Iranian Journal of Fuzzy Systems

Volume 16, Number 1, (2019), pp. 61-73

University of Sistan and Baluchestan

On generalized fuzzy numbers

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Abstract

This paper first improves Chen and Hsieh's definition of generalized fuzzy numbers, which makes it the generalization of definition of fuzzy numbers. Secondly, in terms of the generalized fuzzy numbers set, we introduce two different kinds of orders and arithmetic operations and metrics based on the λ-cutting sets or generalized λ-cutting sets, so that the generalized fuzzy numbers are integrated into a partial ordering set, a semi-ring and a complete non-separable metric space. Again, through two isomorphism theorem, the relationship between generalized fuzzy number space and fuzzy number space is established. Finally, as an application of generalized fuzzy numbers, the concept of continuous generalized fuzzy number-valued functions is introduced. And it is pointed out that many results of trapezoidal generalized fuzzy numbers can also be extended to generalized fuzzy numbers.

Keywords: Fuzzy number, Generalized fuzzy number, Order, Arithmetic operation, Metric.

1 Introduction

Since Zadeh [33] put forward the concept of fuzzy sets, the fuzzy number is introduced by Chang and Zadeh [1] subsequently, and it has been made much deeper by many authors [7, 9, 12, 13, 19, 20, 21, 23, 25, 26] etc.. Now the theory of fuzzy number has become the fundamental of fuzzy analysis [7, 21, 29]. Meanwhile, generalized fuzzy numbers have been defined by Chen and Hsieh [2], and also drawn much attentions in theory and practice. Recently, many investigations on the field such as ranking and similarity measure of generalized fuzzy numbers have been conducted [11, 14, 18, 24]. Their repeatedly applications in risk analysis [3, 4, 5, 16, 22, 30], transportation problems [10, 17], decision making [32] and pattern recognition [8] are also shown. But from the mathematics viewpoint, theoretically, it needs to point out that the existing theory of generalized fuzzy numbers comparing with the theory of fuzzy numbers is obviously lacking, at least, as follows:

(i) The definition of generalized fuzzy numbers [Chen and Hsieh [2]] is not "general", it does not include usual fuzzy numbers;

(ii) The definition of generalized fuzzy numbers [Chen and Hsieh [2]] includes an error, i.e. real number or interval number is not regarded as special generalized fuzzy numbers, since their membership functions do not satisfy the condition—"continuity";

(iii) Most of the results are mainly limited to the special generalized fuzzy numbers, such as generalized trapezoidal fuzzy number, resulting in the lacking of generality.

Based on above considerations, the general theory on generalized fuzzy number needs to be established. At least, they should include the redefinition, orders, arithmetic operations and metrics. They can not only make the usual theory of fuzzy numbers as special one, but also be applied to other aspects practically.

The remainder of the paper consists of three sections. As preliminaries in section 2, we shall remind some basic facts of interval numbers, fuzzy numbers and Chen and Hsieh's generalized fuzzy numbers. Meanwhile, the shortcomings of

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Received: July 2017; Revised: November 2017; Accepted: April 2018.

Chen and Hsieh's are indicated. As the main part, section 3 is divided into four subsections, they are the definition of generalized fuzzy numbers, the set of the same height h-generalized fuzzy numbers, the set of all different heights generalized fuzzy numbers, and a classification of the set of generalized fuzzy numbers. Some results are shown in the section. These include generalized decomposition theorem and generalized representation theorems, two different kinds of orders, arithmetic operations and metrics, and two isomorphism theorems. Some properties such as the metric space of generalized fuzzy numbers are completely proved. All of them are the generalization of the corresponding results of fuzzy numbers. In section 4, as an application, the concept of continuously generalized fuzzy number valued functions is introduced. At last, in the concluding remarks, it is indicated that many results existing for trapezoidal generalized fuzzy numbers can be extended to the case of generalized fuzzy numbers, and the differential and integral for generalized fuzzy number valued functions can be expected.

Throughout the paper, the following symbols will used.

R: the set of all real number;

R^+: the set of non-negative real number;

"$\vee$": "sup", " $\wedge$" :"inf";

$F(R) = \{\tilde{A} : \mu_{\tilde{A}} : R \to [0, 1]\}$: the set of fuzzy sets on R;

A_λ: λ-cutting set of $\tilde{A}$, $A_\lambda = \mu_{\tilde{A}}^{-1}[\lambda, 1] = \{x \in R : \mu_{\tilde{A}}(x) \geq \lambda\}$, for $\lambda \in [0, 1]$;

$\mu_{\tilde{A}}(x)$ or $\tilde{A}(x)$: the membership function.

2 Preliminaries

In this section, we shall remind some concepts on interval numbers, fuzzy numbers and Chen and Hsieh's generalized fuzzy numbers, and point out the shortcomings of Chen and Hsieh's definition [9, 21, 30].

Let X be a nonempty set.

Given a relation "$\leq$" on X, $a, b, c \in X$ and the following conditions:

(i) reflexivity: $a \leq a$;
(ii) anti-symmetric: $a \leq b, b \leq a \Rightarrow a = b$;
(iii) transitivity: $a \leq b, b \leq c \Rightarrow a \leq c$;

If conditions (i) an are d (iii) are satisfied, then $(X, \leq)$ is said to be a quasi-ordering set,

If conditions (i)–(iii) are satisfied, then $(X, \leq)$ is said to be a partial ordering set.

Let $(X, +, \cdot)$ be a closed algebra system, and $a, b, c \in X$. If the following conditions are satisfied:
(i) associative: $(a + b) + c = a + (b + c); (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
(ii) commutative: $a + b = b + a, a \cdot b = b \cdot a$;
(iii) distributive: $(a + b) \cdot c = a \cdot c + b \cdot c$;
(iv) $a + 0 = 0 + a = a, a \cdot 1 = 1 \cdot a = a, 0, 1 \in X$,
then $(X, +, \cdot)$ is called a commutative semi-ring.

Let $d : X \times X \to R^+$ and the following conditions be given:
(i) $d(x, x) \geq 0$, for all $x \in X$
(ii) $d(x, y) = 0 \Leftrightarrow x = y$, for $x, y \in X$
(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$
(iv) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$

If conditions (i), (iii) and (iv) are satisfied, then (X, d) is called a quasi-metric space,

If conditions (i)–(iv) are satisfied, then (X, d) is called a metric space.

2.1 Interval Numbers

An element in $I(R) = \{\bar{a} = [a^-, a^+] : -\infty < a^- \leq a^+ < +\infty\}$ is called an interval number, see e.g. [9, 21, 30]. It is easy to see that $r = [r, r] \in I(R)$ for $r \in R$, so we have $R \subset I(R)$. $I(R^+)$ is the set of non-negative interval numbers.

Let $\bar{a}, \bar{b} \in I(R)$. We define

$\bar{a} \leq \bar{b} \Leftrightarrow a^- \leq b^-, a^+ \leq b^+$;

$\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$;

$\bar{a} \cdot \bar{b} = [\min\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}, \max\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}]$;

$d(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}$,

Then

(i) $(I(R), \leq)$ is a partial ordering set;

(ii)$(I(R^+), +, \cdot)$ is a commutative semi-ring;

(iii) $(I(R), d)$ is a complete metric space.

2.2 Fuzzy Numbers

Definition 2.1. *A fuzzy set* $\tilde{a} \in F(R)$ *is said to be a fuzzy number (see e.g.* [7, 9, 12, 30]*) if it satisfies the following conditions:*

(i) $\tilde{a}$ *is normal, i.e. there exists an* $x_0 \in R$*, such that* $\mu_{\tilde{a}}(x_0) = 1$;
(ii) $\mu_{\tilde{a}}(x)$ *is upper semi-continuous, i.e.* $\mu_{\tilde{a}}^{-1}[\lambda, 1]$ *is closed for all* $\lambda \in [0, 1]$;
(iii) $\tilde{a}$ *is fuzzy convex, i.e.* $\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$ *for all* $x, y \in R$, $\lambda \in [0, 1]$;
(iv) the support of $\tilde{a}$ *is bounded, i.e.* $\bar{a}_0 = cl\{x \in R : \tilde{a}(x) > 0\}$ *is bounded.*

If the support of a fuzzy number $\tilde{a}$ is included in R^+, then the fuzzy number is said to be non-negative. The set of all fuzzy numbers (resp. non-negative fuzzy numbers) on R is written by $\tilde{R}$ (resp. $\tilde{R}^+$).

Let $\tilde{a}, \tilde{b} \in \tilde{R}$. It is defined as
$\tilde{a} \leq \tilde{b} \Leftrightarrow \bar{a}_\lambda \leq \bar{b}_\lambda$, for all $\lambda \in [0, 1]$;
$(\tilde{a} * \tilde{b})_\lambda = \bar{a}_\lambda * \bar{b}_\lambda$, for all $\lambda \in [0, 1]$, $* \in \{+, \cdot\}$;
$d_\infty(\tilde{a}, \tilde{b}) = \sup\{d(\bar{a}_\lambda, \bar{b}_\lambda) : \lambda \in [0, 1]\}$;
Then
(i) $(\tilde{R}, \leq)$ is a partial ordering set;
(ii) $(\tilde{R}^+, +, \cdot)$ is a commutative semi-ring;
(iii) $(\tilde{R}, d_\infty)$ is a complete metric space.

Theorem 2.2 (Representation theorem I, see e.g. [11, 12, 24, 26])**.** *Let* $\tilde{a} \in \tilde{R}$*. Then*
(i) $\bar{a}_\lambda \in I(R)$*, for all* $\lambda \in [0, 1]$*;*
(ii) $\bar{a}_{\lambda_2} \subseteq \bar{a}_{\lambda_1}$*, for* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$*;*
(iii) $\bigcap_{n=1}^{\infty} \bar{a}_{\lambda_n} = \bar{a}_\lambda$*, where* $\lambda_n, \lambda \in (0, 1], n \geq 1$, $\lambda_n \uparrow \lambda$*;*

Conversely, for $\{\bar{a}(\lambda) : \lambda \in [0, 1]\} \subset I(R)$, satisfies conditions (i)-(iii), then there exists $\tilde{a} \in \tilde{R}$, such that $\bar{a}_\lambda = \bar{a}(\lambda)$, for $\lambda \in (0, 1], \bar{a}_0 \subseteq \bar{a}(0)$.

Theorem 2.3 (Representation theorem II, see e.g. [11, 12, 24, 26])**.** *Let* $\tilde{a} \in \tilde{R}$,
$a^-(\lambda) = a_\lambda^-$, $a^+(\lambda) = a_\lambda^+$*, for* $\lambda \in [0, 1]$*. Then*
(i) $a^-(\lambda)$ *is non-decreasing and left continuous on* $(0, 1]$*;*
(ii) $a^+(\lambda)$ *is non-increasing and left continuous on* $(0, 1]$*;*
(iii) $a^-(1) \leq a^+(1)$;
(iv) $a^-(\lambda), a^+(\lambda)$ *is right continuous at* $\lambda = 0$.

Conversely, for the functions satisfying the above conditions (i)–(iv), there exists only one $\tilde{a} \in \tilde{R}$, such that $\bar{a}_\lambda = [a^-(\lambda), a^+(\lambda)], \lambda \in [0, 1]$.

2.3 Chen and Hsieh's Generalized Fuzzy Numbers

Definition 2.4 (Chen and Hsieh, see [2])**.** *Let* $a, b, c, d \in R$*, and* $-\infty < a \leq b \leq c \leq d < +\infty$*,* $h \in [0, 1]$*. Then a fuzzy set* $\tilde{a} \in F(R)$ *is said to be a generalized fuzzy number if it satisfies the following conditions:*
(i) $\mu_{\tilde{a}} : R \to (0, h]$ *is continuous;*
(ii) $\mu_{\tilde{a}}(x)$ *is strictly increasing on* $[a, b)$;
(iii) $\mu_{\tilde{a}}(x) = h$*,* $x \in [b, c]$;
(iv) $\mu_{\tilde{a}}(x)$ *is strictly decreasing on* $(c, d]$;
(v) $\mu_{\tilde{a}}(x) =$ *0, otherwise,* h *is said to be the height of* $\tilde{a}$*.*

If $\mu_{\tilde{a}}(x)$ is linear on $[a, b)$ and $(c, d]$, then $\tilde{a}$ is called a generalized trapezoidal fuzzy number, and is written by $(a, b, c, d; h)$.

For a generalized trapezoidal fuzzy number $(a, b, c, d; h)$.
If $b = c$, then it is called a generalized triangular fuzzy number, and written by $(a, b, d; h)$;
If $a = b, c = d, h = 1$, then it degenerates into an interval number;
If $a = b = c = d, h = 1$, then it degenerates into a real number.

Remark 2.5. *On "arithmetic operations": So far, we can only find the arithmetic operations on generalized trapezoidal fuzzy numbers in the existing literature (See [3, 4, 5, etc.]).*

For two generalized trapezoidal fuzzy numbers,

$$\tilde{a}_1 = (a_1, b_1, c_1, d_1; h_1), \tilde{a}_2 = (a_2, b_2, c_2, d_2; h_2),$$

It is defined as

$$\tilde{a}_1 + \tilde{a}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; h_1 \wedge h_2) \tag{1}$$

$$\tilde{a}_1 \cdot \tilde{a}_2 = (a, b, c, d; h_1 \wedge h_2) \tag{2}$$

, where

$$\begin{aligned} a &= a_1a_2 \wedge d_1d_2, \\ b &= a_1b_2 \wedge a_1c_2 \wedge b_1b_2 \wedge b_1c_2 \wedge b_2c_1 \wedge c_1c_2 \wedge c_1d_2 \wedge b_1d_2 \\ c &= a_1b_2 \vee a_1c_2 \vee b_1b_2 \vee b_1c_2 \vee b_2c_1 \vee c_1c_2 \vee c_1d_2 \vee b_1d_2 \\ d &= a_1a_2 \vee d_1d_2 \end{aligned}$$

All the operations are determined by the four vertex points and the heights.

Remark 2.6. *On "orders": In the existing literatures, e.g. [2, 4, 5, 18, 32, etc.], there exist various kinds of ranking methods on generalized trapezoidal fuzzy numbers, the popular one is constructing a ranking function from the set of generalized trapezoidal fuzzy numbers to the set of real numbers, i.e.* $K : G\tilde{R} \to R$, *and*

$$\begin{aligned} \tilde{a}_1 \prec \tilde{a}_2 &\Leftrightarrow K(\tilde{a}_1) < K(\tilde{a}_2) \\ \tilde{a}_1 \succ \tilde{a}_2 &\Leftrightarrow K(\tilde{a}_1) > K(\tilde{a}_2) \\ \tilde{a}_1 \approx \tilde{a}_2 &\Leftrightarrow K(\tilde{a}_1) = K(\tilde{a}_2) \end{aligned}$$

for $\tilde{a}_1 = (a_1, b_1, c_1, d_1; h_1)$, $\tilde{a}_2 = (a_2, b_2, c_2, d_2; h_2)$

For example, in ref. [16], *there is the following ranking function*

$$K(\tilde{a}_1) = \frac{1}{2}\alpha h_1(a_1 + b_1) + \frac{1}{2}(1 - \alpha)h_1(c_1 + d_1), \quad \alpha \in [0, 1]. \tag{3}$$

Remark 2.7. *On "metrics": In the existing literature* [3, 8, 16, 17, 31], *etc., there exist various kinds of distances or similarity measures for two generalized trapezoidal fuzzy numbers.*

For example, see [29], *Xu et al used distance and COG concepts to present a distance and a similarity measure, i.e.*

$$d(\tilde{a}_1, \tilde{a}_2) = \frac{\sqrt{(x^*_{\tilde{a}_1} - x^*_{\tilde{a}_2})^2 + (y^*_{\tilde{a}_1} - y^*_{\tilde{a}_2})^2}}{\sqrt{1.25}} \tag{4}$$

$$S(\tilde{a}_1, \tilde{a}_2) = 1 - \frac{1}{2} \times (\frac{|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|}{4} + d(\tilde{a}_1, \tilde{a}_2)) \tag{5}$$

where $x^*_{\tilde{a}_i}, y^*_{\tilde{a}_i}$ *determine the COG of* $\tilde{a}_i = (a_i, b_i, c_i, d_i; h_i)$, $i = 1, 2$, *that is,*

$$y^*_{\tilde{a}_i} = \begin{cases} \frac{h_i \times (\frac{c_i - b_i}{d_i - a_i} + 2)}{6}, & if \quad a_i \neq d_i \\ \frac{h_i}{2}, & if \quad a_i = d_i \end{cases} \tag{6}$$

$$x^*_{\tilde{a}_i} = \begin{cases} \frac{y^*_{\tilde{a}_i} \times (b_i + c_i) + (a_i + d_i) \times (h_i - y^*_{\tilde{a}_i})}{2h_i}, & if \quad h_i \neq 0 \\ \frac{a_i + d_i}{2}, & if \quad h_i = 0 \end{cases} \tag{7}$$

All of them are also related to generalized trapezoidal fuzzy numbers, and determined by the four vertex points and the heights.

Remark 2.8. *By comparing the Definitions 2.1 and 2.4, we can find the following facts:*

(i) The generalized fuzzy number is not a generalization of the fuzzy number. Since based Definition 2.2, the membership function must be continuous, strictly increasing on $[a, b)$ and strictly decreasing on $(c, d]$, but by Definition 2.1, it can shown that the membership function is upper semi-continuous, increasing on $[a, b)$ and decreasing on $(c, d]$ but may not strictly.

(ii) There exists an error in Definition 2.4, that is if $a = b, c = d$, then $\mu_{\tilde{a}}(x)$ does not satisfy continuity.

(iii) Based on Definition 2.4, it is known that a real number and an interval number is not generalized fuzzy number, since for $a \in R, \bar{a} \in I(R)$, the membership function is just the characteristic function, i.e.

$$\chi_a(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}, \quad \chi_{\bar{a}}(x) = \begin{cases} 1, & x \in \bar{a} \\ 0, & x \notin \bar{a} \end{cases}.$$

But $\chi_a, \chi_{\bar{a}}$ are not continuous.

From (i)–(iii), we can see that Chen and Hsieh's definition should be improved, such that the generalized fuzzy number can be a generalization of fuzzy numbers.

3 On Generalized Fuzzy Numbers

In this section, we first redefine a generalized fuzzy number, then will discuss the set of generalized fuzzy numbers by in three cases, i.e. the set of the same height h-generalized fuzzy numbers, the set of all different heights generalized fuzzy numbers, and a classification of the set of generalized fuzzy numbers.

3.1 Definitions

Definition 3.1. *A fuzzy set $\tilde{a} \in F(R)$ is said to be a h-generalized fuzzy number if it satisfies the following conditions:*

(i) $\tilde{a}$ is h-normal, i.e. there exists $x_0 \in R$, such that $\mu_{\tilde{a}}(x_0) = h = \sup\{\mu_{\tilde{a}}(x) : x \in R\}$, where h is said to be the height of $\tilde{a}$;

(ii) $\mu_{\tilde{a}}(x)$ is upper semi-continuous, i.e. $\mu_{\tilde{a}}^{-1}[\lambda, h]$ is closed for all $\lambda \in [0, h]$;

(iii) $\tilde{a}$ is fuzzy convex, i.e. $\mu_{\tilde{a}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$ for all $x, y \in R$, $\lambda \in [0, 1]$;

(iv) the support of $\tilde{a}$ is bounded, i.e. $\bar{a}_0 = cl\{x \in R : \tilde{a}(x) > 0\}$ is bounded.

The set of all h-generalized fuzzy numbers on R is written by $G\tilde{R}(h)$. In order to make the height clear, sometimes a h-generalized fuzzy number $\tilde{a}$ is also de noted by $\tilde{a}^h$.

Let $G\tilde{R} = \underset{h \in [0,1]}{\cup} G\tilde{R}(h)$. All elements in $G\tilde{R}$ are called generalized fuzzy numbers.

If the support set of a generalized fuzzy number is included in R^+, then $\tilde{a}$ is called a non-negative generalized fuzzy number. All non-negative generalized fuzzy numbers are de noted by $G\tilde{R}^+$.

Remark 3.2. *It is easy to see that $G\tilde{R}(1) = \tilde{R}$, hence the set of generalized fuzzy numbers is an extension of the set of fuzzy numbers.*

Remark 3.3. *Just like fuzzy numbers', the generalized fuzzy number also holds representation theorems, it only need to use $[0, h]$ instead of $[0, 1]$, for example, $\tilde{a}$ can be represented by $\{\bar{a}_\lambda = [a_\lambda^-, a_\lambda^+] : \lambda \in [0, h]\}$. For simplicity, here omitted.*

3.2 On the Set $G\tilde{R}(h)$

Let $\tilde{a}, \tilde{b} \in G\tilde{R}(h)$, we define
$\tilde{a} \leq \tilde{b} \Leftrightarrow \bar{a}_\lambda \leq \bar{b}_\lambda$, for all $\lambda \in [0, h]$
$(\tilde{a} * \tilde{b})_\lambda = \bar{a}_\lambda * \bar{b}_\lambda$, for all $\lambda \in [0, h]$, $* \in \{+, \cdot\}$
$d_\infty(\tilde{a}, \tilde{b}) = \sup\{d(\bar{a}_\lambda, \bar{b}_\lambda) : \lambda \in [0, h]\}$
Then we have the following isomorphism theorem.

Theorem 3.4 (Isomorphism theorem I)**.** *Let $\phi : G\tilde{R}(h) \to \tilde{R}$, $\tilde{a}^h \to \tilde{a}$, where*

$$\mu_{\tilde{a}}(x) = \begin{cases} \mu_{\tilde{a}^h}(x), & \mu_{\tilde{a}^h}(x) < h \\ 1, & \mu_{\tilde{a}^h}(x) = h \end{cases}$$

Then ϕ is one-to-one mapping, and $(G\tilde{R}(h), *, \leq, d_\infty) \cong (\tilde{R}, *, \leq, d_\infty)$ ("$\cong$" means "isomorphism").

Remark 3.5. *From this theorem, we can see that the set $G\tilde{R}(h)$ is very clear.*

3.3 On the Set $G\tilde{R}$

At first, we comsider generalized fuzzy numbers as a fuzzy sets, and then give a direct discussion.

Let $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}$, we define

(i) $\tilde{a}^h \leq \tilde{b}^k \Leftrightarrow \bar{a}^h_\lambda \leq \bar{b}^k_\lambda$, for all $\lambda \in [0,1]$,

where it is supposed that the empty set is less than any nonempty sets, i.e. $\phi \leq A \subset R$.

(ii)

$$(\tilde{a}^h \tilde{*} \tilde{b}^k)(z) = \sup_{x*y=z} (\tilde{a}^h(x) \wedge \tilde{b}^k(y)), * \in \{+, \cdot\} \tag{8}$$

(note: this is based on the extension principle [9])

(iii) $\hat{d}_\infty(\tilde{a}^h, \tilde{b}^k) = \sup_{\lambda\in[0,1]} d(\bar{a}^h_\lambda, \bar{b}^k_\lambda)$

(iv) $d_\infty(\tilde{a}^h, \tilde{b}^k) = |h-k| \vee \sup_{\lambda\in[0,1]} d(\bar{a}^h_\lambda, \bar{b}^k_\lambda)$

, where it is supposed that $d(\phi, A) = 0$, for any $A \subset R$.

Then we have the following conclusion.

Theorem 3.6.

(i) $(G\tilde{R}, \leq)$ *is a partial ordering set;*

(ii)

$$(\tilde{a}^h \tilde{*} \tilde{b}^k)_\lambda = \begin{cases} \bar{a}^h_\lambda * \bar{b}^k_\lambda, & \lambda \in [0, h\wedge k] \\ \phi, & \lambda \in (h \wedge k, 1] \end{cases} \tag{9}$$

(iii) $(G\tilde{R}, \hat{d}_\infty)$ *is a quasi-metric space;*

(iv) $(G\tilde{R}, d_\infty)$ *is a metric space;*

Proof. (i) is clear. To prove (ii), it is suffices to prove the case "+".

(1) For the case $\lambda \in (h\wedge k, 1]$, it is clear that $(\tilde{a}^h \tilde{+} \tilde{b}^k)_\lambda = \phi$.

(2) For the case $\lambda \in (0, h\wedge k]$, take $z \in (\tilde{a}^h \tilde{+} \tilde{b}^k)_\lambda$, that is, $\bigvee_{x+y=z} (\tilde{a}^h(x) \wedge \tilde{b}^k(y)) \geq \lambda$.

Take $\varepsilon_n \downarrow 0$, and $\lambda - \varepsilon_n > 0$, then there exist $\{x_n\}, \{y_n\} \subset R, x_n + y_n = z$, such that

$$\tilde{a}^h(x_n) \wedge \tilde{b}^k(y_n) \geq \lambda - \varepsilon_n$$

Hence, $x_n \in \bar{a}^h_{\lambda-\varepsilon_n}, y_n \in \bar{b}^k_{\lambda-\varepsilon_n}$. Since λ-cutting sets are closed intervals, there are sub-sequences $\{x_{n_j}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}, x_{n_j} \to x_0, y_{n_j} \to y_0$. By the upper semi-continuity of membership functions, we know

$$\tilde{a}^h(x_0) \wedge \tilde{b}^k(y_0) \geq \lambda$$

Further, $x_0 \in \bar{a}^h_\lambda, y_0 \in \bar{b}^h_\lambda, x_0 + y_0 = z$, that is $z \in \bar{a}^h_\lambda + \bar{b}^k_\lambda$.

On the contrary, for $z \in \bar{a}^h_\lambda + \bar{b}^k_\lambda$, then there exist $x_0 \in \bar{a}^h_\lambda$ and $y_0 \in \bar{b}^h_\lambda$, such that $x_0 + y_0 = z$, then

$$(\tilde{a}^h + \tilde{b}^k)(z) = \bigvee_{x+y=z} (\tilde{a}^h(x) \wedge \tilde{b}^k(y)) \geq \tilde{a}^h(x_0) \wedge \tilde{b}^k(y_0) \geq \lambda$$

That is $z \in (\tilde{a}^h \tilde{+} \tilde{b}^k)\lambda$, then $(\tilde{a}^h \tilde{+} \tilde{b}^k)\lambda = \bar{a}^h_\lambda + \bar{b}^k_\lambda$.

(3) For the case $\lambda = 0$, take $\lambda_n \downarrow 0$, based on the right continuity of cutting functions $a_\lambda^{-(+)}$ at $\lambda = 0$, we have

$$\begin{aligned}(\tilde{a}^h \tilde{+} b^k)_0 &= cl(\bigcup_{\lambda>0} (\tilde{a}^h \tilde{+} \tilde{b}^k)_\lambda) = cl(\lim_{\lambda_n\to 0^+} (\tilde{a}^h \tilde{+} \tilde{b}^k)_{\lambda_n}) \\ &= cl(\lim_{\lambda_n\to 0^+} \bar{a}^h_{\lambda_n} + \lim_{\lambda_n\to 0^+} \bar{b}^k_{\lambda_n}) = cl(\bar{a}_0 + \bar{b}_0) = \bar{a}_0 + \bar{b}_0\end{aligned}$$

Consequently, from (1)–(3), (ii) is proved.

(iii), (iv) is also clear. □

Remark 3.7. *The above arithmetic operations are not the same as Chen and Hsieh's (Remark 2.5).*

Example 3.8. *Let $\tilde{a} = (0,1,2;1)$, $\tilde{b} = (0,1,2;0.5)$. Then by Chen and Hsieh's we have*

$$\tilde{a} + \tilde{b} = (0,2,4;0.5),$$
$$\tilde{a} \cdot \tilde{b} = (0,1,4;0.5)$$

But by equation (8), we have

$$\tilde{a}\tilde{+}\tilde{b} = (0,1.5,2,5,4;0.5)$$

$$(\tilde{a}\tilde{\cdot}\tilde{b})(x) = \begin{cases} \sqrt{\frac{x}{2}}, & x \in [0,0.5) \\ 0.5, & x \in [0.5,1.5] \\ \frac{3-\sqrt{1+2x}}{2}, & x \in (1.5,4] \\ 0, & x \notin [0,4] \end{cases}$$

Obviously, $\tilde{a}\tilde{\cdot}\tilde{b}$ is not a trapezoidal generalized fuzzy number.

Remark 3.9. *Next counter-example shows that $\hat{d}_\infty$ is quasi-metric, but not a metric on $G\tilde{R}$.*

Example 3.10. *Let $\tilde{a}(x) = \begin{cases} 0.5, & x \in [1,2] \\ 0, & x \notin [1,2] \end{cases}$, $\tilde{b}(x) = \begin{cases} 0.3, & x \in [1,2] \\ 0, & x \notin [1,2] \end{cases}$. Then $\tilde{a} \neq \tilde{b}$, but $\hat{d}_\infty(\tilde{a},\tilde{b}) = 0$.*

Corollary 3.11. *$(G\tilde{R}^+, \tilde{+}, \tilde{\cdot})$ is a commutative semi-ring.*

Secondly, another way to discuss the generalized fuzzy number is to build the relationship between the generalized fuzzy number and the fuzzy number by normalizing a generalized fuzzy number.

For $\tilde{a}^h \in G\tilde{R}$, we can define a fuzzy number $\tilde{A} \in \tilde{R}$ by $\tilde{A}(x) = \frac{\tilde{a}^h(x)}{h}$, and $\tilde{A}$ is called the normalization of $\tilde{a}^h$. Then we get a mapping:

$$\psi : G\tilde{R} \to \tilde{R}; \tilde{a}^h \to \frac{\tilde{a}^h}{h} = \tilde{A}$$

Remark 3.12. *The next example shows that ψ is a full mapping, but not a one-to-one mapping.*

Example 3.13. *Let $\tilde{1}^{0.5}(x) = \begin{cases} 0.5, x = 1 \\ 0, x \neq 1 \end{cases}$, $\tilde{1}^{0.8}(x) = \begin{cases} 0.8, x = 1 \\ 0, x \neq 1 \end{cases}$. Then $\tilde{1}^{0.5} \neq \tilde{1}^{0.8}$, but $\psi(\tilde{1}^{0.5}) = \psi(\tilde{1}^{0.8}) = \tilde{1}$.*

We have the λ-cutting set of $\tilde{A}$ as follow:

$$\tilde{A}_\lambda = \{x \in R : \tilde{A}(x) \geq \lambda\} = \{x \in R : \frac{\tilde{a}^h(x)}{h} \geq \lambda\} = \{x \in R : \tilde{a}(x) \geq h\lambda\} = \bar{a}^h_{h\lambda}, where$$

$\bar{a}^h_{h\lambda}$ is called a generalized λ-cutting set of $\tilde{a}^h$, for $\lambda \in [0,1]$.

Theorem 3.14 (Generalized decomposition theorem)**.** *Let $\tilde{a} \in G\tilde{R}(h)$. Then*

$$\tilde{a} = \bigcup_{\lambda\in[0,1]}((h\lambda)\cdot\bar{a}_{h\lambda}), \quad \mu_{\tilde{a}}(x) = \bigvee_{\lambda\in[0,1]}((h\lambda)\wedge\chi_{\bar{a}_{h\lambda}}(x))$$

Where χ_A is the characteristic function of set A.

Proof. Since

$$\chi_{\bar{a}_{h\lambda}}(x) = \begin{cases} 1, & \tilde{a}(x) \geq h\lambda \\ 0, & \tilde{a}(x) < h\lambda \end{cases}, \lambda \in [0,1]$$

then

$$\begin{aligned} \bigcup_{\lambda\in[0,1]}((h\lambda)\bar{a}_{h\lambda}) &= \bigvee_{\lambda\in[0,1]}((h\lambda)\bar{a}_{h\lambda})(x) \\ &= \bigvee_{\lambda\in[0,1]}((h\lambda)\wedge\chi_{\bar{a}_{h\lambda}}(x)) \\ &= \bigvee_{h\lambda\in[0,h]}((h\lambda)\wedge\chi_{\bar{a}_{h\lambda}}(x)) \\ &= (\bigvee_{0\leq h\lambda\leq\tilde{a}(x)}((h\lambda)\wedge\chi_{\bar{a}_{h\lambda}}(x))) \vee (\bigvee_{\tilde{a}(x)<h\lambda\leq h}((h\lambda)\wedge\chi_{\bar{a}_{h\lambda}}(x))) \\ &= \bigvee_{0\leq h\lambda\leq\tilde{a}(x)}(h\lambda) \\ &= \tilde{a}(x) \end{aligned}$$

□

Theorem 3.15 (Generalized representation theorem I). *Let $\tilde{a} \in G\tilde{R}(h)$. Then*

(i) $\bar{a}_{h\lambda} \in I(R)$, *for all* $\lambda \in [0,1]$;

(ii) $\bar{a}_{h\lambda_2} \subseteq \bar{a}_{h\lambda_1}$, *for* $0 \le \lambda_1 \le \lambda_2 \le 1$;

(iii) $\bigcap_{n=1}^{\infty} \bar{a}_{h\lambda_n} = \bar{a}_{h\lambda}$, *where* $\lambda_n, \lambda \in 0,1], n \ge 1,\ \lambda_n \uparrow \lambda$;

Conversely, since $\{\bar{a}(h\lambda) : \lambda \in [0,1]\} \subset I(R)$, *satisfies conditions(i)-(iii), there exists* $\tilde{a} \in G\tilde{R}$, *such that* $\bar{a}_{h\lambda} = \bar{a}(h\lambda)$, *for* $\lambda \in (0,1], \bar{a}_0 \subseteq \bar{a}(0)$.

Theorem 3.16 (Generalized representation theorem II). *Let* $\tilde{a} \in G\tilde{R}(h)$,

$a^-(h\lambda) = a^-_{h\lambda}\quad, a^+(h\lambda) = a^+_{h\lambda}$, *for* $\lambda \in [0,1]$. *Then*

(i) $a^-(h\lambda)$ *is non-decreasing and left continuous on* $(0,1]$;

(ii) $a^+(h\lambda)$ *is non-increasing and left continuous on* $(0,1]$;

(iii) $a^-(h) \le a^+(h)$;

(iv) $a^-(h\lambda), a^+(h\lambda)$ *is right continuous at* $\lambda = 0$.

Conversely, for the functions satisfy conditions (i)–(iv), there exists only one $\tilde{a} \in G\tilde{R}(h)$, *such that* $\bar{a}_{h\lambda} = [a^-(h\lambda), a^+(h\lambda)], \lambda \in [0,1]$.

Corollary 3.17. *Let* $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}$. *Then*

$\tilde{a}^h = \tilde{b}^k \Leftrightarrow \bar{a}^h_{h\lambda} = \bar{b}^k_{k\lambda}, h = k$, *for all* $\lambda \in [0,1]$

Corollary 3.18. *Let* $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}$, *and define*

$\tilde{a}^h \prec \tilde{b}^k \Leftrightarrow \bar{a}^h_{h\lambda} \le \bar{b}^k_{k\lambda}$, *for all* $\lambda \in [0,1]$,

$\tilde{a}^h \le_h \tilde{b}^k \Leftrightarrow \bar{a}^h_{h\lambda} \le \bar{b}^k_{k\lambda}, h \le k$, *for all* $\lambda \in [0,1]$

Then

(i) $(G\tilde{R}, \prec)$ *is a quasi ordering set,*

(ii) $(G\tilde{R}, \le_h)$ *is a partial ordering set.*

Remark 3.19. *The ordering "$\prec$" is not anti-symmetric, i.e.* $\tilde{a}^h \prec \tilde{b}^k, \tilde{b}^k \prec \tilde{a}^h$ *does not imply* $\tilde{a}^h = \tilde{b}^k$.

Example 3.20. *Let*

$$\tilde{a}^{0.6}(x) = \begin{cases} 0.6, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}, \tilde{b}^{0.8}(x) = \begin{cases} 0.8, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Then $\tilde{a}^{0.6} \ne \tilde{b}^{0.8}$, *but* $\bar{a}^{0.6}_{0.6\lambda} = [0,1] = \bar{b}^{0.8}_{0.8\lambda}$, *for all* $\lambda \in [0,1]$.

Remark 3.21. *"$\le_h$" and "$\le$" is different.*

Example 3.22. *Let* $\tilde{a} = (0,1,2;1), \tilde{b} = (0,0.5,1.5,2;0.5)$. *Then* $\tilde{b} \le \tilde{a}$, *but* $\tilde{b} \le_h \tilde{a}$ *does not hold.*

Theorem 3.23.

(i) *If we define*

$$\tilde{a}^h \approx \tilde{b}^k \Leftrightarrow \tilde{a}^h \prec \tilde{b}^k, \tilde{b}^k \prec \tilde{a}^h, \quad {}_f or \tilde{a}^h, \tilde{b}^k \in G\tilde{R}$$

then "$\approx$" is an equivalence relation on $G\tilde{R}$.

(ii) $(G\tilde{R}/\approx, \prec)$ *is a partial ordering set, where* $G\tilde{R}/\approx = \{\bar{\tilde{a}} = \{\tilde{r} : \tilde{r} \approx \tilde{a}\} : \tilde{a} \in G\tilde{R}\}$ *is the quotient set.*

Theorem 3.24. *Let* $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}, * \in \{+, \cdot\}$, *it is defined as*

$$\tilde{a}^h \mathbin{\widehat{*}} \tilde{b}^k = \bigcup_{\lambda \in [0,1]} (((h \wedge k)\lambda) \cdot (\bar{a}^h_{h\lambda} * \bar{b}^k_{k\lambda})) \tag{10}$$

Then

(i)

$$(\tilde{a}^h \mathbin{\widehat{*}} \tilde{b}^k)_{(h\wedge k)\lambda} = \bar{a}^h_{h\lambda} * \bar{b}^k_{k\lambda}, \lambda \in [0,1], \tag{11}$$

(ii) $(G\tilde{R}^+, \widehat{+}, \widehat{\cdot})$ *is a commutative semi-ring.*

Remark 3.25. *So far, we have defined three kinds of arithmetic operations on generalized trapezoidal fuzzy numbers(i.e. equation (1); (2); (8); (9)). It can be easily seen that:*

(i) equation (1) and equation (2) are different from equation (8);

(ii) $\tilde{a}_1 + \tilde{a}_2$ *(equation (1)) is the same as* $\tilde{a}_1 \mathbin{\widehat{+}} \tilde{a}_2$ *(equation (10)), but* $\tilde{a}_1 \cdot \tilde{a}_2$ *and* $\tilde{a}_1 \mathbin{\widehat{\cdot}} \tilde{a}_2$ *are different.*

(ii) equations (8) and (10) are different.

Example 3.26. *Let $\tilde{a}, \tilde{b}$ be as in Example 3.1, i.e. $\tilde{a} = (0,1,2;1)$, $\tilde{b} = (0,1,2;0.5)$. Then*

$$\tilde{a} \mathbin{\widehat{+}} \tilde{b} = (0,2,4;0.5)$$

$$(\tilde{a} \mathbin{\widehat{\cdot}} \tilde{b})(x) = \begin{cases} 0.5\sqrt{x}, & x \in [0,1) \\ 0.5, & x = 1 \\ 0.5(2-\sqrt{x}), & x \in (1,4] \\ 0, & x \notin [0,4] \end{cases}$$

Theorem 3.27. *Let $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}$, it is defined as*

$$\hat{D}_\infty(\tilde{a}^h, \tilde{b}^k) = \sup_{\lambda\in[0,1]} d(\bar{a}^h_{h\lambda}, \bar{b}^k_{k\lambda})$$
$$D_\infty(\tilde{a}^h, \tilde{b}^k) = |h-k| \vee \hat{D}_\infty(\tilde{a}^h, \tilde{b}^k)$$

Then
(i) $(G\tilde{R}, \hat{D}_\infty)$ *is a quasi-metric space,*
(ii) $(G\tilde{R}, D_\infty)$ *is metric space.*

Remark 3.28. *Both Functions $\hat{d}_\infty$ and $\hat{D}_\infty$ (resp. d_∞ and D_∞) are different.*

Example 3.29. *Let $\tilde{a} = (0,4,8;1), \tilde{b} = (0,4,8;0.5)$. Then*

$$\hat{d}_\infty(\tilde{a}, \tilde{b}) = 2, \quad \hat{D}_\infty(\tilde{a}, \tilde{b}) = 0$$
$$d_\infty(\tilde{a}, \tilde{b}) = 2, \quad D_\infty(\tilde{a}, \tilde{b}) = 0.5$$

Theorem 3.30. $(G\tilde{R}, D_\infty)$ *(resp. $(G\tilde{R}, \hat{D}_\infty)$) is a complete and non-separable metric space (resp. quasi-metric space).*

Proof. It is suffices to prove the case $(G\tilde{R}, D_\infty)$.

Let us take a Cauchy sequence $\{\tilde{a}_n^{h_n}\} \subset G\tilde{R}$. For every $\lambda \in [0,1]$, write $\bar{a}_n(\lambda) = \bar{a}^{h_n}_{h_n\lambda}, n \geq 1$. By definition of D_∞, we can see that $\{h_n\}$ and $\{\bar{a}_n(\lambda)\}$ are all Cauchy sequences. Since $(R, ||)$ and $(I(R), d)$ are complete, then $h_n \to h$ and $\bar{a}_n(\lambda) \to \bar{a}(\lambda)$ uniformly with $\lambda \in [0,1]$.

Next, let us prove $\{\bar{a}(h\lambda) : \lambda \in [0,1]\}$ can determine a generalized fuzzy number.

(i) It is obvious that $\{\bar{a}(h\lambda) : \lambda \in [0,1]\}$ is a family of interval number,

(ii) For $\lambda_1 \leq \lambda_2, n \geq 1$, we have $\bar{a}_n(\lambda_2) \subseteq \bar{a}_n(\lambda_1)$, then

$$\bar{a}(h\lambda_2) = \lim_{n\to\infty} \bar{a}_n(\lambda_2) \subseteq \lim_{n\to\infty} \bar{a}_n(\lambda_1) = \bar{a}(h\lambda_1)$$

(iii) For a fixed $\lambda \in (0,1]$, take $\lambda_n \uparrow \lambda$, then

$$\begin{aligned}\bigcap_{n=1}^{\infty} \bar{a}(\lambda_n) &= [\lim_{n\to\infty} a^-(\lambda_n), \lim_{n\to\infty} a^+(\lambda_n)] \\ &= [\lim_{n\to\infty}\lim_{k\to\infty} a_k^-(\lambda_n), \lim_{n\to\infty}\lim_{k\to\infty} a_k^+(\lambda_n)] \\ &= [\lim_{k\to\infty}\lim_{n\to\infty} a_k^-(\lambda_n), \lim_{k\to\infty}\lim_{n\to\infty} a_k^+(\lambda_n)]\end{aligned}$$

(as $\bar{a}_k(\lambda) \to \bar{a}(\lambda)$ uniformly with $\lambda \in [0,1]$, the above two repeated limits can exchange order)

$$= [\lim_{k\to\infty} a_k^-(\lambda), \lim_{k\to\infty} a_k^+(\lambda)]$$

(as $a_k^-(\lambda), a_k^+(\lambda)$ is left continuous)

$$\begin{aligned}&= [a^-(h\lambda), a^+(h\lambda)] \\ &= \bar{a}(h\lambda)\end{aligned}$$

Hence, by generalized representation theorems, there is a $\tilde{a}^h \in G\tilde{R}$, such that $\bar{a}^h_{h\lambda} = \bar{a}(h\lambda)$, for $\lambda \in (0,1]$, meanwhile $\bar{a}^h_0 = cl(\bigcup_{\lambda\in(0,1]} \bar{a}^h_{h\lambda}) \subseteq \bar{a}(0)$.

Since
$cl(\underset{\lambda\in(0,1]}{\cup}\bar{a}_n(\lambda)) = \bar{a}_n(0), \bar{a}_k(\lambda) \to \bar{a}(\lambda)$ uniformly with $\lambda \in [0,1]$, then we have $\bar{a}_0^h = \bar{a}(0)$.

Consequently $D_\infty(\tilde{a}_n^{h_n}, \tilde{a}^h) \to 0$, and $(G\tilde{R}, D_\infty)$ is complete.

We know that $(G\tilde{R}(1), D_\infty)$ is non-separable [7] as a complete subspace, this leads to $(G\tilde{R}, D_\infty)$ which is also non-separable.

For the case $(G\tilde{R}, \hat{D}_\infty)$ is similar,
the proof is end. □

Theorem 3.31. *Let $\tilde{a}^h, \tilde{b}^k \in G\tilde{R}^+$, it is defined*

$$S(\tilde{a}^h, \tilde{b}^k) = \frac{1}{1 + D_\infty(\tilde{a}^h, \tilde{b}^k)}$$

Then S is a similarity measure on $G\tilde{R}^+$, i.e. there holds
(i) $0 \le S(\tilde{a}^h, \tilde{b}^k) \le 1$, $S(\tilde{a}^h, \tilde{b}^k) = 1 \Leftrightarrow \tilde{a}^h = \tilde{b}^k$
(ii) $S(\tilde{a}^h, \tilde{b}^k) = S(\tilde{b}^k, \tilde{a}^h)$
(iii) $\tilde{a}^h \le_h \tilde{b}^k \le_h \tilde{c}^l \Rightarrow S(\tilde{a}^h, \tilde{b}^k) \ge S(\tilde{a}^h, \tilde{c}^l)$

Remark 3.32. *The above theorem is still holding for metric d_∞.*

3.4 On the Set $G\tilde{R}/\approx$

From Theorem 3.23, we know that $G\tilde{R}/\approx = \{\bar{\tilde{a}} = \{\tilde{r} : \tilde{r} \approx \tilde{a}\} : \tilde{a} \in G\tilde{R}\}_{,for} \bar{\tilde{a}}, \bar{\tilde{b}} \in G\tilde{R}/\approx$. Let's define

$$\bar{\tilde{a}} \le_h \bar{\tilde{b}} \Leftrightarrow \tilde{a} \le_h \tilde{b}$$

$$\bar{\tilde{a}} \,\widehat{*}\, \bar{\tilde{b}} = \overline{\tilde{a} \,\widehat{*}\, \tilde{b}}, * \in \{+, \cdot\}$$

$$D_\infty(\bar{\tilde{a}}, \bar{\tilde{b}}) = D_\infty(\tilde{a}, \tilde{b})$$

Then we have the following theorems:

Theorem 3.33 (Isomorphism theorem II)**.**
(i) $(G\tilde{R}/\approx; \le_h) \cong (\tilde{R}; \le)$,
(ii) $(G\tilde{R}^+/\approx; \widehat{*}, D_\infty) \cong (\tilde{R}^+; *, d_\infty)$
Where "≅" means " isomorphism ".

Theorem 3.34. *For a fixed $\bar{\tilde{a}} \in G\tilde{R}/\approx_{.Let}$ $\tilde{a}^h, \tilde{b}^k \in \bar{\tilde{a}}$. Then*
(i) $\tilde{a}^h \le_h \tilde{b}^k \Leftrightarrow h \le k$
(ii) $D_\infty(\tilde{a}^h, \tilde{b}^k) = |h - k|$

4 Applications: An Introduction to Generalized Fuzzy Number-valued Functions

So far, generalized fuzzy numbers have shown repeatedly the applications in risk analysis, see e.g. [3, 4, 5, 14, 21, 29], transportation problems see [9, 16], decision making [30], pattern recognition [7] and many other fields. Meanwhile, it is well-known that fuzzy-valued functions are also useful tool in optimization, decision makings, economic analysis and so on. Many authors, see e.g. [6, 7, 9, 21, 29, 30, 34] have done much work in the field of fuzzy-valued functions, so we should build up the theory on generalized fuzzy number-valued functions.

Definition 4.1. *A mapping $\tilde{F} : [a,b] \to G\tilde{R}$ is said a generalized fuzzy number-valued function. $\tilde{F}$ is said to be left continuous at $t_0 \in (a,b]$ (resp. right continuous at $t_0 \in [a,b)$), if for every $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that $D_\infty(\tilde{F}(t), \tilde{F}(t_0)) < \varepsilon$ for all $t \in [a,b]$ with $0 < t_0 - t < \delta$ (resp. $0 < t - t_0 < \delta$). $\tilde{F}$ is said to be continuous at $t_0 \in (a,b)$, if it is both left and right continuous at $t_0 \in (a,b)$. $\tilde{F}$ is said to be continuous on $[a,b]$, if it is continuous at every $t \in (a,b)$, right continuous at $t = a$ and left continuous at $t = b$.*

Theorem 4.2. *Let $\tilde{F} : [a,b] \to G\tilde{R}$ be a generalized fuzzy number-valued function, $h(t) = h_{\tilde{F}(t)}$ is the height of $\tilde{F}(t)$ and $F_{h(t)\lambda}(t) = [f^-_{h(t)\lambda}(t), f^+_{h(t)\lambda}(t)]$ for $\lambda \in [0,1]$. Then it is continuous if and only if $h(t)$ and $f^{-(+)}_{h(t)\lambda}(t)$ are all continuous for every $\lambda \in [0,1]$.*

This is the beginning of analysis theory of generalized fuzzy number-valued functions.

5 Conclusions

(i) First, many conclusions of trapezoidal generalized fuzzy numbers can be extended to the case of generalized fuzzy numbers.

Example 5.1. *Extensions of equation (3)–(7).*

Let $\tilde{a}^h \in G\tilde{R}$, we define the ranking function as follow:

$$K(\tilde{a}) = \alpha \int_0^h a_\lambda^- d\lambda + (1-\alpha) \int_0^h a_\lambda^+ d\lambda, \quad \alpha \in [0,1]$$

Where "$\int$" is the Riemann integral. since a_λ^-, a_λ^+ are monotone, $K(\tilde{a})$ is well defined and it is an extension of equation (3).

Next, in order to extend equations (4) and (5), it needs the following lemma.

Lemma 5.2. *Let $\tilde{a} \in G\tilde{R}$. Then it's membership function can be represented by*

$$\tilde{a}(x) = \begin{cases} l(x), & a \le x < b \\ h, & b \le x \le c \\ r(x), & c < x \le d \\ 0, & x \notin [a,d] \end{cases},$$

where $a \le b \le c \le d, a, b, c, d \in R, h \in [0,1], l : [a,b) \to [0,h)$ is non-decreasing and right continuous, $r : (c,d] \to (0,h]$ is non-increasing left continuous.

By the lemma, we suppose $\tilde{a}_i \in G\tilde{R}, i = 1,2$, and

$$\tilde{a}_i(x) = \begin{cases} l_i(x), & a_i \le x < b_i \\ h_i, & b_i \le x \le c_i \\ r_i(x), & c_i < x \le d_i \\ 0, & x \notin [a_i,d_i] \end{cases}$$

We define

$$d(\tilde{a}_1,\tilde{a}_2) = \frac{\sqrt{(x^*_{\tilde{a}_1} - x^*_{\tilde{a}_2})^2 + (y^*_{\tilde{a}_1} - y^*_{\tilde{a}_2})^2}}{\sqrt{1.25}}$$

$$S(\tilde{a}_1,\tilde{a}_2) = 1 - \frac{1}{2} \times \left(\frac{|a_1-a_2| + |b_1-b_2| + |c_1-c_2| + |d_1-d_2|}{4} + d(\tilde{a}_1,\tilde{a}_2)\right)$$

where $x^*_{\tilde{a}_i}, y^*_{\tilde{a}_i}$ determine the COG of $\tilde{a}_i$, $i = 1,2$, that is,

$$x_{i*} = \frac{\int_{a_i}^{b_i} x l_i(x)dx + \int_{b_i}^{c_i} x h_i dx + \int_{c_i}^{d_i} x r_i(x)dx}{\int_{a_i}^{b_i} l_i(x)dx + \int_{b_i}^{c_i} h_i dx + \int_{c_i}^{d_i} r_i(x)dx},$$

$$y_{i*} = \frac{\int_0^{h_i} \lambda(l_i^{-1}(\lambda) - r_i^{-1}(\lambda))d\lambda}{\int_0^{h_i} (l_i^{-1}(\lambda) - r_i^{-1}(\lambda))d\lambda}$$

$$l_i^{-1}(\lambda) = \inf\{x : l_i(x) = \lambda\}, \quad r_i^{-1}(\lambda) = \sup\{x : r_i(x) = \lambda\}.$$

Then equations (4) and (5) are extended.

(ii) since now, the general theory on generalized fuzzy numbers has been established. These including orders, arithmetic operations and metrics, and some properties are obtained. It is not only the deeper study of Chen and Hsieh's [2], but also a good generalization of the corresponding theory of fuzzy numbers.

(iii) There are some other definitions of different kinds of generalized fuzzy numbers, such as Wang's generalized discrete fuzzy number [27, 28]. It is easy to see that Wang's can not be included in ours. It is necessary to point out that an interesting work about Huang and Wu's general fuzzy sets which had no the assumption of fuzzy convexity or compact support, is also the generalization of fuzzy numbers not being included in ours, for the details, see [15].

With the basic theory of generalized fuzzy numbers shown in the paper, a new branch of fuzzy analysis with generalized fuzzy number-valued functions can be expected. As an application, we have put forward the basic issues of functions with values in generalized fuzzy numbers, its differential and integral can be investigated subsequently.

Acknowledgement

The authors are grateful to the unknown reviewers for their valuable suggestions and Editor-in-Chief, Prof. Mashinchi's kind help. The work is supported by the national natural science foundation of China (no.11271063) and the natural science foundation of Jilin Province(no.20190201014JC).

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