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M-FUZZIFYING TOPOLOGICAL CONVEX SPACES

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ABSTRACT. The main purpose of this paper is to introduce the compatibility of M-fuzzifying topologies and M-fuzzifying convexities, define an M-fuzzifying topological convex space, and give a method to generate an M-fuzzifying topological convex space. Some characterizations of M-fuzzifying topological convex spaces are presented. Finally, the notion of M-fuzzifying weak topologies is obtained from M-fuzzifying topological convex spaces.

1. **Introduction**

Referring to [17], a topology and a convexity on the same set are compatible if all polytopes of convexity are closed. The resulting combined structure is called a topological convex structure. The notion of topological convex structures has been introduced by Jamison [5] in 1974. Based on a topological convex structure, the notion of weak topologies was introduced by M.L.J. van de Vel [18] in 1983. So it is natural to consider the reasonable generalization of the topological convex structure in the setting of many-valued sets.

In 1994, Rosa [11] introduced fuzzy topology fuzzy convexity spaces on the same set. However, the compatibility of fuzzy topology and fuzzy convexity on the same underlying set was not discussed in his paper. In 2009, Maruyama [9] generalized Rosa's definition of fuzzy convexity to L-fuzzy setting. In Rosa and Maruyama's definition, an L-fuzzy convex structure is a crisp family of L-fuzzy subsets of X satisfying certain axioms. Above approach to the study of convex spaces in the fuzzy context was similar to the study of L-topology in the sense of Chang and Lowen. Another approach to the study of topological spaces in the fuzzy context different from that of L-topology was due to Höhle. He introduced the notion of fuzzy measurable spaces with the idea of giving degrees in [0,1] to some topological terms in [4]. By the similar ideas, Ying [23, 24, 25] gave a new approach for fuzzy topology from a logical point of view, i.e., a concept of fuzzifying topologies was given; recently, Shi and Xiu introduced the concept of M-fuzzifying convex structures in [15]. Based on concepts of fuzzifying topologies and M-fuzzifying convex structures, many results are obtained in [2, 7, 8, 12, 16, 20, 21].

In this paper, we will combine M-fuzzifying topological structures with M-fuzzifying convex structures and introduce the compatibility of M-fuzzifying topological structures and M-fuzzifying convex structures on the same underlying set,

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whence give the notion of M-fuzzifying topological convex spaces. Then we will introduce the concept of M-fuzzifying weak topologies.

2. Preliminaries

Throughout this paper, $(M, \vee, \wedge, ')$ denotes a completely distributive lattice with an order-reversing involution $'$, i.e., M is a De Morgan algebra. The smallest element and the largest element in M are denoted by $\perp$ and $\top$, respectively. For $B \subseteq M$, write $\bigvee B$ for the least upper bound of B and $\bigwedge B$ for the greatest lower bound of B. In particular, $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$. The wedge below relation $\prec$ in M is defined as follow: for $a, b \in M, a \prec b$ (a is wedge below b) if and only if for each subset $D \subseteq M$, $b \leq \bigvee D$ always implies $a \leq d$ for some $d \in D$ [1]. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M : a \prec b\}$ for every $b \in M$. The wedge below relation in M has the interpolation property, that is, $a \prec c \Rightarrow \exists\ b \in M$ such that $a \prec b \prec c$. Moreover, we know that $a \prec \bigvee_{j \in J} b_j$ $\Rightarrow \exists\ b_j$ such that $a \prec b_j$. An element $a \in M$ is called a co-prime element if $b \vee c \geq a$ implies $b \geq a$ or $c \geq a$ [3]. The set of non-zero co-prime elements in M is denoted by $J(M)$. $\{a \in J(M) : a \prec b\}$ is a minimal family of b in [19].

We denote the set of all subsets (all finite subsets, respectively) of a set X by 2^X (2^X_{fin}, respectively). A family $\{A_j : j \in J\}$ is up-directed provided for each $A_1, A_2 \in \{A_j : j \in J\}$ there is a third element $A_3 \in \{A_j : j \in J\}$ such that $A_1 \subseteq A_3$ and $A_2 \subseteq A_3$.

For a set X, M^X denotes the set of all M-subsets on X. Let $A \in M^X$. For $a \in M$, we can define $A_{[a]} = \{x \in X \mid A(x) \geq a\}$.

Definition 2.1. [4, 23] An M-fuzzifying topology is a mapping $\mathcal{T} : 2^X \to M$ such that

(FY1): $\mathcal{T}(X) = \mathcal{T}(\emptyset) = \top$;
(FY2): $\mathcal{T}(U \cap V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$ for any $U, V \in 2^X$;
(FY3): $\mathcal{T}(\bigcup\limits_{j \in J} U_j) \geq \bigwedge\limits_{j \in J} \mathcal{T}(U_j)$ for each family $\{U_j | j \in J \subseteq 2^X\}$.

If $\mathcal{T}$ is an M-fuzzifying topology on X, then the pair $(X, \mathcal{T})$ is called an M-fuzzifying topological space. For $\mathcal{T}$, let $\mathcal{T}^{co}(U) = \mathcal{T}(X - U)$ for each $U \in 2^X$, the value $\mathcal{T}^{co}(U)$ is interpreted as the degree of closeness of U. Then the mapping $\mathcal{T}^{co} : 2^X \to M$ is called an M-fuzzifying co-topology.

In following sections, $\mathcal{T}^{co}$ denotes M-fuzzifying co-topology of the corresponding M-fuzzifying topology $\mathcal{T}$.

Definition 2.2. [15] A mapping $\mathcal{C} : 2^X \to M$ is called an M-fuzzifying convexity on a set X if it satisfies the following conditions:

(FC1): $\mathcal{C}(\emptyset) = \mathcal{C}(X) = \top$;
(FC2): if $\{A_i : i \in \Lambda\} \subseteq 2^X$ is non-empty, then $\mathcal{C}(\bigcap\limits_{i \in \Lambda} A_i) \geq \bigwedge\limits_{i \in \Lambda} \mathcal{C}(A_i)$;
(FC3): if $\{A_i : i \in \Lambda\} \subseteq 2^X$ is non-empty and totally ordered by inclusion, then $\mathcal{C}(\bigcup\limits_{i \in \Lambda} A_i) \geq \bigwedge\limits_{i \in \Lambda} \mathcal{C}(A_i)$.

If $\mathcal{C}$ is an M-fuzzifying convexity on X, then the pair $(X, \mathcal{C})$ is called an M-fuzzifying convex space.

Definition 2.3. [14, 23] An M-fuzzifying closure operator on X is a mapping $cl : 2^X \to M^X$ satisfying the following conditions for any $A, B \in 2^X, x \in X$:

(FCL0): $cl(\emptyset)(x) = \bot$;
(FCL1): $cl(A)(x) = \top$ if $x \in A$;
(FCL2): $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$;
(FCL3): $cl(A)(x) = \bigwedge\limits_{x \notin B \supseteq A} \left(\bigvee\limits_{y \notin B} cl(B)(y) \right)$.

If a mapping $cl : 2^X \to M^X$ satisfies (FCL1) and (FCL2), then (FCL3) is equivalent to (FCL3*): $cl(cl(A)_{[\lambda]})_{[\lambda]} \subseteq cl(A)_{[\lambda]}$ for all $\lambda \in M$.

Definition 2.4. [12, 23] An M-fuzzifying closure operator $cl : 2^X \to M^X$ is called M-fuzzifying topological closure operator if it satisfies the following condition:

(FCL4): $cl(A \cup B) = cl(A) \vee cl(B)$.

In [23], Ying proved that $cl_{\mathcal{T}} : 2^X \to M^X$ is an M-fuzzifying topological closure operator induced by the M-fuzzifying topology $\mathcal{T}$, where $cl_{\mathcal{T}}$ is defined by $cl_{\mathcal{T}}(A)(x) = \bigwedge\limits_{x \notin B \supseteq A} (\mathcal{T}(X - B))'$; and $\mathcal{T}_{cl}$ is an M-fuzzifying topology induced by the M-fuzzifying topological closure operator cl, in which it is defined by $\mathcal{T}_{cl}(A) = \bigwedge\limits_{x \notin (X-A)} (cl(X - A)(x))'$. And $cl_{\mathcal{T}_{cl}} = cl, \mathcal{T}_{cl_{\mathcal{T}}} = \mathcal{T}$.

Definition 2.5. [15] An M-fuzzifying closure operator $co : 2^X \to M^X$ is called M-fuzzifying hull (or algebraic closure) operator if it satisfies the following condition:

(FCL5): $co_{\mathcal{C}}(A)(x) = \bigvee\{co_{\mathcal{C}}(F)(x) : F \in 2^A_{fin}\}$.

In [15], Shi and Xiu proved that $co_{\mathcal{C}}$ is an M-fuzzifying hull operator if $\mathcal{C}$ is an M-fuzzifying convexity, where $co_{\mathcal{C}}$ is defined by $co_{\mathcal{C}}(A)(x) = \bigwedge\limits_{x \notin B \supseteq A} (\mathcal{C}(B))'$; and $\mathcal{C}_{co}$, defined by $\mathcal{C}_{co}(A) = \bigwedge\limits_{x \notin A} (co(A)(x))'$, is an M-fuzzifying convexity if co is an M-fuzzifying hull operator. And $co_{\mathcal{C}_{co}} = co, \mathcal{C}_{co_{\mathcal{C}}} = \mathcal{C}$.

Definition 2.6. [16] A mapping $h : 2^X_{fin} \to M^X$ is called an M-fuzzifying restricted hull operator if it satisfies the following conditions:

(MH1): $h(\emptyset)(x) = \bot$, for each $x \in X$;
(MH2): $h(F)(x) = \top$ if $x \in F$, for $F \in 2^X_{fin}$;
(MH3): $h(G)(x) \wedge [\bigwedge\limits_{y \in G} h(F)(y)] \leq h(F)(x)$, for all $F, G \in 2^X_{fin}$, for all $x \in X$.

In [16], Shi and Li proved that $h_{\mathcal{C}}$ is an M-fuzzifying restricted hull operator if $\mathcal{C}$ is an M-fuzzifying convexity, where $h_{\mathcal{C}} = co_{\mathcal{C}}|_{2^X_{fin}}$, $co_{\mathcal{C}}$ is defined as above; and $\mathcal{C}_h$, defined by $\mathcal{C}_h(A) = \bigwedge\limits_{x \notin A, F \in 2^A_{fin}} (h(F)(x))'$, is an M-fuzzifying convexity if h is an M-fuzzifying restricted hull operator. And $h_{\mathcal{C}_h} = h, \mathcal{C}_{h_{\mathcal{C}}} = \mathcal{C}$.

3. M-fuzzifying Topology and M-fuzzifying Convexity on the Same Set

Firstly, let us recall some definitions and results. $\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda$ denotes the union of an up-directed set $\{B_\lambda : \lambda \in \Lambda\} \subseteq 2^X$.

In the classical situation, we know that there exist a topology and a compatible convexity on the same underlying set. Referring to [17], let X be a set equipped with a topology $\mathcal{T}$ and with a convexity $\mathcal{C}$. We say that $\mathcal{T}$ is compatible with the convexity $\mathcal{C}$ provided all polytopes of $\mathcal{C}$ are closed in $\mathcal{T}$. So the triple $(X, \mathcal{T}, \mathcal{C})$ is called a topological convex space. From the definition of $(X, \mathcal{T}, \mathcal{C})$, it is easy to see that $\mathcal{C}$ can be generated by closed sets of $\mathcal{T}$.

In the following, we will give the compatibility of M-fuzzifying topologies and M-fuzzifying convexities in the many-valued situation.

Definition 3.1. Let X be a set equipped with an M-fuzzifying topology $\mathcal{T}$ and with an M-fuzzifying convexity $\mathcal{C}$, and let h be the M-fuzzifying restricted hull operator of $\mathcal{C}$. Then $\mathcal{T}$ is called compatible with the M-fuzzifying convexity $\mathcal{C}$ provided for each $F \in 2^X_{fin}$ and for each $x \in X$,

$$h(F)(x) = \bigwedge_{x\notin B\supseteq F} [\mathcal{T}^{co}(B) \wedge \mathcal{C}(B)]'.$$

Definition 3.2. A triple $(X, \mathcal{T}, \mathcal{C})$ consisting of a set X, an M-fuzzifying topology $\mathcal{T}$ on X, and an M-fuzzifying convexity $\mathcal{C}$ on X, is called an M-fuzzifying topological convex space provided $\mathcal{T}$ is compatible with $\mathcal{C}$.

Theorem 3.3. *Let cl and co (h, respectively) be the M-fuzzifying topological closure operator and the M-fuzzifying (restricted, respectively) hull operator of $(X, \mathcal{T}, \mathcal{C})$. Then the following conditions are equivalent.*

(1): $(X, \mathcal{T}, \mathcal{C})$ *is an M-fuzzifying topological convex space.*

(2): $h(F)(x) = \bigwedge_{x\notin B\supseteq F} \left(\bigvee_{y\notin B} [cl(B)(y) \vee co(B)(y)] \right), \ \forall F \in 2^X_{fin}, \forall x \in X.$

(3): $cl(h(F)_{[a]})_{[a]} \subseteq h(F)_{[a]}, \ \forall F \in 2^X_{fin}, \forall a \in M.$

Proof. (1) $\Leftrightarrow$ (2) By Definition 2.4-2.5, and Definition 3.1, for any $F \in 2^X_{fin}$ and for any $x \in X$,

$$\begin{aligned}
& h(F)(x) \\
= & \bigwedge_{x\notin B\supseteq F} \left(\bigvee_{y\notin B} [cl(B)(y) \vee co(B)(y)] \right) \\
= & \bigwedge_{x\notin B\supseteq F} \left(\bigwedge_{y\notin B} [cl(B)(y)]' \wedge [co(B)(y)]' \right)' \\
= & \bigwedge_{x\notin B\supseteq F} \left(\bigwedge_{y\notin B} [cl(B)(y)]' \wedge \bigwedge_{y\notin B} [co(B)(y)]' \right)'
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{x\notin B\supseteq F} [\mathcal{T}(X-B)\wedge \mathcal{C}(B)]' \\
&= \bigwedge_{x\notin B\supseteq F} [\mathcal{T}^{co}(B)\wedge \mathcal{C}(B)]'.
\end{aligned}
$$

(2) $\Rightarrow$ (3) For all $F\in 2^X_{fin}$ and for all $a\in M$,

$$
\begin{aligned}
& x\notin h(F)_{[a]} \\
\Rightarrow\;& a\not\leq h(F)(x) \\
\Rightarrow\;& h(F)(x)'\not\leq a' \\
\Rightarrow\;& \bigvee_{x\notin B\supseteq F}\left(\bigwedge_{y\notin B}[cl(B)(y)\vee co(B)(y)]'\right)\not\leq a' \\
\Rightarrow\;& \text{there exists } B\supseteq F \text{ with } x\notin B \text{ such that } \bigwedge_{y\notin B}[cl(B)(y)'\wedge co(B)(y)']\not\leq a' \\
\Rightarrow\;& \forall y\notin B, cl(B)(y)\not\geq a \text{ and } co(B)(y)\not\geq a \\
\Rightarrow\;& \forall y\notin B, y\notin cl(B)_{[a]} \text{ and } y\notin co(B)_{[a]} \\
\Rightarrow\;& cl(B)_{[a]}\subseteq B \text{ and } co(B)_{[a]}\subseteq B \\
\Rightarrow\;& cl(B)_{[a]}=co(B)_{[a]}=B \\
\Rightarrow\;& h_{[a]}(F)\subseteq co(B)_{[a]}\subseteq B \\
\Rightarrow\;& cl(h(F)_{[a]})_{[a]}\subseteq cl(cl(F)_{[a]})_{[a]}=B \\
\Rightarrow\;& x\notin cl(h(F)_{[a]})_{[a]}.
\end{aligned}
$$

(3) $\Rightarrow$ (2) From (FCL2), $h(F)(x)\leq \bigwedge_{x\notin B\supseteq F}\left(\bigvee_{y\notin B}[cl(B)(y)\vee co(B)(y)]\right)$ for each $F\in 2^X_{fin}, x\in X$. It suffices to show that

$$
h(F)(x)\geq \bigwedge_{x\notin B\supseteq F}\left(\bigvee_{y\notin B}[cl(B)(y)\vee co(B)(y)]\right).
$$

Let a be a co-prime element with $a\prec \bigwedge_{x\notin B\supseteq F}\left(\bigvee_{y\notin B}[cl(B)(y)\vee co(B)(y)]\right)$. If $a\not\leq h(F)(x)$, then $x\notin h(F)_{[a]}$. $h(F)_{[a]}$ is denoted by A. By (FCL1), $A\supseteq F$ and $a\prec \bigvee_{y\notin A}[cl(A)(y)\vee co(A)(y)]$. Hence there exists $y_0\notin A$ with $a\leq cl(A)(y_0)\vee co(A)(y_0)$. Since $a\in J(M)$, then either $a\leq cl(A)(y_0)$ or $a\leq co(A)(y_0)$. If $a\leq co(A)(y_0)$, then $y_0\in co(A)_{[a]}=co(h(F)_{[a]})_{[a]}=A$ by (FCL3*), whence this is a contradiction. If $a\leq cl(A)(y_0)$, then it follows from (3) that $y_0\in cl(A)_{[a]}=co(h(F)_{[a]})_{[a]}=A$, and hence this is a contradiction. Then $a\leq h(F)(x)$ holds. By the arbitrariness of a, $h(F)(x)\geq \bigwedge_{x\notin B\supseteq F}\left(\bigvee_{y\notin B}[cl(B)(y)\vee co(B)(y)]\right)$. □

Example 3.4. Referring to [22], M-fuzzifying topological vector space can be defined as follows. Let X be a vector space over $\mathbb{K}$ ($\mathbb{K}$ is a non-discrete-valued field) and $\mathcal{T}$ an M-fuzzifying topology on X. The pair $(X, \mathcal{T})$ is said to be an M-fuzzifying topological vector space, if the following mappings

(1): $f : X \times X \to X, (x, y) \mapsto x + y,$
(2): $g : \mathbb{K} \times X \to X, (k, x) \mapsto kx,$

are continuous, where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding M-fuzzifying product topologies $\mathcal{T} \times \mathcal{T}$ and $\mathfrak{I}_{\mathbb{K}} \times \mathcal{T}$ (here $\mathfrak{I}_{\mathbb{K}}$ is an M-fuzzifying topology determined by the crisp neighborhood structure on $\mathbb{K}$), respectively.

Now, we can construct an M-fuzzifying topological convex space as follows. Using $(X, \mathcal{T})$, we can define the corresponding M-fuzzifying topological closure operator $cl_{\mathcal{T}}$ by $cl_{\mathcal{T}}(A)(x) = \bigwedge\limits_{x \notin B \supseteq A} (\mathcal{T}(X - B))'$ for each $A \in 2^X$ and for each $x \in X$. $cl_{\mathcal{T}}(A)$ is called an M-fuzzifying convex closed set if it satisfies the following condition

(FCC): $cl_{\mathcal{T}}(A)(u) \wedge cl_{\mathcal{T}}(A)(v) \leq cl_{\mathcal{T}}(A)(tu + (1 - t)v), \ \forall u, v \in X, \forall t \in [0, 1],$

for each $A \in 2^X$.

Next, we can define a mapping $h : 2^X_{fin} \to M^X$ by

$$h(F)(x) = \wedge\{cl_{\mathcal{T}}(A)(x) \mid F \subseteq A, cl_{\mathcal{T}}(A) \text{ satisfies (FCC) }\}$$

for each $F \in 2^X_{fin}, x \in X$. Then h is an M-fuzzifying restricted hull operator. In following, we need to verify that it satisfies (MH1)-(MH3).

(MH1): $h(\emptyset)(x) = cl_{\mathcal{T}}(\emptyset)(x) = \bot$, for each $x \in X$.

(MH2): If $F \in 2^X_{fin}, x \in F$, then $h(F)(x) = \top$ since $cl_{\mathcal{T}}$ satisfies (FCL1).

(MH3): We need to show that $h(G)(x) \wedge [\bigwedge\limits_{y \in G} h(F)(y)] \leq h(F)(x)$ for all $F, G \in 2^X_{fin}$, for all $x \in X$. Since $cl_{\mathcal{T}}$ satisfies (FCL3),

$$\begin{aligned}
& h(F)(x) \\
= \ & \bigwedge\{cl_{\mathcal{T}}(A)(x) \mid F \subseteq A, cl_{\mathcal{T}}(A) \text{ satisfies (FCC) }\} \\
= \ & \bigwedge\left\{\bigwedge_{x \notin B \supseteq A}\left(\bigvee_{z \notin B} cl_{\mathcal{T}}(B)(z)\right) \mid F \subseteq A, cl_{\mathcal{T}}(A) \text{ satisfies (FCC)}\right\}
\end{aligned}$$

Observe that, for any $B \supseteq A$ with $x \notin B$, if $G \nsubseteq B$, then

$$\begin{aligned}
\bigvee_{z \notin B} cl_{\mathcal{T}}(B)(z) &\geq \bigvee_{z \in (G - B)} cl_{\mathcal{T}}(B)(z) \\
&\geq \bigvee_{z \in (G - B)} h(F)(z) \\
&\geq \bigwedge_{y \in G} h(F)(y).
\end{aligned}$$

Otherwise, if $G \subseteq B$, then $\bigvee_{z \notin B} cl_{\mathcal{T}}(B)(z) \geq cl_{\mathcal{T}}(B)(x) \geq h(G)(x)$. Thus, by the arbitrariness of A, B (with $x \notin B \supseteq A \supseteq F$), $h(G)(x) \wedge [\bigwedge_{y \in G} h(F)(y)] \leq h(F)(x)$ holds.

Using the mapping h introduced above, an M-fuzzifying convexity $\mathcal{C}_h$ can be defined by

$$\mathcal{C}_h(A) = \bigwedge_{F \in 2^X_{fin}, x \notin A} [h(F)(x)]' \text{ for each } A \in 2^X.$$

Moreover, the M-fuzzifying hull operator $co_{\mathcal{C}_h}$ of $\mathcal{C}_h$ satisfies $co_{\mathcal{C}_h}(F) = h(F)$.

Finally, we will show that $\mathcal{C}_h$ is compatible with M-fuzzifying topological vector space $(X, \mathcal{T})$, i.e., $cl(h(F)_{[\lambda]})_{[\lambda]} \subseteq h(F)_{[\lambda]}$, $\forall F \in 2^X_{fin}, \forall \lambda \in J(M)$.

In fact, for any $x \in X$,

$$\begin{aligned}
cl(h(F)_{[\lambda]})(x) &\leq cl\left(\left(\bigwedge\{cl(A) \mid F \subseteq A, cl(A) \text{ satisfies (FCC) }\}\right)_{[\lambda]}\right)(x) \\
&\leq \bigwedge\{cl(cl(A)_{[\lambda]})(x) \mid F \subseteq A, cl(A) \text{ satisfies (FCC) }\} \\
&\quad [\text{We denote the type } cl(A) \text{ that satisfies (FCC) by } cl(A_F)] \\
&= \bigwedge_{F \subseteq A_F} cl(cl(A_F)_{[\lambda]})(x)
\end{aligned}$$

Hence $cl(h(F)_{[\lambda]})_{[\lambda]} \subseteq \bigcap_{F \subseteq A_F} cl(cl(A_F)_{[\lambda]})_{[\lambda]} \overset{(FCL3^*)}{=} \bigcap_{F \subseteq A_F} cl(A_F)_{[\lambda]} = h(F)_{[\lambda]}$. Thus, the triple $(X, \mathcal{T}, \mathcal{C}_h)$ is an M-fuzzifying topological convex space.

Next, we give a method allowing to construct a compatible M-fuzzifying convexity in an M-fuzzifying topological convex space.

Theorem 3.5. *Let $(X, \mathcal{T})$ be an M-fuzzifying topological space and define a mapping $\mathcal{C} : 2^X \to M$ by*

$$\mathcal{C}(A) = \bigvee_{\bigcup^{dir}_{\lambda \in \Lambda} B_\lambda = A} \left(\bigwedge_{\lambda \in \Lambda} \mathcal{T}(X - B_\lambda)\right)$$

for each $A \in 2^X$. Then $\mathcal{C}$ is an M-fuzzifying convexity on X and $cl_{\mathcal{T}}(F) = co_{\mathcal{C}}(F)$ for each $F \in 2^X_{fin}$. Moreover, $(X, \mathcal{T}, \mathcal{C})$ is an M-fuzzifying topological convex space.

Proof. It suffices to verify that $\mathcal{C}$ satisfies (FC1)-(FC3).

With respect to (FC1), $\mathcal{C}(\emptyset) = \mathcal{C}(X) = \top$ since $\mathcal{T}$ satisfies the condition (FY1).

(FC2): For any family $\{A_j\}_{j \in J} \subseteq 2^X$, we need to show $\mathcal{C}(\bigcap_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathcal{C}(A_j)$, i.e.,

$$\bigvee_{\bigcup^{dir}_{\lambda \in \Lambda} B_\lambda = \bigcap_{j \in J} A_j} \left(\bigwedge_{\lambda \in \Lambda} \mathcal{T}(X - B_\lambda)\right) \geq \bigwedge_{j \in J} \left(\bigvee_{\bigcup^{dir}_{i \in \Lambda_j} H_{i,j} = A_j} \left(\bigwedge_{i \in \Lambda_j} \mathcal{T}(X - H_{i,j})\right)\right).$$

Let $a \in J(M)$ with $a \leq \bigwedge\limits_{j\in J} \mathcal{C}(A_j)$. Then, for any $b \prec a$, there exists an up-directed family $\{H_{i,j}\}_{i\in J_j} \subseteq 2^X$ with $\bigcup_{i\in\Lambda_j}^{dir} H_{i,j} = A_j$ such that $\bigwedge\limits_{i\in\Lambda_j} \mathcal{T}(X - H_{i,j}) \geq b$ for all $j \in J$, whence $\mathcal{T}(X - H_{i,j}) \geq b$ for any $i \in \Lambda_j, j \in J$. Further, we can get $\bigcap\limits_{j\in J} A_j = \bigcap\limits_{j\in J} (\bigcup\limits_{i\in\Lambda_j} H_{i,j}) = \bigcup\{\bigcap\limits_{j\in J} H_{f(j),j} \mid f \in \prod\limits_{j\in J} \Lambda_j\}$. The family $\{\bigcap\limits_{j\in J} H_{f(j),j} \mid f \in \prod\limits_{j\in J} \Lambda_j\}$ is up-directed since the family $\{H_{i,j}\}_{i\in J_j}$ is up-directed for any $j \in J$. Since $\mathcal{T}$ satisfies (FY2), we see

$$\begin{aligned}
\bigvee_{\bigcup\limits_{\lambda\in\Lambda} B_\lambda = \bigcap\limits_{j\in J} A_j} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}(X - B_\lambda)\right) &\geq \bigwedge_{f\in\prod\limits_{j\in J}\Lambda_j} \mathcal{T}(X - \bigcap_{j\in J} H_{f(j),j}) \\
&\geq \bigwedge_{f\in\prod\limits_{j\in J}\Lambda_j} \left(\bigwedge_{j\in J} \mathcal{T}(X - H_{f(j),j})\right) \\
&\geq b,
\end{aligned}$$

whence $\mathcal{C}(\bigcap\limits_{j\in J} A_j) \geq \bigvee\{b : b \prec a\} = a$. Then, we can obtain $\mathcal{C}(\bigcap\limits_{j\in J} A_j) \geq \bigwedge\limits_{j\in J} \mathcal{C}(A_j)$.

(FC3): For any up-directed family $\{A_j\}_{j\in J} \subseteq 2^X$, we need to verify $\mathcal{C}(\bigcup_{j\in J}^{dir} A_j) \geq \bigwedge\limits_{j\in J} \mathcal{C}(A_j)$, i.e.,

$$\bigvee_{\bigcup_{\lambda\in\Lambda}^{dir} B_\lambda = \bigcup_{j\in J}^{dir} A_j} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}(X - B_\lambda)\right) \geq \bigwedge_{j\in J} \left(\bigvee_{\bigcup_{i\in\Lambda_j}^{dir} H_{i,j} = A_j} \left(\bigwedge_{i\in\Lambda_j} \mathcal{T}(X - H_{i,j})\right)\right).$$

Let $a \in L$ with $a \leq \bigwedge\limits_{j\in J} \mathcal{C}(A_j)$. Then, for any $b \prec a$, there exists an up-directed family $\{H_{i,j}\}_{i\in J_j} \subseteq 2^X$ with $\bigcup\limits_{i\in\Lambda_j} H_{i,j} = A_j$ such that $\bigwedge\limits_{i\in\Lambda_j} \mathcal{T}(X - H_{i,j}) \geq b$ for all $j \in J$, whence $\mathcal{T}(X - H_{i,j}) \geq b$ for any $i \in \Lambda_j, j \in J$. The set of all $H_{i,j}$ for any $j \in J, i \in \Lambda_j$, is denoted by $\mathcal{D}_0$. And we see

$$\bigcup \mathcal{D}_0 = \bigcup\{H_{i,j} : \forall j \in J, \forall i \in \Lambda_j\} = \bigcup_{j\in J} A_j.$$

Consider the family $\mathcal{D}_1$ of all sets of type $\bigcup\limits_{1\leq\gamma\leq n} H_\gamma$, where $H_\gamma \in \mathcal{D}_0$. Then $\mathcal{D}_1$ is up-directed and $\bigcup \mathcal{D}_1 = \bigcup\limits_{j\in J} A_j$. Since $\mathcal{T}$ satisfies (FY2), we see

$$\begin{aligned}
\bigvee_{\bigcup\limits_{\lambda\in\Lambda} B_\lambda = \bigcup\limits_{j\in J} A_j} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}(X - B_\lambda)\right) &\geq \bigwedge\{\mathcal{T}(X - \bigcup_{1\leq\gamma\leq n} H_\gamma) \mid \bigcup_{1\leq\gamma\leq n} H_\gamma \in \mathcal{D}_1\} \\
&\geq \bigwedge\{\bigwedge_{1\leq\gamma\leq n} \mathcal{T}(X - H_\gamma) \mid \bigcup_{1\leq\gamma\leq n} H_\gamma \in \mathcal{D}_1\} \\
&\geq b,
\end{aligned}$$

and hence $\mathcal{C}(\bigcup_{j\in J} A_j) \geq \bigvee\{b : b \prec a\} = a$. Then, we can obtain $\mathcal{C}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathcal{C}(A_j)$.

To complete the proof of the theorem we have to establish the equality $cl_{\mathcal{T}}(F) = co_{\mathcal{C}}(F)$ for all $F \in 2^X_{fin}$. On one hand, by Definition 2.4-2.5, we can get

$$\begin{aligned} co_{\mathcal{C}}(F)(x) &= \bigwedge_{x\notin B\supseteq F} [\mathcal{C}(B)]' \\ &= \bigwedge_{x\notin B\supseteq F} \left(\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = B} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}_{cl_{\mathcal{T}}}(X - B_\lambda) \right) \right)' \\ &\leq \bigwedge_{x\notin B\supseteq F} [\mathcal{T}_{cl_{\mathcal{T}}}(B)]' \\ &= cl_{\mathcal{T}}(F)(x). \end{aligned}$$

On the other hand, by Definition 2.4-2.5 and the definition of $\mathcal{C}$, We prove the converse inequality as follows.

$$\begin{aligned} co_{\mathcal{C}}(F)(x) &= \bigwedge_{x\notin B\supseteq F} \left(\bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = B} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}_{cl_{\mathcal{T}}}(X - B_\lambda) \right) \right)' \\ &= \bigwedge_{x\notin B\supseteq F} \left(\bigwedge_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = B} \left(\bigvee_{\lambda\in\Lambda} \left(\bigwedge_{y\notin B_\lambda} [cl_{\mathcal{T}}(B_\lambda)]' \right)' \right) \right) \\ &= \bigwedge_{x\notin B\supseteq F} \left(\bigwedge_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = B} \left(\bigvee_{\lambda\in\Lambda} \left(\bigvee_{y\notin B_\lambda} cl_{\mathcal{T}}(B_\lambda)(y) \right) \right) \right). \end{aligned}$$

For any $x \notin B \supseteq F$, the up-directed family $\{B_\lambda : \lambda \in \Lambda\}$ with $\bigcup_{\lambda\in\Lambda} B_\lambda = B$, it follows from the finiteness of F and from the up-directed property of $\{B_\lambda : \lambda \in \Lambda\}$ that there exits B_λ such that $F \subseteq B_\lambda$. Since $cl_{\mathcal{T}}$ is order-preserving and $x \notin B \supseteq B_\lambda$, we have $\bigvee_{y\notin B_\lambda} cl_{\mathcal{T}}(B_\lambda)(y) \geq cl_{\mathcal{T}}(B_\lambda)(x) \geq cl_{\mathcal{T}}(F)(x)$. Then, by the arbitrariness of B and $\{B_\lambda : \lambda \in \Lambda\}$,

$$\begin{aligned} co_{\mathcal{C}}(F)(x) &= \bigwedge_{x\notin B\supseteq F} \left(\bigwedge_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = B} \left(\bigvee_{\lambda\in\Lambda} \left(\bigvee_{y\notin B_\lambda} cl_{\mathcal{T}}(B_\lambda)(y) \right) \right) \right) \\ &\geq cl_{\mathcal{T}}(F)(x). \end{aligned}$$

Thus, we obtain $cl_{\mathcal{T}}(F) = co_{\mathcal{C}}(F)$.

From $cl_{\mathcal{T}}(F) = co_{\mathcal{C}}(F) = h_{\mathcal{C}}(F)$, it is clear that $cl_{\mathcal{T}}(h_{\mathcal{C}}(F)_{[\lambda]})_{[\lambda]} \subseteq h_{\mathcal{C}}(F)_{[\lambda]}$ for each $\lambda \in M$, whence the triple $(X, \mathcal{T}, \mathcal{C})$ is an M-fuzzifying topological convex space. $\square$

Example 3.6. In [13], Shi introduced the notion of (L, M)-fuzzy metrics. When $L = 2 = \{0, 1\}$, a $(2, I)$-fuzzy metric is equivalent to a Kramosil and Michalek [6] fuzzy metric and it is also equivalent to a Morsi [10] fuzzy metric. In the sequel, it will be called an M-fuzzifying metric and it can be stated as follows.

An M-fuzzifying pseudo-metric on a non-empty set X is a mapping $d : X \times X \to [0, +\infty)(M)$ which satisfies for each $x, y, z \in X$:

(FM1): $d(x, x)(0+) = \bigvee_{t>0} d(x, x)(t) = \perp$;
(FM2): $d(x, y) = d(y, x)$;
(FM3): $d(x, z)(r + s) \leq d(x, y)(r) \vee d(y, z)(s)$, for all $r, s > 0$.

We can define a mapping $cl_d : 2^X \to M^X$ by for each $A \in 2^X$ and for each $x \in X$,

$$cl_d(A)(x) = \bigwedge_{\varepsilon>0} \left(\bigvee_{a\in A} [d(x, a)(\varepsilon)]' \right).$$

Then cl_d is an M-fuzzifying topological closure operator. Now, we need to verify that it satisfies (FCL1)-(FCL4). By the method of Theorem 3.7 in [13], (FCL0)-(FCL3) can be proved. We only verify (FCL4) as follows.

(FCL4): From the definition of cl_d, we see $cl_d(A) \vee cl_d(B) \leq cl_d(A \cup B)$, and hence only need to verify the invertible inequality as follows.

$$\begin{aligned} cl_d(A)(x) \vee cl_d(B)(x) &= \bigwedge_{\varepsilon>0} \left(\bigvee_{a\in A} [d(x, a)(\varepsilon)]' \right) \vee \bigwedge_{r>0} \left(\bigvee_{b\in B} [d(x, b)(r))]' \right) \\ &= \left[\bigvee_{\varepsilon>0, r>0} \left(\bigwedge_{a\in A} d(x, a)(\varepsilon) \wedge \bigwedge_{b\in B} d(x, b)(r) \right) \right]' \\ &= \bigwedge_{\varepsilon>0, r>0} \left[\bigwedge_{a\in A} d(x, a)(\varepsilon) \wedge \bigwedge_{b\in B} d(x, b)(r) \right]' \\ &\geq \bigwedge_{s>0} \left[(\bigwedge_{y\in A\cup B} d(x, y)(s)) \right]' \\ &= cl_d(A \cup B)(x). \end{aligned}$$

Hence cl_d can induced an M-fuzzifying topology $\mathcal{T}_d$, where $\mathcal{T}_d$ is defined by

$$\mathcal{T}_d(A) = \bigwedge_{x\notin(X-A)} cl_d(X - A)(x)', \text{ for each } A \in 2^X.$$

By Theorem 3.5, we can define an M-fuzzifying convexity $\mathcal{C}_d$ as follows:

$$\mathcal{C}_d(A) = \bigvee_{\bigcup^{dir}_{\lambda\in\Lambda} B_\lambda = A} \left(\bigwedge_{\lambda\in\Lambda} \mathcal{T}_d(X - B_\lambda) \right), \text{ for each } A \in 2^X.$$

Thus, the triple $(X, \mathcal{T}_d, \mathcal{C}_d)$ is an M-fuzzifying topological convex space.

4. M-fuzzifying Weak Topology

Referring to [17], a co-topology $\mathcal{T}$ of (closed sets of) the underlying set of a convex space $(X, \mathcal{C})$ is a weak topology provided it has a subbase of $\mathcal{C}$-convex sets. Observe the duality with the concept of compatibility (see Section 3). It appears that the term subbase can be used with two different meanings: a subbase (of closed sets) for a co-topology will be referred to as a closed subbase; a subbase for a convexity is called a convex subbase. Weak topologies are usually obtained from a topological convex structure $(X, \mathcal{T}, \mathcal{C})$ by considering the coarser topology $\mathcal{T}_w$, generated by $(X, \mathcal{T}, \mathcal{C})$. It is called the weak topology of $(X, \mathcal{T}, \mathcal{C})$. Moreover, in a topological convex structure a convex set is closed if and only if it is weakly closed (that is, closed in the corresponding weak topology).

Next, we generalize the notion of weak topology to M-fuzzy setting. Firstly, refer to concepts of base and subbase in [2, 24], we give notions of closed base and closed subbase of M-fuzzifying topology in M-fuzzifying situation.

Definition 4.1. Let $(X, \mathcal{T})$ be an M-fuzzifying topological space. Then a mapping $\mathcal{B} : 2^X \to M$ is called a closed base of $\mathcal{T}$ if $\mathcal{B}$ satisfies the following condition

$$\mathcal{T}^{co}(A) = \bigvee_{\bigcap_{i\in\Lambda} B_i = A} \left(\bigwedge_{i\in\Lambda} \mathcal{B}(B_i) \right), \ \forall A \in 2^X.$$

Definition 4.2. Let $(X, \mathcal{T})$ be an M-fuzzifying topological space. Then a mapping $\varphi : 2^X \to M$ is called a closed subbase of $(X, \mathcal{T})$ if $\varphi^{\sqcup} : 2^X \to M$ is a closed base, where

$$\varphi^{\sqcup}(A) = \bigvee_{\sqcup_{i\in\Lambda} B_i = A} \left(\bigwedge_{i\in\Lambda} \varphi(B_i) \right)$$

for each $A \in 2^X$ with $\sqcup$ stands for "finite joins".

Lemma 4.3. *Let $\mathcal{T}$ be an M-fuzzifying topology on X, let $\mathcal{C}$ be an M-fuzzifying convexity on X, and let $\varphi = \mathcal{T}^{co} \wedge \mathcal{C}$. Then a mapping $\varphi^{\sqcup} : 2^X \to M$, defined as Definition 4.2, satisfies the following conditions*

(1): $\varphi^{\sqcup}(\emptyset) = \varphi^{\sqcup}(X) = \top$,

(2): $\varphi^{\sqcup}(A \cup B) \geq \varphi^{\sqcup}(A) \wedge \varphi^{\sqcup}(B)$,

for each $A, B \in 2^X$.

Proof. (1): By (FY1) and (FC1), $\varphi^{\sqcup}(\emptyset) = \varphi^{\sqcup}(X) = \top$.

(2): Let $a \in J(M)$ with $\varphi^{\sqcup}(A) \wedge \varphi^{\sqcup}(B) \succ a$. Then $\varphi^{\sqcup}(A) \succ a$ and $\varphi^{\sqcup}(B) \succ a$. By $\varphi^{\sqcup}(A) = \bigvee_{\sqcup_{i\in\Lambda} B_i = A} \left(\bigwedge_{i\in\Lambda} \varphi(B_i) \right)$, there exists a family $\{H_i\}_{i\in\Lambda_\alpha}$ with $\sqcup_{i\in\Lambda_\alpha} H_i = A$ such that $\varphi(H_i) \geq a$ for each $i \in \Lambda_\alpha$. Also, using the definition of $\varphi^{\sqcup}(B)$, there exists a family $\{H_j\}_{j\in\Lambda_\beta}$ with $\sqcup_{j\in\Lambda_\beta} H_j = B$ such that $\varphi(H_j) \geq a$ for each $j \in \Lambda_\beta$. Hence $\sqcup\{H_k \mid k \in \Lambda_\alpha \cup \Lambda_\beta\} = A \cup B$, whence

$$\begin{aligned} \varphi^{\sqcup}(A \cup B) &= \bigvee_{\sqcup_{k\in\Lambda} H_k = A\cup B} \left(\bigwedge_{k\in\Lambda} \varphi(H_k) \right) \\ &\geq \bigwedge\{ \bigwedge_{k\in\Lambda} \varphi(H_k) \mid k \in \Lambda_\alpha \cup \Lambda_\beta \} \\ &\geq a. \end{aligned}$$

Thus, $\varphi^{\sqcup}(A \cup B) \geq \varphi^{\sqcup}(A) \wedge \varphi^{\sqcup}(B)$ holds. □

Lemma 4.4. *Let φ and $\varphi^{\sqcup}$ be defined as Lemma 4.3 and define a mapping $\mathcal{T}_w : 2^X \to M$ by $\mathcal{T}_w(X-A) = \mathcal{T}_w^{co}(A) = \bigvee_{\bigcap_{i\in\Lambda} B_i = A} \left(\bigwedge_{i\in\Lambda} \varphi^{\sqcup}(B_i)\right)$ for each $A \in 2^X$. Then $\mathcal{T}_w$ is an M-fuzzifying topology.*

Proof. (FY1): $\mathcal{T}_w(\emptyset) = \mathcal{T}_w(X) = \top$ by (1) of Lemma 4.3.

(FY2): For a family $\{A_j\}_{j\in J}$, we need to show that $\mathcal{T}_w(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathcal{T}_w(A_j)$, i.e., $\mathcal{T}_w^{co}(\bigcap_{j\in J}(X-A_j)) \geq \bigwedge_{j\in J} \mathcal{T}_w^{co}(X-A_j)$.

Let $a \in J(M)$ with $\bigwedge_{j\in J} \mathcal{T}_w^{co}(X-A_j) \succ a$. Then, for each $j \in J$, there exists $\{B_{ji} \mid i \in \Lambda_j\}$ with $\bigcap_{i\in\Lambda_j} B_{ji} = X - A_j$ such that $\varphi^{\sqcup}(B_{ji}) \geq a$ for all $i \in \Lambda_j$. And we know that $\bigcap_{j\in J}(X-A_j) = \bigcap_{j\in J}\left(\bigcap_{i\in\Lambda_j} B_{ji}\right)$. Hence

$$\begin{aligned}
\mathcal{T}_w^{co}(\bigcap_{j\in J}(X-A_j)) &= \bigvee_{\bigcap\limits_{k\in\Lambda} H_k = \bigcap\limits_{j\in J}(X-A_j)} \left(\bigwedge_{k\in\Lambda} \varphi^{\sqcup}(H_k)\right) \\
&\geq \bigwedge\{\bigwedge_{j\in J, i\in\Lambda_j} \varphi^{\sqcup}(B_{ji}) \mid \bigcap_{j\in J}(X-A_j) = \bigcap_{j\in J}\left(\bigcap_{i\in\Lambda_j} B_{ji}\right)\} \\
&\geq a.
\end{aligned}$$

(FY3): For $A, B \in 2^X$, it suffices to show that $\mathcal{T}_w^{co}(A\cup B) \geq \mathcal{T}_w^{co}(A) \wedge \mathcal{T}_w^{co}(B)$.

Let $a \in J(M)$ with $\mathcal{T}_w^{co}(A) \wedge \mathcal{T}_w^{co}(B) \succ a$. Then, there exists $\{H_i\}_{i\in\Lambda_\alpha}, \{H_j\}_{j\in\Lambda_\beta}$ with $\bigcap_{i\in\Lambda_\alpha} H_i = A$, $\bigcap_{j\in\Lambda_\beta} H_j = B$ such that $\varphi^{\sqcup}(H_\lambda) \geq a$ for each $\lambda \in \Lambda_\alpha \cup \Lambda_\beta$. And $\bigcap_{i\in\Lambda_\alpha, j\in\Lambda_\beta}(B_{ij}) = A \cup B$ with $B_{ij} = H_i \cup H_j$ holds. By (2) of Lemma 4.3, $\varphi^{\sqcup}(B_{ij}) = \varphi^{\sqcup}(H_i \cup H_j) \geq \varphi^{\sqcup}(H_i) \wedge \varphi^{\sqcup}(H_j) \geq a$. Hence

$$\begin{aligned}
\mathcal{T}_w^{co}(A\cup B) &= \bigvee_{\bigcap\limits_{k\in\Lambda} H_k = A\cup B} \left(\bigwedge_{k\in\Lambda} \varphi^{\sqcup}(H_k)\right) \\
&\geq \bigwedge\{\bigwedge_{i\in\Lambda_\alpha, j\in\Lambda_\beta} \varphi^{\sqcup}(B_{ji}) \mid \bigcap_{i\in\Lambda_\alpha, j\in\Lambda_\beta}(B_{ij}) = A\cup B\} \\
&\geq a.
\end{aligned}$$

□

By Definition 4.1-4.2 and Lemma 4.3-4.4, we see the following conclusion.

Theorem 4.5. *Let $\mathcal{T}$ be an M-fuzzifying topology on X, let $\mathcal{C}$ be an M-fuzzifying convexity on X, and let $\varphi = \mathcal{T}^{co} \wedge \mathcal{C}$. Then φ is a closed subbase of $\mathcal{T}_w$.*

By the foregoing results, we generalize the concept of weak topology to M-fuzzy setting.

Definition 4.6. Let $(X,\mathcal{T},\mathcal{C})$ be an M-fuzzifying topological convex space and let $\varphi=\mathcal{T}^{co}\wedge\mathcal{C}$. Then $\mathcal{T}_w$ is called an M-fuzzifying weak topology, which is generated by the closed subbase φ.

Connecting Example 3.4 and Example 3.6, we can easily give the correspondence M-fuzzifying weak topology. Next, we will study the compatibility of M-fuzzifying weak topology and M-fuzzifying convexity.

Theorem 4.7. *$(X,\mathcal{T},\mathcal{C})$ be an M-fuzzifying topological convex space if and only if $\mathcal{T}_w$ is compatible with $\mathcal{C}$, where $\mathcal{T}_w$ be define by Lemma 4.4.*

Proof. Sufficiency. Suppose that $\mathcal{T}_w$ is compatible with $\mathcal{C}$, i.e., $cl_w(h(F)_{[a]})_{[a]}\subseteq h(F)_{[a]},\ \forall F\in 2^X_{fin},\forall a\in M$. For $F\in 2^X_{fin}$, it suffices to prove that $cl(h(F)_{[a]})_{[a]}\subseteq h(F)_{[a]}$ for each $a\in M$. By the definition of $\mathcal{T}_w$ in Lemma 4.4, for any $A\in 2^X$,

$$\begin{aligned}
&\mathcal{T}_w^{co}(A)\\
=&\bigvee_{\bigcap_{i\in\Lambda}B_i=A}\left(\bigwedge_{i\in\Lambda}\varphi^{\sqcup}(B_i)\right)\\
=&\bigvee_{\bigcap_{i\in\Lambda}B_i=A}\left(\bigwedge_{i\in\Lambda}\left(\bigvee_{\sqcup_{j\in\Lambda_i}B_{ij}=B_i}\left(\bigwedge_{j\in\Lambda_i}\varphi(B_{ij})\right)\right)\right)\\
=&\bigvee_{\bigcap_{i\in\Lambda}B_i=A}\left(\bigwedge_{i\in\Lambda}\left(\bigvee_{\sqcup_{j\in\Lambda_i}B_{ij}=B_i}\left(\bigwedge_{j\in\Lambda_i}(\mathcal{T}^{co}\wedge\mathcal{C})(B_{ij})\right)\right)\right)\\
\leq&\bigvee_{\bigcap_{i\in\Lambda}B_i=A}\left(\bigwedge_{i\in\Lambda}\left(\bigvee_{\sqcup_{j\in\Lambda_i}B_{ij}=B_i}\left(\bigwedge_{j\in\Lambda_i}\mathcal{T}^{co}(B_{ij})\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
\leq&\bigvee_{\bigcap_{i\in\Lambda}B_i=A}\left(\bigwedge_{i\in\Lambda}\mathcal{T}^{co}(B_i)\right)\quad\text{by (FY3)}\\
\leq&\ \mathcal{T}^{co}(A)\quad\text{by (FY2).}
\end{aligned}$$

Hence for any $A\in 2^X$ and for any $x\in X$,

$$\begin{aligned}
cl_w(A)(x)&=\bigwedge_{x\notin B\supseteq A}\mathcal{T}_w^{co}(B)'\\
&\geq\bigwedge_{x\notin B\supseteq A}\mathcal{T}^{co}(B)'=cl(A)(x).
\end{aligned}$$

Thus, it follows from the compatibility of $(\mathcal{T}_w,\mathcal{C})$ that

$$cl(h(F)_{[a]})_{[a]}\subseteq cl_w(h(F)_{[a]})_{[a]}\subseteq h(F)_{[a]}.$$

Necessity. For each $F \in 2^X_{fin}$, we need to verify that $cl_w(h(F)_{[a]})_{[a]} \subseteq h(F)_{[a]}$ for each $a \in M$, i.e., if $x \notin h(F)_{[a]}$ for every $x \in X$, then $x \notin cl_w(h(F)_{[a]})_{[a]}$.

$$
\begin{aligned}
& x \notin h(F)_{[a]} \\
\Leftrightarrow \quad & h(F)(x) \not\geq a \\
\Leftrightarrow \quad & h(F)(x) = \bigwedge_{x \notin B \supseteq F} \left(\bigvee_{y \notin B} [cl(B)(y) \vee co(B)(y)] \right) \not\geq a
\end{aligned}
$$

By the proof of sufficiency in Theorem 3.3, we see that

$\Rightarrow$ there exists $B_0 \supseteq F$ with $x \notin B_0$ s.t. $\bigwedge_{y \notin B_0} [cl(B_0)(y)' \wedge co(B_0)(y)'] \not\leq a'$

and hence $h_{[a]}(F) \subseteq cl(B)_{[a]} = co(B)_{[a]} = B$.

$\Leftrightarrow$ there exists $B_0 \supseteq h_{[a]}(F)$ with $x \notin B_0$ s.t. $[\mathcal{T}^{co}(B_0) \wedge \mathcal{C}(B_0)] \not\leq a'$

$\Rightarrow$ $\exists \widetilde{B}$ with $x \notin \widetilde{B} \supseteq h_{[a]}(F); \exists \{B_i\}_{i\in\Lambda}$ with $\bigcap_{i\in\Lambda} B_i = \widetilde{B}, \forall i \in \Lambda,\ \exists \{H_{ij}\}_{i\in\Lambda}$

with $\sqcup_{j\in\Lambda_i} H_{ij} = B_i, \forall j \in \Lambda_i, [\mathcal{T}^{co}(H_{ij}) \wedge \mathcal{C}(H_{ij})]' \not\geq a$,

where $\widetilde{B} = B_0, B_i = \widetilde{B},$ and $H_{ij} = \widetilde{B},\ \forall i \in \Lambda, \forall j \in \Lambda_i$.

$$
\begin{aligned}
\Rightarrow \quad & \bigwedge_{x \notin B \supseteq h_{[a]}(F)} \left(\bigvee_{\bigcap_{i\in\Lambda} B_i = B} \left(\bigwedge_{i\in\Lambda} \left(\bigvee_{\sqcup_{j\in\Lambda_i} H_{ij} = B_i} \left(\bigwedge_{j\in\Lambda_i} [\mathcal{T}^{co} \wedge \mathcal{C}](H_{ij}) \right) \right) \right) \right)' \not\geq a \\
\Leftrightarrow \quad & \bigwedge_{x \notin B \supseteq h_{[a]}(F)} \mathcal{T}_w(X - B)' \not\geq a \\
\Leftrightarrow \quad & cl_w(h_{[a]}(F))(x) \not\geq a \\
\Leftrightarrow \quad & x \notin cl_w(h_{[a]}(F))_{[a]}.
\end{aligned}
$$

□

Using the proof of Sufficiency in above theorem, $\mathcal{T}^{co}_w \leq \mathcal{T}^{co}$ holds, that is to say, $\mathcal{T}^{co}_w$ is coarser than $\mathcal{T}^{co}$. In addition, by analysis of proof of Theorem 4.7, we do not use the finiteness of F, hence see the following corollary.

Corollary 4.8. *$cl(co(A)_{[a]})_{[a]} \subseteq co(A)_{[a]}$ if and only if $cl_w(co(A)_{[a]})_{[a]} \subseteq co(A)_{[a]}$, for each $A \in 2^X$ and for each $a \in M$.*

5. Conclusions

In this paper, the notion of M-fuzzifying topological convex spaces is defined and several equivalent characterizations are obtained. In future work, we will consider to discuss the compatibility of M-fuzzifying uniformity and M-fuzzifying convexity.

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