that the following matrices:

$$\begin{aligned} &A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, && A_2 = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}, && A_3 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}. \\ &B_1 = \begin{bmatrix} -0.7 & 0 \\ 0.7 & 0 \end{bmatrix}, && B_2 = \begin{bmatrix} 4 & 4 \\ 6 & 6 \end{bmatrix}, && B_3 = \begin{bmatrix} 0 & 0.7 \\ 0 & 0.7 \end{bmatrix} \\ &C_1 = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}, && C_2 = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}, && C_3 = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}. \end{aligned}$$

So, we obtain the following matrices:

$$\begin{aligned} &A_1^+ = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, && A_1^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, && A_3^+ = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, && A_3^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ &B_1^+ = \begin{bmatrix} 0 & 0 \\ 0.7 & 0 \end{bmatrix}, && B_1^- = \begin{bmatrix} -0.7 & 0 \\ 0 & 0 \end{bmatrix}, && B_3^+ = \begin{bmatrix} 0 & 0.7 \\ 0 & 0.7 \end{bmatrix}, && B_3^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ &C_1^+ = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, && C_1^- = \begin{bmatrix} -2 & 0 \\ -4 & 0 \end{bmatrix}, && C_3^+ = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, && C_3^- = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, applying system (11) implies:

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_1'' \\ \dot{x}_2'' \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \\ x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_1 \\ u_2 \\ u_1'' \\ u_2'' \end{bmatrix}$$

The eigenvalues of the first matrix are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 2, \lambda_5 = 3, \lambda_6 = 4$. Then,

$$P = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

So,

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.35 & 0 & 0 & 0 & 0.35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0.35 & 0 & 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 \end{bmatrix}.$$

As one can see, three rows of the matrix $P^{-1}B$ corresponding to u_1', u_1, u_1'' and three rows of the matrix $P^{-1}B$ corresponding to u_2', u_2, u_2'' are not zero. Therefore, the above system is the controllable. Also, by using system (12) we have:

$$(y_1', y_2', y_1, y_2, y_1'', y_2'') = \begin{bmatrix} 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ -2 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_1 \\ x_2 \\ x_1'' \\ x_2'' \end{bmatrix}.$$

So, we get:

$$CP = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & -2 \\ 4 & 3 & 0 & 0 & 3 & -4 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 3 \\ 5 & 2 & 0 & 0 & -2 & 5 \end{bmatrix}.$$

Since three columns of the matrix CP corresponding to y_1', y_1, y_1'' and three columns of the matrix CP corresponding to y_2', y_2, y_2'' are not zero thus, the above system is the observable. Also, in other way,

$$\operatorname{rank}(B', A'B', A'^2B', A'^3B', A'^4B', A'^5B') = 6, \ \operatorname{rank}([C'\ C'A'\ C'A'^2\ C'A'^3\ C'A'^4\ C'A'^5]^T) = 6.$$

Based on Theorem 5.2 the above system is both controllable and observable.

Example 7.4. Consider the fuzzy inverted pendulum with the parameters setting in the following table (Table 2). Therefore, we get the following fuzzy state-space

System Parameters	Values
M (kg)	1.096
m (kg)	0.109
b (Nm/s)	0.1
l (m)	0.25
I (kg.m.m)	0.0034

Table 2. List of System Parameters

equation cart-inverted pendulum:

$$\begin{bmatrix} \dot{\tilde{x}} \\ \ddot{\tilde{x}} \\ \dot{\tilde{\phi}} \\ \ddot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \\ \tilde{\phi} \\ \dot{\tilde{\phi}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.18182 \\ 0 \\ 4.5455 \end{bmatrix} \tilde{u},$$

$$\tilde{y} = \begin{bmatrix} \tilde{x} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \\ \tilde{\phi} \\ \dot{\tilde{\phi}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tilde{u}.$$

Matrices P and P^{-1} are obtained as follow:

$$P = \begin{bmatrix} 1 & 0.0154 & -0.990 & 0.0147 \\ 0 & -0.0863 & 0.1414 & 0.0820 \\ 0 & 0.1750 & 0.0021 & 0.1762 \\ 0 & -0.9807 & 0.0003 & 0.9808 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 7.0007 & 0 & -0.6001 \\ 0 & -0.0426 & 2.8468 & -0.5080 \\ 0 & 7.0705 & 0.0866 & -0.6064 \\ 0 & -0.0405 & 2.8464 & 0.5115 \end{bmatrix}.$$

Now, we define:

$$S = \begin{bmatrix} 1 & 0.0154 & 0 & 0.0147 & 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0.1414 & 0.0820 & 0 & 0.0863 & 0 & 0 \\ 0 & 0.1750 & 0.0021 & 0.1762 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0003 & 0.9808 & 0 & 0.9807 & 0 & 0 \\ 0 & 0 & 0.99 & 0 & 1 & 0.0154 & 0 & 0.0147 \\ 0 & 0.0863 & 0 & 0 & 0 & 0 & 0.1414 & 0.0820 \\ 0 & 0 & 0 & 0 & 0 & 0.1750 & 0.0021 & 0.1762 \\ 0 & 0.9807 & 0 & 0 & 0 & 0 & 0.0003 & 0.9808 \end{bmatrix},$$

$$C' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B' = \begin{bmatrix} 0 & 0 \\ 0.18182 & 0 \\ 0 & 0 \\ 4.5554 & 0 \\ 0 & 0 \\ 0 & 0.18182 \\ 0 & 0 \\ 0 & 0.4.5455 \end{bmatrix}$$

$$S' = \begin{bmatrix} 1 & 7.0007 & 0 & 0 & 0 & 0 & 0 & 0.6001 & \\ 0 & 0 & 2.8468 & 0 & 0 & 0.0426 & 0 & 0.5080 & \\ 0 & 7.0705 & 0.0866 & 0 & 0 & 0 & 0 & 0.6064 & \\ 0 & 0 & 2.8464 & 0.5115 & 0 & 0.0405 & 0 & 0 & \\ 0 & 0 & 0 & 0.6001 & 1 & 7.0007 & 0 & 0 & \\ 0 & 0.0426 & 0 & 0.5080 & 0 & 0 & 2.8468 & 0 & 0 \\ 0 & 0 & 0 & & 0.6064 & 0 & 7.0705 & 0.0866 & 0 \\ 0 & 0.0405 & 0 & 0 & 0 & 0 & 2.8464 & 0.5115 & \end{bmatrix}.$$

Consequently,

$$S'B' = \begin{bmatrix} 1.2729 & 2.7278 \\ 0 & 2.3169 \\ 1.2856 & 2.7564 \\ 2.3250 & 0.0074 \\ 2.7278 & 1.2729 \\ 2.3169 & 0 \\ 2.7564 & 1.2856 \\ 0.0074 & 2.3250 \end{bmatrix}$$

$$C'S = \begin{bmatrix} 1 & 0.0154 & 0 & 0.0147 & 0 & 0 & 0.99 & 0 \\ 0 & 0.1750 & 0.0021 & 0.1726 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & 0 & 1 & 0.0154 & 0 & 0.0147 \\ 0 & 0 & 0 & 0 & 0 & 0.1750 & 0.0021 & 0.1726 \end{bmatrix}.$$

As one can check,

$$F = \begin{bmatrix} 1.2729 \\ 0 \\ 1.2856 \\ 2.3250 \end{bmatrix}, \qquad G = \begin{bmatrix} 2.7278 \\ 2.3169 \\ 2.7564 \\ 0.0074 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0.0154 & 0 & 0.0147 \\ 0 & 0.1750 & 0.0021 & 0.1726 \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

And, finally we get,

$$F - G = \begin{bmatrix} -1.4549 \\ -2.3169 \\ -1.3708 \\ 2.3176 \end{bmatrix}, \quad H - I = \begin{bmatrix} 1 & 0.0154 & -0.99 & 0.0147 \\ 0 & 0.1750 & 0.0021 & 0.1726 \end{bmatrix}.$$

As a result, none of the rows of the matrix $F-G$ or $P^{-1}B$ and none of the columns of the matrix $H-I$ or CP are not zero, so the system is both the controllable and the observable.

Example 7.5. Consider the fuzzy inverted pendulum with the parameters setting in the following table (Table 3). Therefore, we get the following fuzzy state-space

System Parameters	Values
M (kg)	2.081
m (kg)	0.54
b (Nm/s)	0.15
l (m)	0.37
I (kg.m.m)	0.01

TABLE 3. List of System Parameters

equation cart-inverted pendulum:

$$\begin{bmatrix}\dot{\tilde{x}}\\ \ddot{\tilde{x}}\\ \dot{\tilde{\phi}}\\ \ddot{\tilde{\phi}}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -0.078 & 35.95 & 0\\ 0 & 0 & 0 & 1\\ 0 & -0.17 & 12.752 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \dot{\tilde{x}}\\ \tilde{\phi}\\ \dot{\tilde{\phi}}\end{bmatrix}+\begin{bmatrix}0\\ 0.47\\ 0\\ 1.11\end{bmatrix}\tilde{u},$$

$$\tilde{y}=\begin{bmatrix}\tilde{x}\\ \tilde{\phi}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \dot{\tilde{x}}\\ \tilde{\phi}\\ \dot{\tilde{\phi}}\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\tilde{u}.$$

Matrices P and P^{-1} are obtained as follow:

$$P=\begin{bmatrix}1 & 0.1005 & 0.9968 & 0.1043\\ 0 & -0.6492 & 0.0797 & 0.6581\\ 0 & 0.1154 & -0.0003 & 0.1168\\ 0 & -0.745 & 0.000 & 0.7365\end{bmatrix},\ P^{-1}=\begin{bmatrix}1 & -12.5038 & 0 & 11.031\\ 0 & -0.0177 & 4.2844 & -0.6634\\ 0 & 12.5475 & -0.8857 & -11.0712\\ 0 & -0.0183 & 4.3341 & 0.6871\end{bmatrix}.$$

Now, we define:

$$S=\begin{bmatrix}1 & 0.1005 & 0.9968 & 0.1043 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.0797 & 0.6581 & 0 & 0.6492 & 0 & 0\\ 0 & 0.1154 & 0 & 0.1168 & 0 & 0 & 0.0003 & 0\\ 0 & 0 & 0.000 & 0.7365 & 0 & 0.745 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0.1005 & 0.9968 & 0.1043\\ 0 & 0.6492 & 0 & 0 & 0 & 0 & 0.0797 & 0.6581\\ 0 & 0 & 0.0003 & 0 & 0 & 0.1154 & 0 & 0.1168\\ 0 & 0.745 & 0 & 0 & 0 & 0 & 0.000 & 0.7365\end{bmatrix},$$

$$C'=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix},\qquad B'=\begin{bmatrix}0 & 0\\ 0.47 & 0\\ 0 & 0\\ 1.11 & 0\\ 0 & 0\\ 0 & 0.47\\ 0 & 0\\ 0 & 1.11\end{bmatrix}$$

$$S'=\begin{bmatrix}1 & 0 & 0 & 11.031 & 0 & 12.5038 & 0 & 0\\ 0 & 0 & 4.28448 & 0 & 0 & 0.0177 & 0 & 0.6634\\ 0 & 12.5475 & 0 & 0 & 0 & 0 & 0.8857 & 11.0712\\ 0 & 0 & 4.3341 & 0.6871 & 0 & 0.0183 & 0 & 0\\ 0 & 12.5038 & 0 & 0 & 1 & 0 & 0 & 11.031\\ 0 & 0.0177 & 0 & 0.6634 & 0 & 0 & 4.28448 & 0\\ 0 & 0 & 0.8857 & 11.0712 & 0 & 12.5475 & 0 & 0\\ 0 & 0.0183 & 0 & 0 & 0 & 0 & 4.3341 & 0.6871\end{bmatrix}.$$

Consequently,

$$S'B' = \begin{bmatrix} 12.2444 & 5.8768 \\ 0 & 0.7447 \\ 0.8973 & 12.2890 \\ 0.7627 & 0.0086 \\ 5.8768 & 12.2444 \\ 0.7447 & 0 \\ 12.2890 & 0.8973 \\ 0.0086 & 0.7627 \end{bmatrix}$$

$$C'S = \begin{bmatrix} 1 & 0.1005 & 0.9968 & 0.1043 & 0 & 0 & 0 & 0 \\ 0 & 0.1154 & 0 & 0.1168 & 0 & 0 & 0.0003 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.1005 & 0.9968 & 0.1043 \\ 0 & 0 & 0.0003 & 0 & 0 & 0.1154 & 0 & 0.1168 \end{bmatrix}.$$

As one can check,

$$F = \begin{bmatrix} 12.2444 \\ 0 \\ 0.8973 \\ 0.7627 \end{bmatrix}, \qquad G = \begin{bmatrix} 5.8768 \\ 0.7447 \\ 12.2890 \\ 0.0086 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0.1005 & 0.9968 & 0.1043 \\ 0 & 0.1154 & 0 & 0.1168 \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0003 & 0 \end{bmatrix}.$$

And, finally we get,

$$F - G = \begin{bmatrix} 6.3676 \\ -0.7447 \\ -11.3917 \\ 0.7541 \end{bmatrix}, \quad H - I = \begin{bmatrix} 1 & 0.1005 & 0.9968 & 0.1043 \\ 0 & 0.1154 & -0.0003 & 0.1168 \end{bmatrix}.$$

As a result, none of the rows of the matrix $F-G$ or $P^{-1}B$ and none of the columns of the matrix $H-I$ or CP are not zero, so the system is both the controllable and the observable.

Example 7.6. Consider the fuzzy inverted pendulum with the parameters setting in the following table (Table 4). Therefore, we get the following fuzzy state-space

System Parameters	Values
M (kg)	2000
m (kg)	850
b (Nm/s)	140
l (m)	20
I (kg.m.m)	80

Table 4. List of System Parameters

equation cart-inverted pendulum:

$$\begin{bmatrix}\dot{\tilde{x}}\\ \ddot{\tilde{x}}\\ \dot{\tilde{\phi}}\\ \ddot{\tilde{\phi}}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -0.07 & 0.00002 & 0\\ 0 & 0 & 0 & 1\\ 0 & -0.003 & 0.000003 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \dot{\tilde{x}}\\ \tilde{\phi}\\ \dot{\tilde{\phi}}\end{bmatrix} + \begin{bmatrix}0\\ 0.00005\\ 0\\ 0.00003\end{bmatrix}\tilde{u},$$

$$\tilde{y} = \begin{bmatrix}\tilde{x}\\ \tilde{\phi}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \dot{\tilde{x}}\\ \tilde{\phi}\\ \dot{\tilde{\phi}}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}\tilde{u}.$$

Matrices P and P^{-1} are obtained as follow:

$$P = \begin{bmatrix}1 & -0.9966 & 0.1963 & 0.1870\\ 0 & 0.0698 & -0.0003 & 0.0003\\ 0 & -0.0427 & -0.9805 & 0.9824\\ 0 & 0.0030 & 0.0014 & 0.0014\end{bmatrix}, \; P^{-1} = \begin{bmatrix}1 & 20 & 0 & -133.3333\\ 0 & 14.3263 & -0.0041 & 0.0585\\ 0 & -15.2449 & -0.5077 & 348.3072\\ 0 & -14.5936 & 0.5110 & 347.6683\end{bmatrix}.$$

Now, we define:

$$S = \begin{bmatrix}1 & 0 & 0.1963 & 0.1870 & 0 & 0.9966 & 0 & 0\\ 0 & 0.0698 & 0 & 0.0003 & 0 & 0 & 0.0003 & 0\\ 0 & 0 & 0 & 0.9824 & 0 & 0.0427 & 0.9805 & 0\\ 0 & 0.0030 & 0.0014 & 0.0014 & 0 & 0 & 0 & 0\\ 0 & 0.9966 & 0 & 0 & 1 & 0 & 0.1963 & 0.1870\\ 0 & 0 & 0.0003 & 0 & 0 & 0.0698 & 0 & 0.0003\\ 0 & 0.0427 & 0.9805 & 0 & 0 & 0 & 0 & 0.9824\\ 0 & 0 & 0 & 0 & 0 & 0.0030 & 0.0014 & 0.0014\end{bmatrix},$$

$$C' = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}, \qquad B' = \begin{bmatrix}0 & 0\\ 0.00005 & 0\\ 0 & 0\\ 0.00003 & 0\\ 0 & 0\\ 0 & 0.00005\\ 0 & 0\\ 0 & 0.00003\end{bmatrix}$$

$$S' = \begin{bmatrix}1 & 20 & 0 & 0 & 0 & 0 & 0 & -133.3333\\ 0 & 14.3263 & 0 & 0.0585 & 0 & 0 & 0.0041 & 0\\ 0 & 0 & 0 & 348.3072 & 0 & 15.2449 & 0.5077 & 0\\ 0 & 0 & 0.5110 & 347.6683 & 0 & 14.5936 & 0 & 0\\ 0 & 0 & 0 & 133.3333 & 1 & 20 & 0 & 0\\ 0 & 0 & 0.0041 & 0 & 0 & 14.3263 & 0 & 0.0585\\ 0 & 15.2449 & 0.5077 & 0 & 0 & 0 & 0 & 348.3072\\ 0 & 14.5936 & 0 & 0 & 0 & 0 & 0.5110 & 347.6683\end{bmatrix}.$$

Consequently,

$$S'B' = \begin{bmatrix} 0.0010 & 0.0040 \\ 0.0007 & 0 \\ 0.0104 & 0.0008 \\ 0.0104 & 0.0007 \\ 0.0040 & 0.0010 \\ 0 & 0.0007 \\ 0.0008 & 0.0104 \\ 0.0007 & 0.0104 \end{bmatrix}$$

$$C'S = \begin{bmatrix} 1.0000 & 0 & 0.1963 & 0.1870 & 0 & 0.9966 & 0 & 0 \\ 0 & 0 & 0 & 0.9824 & 0 & 0.0427 & 0.9805 & 0 \\ 0 & 0.9966 & 0 & 0 & 1.0000 & 0 & 0.1963 & 0.1870 \\ 0 & 0.0427 & 0.9805 & 0 & 0 & 0 & 0 & 0.9824 \end{bmatrix}.$$

As one can check,

$$F = \begin{bmatrix} 0.0010 \\ 0.0007 \\ 0.0104 \\ 0.0104 \end{bmatrix}, \qquad G = \begin{bmatrix} 0.0040 \\ 0 \\ 0.0008 \\ 0.0007 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0.1963 & 0.1870 \\ 0 & 0 & 0 & 0.9824 \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 0.9966 & 0 & 0 \\ 0 & 0.0427 & 0.9805 & 0 \end{bmatrix}.$$

And, finally we get,

$$F - G = \begin{bmatrix} -0.003 \\ 0.0007 \\ 0.0096 \\ 0.0097 \end{bmatrix}, \quad H - I = \begin{bmatrix} 1 & -0.9966 & 0.1963 & 0.1870 \\ 0 & -0.0427 & -0.9805 & 0.9824 \end{bmatrix}.$$

As a result, none of the rows of the matrix $F-G$ or $P^{-1}B$ and none of the columns of the matrix $H-I$ or CP are not zero, so the system is both the controllable and the observable.

8. **Conclusions**

In this paper, we used operations on fuzzy numbers and solve two fuzzy system, that the first system had fuzzy variable and the second system was fully fuzzy. In the both systems we changed them to the crisp system then examined the controllability and the observability of the system. All simulation results that are presented as support to the theories in this work have been for a continuous, linear time invariant systems. The proposed theory also holds for time variant systems which still lack simulation evidences. Therefore, simulation results that support the theory for such systems would be particularly interesting. Also, the simulation results shows the applicability and the effectiveness of the method. Finally, the work is in progress to extend the method to investigate the controllability and the observability of the time-variant systems.

References

[1] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, Berlin, 2013.
[2] M. Biglarbegian, A. Sadeghian and W. Melek, *On the accessibility/ controllability of fuzzy control systems*, Information Science, **202** (2012), 58-72.
[3] Z. Cai and S. Tang, *Controllability and robustness of T-fuzzy control systems under directional disturbance*, Fuzzy Sets and Systems, **115** (2000), 279-285.
[4] Z. Ding and A. Kandel, *On the controllability of fuzzy dynamical systems (I)*, Journal of Fuzzy Mathematics, **8(1)** (2000), 203-214.
[5] Z. Ding and A. Kandel, *On the controllability of fuzzy dynamical systems (II)*, Journal of Fuzzy Mathematics, **8(2)** (2000), 295-306.
[6] Z. Ding, M. Ma and A. Kandel, *On the observability of fuzzy dynamical control systems (I)*, Fuzzy Sets and Systems, **111(2)** (2000), 225-236.
[7] S. S. Farinwata and G. Vachtsevanos, *Survey on the controllability of fuzzy logic systems*, Proceedings of 32th IEEE Conference on Decision and Control, (1993), 1749-1750.
[8] Y. Feng and L. Hua, *On the quasi-controllability of continuous-time dynamic fuzzy control systems*, Chaos, Solitons and Fractals, **30(1)** (2006), 177-188.
[9] M. Friedman, Ma Ming and A. Kandel, *Fuzzy linear systems*, Fuzzy Sets and Systems, **96** (1998), 201-209.
[10] W. I. Gabr, *A new approach for automatic control modelling, analysis and design in fully fuzzy environment*, Ain Shams Engineering Journal, **6** (2015), 835-850.
[11] M. M. Gupta, *Controllability of fuzzy control systems*, IEEE Transactions on Systems, Man, and Cybernetics, **16** (1985), 576-582.
[12] A. Kumar, J. Kaur and P. Singh, *A new method for solving fully fuzzy linear programming problems*, Applied Mathematics and Computation, **35** (2011), 817-823.
[13] A. Mansoori, S. Effati and M. Eshaghnezhad, *A neural network to solve quadratic programming problems with fuzzy parameters*, Fuzzy Optimization and Decision Making, **17(1)** (2018), 75-101.
[14] M. S. N. Murty and G. Suresh Kumar, *On Controllability and Observability of Fuzzy Dynamical Matrix Lyapunov Systems*, Advances in Fuzzy Systems, **8(1)** (2008), 1-16.
[15] K. Ogata, *Modern Control Engineering*, Fourth ed., Prentice Hall, New Jersey, 2002.
[16] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems **161(11)** (2010), 1564-1584.
[17] C. Tetiana and Z. Alexander, *Application of the return method to the steering of nonlinear systems*, in: Krzysztof Kozlowski (Ed.), Robot Motion and Control 2009, Lecture Notes in Control and Information Sciences, vol. 396, Springer, Berlin/Heidelberg, (2009), 83-91.
[18] L. X. Wang, *A Course in Fuzzy Systems and Control*, Prentice Hall PTR, 1997.

Roya Mastiani, Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: roya_mastiani@yahoo.com

Sohrab Effati*, Department of Applied Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran and The Center of Excellence on Soft Computing and Intelligent Information Processing (SCIIP), Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: s-effati@um.ac.ir

*Corresponding Author