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Separable programming problems with the max-product fuzzy relation equation constraints

B. Hedayatfar[1], A. Abbasi Molai[2] and S. Aliannezhadi[3]

[1,2,3] *School of Mathematics and Computer Sciences, Damghan University, P.O.Box 36715-364, Damghan, Iran*

b.hedayatfar@std.du.ac.ir, a_ abbasimolai@du.ac.ir, s.aliannezhadi@std.du.ac.ir

Abstract

In this paper, the separable programming problem subject to Fuzzy Relation Equation (FRE) constraints is studied. It is decomposed to two subproblems with decreasing and increasing objective functions with the same constraints. They are solved by the maximum solution and one of minimal solutions of its feasible domain, respectively. Their combination produces the original optimal solution. The detection of the optimal solution of the second subproblem by finding all the minimal solutions will be very time-consuming because of its NP-hardness. To overcome such difficulty, two types of sufficient conditions are proposed to find some of its optimal components or all of them. Under the first type sufficient conditions, some procedures are given to simplify the original problem. Also, a value matrix is defined and an algorithm is proposed to compute an initial upper bound on its optimal objective value using the matrix. Then, a branch-and-bound method is extended using the matrix and initial upper bound to solve the simplified problem without finding all the minimal solutions.

Keywords: Separable programming problem, Fuzzy relation equations, Max-product composition, Fuzzy optimization.

The max-min Fuzzy Relation Equations (FREs) and its associated problems were firstly suggested by Sanchez [25] in 1976. Its structure of solution set was then studied in [26]. The researches were extended to the max-T FREs in [12, 13]. It was shown that the solution set of a consistent finite system of max-T FREs can be specified by a maximum solution and a finite number of minimal solutions. The maximum solution can easily be obtained due to the existence of a formula for its computation. However, the determination of all the minimal solutions is an NP-hard problem [9, 10, 22, 23]. Hence, many researchers have been proposed various procedures to detect the minimal solutions. The most well-known methods are the rule-based methods [5, 6], the iterative method [4], the matrix pattern method [21], algebraic method [18] and so on. A complete review of the methods can be found in [11].

The fuzzy relation programming is an important research area in the fuzzy optimization [1, 2, 3, 8, 14, 15, 16, 17, 19, 20, 24, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. The problem of minimizing a linear function provided to a consistent finite system of the max-min FREs was firstly considered in the textile industry by Fang and Li [14]. They decomposed it into two subproblems with regard to the nonnegative and negative coefficients of the objective function with the same constraints. The second subproblem is solved by the maximum solution of the feasible domain. The first subproblem obtains its optimum at one of the minimal solutions of the feasible domain. To find the optimum, the first subproblem was equivalently converted to an 0-1 integer programming problem and the branch-and-bound method was applied to solve the subproblem. The problem with the max-product operator was considered by Loetamonphong and Fang [19] and a similar procedure was applied to solve the problem. Some researchers tried to present several simplification procedures and rules to reduce the dimension of the original problem or search domain. Wu et al. [32] provided a suitable upper bound for its optimal objective value and presented a procedure based on this topic. The procedure meets much fewer nodes to obtain the optimal solution. Chang and Shieh [8] improved upper bound on the optimal objective value and the rules for simplifying in [32]. Wu and Guu [31] presented a necessary condition for the optimal

Corresponding Author: A. Abbasi Molai

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solution of the first subproblem. They proposed three rules to simplify the process of finding the optimal solution based on the necessary condition. Guu and Wu [15] considered the problem with nonnegative coefficients subject to the max-product FREs. They obtained a necessary condition for an optimal solution in terms of the maximum solution of its feasible domain. An efficient algorithm was designed to solve the problem based on the necessary condition in [15]. The necessary condition was also extended to the optimization problem with the max-strict t-norm composition operator in [30]. However, we cannot formulate many the real-world problems in terms of linear functions. The optimization problem of a nonlinear objective function provided to the max-min FREs was investigated by Lu and Fang [20]. They designed a genetic algorithm to solve the problem. The problem with the max-average composition was considered by Khorram and Hassanzadeh [17]. They designed a modified genetic algorithm by changing some of its components. Also, Hassanzadeh et al. proposed a genetic algorithm to solve the problem with the max-product operator in [16]. However, the above methods for resolution of the nonlinear optimization problems produce a near optimal solution. Moreover, these algorithms may be time-consuming for some problems. To overcome these difficulties, some researchers focussed on designing algorithms for resolution of a special class of the nonlinear optimization problems. The latticized linear programming problem provided to the max-min Fuzzy Relation Inequality (FRI) constraints was firstly studied by Wang et al. [28]. They used the properties of conservative paths to solve the problem. The same problem was solved by finding all the minimal solutions of FRI in [24]. Zhou and his colleagues [34, 35, 36, 37] formulated fuzzy relation geometric programming provided to FRE or FRI constraints. They developed a matrix approach to solve the problem in [34]. Also, some rules for simplification of the problem were proposed and an algorithm was presented to solve the problem based on a branch-and-bound method and FRE path [37]. A special case of the fuzzy relation geometric programming problems as monomial geometric programming problem subject to FRI constraints was studied by Shivanian and Khorram [27].

A linear fractional programming problem subject to the max -Archimedean t-norm FRE constraints was investigated by Wu et al. [33]. They presented some corollaries and reduced the search domain of the optimal solution. Then, the reduced problem was converted to a linear programming problem. The quadratic programming problem subject to the max-product FRI constraints was studied by Abbasi Molai [2]. He firstly presented some sufficient conditions to find some of the components of its optimal solutions. Some procedures were also given to simplify the problem. The modified simplex method was then applied to solve the simplified problem due to convexity of its objective function. The same problem was considered in a more general case [1] where its objective function is not necessarily convex, in a general case. The problem was converted to a separable programming problem. Using the resolution of the recent problem, the approximate optimal solution of the original problem was obtained.

One of the most important problems, which arises repeatedly in applications and practice, is separable programming problems. The time dependent optimization is one of the problems. In this model, each variable denotes an activity level in a determined time period. In these problems, the decision-makings based on the resources and cost (or profit) have an additive property with respect to time. Another instance of their applications is in the area of water-resource planning. In this problem, the optimization is done over separated zones. Also, many nonseparable problems can be converted to some separable problems by various techniques. Hence, the study of the separable optimization problems provided to FRE constraints is motivated.

In this paper, the separable programming provided to FRE constraints is studied. Its objective function can be written as summation of some single-variable functions. These single-variable functions are decreasing or increasing. In fact, the problem is also an extension of the models presented in [14, 19]. We firstly decompose the problem into two subproblems with objective functions including the summation of decreasing single-variable functions and the summation of increasing single-variable functions, respectively. It is shown that the maximum solution and one of the minimal solutions of its feasible domain solve the first and second subproblem, respectively. The combination of two solutions produces the optimal solution of the original problem. The main difficulty is finding the minimal solution. To overcome it, two types of sufficient conditions are proposed to simplify the original problem. Under the conditions, some of the optimal components of the problem or all of them are determined without finding all the minimal solutions of its feasible domain. Then a value matrix is defined to capture all the properties of the simplified problem. Applying the value matrix, we design an algorithm to find an initial upper bound on the optimal objective value of the simplified problem. In fact, the algorithm is an extension of Rule 6 in [8]. Finally, a branch-and-bound approach is extended to solve the simplified problem without finding all the minimal solutions. The initial upper bound may be improved by a better solution during the branch-and-bound method.

The rest of the paper is organized as follows: Section 2 investigates the structure of solution set of a system of the max-product FREs. The separable programming problem provided to the max-product FRE constraints is formulated in Section 3. In Section 4, the problem is decomposed to two subproblems based on decreasing or increasing single-variable functions in the objective function of the original problem. Section 5 presents two types of sufficient conditions to simplify or solve the problem without finding all the minimal solutions of its feasible domain. Section 6 designs an algorithm to find an initial upper bound on the optimal objective value of the simplified problem. Section 7 presents the resolution process of the original problem as an algorithm. The algorithm is illustrated by some numerical examples in Section 8. Finally, conclusions are given in Section 9.

1 The structure of solution set of a system of FREs

In this section, some basic concepts and important properties about the max-product FRE are reminded. The system of the max-product FREs is denoted by $X(A,b)=\{x\in[0,1]^n \mid A\circ x=b\}$, where the matrices $A=[a_{ij}]$ and $b=[b_i]$ are $m\times n$ and $m\times 1$ dimensional, respectively, and $a_{ij},b_i\in[0,1]$, for each $i\in I=\{1,\dots,m\}$ and $j\in J=\{1,\dots,n\}$. The operator " $\circ$ " is the max-product composition. The system $A\circ x=b$ is to find vectors with $n\times 1$-dimensional as $x=[x_j]$ satisfying the following relation:

$$\max_{j\in J}\{a_{ij}.x_j\}=b_i,\quad \forall i\in I. \tag{1}$$

Assume $X=[0,1]^n$. Then for $x^1,x^2\in X$, we define $x^1\le x^2$ if and only if $x_j^1\le x_j^2$, for each $j\in J$. The relation " $\le$ " defines a partial order relation on X. Hence, $(X,\le)$ will be a lattice with regard to the above relation. The maximum solution and the minimal solution is defined as follows.

Definition 1.1. [14] *The vector $\hat{x}\in X(A,b)$ is called a maximum solution if for each $x\in X(A,b)$, we have $x\le\hat{x}$. Also, the vector $\check{x}\in X(A,b)$ is called a minimal solution if $x\le\check{x}$, for any $x\in X(A,b)$, implies $x=\check{x}$.*

When $X(A,b)\neq\emptyset$, we can completely determine it by a unique maximum solution and a finite number of minimal solutions. The maximum solution is easily computed by the following relation [7]:

$$\hat{x}=A@b=\left[\bigwedge_{i=1}^{m}(b_i@a_{ij})\right]_{j\in J}, \tag{2}$$

where

$$b_i@a_{ij}=\begin{cases}1, & \text{if}\quad a_{ij}\le b_i,\\ \frac{b_i}{a_{ij}}, & \text{if}\quad a_{ij}>b_i.\end{cases} \tag{3}$$

We can easily check the consistency of the system by the maximum solution. If $A\circ\hat{x}=b$, then $X(A,b)\neq\emptyset$. Otherwise, the system is infeasible. If the set of all the minimal solutions is denoted by $\check{X}(A,b)$, then the set $X(A,b)$ is determined by the following relation [7]:

$$X(A,b)=\bigcup_{\check{x}\in\check{X}(A,b)}\{x\in X\mid \check{x}\le x\le\hat{x}\}. \tag{4}$$

Lemma 1.2. [19] *If $x\in X(A,b)$, then $\forall i\in I$, $\exists j(i)\in J$ s.t. $a_{ij(i)}.x_{j(i)}=b_i$ and $\forall j\in J$: $a_{ij}.x_j\le b_i$.*

Notation. When $X(A,b)\neq\emptyset$, we introduce the following relations:

$$I_j(x)=\{i\in I\mid a_{ij}.x_j=b_i\},\qquad \forall j\in J, \tag{5}$$

and

$$J_i(x)=\{j\in J\mid a_{ij}.x_j=b_i\},\qquad \forall i\in I. \tag{6}$$

The following lemma and remark are directly obtained from Lemma 1.2 and Notation.

Lemma 1.3. *If there exists some $i_0\in I$ such that $J_{i_0}(\hat{x})=\{j_0\}$, then for each vector $x=[x_1,\dots,x_n]^T\in X(A,b)$, we have: $x_{j_0}=\hat{x}_{j_0}$.*

Proof. It is similar to Rule 2 in [29]. □

Remark 1.4. [29] *If $J_p(\hat{x})\subseteq J_q(\hat{x})$ for some $p,q\in I$, then the q^{th} row of matrices A and b can be removed.*

Simplification 1. Using Remark 1.4, we can remove the row q from matrices A and b. Also, we can update index set I as $I:=I-\{q\}$.

It is necessary to remind the following remark.

Remark 1.5. *If there exists some $i_0\in I$ such that $b_{i_0}=0$, then two cases can occur as follows: (i) $a_{i_0j}=0$, then the variable x_j can be assigned any value of interval [0,1]. (ii) $a_{i_0j}>0$, then the variable x_j should be assigned zero value. We exclude this obvious case from our considerations.*

We are now ready to formulate the separable programming problem provided to the max-product FRE constraints in the next section.

2 Formulation of the Separable Programming Problem with FRE Constraints

In this section, the separable programming problem with the max-product FRE constraints is formulated as follows:

$$\min \quad Z(x) = \sum_{j=1}^{n} f_j(x_j), \tag{7}$$

$$\text{s.t.} \quad A \circ x = b, \tag{8}$$

$$x \in [0,1]^n, \tag{9}$$

Moreover, the single-variable functions $f_j(x_j)$ are increasing or decreasing with respect to the variable x_j. The aim of problem (7)-(9) is to find a vector x^* satisfying in the constraints (8)-(9) and minimize the objective function Z. We can now express the following simplification with regard to Lemma 1.3 and the problem (7)-(9).

Simplification 2. Under the assumptions of Lemma 1.3, for each optimal solution $x^* = [x_1^*, \ldots, x_n^*]^T$ of problem (7)-(9), we have: $x_{j_0}^* = \hat{x}_{j_0}$. Also, the column j_0 from matrix A and row i_0 from matrices A and b can be removed and the index sets I and J as $I := I - \{i_0\}$ and $J := J - \{j_0\}$ can be updated.

First of all, we remind the following remark.

Remark 2.1. *If there exists some $j \in J$ such that $f_j(x_j)$ is a fixed function, then we can consider each feasible x_j as the optimal component, say $\hat{x}_j$. Therefore, without loss of generality, we assume that the functions $f_j(x_j)$, for each $j \in J$, are not fixed. Hence, the index set of J can be partitioned to two index sets P' and P'' as follows: $P' = \{j \in J | f_j(x_j) \text{ is decreasing}\}$ and $P'' = \{j \in J | f_j(x_j) \text{ is increasing}\}$ such that $P' \cap P'' = \emptyset$ and $P' \cup P'' = J$.*

3 Decomposition of Problem (7)-(9) to Two Subproblems

In this section, the problem (7)-(9) is decomposed to two subproblems based on decreasing or increasing single-variable functions in the objective function with the same constraints. It is shown that the maximum solution and one of the minimal solutions solve two subproblems, respectively. Using the fact, we can construct the optimal solution of the problem (7)-(9) by combining the optimal solutions of two subproblems. In this section, we assume that $X(A,b) \neq \emptyset$. With regard to this assumption, we express the results in this section. Assume that the single-variable functions $f_1(x_1), \ldots, f_n(x_n)$ have been given where the functions are increasing or decreasing. Define $f_1'(x_1), \ldots, f_n'(x_n)$ and $f_1''(x_1), \ldots, f_n''(x_n)$ as follows:

$$f_j'(x_j) = \begin{cases} f_j(x_j), & \text{if } j \in P', \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_j''(x_j) = \begin{cases} f_j(x_j), & \text{if } j \in P'', \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

It is obvious that $f_j(x_j) = f_j'(x_j) + f_j''(x_j)$. Now, two following subproblems are considered.

$$\min \quad Z_1(x) = \sum_{j=1}^{n} f_j'(x_j), \tag{11}$$

$$\text{s.t.} \quad A \circ x = b, \tag{12}$$

$$x \in [0,1]^n, \tag{13}$$

and

$$\min \quad Z_2(x) = \sum_{j=1}^{n} f_j''(x_j), \tag{14}$$

$$\text{s.t.} \quad A \circ x = b,$$

$$ \tag{15}$$

$$x \in [0,1]^n. \tag{16}$$

The following lemma presents the optimal solution of subproblem (11)-(13).

Lemma 3.1. *The maximum solution $\hat{x}$ solves subproblem* (11)-(13).

Proof. Since all the functions $f_j(x_j)$, $j \in P'$, with respect to the variable x_j are decreasing, the function $Z_1(x)$ becomes decreasing. On the other hand, the vector $\hat{x}$ is the maximum solution. Hence, for each $x \in X(A,b)$, we have $x \leq \hat{x}$. So, it is concluded that $Z_1(x) \geq Z_1(\hat{x})$. Therefore, vector $\hat{x}$ solves subproblem (11)-(13). □

The following corollary is a direct result of Lemma 3.1. Applying this corollary, we can directly obtain an optimal solution of problem (7)-(9).

Corollary 3.2. *If* $\bigcup_{j \in P'} I_j(\hat{x}) = I$, *then there exists an optimal solution* $x^* = [x_j^*]_{j \in P' \cup P''}$ *for the optimization problem* (7)-(9) *such that*

$$x_j^* = \begin{cases} \hat{x}_j, & \text{if } j \in P', \\ 0, & \text{if } j \in P''. \end{cases} \tag{17}$$

Proof. With regard to Lemma 3.1, we can put $x_j^* = \hat{x}_j$, for each $j \in P'$. Then, we can remove all rows i, $i \in I_j(\hat{x})$ where $j \in P'$. Since $\bigcup_{j \in P'} I_j(\hat{x}) = I$, all rows i, $i \in I$ are removed, i.e., all equations can be satisfied by $\{\hat{x}_j \mid j \in P'\}$. Hence, we can assign zero to x_j^*, for each $j \in P''$, because $f_j(x_j)$ is an increasing function with respect to x_j and $0 \leq x_j \leq \hat{x}_j$, for each $j \in P''$. □

The optimal solution of subproblem (14)-(16) is given by the following lemma.

Lemma 3.3. *One of the minimal solutions of $X(A,b)$, i.e., $\check{x}^*$, solves subproblem* (14)-(16).

Proof. Since all the functions $f_j(x_j)$, $j \in P''$, with respect to the variable x_j are increasing, the function $Z_2(x)$ becomes increasing. Let $\check{X}(A,b) = \{\check{x}^1, \ldots, \check{x}^k\}$ be all the minimal solutions of $X(A,b)$. Assume that $Z_2(\check{x}^*) = \min\{Z_2(\check{x}^1), \ldots, Z_2(\check{x}^k)\}$. Then, for each $x \in X(A,b)$, there exists some $\check{x}^L$ such that $\check{x}^L \leq x$. Since the function $Z_2(x)$ is increasing, it is concluded that $Z_2(\check{x}^*) \leq Z_2(\check{x}^L) \leq Z_2(x)$. Therefore, for each $x \in X(A,b)$, we have $Z_2(\check{x}^*) \leq Z_2(x)$. Hence, one of the minimal solutions, i.e., $\check{x}^*$ solves subproblem (14)-(16).

We now construct the solution $x^* = [x_1^*, \ldots, x_n^*]^T$ with respect to the obtained solutions from Lemmas 3.1 and 3.3 as follows:

$$x_j^* = \begin{cases} \check{x}_j^*, & \text{if } j \in P'', \\ \hat{x}_j, & \text{otherwise.} \end{cases} \tag{18}$$

□

In the next theorem, it is established that the vector x^* is the optimal solution of problem (7)-(9).

Theorem 3.4. *The vector x^* defined by relation* (18) *is an optimal solution of problem* (7)-(9).

Proof. With regard to subproblems (11)-(13) and (14)-(16) and Lemmas 3.1 and 3.3, we have:

$$Z(x) = \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n} (f_j'(x_j) + f_j''(x_j)) = \sum_{j=1}^{n} f_j'(x_j) + \sum_{j=1}^{n} f_j''(x_j)$$

$$\geq \sum_{j=1}^{n} f_j'(\hat{x}_j) + \sum_{j=1}^{n} f_j''(\check{x}_j^*) = \sum_{j=1}^{n} f_j(x_j^*) = Z(x^*).$$

The above relation is satisfied for each $x \in X(A,b)$. Therefore, the vector x^* is an optimal solution of problem (7)-(9). □

With regard to Theorem 3.4, we can find some of components of the optimal solution of the problem (7)-(9) and update the feasible domain of problem (7)-(9) as follows:

Simplification 3. We can put $x_j^* = \hat{x}_j$, for each $j \in P'$, using Theorem 3.4. Doing this, we can remove the columns j, $j \in P'$, from matrix A and the variable x_j from vector x. Also, the rows i, $i \in I_j(\hat{x})$ where $j \in P'$, are removed from matrices A and b. Update $J := J - P'$, $I := I - \bigcup_{j\in P'} I_j(\hat{x})$, $A = [a_{ij}]_{i\in I, j\in J}$, $x = [x_j]_{j\in J}$, and $b = [b_i]_{i\in I}$.

Finding the optimal solution of subproblem (11)-(13) with regard to formulae (2)-(3) is easy. The main difficulty is to compute the minimal solution $\check{x}^*$ because the number of minimal solutions may be very large. Hence, finding vector $\check{x}^*$ can be a huge work. To overcome this difficulty, some simplification procedures are presented to reduce the search domain of the optimal solution of the simplified problem.

4 Two Types of Sufficient Conditions for Simplified Problem: Further Simplifications

This section is divided to two subsections. Two types of sufficient conditions are proposed. The first type sufficient conditions are discussed in the first subsection. Under the first type sufficient conditions, some of the components of an optimal solution of the simplified problem is found. Doing this work, the search domain of the optimal solution of the simplified problem becomes more limited. Hence, the required time and the computational complexity are decreased considerably. In the second subsection, there are other sufficient conditions (the second type sufficient conditions) that explicitly determine one of the optimal solutions of the simplified problem.

4.1 The First Type Sufficient Conditions

We now present the first type sufficient conditions in terms of a theorem and a corollary to determine some of the optimal components under the conditions.

Theorem 4.1. *If there exists a pair index sets $J', J'' \subset J$ such that the following conditions are satisfied:*
1. $J' \cap J'' = \emptyset$,
2. $\bigcup_{j\in J'} I_j(\hat{x}) \subseteq \bigcup_{j\in J''} I_j(\hat{x})$, *and*
3. $\sum_{j\in J''} (f_j(\hat{x}_j) - f_j(0)) \leq f_j(\hat{x}_j) - f_j(0), \quad \forall j \in J'$,
then there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ for the simplified problem such that $x_j^* = 0$, for each $j \in J'$.

Proof. Let $x^* = [x_j^*]_{j\in J}$ be an optimal solution of the simplified problem. If $x_j^* = 0$, for each $j \in J'$, then the result is obtained. Otherwise, there exists an index set $\emptyset \neq K' \subseteq J'$ such that $0 < x_j^* \leq \hat{x}_j$, for each $j \in K'$. Without loss of generality, assume that $x_j^* = 0$, for each $j \in J' \setminus K'$. If we have $0 < x_j^* < \hat{x}_j$, for each $j \in K'$, then we can change some components of vector x^* to obtain a feasible solution x^{**} with $Z(x^{**}) \leq Z(x^*)$ as follows: Set $x_j^{**} = 0$, for each $j \in K'$ and $x_j^{**} = x_j^*$, for each $j \in J' \setminus K'$. It is clear that x^{**} is a feasible solution and $Z(x^{**}) \leq Z(x^*)$ because all functions $f_j(x_j)$ are increasing with respect to x_j, for each $j \in J$. Otherwise, we have $x_j^* = \hat{x}_j$, for each $j \in E'$, where $\emptyset \neq E' \subseteq K'$, $0 < x_j^* < \hat{x}_j$, for each $j \in K' \setminus E'$, and $x_j^* = 0$, for each $j \in J' \setminus K'$. Now, we have two cases to be considered. In each case, we will introduce a feasible solution with $Z(x^{**}) \leq Z(x^*)$.

Case 1: $x_j^* = \hat{x}_j$, for each $j \in J''$.
In this case, set $x_j^{**} = 0$, for each $j \in K'$ and $x_j^{**} = x_j^*$, for each $j \in J \setminus K'$. With regard to Condition 2, we have:

$$\begin{aligned}\bigcup_{j\in J} I_j(x^*) &= \left(\bigcup_{j\in E'} I_j(\hat{x})\right) \cup \left(\bigcup_{j\in J''} I_j(\hat{x})\right) \cup \left(\bigcup_{j\in J\setminus J'\cup J''} I_j(x^*)\right) \\ &\subseteq \left(\bigcup_{j\in J''} I_j(\hat{x})\right) \cup \left(\bigcup_{j\in J''} I_j(\hat{x})\right) \cup \left(\bigcup_{j\in J\setminus J'\cup J''} I_j(x^*)\right) = \bigcup_{j\in J} I_j(x^{**}).\end{aligned}$$

Since we have $\bigcup_{j\in J} I_j(x^{**}) \supseteq \bigcup_{j\in J} I_j(x^*)$, then vector x^{**} is a feasible solution. On the other hand, we have:

$$Z(x^*) = \sum_{j\in E'} f_j(\hat{x}_j) + \sum_{j\in K'\setminus E'} f_j(x_j^*) + \sum_{j\in J'\setminus K'} f_j(0) + \sum_{j\in J\setminus J'} f_j(x_j^*)$$

$$\geq \sum_{j\in E'} f_j(0) + \sum_{j\in K'\setminus E'} f_j(0) + \sum_{j\in J'\setminus K'} f_j(0) + \sum_{j\in J\setminus J'} f_j(x_j^*) = Z(x^{**}).$$

Hence, such an optimal solution x^{**} must exist with $x_j^{**}=0$, for each $j\in J'$, in this case.

Case 2: $x_j^* \neq \hat{x}_j$, for each $j\in J''$.

More exactly, suppose that $0<x_j^*<\hat{x}_j$, for each $j\in K''$, $x_j^*=0$, for each $j\in E''$, and $x_j^*=\hat{x}_j$, for each $j\in J''\setminus K''\cup E''$ where $K''\subseteq J''$ and $E''\subseteq J''$. With regard to Condition 1, we can set $x_j^{**}=0$, for each $j\in K'$, $x_j^{**}=\hat{x}_j$, for each $j\in K''\cup E''$, and $x_j^{**}=x_j^*$, for each $j\in J\setminus K'\cup K''\cup E''$. Apply Condition 2. Vector x^{**} is a feasible solution because we have:

$$\begin{aligned}\bigcup_{j\in J} I_j\left(x^*\right) &= \left(\bigcup_{j\in E'} I_j\left(\hat{x}\right)\right) \cup \left(\bigcup_{j\in J''\setminus K''\cup E''} I_j\left(\hat{x}\right)\right) \cup \left(\bigcup_{j\in J\setminus J'\cup J''} I_j\left(x^*\right)\right)\\ &\subseteq \left(\bigcup_{j\in J''} I_j\left(\hat{x}\right)\right) \cup \left(\bigcup_{j\in J''\setminus K''\cup E''} I_j\left(\hat{x}\right)\right) \cup \left(\bigcup_{j\in J\setminus J'\cup J''} I_j\left(x^*\right)\right)\\ &= \bigcup_{j\in J} I_j\left(x^{**}\right).\end{aligned}$$

Furthermore, since $f_j(x_j)$ is an increasing function with respect to x_j, for each $j\in J$, we have $f_j(\hat{x}_j)-f_j(0)\geq 0$. So, with regard to Conditions 2 and 3, we have: $\sum\limits_{j\in E''}(f_j(\hat{x}_j)-f_j(0))+\sum\limits_{j\in K''}(f_j(\hat{x}_j)-f_j(0))\leq \sum\limits_{j\in J''}(f_j(\hat{x}_j)-f_j(0))\leq f_j(\hat{x}_j)-f_j(0)$, for each $j\in E'$, and $f_j(\hat{x}_j)-f_j(0)\leq \sum\limits_{j\in E'}(f_j(\hat{x}_j)-f_j(0))$, for each $j\in E'$. Therefore, we have:

$$\begin{aligned}Z(x^*) &= \sum_{j\in K'\setminus E'} f_j(x_j^*) + \sum_{j\in E'} f_j(\hat{x}_j) + \sum_{j\in K''} f_j(x_j^*) + \sum_{j\in E''} f_j(0) + \sum_{j\in J\setminus K'\cup K''\cup E''} f_j(x_j^*)\\ &\geq \sum_{j\in K'\setminus E'} f_j(0) + \sum_{j\in E''} f_j(\hat{x}_j) - \sum_{j\in E''} f_j(0) + \sum_{j\in K''} f_j(\hat{x}_j) - \sum_{j\in K''} f_j(0) +\\ &\quad \sum_{j\in E'} f_j(0) + \sum_{j\in K''} f_j(0) + \sum_{j\in E''} f_j(0) + \sum_{j\in J\setminus K'\cup K''\cup E''} f_j(x_j^*) = Z(x^{**}).\end{aligned}$$

Hence, such an optimal solution x^{**} must exist with $x_j^{**}=0$, for each $j\in J'$, in this case. □

The simplification corresponding Theorem 4.1 is as follows:

Simplification 4. If the conditions of Theorem 4.1 are satisfied in the process of resolution of the simplified problem, then x_j^* can be set zero for each $j\in J'$ and all columns $j\in J'$ can be deleted from matrix A.

The following corollary is a direct result of Theorem 4.1.

Corollary 4.2. *If there exists $s,t\in J$ such that $s\neq t$, $I_s(\hat{x})\subseteq I_t(\hat{x})$, and $f_t(\hat{x}_t)-f_t(0)\leq f_s(\hat{x}_s)-f_s(0)$, then there exists an optimal solution x^* for the simplified problem such that $x_s^*=0$.*

Simplification 5. Using Corollary 4.2, we can remove column $s\in J$ from matrix A and update index set J as: $J:=J-\{s\}$.

4.2 The Second Type Sufficient Conditions

In this subsection, we present some other sufficient conditions that completely determine an optimal solution of the simplified problem. First of all, we remind the following point.

Remark 4.3. *Note that the value of each single-variable function $f_j(x_j)$ may be a nonzero value, when $x_j=0$, for each $j\in J$. We hereinafter rearrange the functions $f_j(x_j)$, for each $j\in J$, in an increasing order with respect to the values $f_j(\hat{x}_j)-f_j(0)$, i.e., $f_1(\hat{x}_1)-f_1(0)\leq f_2(\hat{x}_2)-f_2(0)\leq\ldots\leq f_{|J|}(\hat{x}_{|J|})-f_{|J|}(0)$.*

Three sufficient conditions of the second type are given in this subsection. The following theorem presents such conditions.

Theorem 4.4. *If there exists $J'' = \{1, 2, \ldots, j''\} \subseteq J$ such that the following conditions are satisfied:*
1. $\bigcup_{j \in J''} I_j(\hat{x}) = I$,
2. $\bigcup_{j \in J'' \setminus \{k\}} I_j(\hat{x}) \neq I, \qquad \forall k \in J''$, *and*
3. $\sum_{j \in J''} (f_j(\hat{x}_j) - f_j(0)) \leq f_{j''+1}(\hat{x}_{j''+1}) - f_{j''+1}(0)$,
then there exists an optimal solution for the simplified problem as $x^* = [x_j^*]_{j \in J}$ such that

$$x_j^* = \begin{cases} \hat{x}_j, & \text{if } j \in J'', \\ 0, & \text{if } j \in J \setminus J''. \end{cases} \tag{19}$$

Proof. Set $J' = J \setminus J''$. Since $\bigcup_{j \in J''} I_j(\hat{x}) = I$ and

$$f_{j''+1}(\hat{x}_{j''+1}) - f_{j''+1}(0)) \leq f_{j''+2}(\hat{x}_{j''+2}) - f_{j''+2}(0) \leq \ldots \leq f_{|J|}(\hat{x}_{|J|}) - f_{|J|}(0)$$

we can assign zero to x_j^*, for each $j \in J' = J \setminus J''$, with regard to Theorem 4.1. Now, according to Conditions 1 and 2, we can easily set $x_j^* = \hat{x}_j$, for each $j \in J''$ and the proof is completed. □

The following lemma also considers other sufficient conditions of the second type, independently.

Lemma 4.5. *Assume that $X(A, b) \neq \emptyset$. If there exists $k \in J$ such that $\bigcup_{j \in J - \{k\}} I_j(\hat{x}) \subset I_k(\hat{x})$, then an optimal solution $x^* = [x_1^*, \ldots, x_n^*]^T$ of the simplified problem is as follows:*

$$x_j^* = \begin{cases} \hat{x}_k, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Proof. Since $X(A, b) \neq \emptyset$, with regard to the assumption, it is concluded that $I_k(\hat{x}) = I$. Otherwise, if there exists some $i_0 \in I$ such that $J_{i_0}(\hat{x}) = \emptyset$, then $X(A, b) = \emptyset$. This is a contradiction with the assumption. Therefore, for each $i \in I_k(\hat{x})$, we should have $a_{ik}.\hat{x}_k = b_i$ or $\hat{x}_k = \frac{b_i}{a_{ik}}$. Since the functions $f_j(x_j)$ are increasing and the objective of the simplified problem is minimization, the vector $x^* = (x_j^*)_{j \in J}$ with $x_k^* = \frac{b_i}{a_{ik}}$ and $x_j^* = 0$, for each $j \in J - \{k\}$, is the optimal solution of the simplified problem. □

The following corollary is a direct result of Lemma 4.5. It presents some sufficient conditions which under them, the optimal solution is unique.

Corollary 4.6. *Under the assumption of above lemma, if the functions $f_j(x_j)$ are strictly increasing, then the optimal solution mentioned in the above lemma is unique, for the simplified problem.*

5 An Initial Upper Bound on the Optimal Objective Value of Subproblem (14)-(16)

We firstly express a useful property for an optimal solution of subproblem (14)-(16). This property says that each of its components is either zero or the corresponding component value of the maximum solution. First of all, we remind the following lemma from [33].

Lemma 5.1. [33] *Let $\check{x} = [\check{x}_1, \ldots, \check{x}_n]^T$ be a minimal solution of set $X(A, b)$. If $X(A, b) \neq \emptyset$, then $\check{x}_j \in \{0, \hat{x}_j\}$, for all $j \in J$.*

We are now ready to express the property about an optimal solution of subproblem (14)-(16) as follows.

Lemma 5.2. *Consider subproblem* (14)-(16)*. Assume that $X(A, b) \neq \emptyset$. Then there exists an optimal solution $x^* = [x_j^*]_{j \in J}$ such that for each $j \in P''$, we have either $x_j^* = \hat{x}_j$ or $x_j^* = 0$.*

Proof. With regard to Lemma 3.3, one of the minimal solutions is an optimal solution of subproblem (14)-(16). Since for any minimal solution $\check{x} = [\check{x}_j]_{j \in J}$, we have $\check{x}_j \in \{0, \hat{x}_j\}$, for all $j \in J$, with regard to Lemma 5.1, the proof is easily concluded. □

With regard to Lemma 5.2, there exists an optimal solution $x^* = [x_j^*]_{j\in J}$ for the subproblem such that either $x_j^* = \hat{x}_j$ or $x_j^* = 0$, for each $j \in J$. Also, as it has been stated before, each function $f_j(x_j)$ may be nonzero when $x_j = 0$. Based on the above concepts, we can define a value matrix $M = [m_{ij}]$ to record all the critical information about the contributions of the objective value and the equality requirements.

Definition 5.3. *Define the value matrix $M = [m_{ij}]$ where*

$$m_{ij} = \begin{cases} f_j(\hat{x}_j) - f_j(0), & \textit{if } j \in J_i(\hat{x}), \\ \infty, & \textit{otherwise.} \end{cases}$$

Now, we are ready to develop Rule 6 in [8] to compute an initial upper bound on the optimal objective value of subproblem (14)-(16) as follows:

Algorithm 5.4. *An algorithm for computing an initial upper bound on the optimal objective value of subproblem* (14)-(16).

Step 1: *Set $U := 0$.*

Step 2: *If $I \neq \emptyset$, then*

1.2. *Set $U := U + m_{\underline{i}\underline{j}}$, where $\underline{i} = \min\{i \mid i \in I\}$ and $\underline{j} = \min\{j \mid j \in J_{\underline{i}}(\hat{x})\}$.*

2.2. *Set $I := I - I_{\underline{j}}(\hat{x})$.*

3.2. *Go to Step 2.*

Step 3: *Stop.*

The following lemma shows the validity of Algorithm 5.4.

Lemma 5.5. *U is an upper bound on the optimal objective value of subproblem* (14)-(16).

Proof. If we set $x = [0, \dots, 0]^T$ in Step 1 and update vector x in Sub-step 1.2 such that $x_{\underline{j}} := \hat{x}_{\underline{j}}$, then it can be easily seen that the obtained vector x during Algorithm 5.4 is a feasible solution. Hence, its objective value U is an upper bound on the optimal objective value of the subproblem. $\square$

6 The Resolution Process of Problem (7)-(9): an Algorithm

Based on the concepts discussed up to now, we design an algorithm to solve the optimization problem (7)-(9). The main idea for resolution of the problem is employing the branch-and-bound method with the jump-tracking technique to solve the simplified problem on the value matrix M. Note that the initial upper bound may be improved by a better solution during the branch-and-bound method (if possible). Further details will be illustrated in Examples 1 and 2.

Algorithm 6.1. *An algorithm for resolution of problem (7)-(9)*

Step 1: *Compute the maximum solution $\hat{x}$ using relations (2) and (3).*

Step 2: *If $A \circ \hat{x} = b$, then go to Step 3. Otherwise, the problem (7)-(9) is infeasible. Stop!*

Step 3: *Compute the index sets $J_i(\hat{x})$ and $I_j(\hat{x})$, for each $i \in I$ and $j \in J$, applying Notation.*

Step 4: *If $\exists i_0 \in I$ s.t. $J_{i_0}(\hat{x}) = \{j_0\}$, then according to Simplification 2, let $x_{j_0}^* = \hat{x}_{j_0}$. Also, remove column j_0 from matrix A and row i_0 from matrices A and b and update $I := I - \{i_0\}$ and $J := J - \{j_0\}$.*

Step 5: *If for some $p, q \in I$, we have $J_p(\hat{x}) \subseteq J_q(\hat{x})$, then according to Simplification 1, row q can be removed from matrices A and b. Update $I := I - \{q\}$.*

Step 6: *Compute index sets P' and P'' using Remark 2.1.*

Step 7: *If $\bigcup_{j\in P'} I_j(\hat{x}) = I$, then an optimal solution of the (simplified) problem (7)-(9) is computed by relation (17) using Corollary 3.2. Stop!*

Step 8: *For problem (7)-(9), let $x_j^* = \hat{x}_j$, for each $j \in P'$, with regard to Theorem 3.4.*

Step 9: *Remove the columns j, $j \in P'$, from matrix A and rows i, $i \in I_j(\hat{x})$ where $j \in P'$, from matrices A and b. Update $J := J - P'$ and $I := I - \bigcup_{j\in P'} I_j(\hat{x})$ with regard to Simplification 3.*

Step 10: *Check the second type sufficient conditions for the (simplified) problem (7)-(9):*

10.1. *Rearrange the columns of reduced matrix A according to Remark 4.3.*

10.2. *If the conditions of Theorem 4.4 are satisfied, then an optimal solution of the (simplified) problem is computed by relation (19). Stop!*

10.3. *If the conditions of Lemma 4.5 are satisfied, then an optimal solution of the (simplified) problem can be* *obtained using relation (20). Furthermore, under the conditions of Corollary 4.6, this optimal solution is unique. Stop!*

Step 11: *Check the first type sufficient conditions for the (simplified) problem (7)-(9):*

11.1. *If* $\exists s, t \in J$ *s.t.* $s \neq t$, $I_s(\hat{x}) \subseteq I_t(\hat{x})$, *and* $f_t(\hat{x}_t) - f_t(0) \leq f_s(\hat{x}_s) - f_s(0)$, *then according to Corollary 4.2, let* $x_s^* = 0$. *Also, remove column* $s \in J$ *from matrix* A *and update index set* J *with regard to Simplification 5.*

11.2. *If the conditions of Theorem 4.1 are satisfied, then let* $x_j^* = 0$, *for each* $j \in J'$. *Also, remove all columns* $j \in J'$ *from matrix* A *with regard to Simplification 4. Update index set* J.

Note: It is noticeable that if we simplified the problem (7)-(9) by one of Steps 4,5, and 11, then we can again use their results for further simplifications (if possible).

Step 12: *Generate the value matrix* M *using Definition 5.3.*

Step 13: *Compute the initial upper bound* U *on the optimal objective value of the (simplified) problem (7)-(9) using Algorithm 5.4.*

Step 14: *Employ the branch-and-bound method with the jump-tracking technique on the matrix* M *to solve the (simplified) optimization problem (7)-(9). In addition, the initial upper bound* U *may be improved by a better solution during the procedure.*

Step 15: *Produce the optimal solution* x^* *and the optimal objective value* $Z(x^*)$ *of the optimization problem (7)-(9). End.*

Remark 6.2. *All results presented in this paper can easily be rewritten for other types of composition operators such as e.g., max-average and max-min. Therefore, we can easily extend the results for the problem (7)-(9) with max-t composition operator.*

7 Numerical Examples

We now illustrate our algorithm by two numerical examples.

Example 7.1. *Consider the following optimization problem:*

$$\begin{aligned} \min \quad Z(y) = & \frac{1}{y_1+2} + y_2\sqrt{y_2+3} + (y_3+2)^2 + \sqrt{32y_4+1} + e^{y_5} + 3y_6^5 + 6y_6 \\ & + 3^{y_7+1} + e^{y_8} + 3y_8^2 + 10y_9^2 + \frac{1}{2}\sqrt{6y_{10}} + 4^{y_{10}}, \end{aligned} \tag{21}$$

$$\text{s.t.} \quad A \circ y = b, \tag{22}$$

$$y \in [0,1]^{10}, \tag{23}$$

where

$$A = \begin{pmatrix} 0.07 & 0.18 & 0.04 & 0.12 & 0.06 & 0.08 & 0.03 & 0.01 & 0.05 & 0.08 \\ 0.4 & 0.96 & 0.8 & 0.27 & 0.12 & 0.6 & 0.19 & 0.46 & 0.2 & 0.35 \\ 0.17 & 0.04 & 0.3 & 0.24 & 0.24 & 0.09 & 0.15 & 0.2 & 0.07 & 0.16 \\ 0.1 & 0.09 & 0.12 & 0.03 & 0.11 & 0.08 & 0.2 & 0.05 & 0.7 & 0.1 \\ 0.22 & 0.31 & 0.34 & 0.25 & 0.33 & 0.45 & 0.29 & 0.4 & 0.21 & 0.36 \\ 0.6 & 0.51 & 0.42 & 0.54 & 0.69 & 0.5 & 1 & 0.48 & 0.36 & 0.7 \\ 1 & 0.5 & 0.21 & 0.1 & 0.18 & 0.22 & 0.14 & 0.17 & 0.2 & 0.24 \end{pmatrix},$$

$y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10})^T$, $b = (0.09, 0.48, 0.18, 0.14, 0.36, 0.7, 0.25)^T$. *Now, we want to solve this problem by Algorithm 6.1.*

Step 1: *The maximum solution of the feasible domain can be obtained as follows:* $\hat{y} = (0.25, 0.5, 0.6, 0.75, 0.75, 0.8, 0.7, 0.9, 0.2, 1)^T$.

Step 2: *Since* $A \circ \hat{y} = b$, *the problem is feasible. So, we go to Step 3.*

Step 3: *The index sets* $J_i(\hat{y})$ *and* $I_j(\hat{y})$, *for each* $i \in I$ *and* $j \in J$, *according to Notation are as follows:*
$J_1(\hat{y}) = \{2,4\}$, $J_2(\hat{y}) = \{2,3,6\}$, $J_3(\hat{y}) = \{3,4,5,8\}$, $J_4(\hat{y}) = \{7,9\}$,
$J_5(\hat{y}) = \{6,8,10\}$, $J_6(\hat{y}) = \{7,10\}$, $J_7(\hat{y}) = \{1,2\}$.
$I_1(\hat{y}) = \{7\}$, $I_2(\hat{y}) = \{1,2,7\}$, $I_3(\hat{y}) = \{2,3\}$, $I_4(\hat{y}) = \{1,3\}$, $I_5(\hat{y}) = \{3\}$,
$I_6(\hat{y}) = \{2,5\}$, $I_7(\hat{y}) = \{4,6\}$, $I_8(\hat{y}) = \{3,5\}$, $I_9(\hat{y}) = \{4\}$, *and* $I_{10}(\hat{y}) = \{5,6\}$.

Steps 4 and 5: *The conditions of Steps 4 and 5 are not satisfied. So, we go to Step 6.*

Step 6: *Two index sets* P' *and* P'' *are as follows:* $P' = \{1\}$ *and* $P'' = \{2,3,\ldots,10\}$.

Step 7: *The condition of Step 7 is not satisfied. So, we go to Step 8.*

Step 8: *With regard to Theorem 3.4, we can set* $y_1^* = \hat{y}_1 = 0.25$.

Step 9: *With regard to Simplification 3, the column 1 can be removed from matrix A and the row 7 can be removed from matrices A and b. Also, two index sets I and J can be updated as* $I := \{1, \ldots, 6\}$ *and* $J := \{2, \ldots, 10\}$.

Step 10: *Check the second type sufficient conditions for the simplified problem in Step 9:*

10.1. *Rearrange the columns of the reduced matrix A according to Remark 4.3. Since functions* $f_j(y_j)$ *are increasing, for each* $j \in \{2, 3, \ldots, 10\}$, *and* $f_9(\hat{y}_9) - f_9(0) \le f_2(\hat{y}_2) - f_2(0) \le f_5(\hat{y}_5) - f_5(0) \le f_3(\hat{y}_3) - f_3(0) \le f_7(\hat{y}_7) - f_7(0) \le f_8(\hat{y}_8) - f_8(0) \le f_4(\hat{y}_4) - f_4(0) \le f_{10}(\hat{y}_{10}) - f_{10}(0) \le f_6(\hat{y}_6) - f_6(0)$, *we can rename the variables* y_j, *for each* $j \in J$, *as follows:*

$$y_9 \to x_1, y_2 \to x_2, y_5 \to x_3, y_3 \to x_4, y_7 \to x_5, y_8 \to x_6, y_4 \to x_7, y_{10} \to x_8, y_6 \to x_9.$$

Now, we obtain the following simplified optimization problem.

$$\min \quad 10x_1^2 + x_2\sqrt{x_2+3} + e^{x_3} + (x_4+2)^2 + 3^{x_5+1} + e^{x_6} + 3x_6^2$$

$$+\sqrt{32x_7+1} + \frac{1}{2}\sqrt{6x_8} + 4^{x_8} + 3x_9^5 + 6x_9, \tag{24}$$

$$s.t. \quad A \circ x = b, \tag{25}$$

$$x \in [0,1]^9, \tag{26}$$

where

$$A = \begin{pmatrix} 0.05 & 0.18 & 0.06 & 0.04 & 0.03 & 0.01 & 0.12 & 0.08 & 0.08 \\ 0.2 & 0.96 & 0.12 & 0.8 & 0.19 & 0.46 & 0.27 & 0.35 & 0.6 \\ 0.07 & 0.04 & 0.24 & 0.3 & 0.15 & 0.2 & 0.24 & 0.16 & 0.09 \\ 0.7 & 0.09 & 0.11 & 0.12 & 0.2 & 0.05 & 0.03 & 0.1 & 0.08 \\ 0.21 & 0.31 & 0.33 & 0.34 & 0.29 & 0.4 & 0.25 & 0.36 & 0.45 \\ 0.36 & 0.51 & 0.69 & 0.42 & 1 & 0.48 & 0.54 & 0.7 & 0.5 \end{pmatrix},$$

$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)^T$, *and* $b = (0.09, 0.48, 0.18, 0.14, 0.36, 0.7)^T$. *Moreover, we have* $\hat{x} = (0.2, 0.5, 0.75, 0.6, 0.7, 0.9, 0.75, 1, 0.8)^T$. *The conditions of Steps 10.2 and 10.3 are not satisfied. Therefore, we go to Step 11.*

Step 11: *Check the first type sufficient conditions for the simplified problem (24)-(26):*
The conditions of Step 11.1 are not satisfied. So, we go to Step 11.2.

11.2. *a) If we set* $J' = \{4, 7\}$ *and* $J'' = \{2, 3\}$, *then we have:*
1. $J' \cap J'' = \emptyset$,
2. $\bigcup_{j \in J'} I_j(\hat{x}) = \{1, 2, 3\} \subseteq \{1, 2, 3\} = \bigcup_{j \in J''} I_j(\hat{x})$, *and*
3. $\sum_{j \in \{2,3\}} (f_j(\hat{x}_j) - f_j(0)) = 2.04 \le 2.76 = f_4(\hat{x}_4) - f_4(0)$ *and* $\sum_{j \in \{2,3\}} (f_j(\hat{x}_j) - f_j(0))$ $= 2.04 \le 4 = f_7(\hat{x}_7) - f_7(0)$.

Since all the conditions of Theorem 4.1 are satisfied, we can easily set $x_4^* = 0$ *and* $x_7^* = 0$. *Also, the columns 4 and 7 can be removed from matrix A and the index set J can be updated as* $J := \{1, 2, 3, 5, 6, 8, 9\}$ *with regard to Simplification 4.*

b) Moreover, if we set $J' = \{9\}$ *and* $J'' = \{2, 6\}$, *then we have:*
1. $J' \cap J'' = \emptyset$,
2. $\bigcup_{j \in J'} I_j(\hat{x}) = \{2, 5\} \subseteq \{1, 2, 3, 5\} = \bigcup_{j \in J''} I_j(\hat{x})$, *and*
3. $\sum_{j \in \{2,6\}} (f_j(\hat{x}_j) - f_j(0)) = 4.81 \le 5.78 = f_9(\hat{x}_9) - f_9(0)$.

Hence, we can set $x_9^* = 0$, *with regard to Theorem 4.1. Also, the column 9 can be deleted from matrix A and the index set J can be updated as* $J := \{1, 2, 3, 5, 6, 8\}$ *with regard to Simplification 4.*
On the other hand, since $\bigcup_{j \in J'} I_j(\hat{x}) = \{5, 6\} \subseteq \{3, 4, 5, 6\} = \bigcup_{j \in J''} I_j(\hat{x})$ *and* $\sum_{j \in J''} (f_j(\hat{x}_j) - f_j(0)) = 7.35 > 4.22 = f_8(\hat{x}_8) - f_8(0)$, *where* $J' = \{8\}$ *and* $J'' = \{5, 6\}$, *we cannot use Theorem 4.1 in this case. Now, we can apply Step 4 for further simplification.*

c) Since $J_1 = \{2\}$ *and* $J_2 = \{2\}$, *we can set* $x_2^* = \hat{x}_2 = 0.5$, *with regard to Step 4. Thus, according to Simplification 2, the column 2 can be removed from matrix A and the rows 1 and 2 can be deleted from matrices A and b. Also, two index sets I and J can be updated as* $I := \{3, 4, 5, 6\}$ *and* $J := \{1, 3, 5, 6, 8\}$. *Since the simplified optimization problem (24)-(26) cannot be reduced further applying Steps 4,5,10, and 11, we go to Step 12.*

Step 12: *For the reduced matrix A, the value matrix M can be generated as follows:*

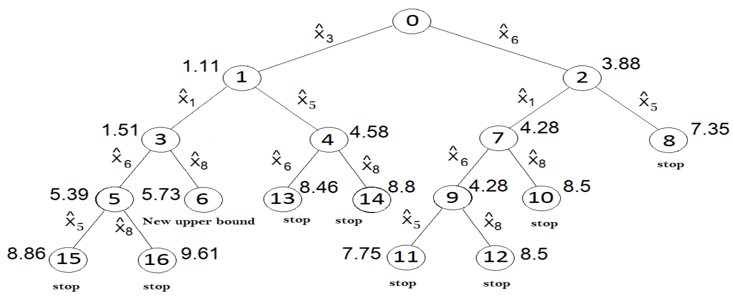

Figure 1: Branch-and-bound Method

$$M = \begin{array}{c c} & \begin{array}{ccccc} 1 & 3 & 5 & 6 & 8 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} \infty & 1.11 & \infty & 3.88 & \infty \\ 0.4 & \infty & 3.47 & \infty & \infty \\ \infty & \infty & \infty & 3.88 & 4.22 \\ \infty & \infty & 3.47 & \infty & 4.22 \end{pmatrix} \end{array}.$$

Step 13: *Compute the initial upper bound U on the optimal objective value of the simplified problem using Algorithm 5.4 as follows:*
$U = m_{33} + m_{41} + m_{56} + m_{65} = 1.11 + 0.4 + 3.88 + 3.47 = 8.86$.

Step 14: *Now, we are ready to solve the simplified problem (24)-(26) by the branch-and-bound method with the jump-tracking technique on the matrix M. We denote the objective function of the simplified problem (24)-(26) by Z'. The details of this step is summarized in Figure 1. We begin from the first equation. Since $J_3(\hat{x}) = \{3,6\}$, we have two candidates for satisfying the third equation, i.e., we branch two Nodes 1 ($\hat{x}_3$) and 2 ($\hat{x}_6$) with the values $Z'_1 = 1.11$ and $Z'_2 = 3.88$, respectively. Since $\hat{x}_3$ or $\hat{x}_6$ does not satisfy all the equations, we branch further to generate feasible solutions. We branch further on Node 1 with regard to the least objective value. We have $J_4(\hat{x}) = \{1,5\}$. So, $\hat{x}_1$ and $\hat{x}_5$ are two candidates for satisfying the fourth equation. If we select $\hat{x}_1$, then the value Z'_3 is 1.51. If we select $\hat{x}_5$, then the value Z'_4 is 4.58. Now, we have three nodes to select for the next branching process. By the jump-tracking technique, we select Node 3 to branch further because of the least objective value there.*

Consider Node 3. $\hat{x}_6$ and $\hat{x}_8$ are two candidates for satisfying the fifth equation. Thus, we generate Node 5 ($\hat{x}_6$) and Node 6 ($\hat{x}_8$) with the values 5.39 and 5.73, respectively. Since $\hat{x}_3$, $\hat{x}_1$, and $\hat{x}_8$ satisfy all equations with the value 5.73, which is better than the initial upper bound 8.86, we update the current upper bound as 5.73. Continuing this process, a tree with 16 nodes is generated as Figure 1. So, the optimal solution of the simplified problem is as $x_1^* = 0.2$, $x_3^* = 0.75$, $x_5^* = 0$, $x_6^* = 0$, and $x_8^* = 1$. Therefore, the optimal solution x^* of simplified problem (24)-(26) is as follows:
$x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*)^T = (0.2, 0.5, 0.75, 0, 0, 0, 0, 1, 0)^T$.

Step 15: *The optimal solution y^* and the optimal objective value $Z(y^*)$ of the optimization problem (21)-(23) are as follows:*
$y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*, y_6^*, y_7^*, y_8^*, y_9^*, y_{10}^*)^T = (0.25, 0.5, 0, 0, 0.75, 0, 0, 0, 0.2, 1)^T$ *with* $Z(y^*) = 18.12$.

Example 7.2. *Consider the following optimization problem:*

$$\min \quad Z(y) = y_1^3 + y_2\sqrt{y_2+3} + e^{y_3} - \sqrt{y_4} + 6^{y_5} + 5y_6^2 \tag{27}$$

$$\text{s.t.} \quad A \circ y = b, \tag{28}$$

$$y \in [0,1]^6, \tag{29}$$

where

$$A = \begin{pmatrix} 0.7 & 0.2 & 0.11 & 0.15 & 0.08 & 0.3 \\ 0.05 & 0.21 & 0.7 & 0.14 & 0.35 & 0.4 \\ 0.09 & 0.32 & 0.17 & 0.23 & 0.3 & 0.19 \\ 1 & 0.25 & 0.75 & 0.16 & 0.21 & 0.14 \\ 0.03 & 0.12 & 0.07 & 0.1 & 0.05 & 0.06 \end{pmatrix},$$

$y = (y_1, y_2, y_3, y_4, y_5, y_6)^T$, *and* $b = (0.21, 0.28, 0.24, 0.3, 0.09)^T$. *To solve problem (27)-(29), we apply Algorithm 6.1 as follows.*

Step 1: *The maximum solution of the feasible domain is as follows:*
$\hat{y} = (0.3, 0.75, 0.4, 0.9, 0.8, 0.7)^T$.

Step 2: *Since $A \circ \hat{y} = b$, then the feasible domain of this problem is not empty.*

Step 3: *In this example, the index sets $J_i(\hat{y})$ and $I_j(\hat{y})$, for each $i \in I$ and $j \in J$, are as follows:*
$J_1(\hat{y}) = \{1,6\}$, $J_2(\hat{y}) = \{3,5,6\}$, $J_3(\hat{y}) = \{2,5\}$, $J_4(\hat{y}) = \{1,3\}$, $J_5(\hat{y}) = \{2,4\}$, $I_1(\hat{y}) = \{1,4\}$, $I_2(\hat{y}) = \{3,5\}$, $I_3(\hat{y}) = \{2,4\}$, $I_4(\hat{y}) = \{5\}$, $I_5(\hat{y}) = \{2,3\}$, *and* $I_6(\hat{y}) = \{1,2\}$.

Steps 4 and 5: *The conditions of Steps 4 and 5 are not satisfied. Thus, we go to Step 6.*

Step 6: *Two index sets P' and P'' are as $P' = \{4\}$ and $P'' = \{1,2,3,5,6\}$.*

Step 7: *Since $\bigcup_{j \in \{4\}} I_j(\hat{y}) = I_4(\hat{y}) = \{5\} \neq I$, we go to Step 8.*

Step 8: *According to Theorem 3.4, we can set $y_4^* = \hat{y}_4 = 0.9$.*

Step 9: *According to Simplification 3, we can remove the column 4 from matrix A and the row 5 from matrices A and b. So, two index sets I and J can be updated as $I := \{1,2,3,4\}$ and $J := \{1,2,3,5,6\}$.*

Step 10: *Check the second type sufficient conditions for the simplified problem in Step 9:*

10.1. *Rearrange the columns of the reduced matrix A according to Remark 4.3. Since functions $f_j(y_j)$ are increasing, for each $j \in \{1,2,3,5,6\}$, and $f_1(\hat{y}_1) - f_1(0) \leq f_3(\hat{y}_3) - f_3(0) \leq f_2(\hat{y}_2) - f_2(0) \leq f_6(\hat{y}_6) - f_6(0) \leq f_5(\hat{y}_5) - f_5(0)$, then we can rearrange and rename the variables y_j, for each $j \in J$, as follows:*

$$y_1 \to x_1, \ y_3 \to x_2, \ y_2 \to x_3, \ y_6 \to x_4, \ \text{and} \ \ y_5 \to x_5.$$

We now focus on the following simplified optimization problem.

$$\min \quad x_1^3 + e^{x_2} + x_3\sqrt{x_3+3} + 5x_4^2 + 6^{x_5} \tag{30}$$
$$\text{s.t.} \quad A \circ x = b, \tag{31}$$
$$x \in [0,1]^5, \tag{32}$$

where

$$A = \begin{pmatrix} 0.7 & 0.11 & 0.2 & 0.3 & 0.08 \\ 0.05 & 0.7 & 0.21 & 0.4 & 0.35 \\ 0.09 & 0.17 & 0.32 & 0.19 & 0.3 \\ 1 & 0.75 & 0.25 & 0.14 & 0.21 \end{pmatrix},$$

$x = (x_1, x_2, x_3, x_4, x_5)^T$, *and* $b = (0.21, 0.28, 0.24, 0.3)^T$.
Also, we have $\hat{x} = (0.3, 0.4, 0.75, 0.7, 0.8)^T$.

10.2. *If we set $J'' = \{1,2,3\}$, then the following conditions are satisfied:*

1. $\bigcup_{j \in \{1,2,3\}} I_j(\hat{x}) = I$,
2. $\bigcup_{j \in \{2,3\}} I_j(\hat{x}) \neq I$, $\bigcup_{j \in \{1,3\}} I_j(\hat{x}) \neq I$, *and* $\bigcup_{j \in \{1,2\}} I_j(\hat{x}) \neq I$,
3. $\sum_{j \in \{1,2,3\}} (f_j(\hat{x}_j) - f_j(0)) = 0.027 + 0.49 + 1.45 = 1.967 \leq 2.45 = f_4(\hat{x}_4) - f_4(0)$.

Since all the conditions of Theorem 4.4 are satisfied, $x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)^T = (0.3, 0.4, 0.75, 0, 0)^T$ is an optimal solution of the simplified problem (30)-(32) with regard to the relation (19).
Hence, $y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*, y_6^*)^T = (0.3, 0.75, 0.4, 0.9, 0, 0)^T$ is an optimal solution of the problem (27)-(29) with the objective value $Z(y^*) = 3.02$.

8 Conclusions

The separable programming problem with the max-product $FREs$ was studied in this paper. Its feasible solution set was determined and the problem was divided into two subproblems based on decreasing or increasing single-variable functions in the objective function. Finding an optimal solution of the first subproblem was easy. Conversely, the main difficulty was the resolution of the second subproblem. To overcome it, some sufficient conditions were presented, under which some of the optimal components or all of them were determined without finding all the minimal solutions of its feasible domain. An algorithm was then designed to obtain an initial upper bound on the second subproblem. Also, a branch-and-bound algorithm was extended to solve the second subproblem without finding all the minimal solutions of its feasible domain. Finally, an efficient algorithm was designed based on the simplifications, the above algorithm, and the branch-and-bound method. Two numerical examples were given to illustrate the proposed algorithm.

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