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# Restricted cascade and wreath products of fuzzy finite switchboard state machines

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**Abstract**

A finite switchboard state machine is a specialized finite state machine. It is built by binding the concepts of switching state machines and commutative state machines. The main purpose of this paper is to give a specific algorithm for fuzzy finite switchboard state machine and also, investigates the concepts of switching relation, covering, restricted cascade products and wreath products of fuzzy finite switchboard state machines. More precisely, we study that the direct products/Cartesian compositions of two such fuzzy finite switchboard state machines is again a fuzzy finite switchboard state machine. In addition, we introduce the perfect switchboard machine and establish its Cartesian composition. The relations among the products also been examined. Finally, we introduce asynchronous fuzzy finite switchboard state machine and study the switching homomorphic image of asynchronous fuzzy finite switchboard state machine. We illustrate the definition of a restricted product of fuzzy finite switchboard state machine with the single pattern example.

*Keywords:* Fuzzy finite state machine, Switchboard, Direct product, Cascade product, Wreath product, Asynchronous.

## 1 Introduction

Automata theory is one of the topics from the general system theory which provides mechanisms for the formulation and solution of general problems which can be applied to real-world problems in the future. A different class of switching mechanisms has been used for controlling more complex systems. It is necessary to understand the significance of the modeling of switching mechanisms as a control device for any electronic system. In 2002, according to Inagaki [10], Genetic algorithms (GAs), an evolutionary computation method, was used for generating more complex deterministic finite automata (DFA) through the use of a switching device to make correct predictions on the next input symbol. Within the context of a Design Pattern, Ramnath and Dathan [33] studied the switchboard behavior which is similar to a mediator in a finite state machine (FSM) and also highlighted that FSM events allow anyone to design and modify the two subsystems independently. An FSM model exhibits a behavior where responses to future events depend on previous events. A classical problem of the finite state machine is to navigate or to predict the flow of the next input information into a designated output when it receives a given input information from a sequence of integers. The purpose of the switchboard in a finite state machine is that a direct flow of information from one state to another is able to be controlled and any sudden failure will not cause the information to be entirely lost. However, the switchboard

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does not attempt to predict the next state input. Instead, it performs a correct prediction of the next input for the interaction between the subsystems. Therefore, in this research, the designed switchboard for a finite state machine is used for communication between the subsystems of the whole system and also maintains the mapping between the objects within one subsystem and another subsystem. As an application in pattern recognition, the switchboard allows communication with any two subsystems on a particular system or different systems. Moreover, the theory of switching state machines can be applied to computing systems in microwave, water phase, dash control and automatic braking systems.

Theoretical computer science uses models and analysis to examine computers and computation. It covers many areas of computer science to develop models and methods of analysis. Among the many mathematical patterns found in numerical computation, automata is one of the most fundamental and significant theories in computer science. Algebraic techniques used to build the structure of automata are also widely available. Investigating finite automata or finite state machines is important as it can reduce the gap between the precision of formal languages and the imprecision of natural languages. When dealing with imprecision due to the fuzziness in modeling certain vague systems, fuzzy automata and fuzzy languages are used as a systematic extension of classical automata and formal language theory. Unlike traditional case, the principal idea in the formulation is that a fuzzy automaton can be changed from one state to another to a certain degree. Thus, it is capable of overwhelming the uncertainty appearing in states or state transitions of a system. This actually motivates the present study to examine the fuzzy automata theory by binding the concept of switching and commutative state machines as a specialization of fuzzy finite state machines.

After the advent of fuzzy set theory by Zadeh [39] in 1965, the first mathematical formulation of fuzzy automata was proposed by Wee [37] in 1967 which was considered as a generalization of finite-automata theory. Malik *et al.* applied algebraic techniques to introduce the concept of submachines in [22] and subsystems in [23] for fuzzy finite state machines (FFSMs) where they considered state membership as fuzzy and discussed their fundamental properties. Besides that, products of fuzzy finite state machines were introduced in [24] by Malik *et al.* who also studied fuzzy transformation semigroups, covering, cascade and wreath products. Furthermore, Kumbhojkar and Chaudhari [15] developed the notion of a product of fuzzy finite state machines and analyzed their mutual relationship along with isomorphism and covering. Furthermore, Qiu [29] investigated the idea of a characterization of fuzzy finite automata. In that research paper, Qiu focused on the concept of a fuzzy automaton with a bi-fuzzy property. Many researcher's contributions related to classical fuzzy automata with membership grades are from the unit interval [0,1] but in [27, 28, 31], Qiu enhanced the membership grades in a more general algebraic structure such as complete residuated lattice-valued logic and established a remarkable theoretical framework of fuzzy automata based on complete residuated lattice-valued logic. Besides Qiu and his coworkers studied the state minimization problem of fuzzy finite automata underlying the structure of complete residuated lattices-valued logic in [38] and totally ordered lattices in [16]. Subsequently, Li and Pedrycz [17] found that the algebraic properties of fuzzy automata depended heavily on the properties of underlying structures such as lattice-ordered monoids. Meanwhile, the fuzzy finite automata theory on po-monoids from the algebraic viewpoint was introduced by Jin et.al [11]. Li and Wang [18], on the other hand, introduced the concept of the universal fuzzy automaton of a fuzzy language with membership values in a complete residuated lattice. In that particular study, the factorizations of fuzzy language and transition function are defined using the inclusion degree of related fuzzy languages. Moreover, Das [4] invented the concept of fuzzy topology which is associated with fuzzy finite state machines. In addition, he also established the concept of fuzzy subsystems and the connectedness of fuzzy finite state machines. Furthermore, Srivastava and Tiwari [35] established a relationship among fuzzy rough set theory, fuzzy topology and fuzzy automata and showed that the homomorphisms between fuzzy automata are continuously mapped in the sense of fuzzy topologies. Bisimulations usually express the model equivalence between states of the same system and allow the reduction of the number of states by combining bisimilar states. Recently, Cao [1] introduced bisimulation techniques for (deterministic) fuzzy transition systems. In addition, bisimulation can be regarded as a kind of behavioral distance for non-deterministic fuzzy transition systems [3]. Ignjatović *et al.* in a recent work [8] presented a more comprehensive notion of subsystems of the fuzzy transition systems. They have also studied fuzzy relation equations based on fuzzy transition systems. In view of an application, fuzzy finite automata have been used in model checking [19, 20], fuzzy discrete event system [5, 30, 32, 21] and bi-fuzzy discrete event system [6, 7]. The notion and its representations of asynchronous automata and commutative asynchronous automata were studied in [9] and [36].

Some interesting results and the classification of finite state machines with switchboards have been established by Sato and Kuroki [34]. In that study, there were attempts to describe the theory of finite switchboard state machines in both switching and commutative states. Based on prior research, Jun[12] explored and conducted research about the intuitionistic fuzzy finite switchboard state machines. On the other hand, the bipolar fuzzy finite switchboard state machine model was introduced by Jun *et al.*[13] and Kavikumar *et al.*[14]. However, most if the work found in the literature considered only the algebraic study of fuzzy finite state machines, whereas the incorporation of switchboards in the machine as well as the algebraic properties for fuzzy finite switchboard state machines have yet to be explored. Not much research has been performed by other researchers in the domain of fuzzy finite switchboard state machines because there is a lack of algebraic approaches. The domain of fuzzy finite switchboard state machines is still open to many possibilities for innovative research work. Motivated by the work of Sato and Kuroki[34] and Jun[12], we continue the study of fuzzy finite switchboard state machines in this paper. The aim of this paper is to solve this problem and to propose an efficient algebraic technique to study finite switchboard state machine. In particular, we examine the direct product, Cartesian product, the covering and other products of fuzzy finite switchboard state machines.

### 1.1 Main Contribution & Paper Outline

In the present work, we are mainly concerned with restricted types of fuzzy finite state machines. We have first attempted to introduce switching relations to fuzzy finite switchboard state machines. Using the concept of switching relation, we defined switching classes which further led to the analysis of quotient fuzzy finite switchboard state machines. We show that the quotient fuzzy finite switchboard state machines can be covered by the fuzzy finite switchboard state machines. In addition, we study the concepts of wreath products, restricted cascade products, restricted direct products and the Cartesian product of the fuzzy finite switchboard state machines which is similar to the study by Malik *et al.* [24] but with the restricted condition. We show that the Cartesian compositions of two such fuzzy finite switchboard state machines are fuzzy finite switchboard state machine. The remainder of this paper organized as follows. Section 2 provides the results and definitions concerning fuzzy finite state machines. Section 3 describes the wreath products, restricted cascade products, restricted direct products, Cartesian products of the fuzzy finite switchboard state machines and Cartesian products of prefect switchboard machines. Section 4, we introduce asynchronous fuzzy finite switchboard state machine and study their related properties. In section 5, we give a single pattern one-minute microwave as an example for the definition of the restricted product of fuzzy finite switchboard state machine. Finally, this paper ends with a conclusion of this study.

## 2 Preliminaries

Now, we provide some definitions associated with fuzzy finite state machines which are useful in the next section.

Let $\mathcal{M} = (Q, X, \mu)$ be a fuzzy finite state machine (ffsm)[26], where $Q$ and $X$ are finite non-empty sets and $\mu$ is a fuzzy subset of $\mu : Q \times X \times Q \to [0, 1]$, where $Q$ is called the set of states, $X$ is called the set of inputs and $\mu$ is called the transition function. Let $X^*$ be the set of all word elements of $X$ of finite length. Let $\beta$ be the empty words in $X^*$ and $\mid x \mid$ be the length of finite length. Define $\mu^* : Q \times X^* \times Q \to [0, 1]$ by

$$\mu^*(q, \beta, p) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases} \quad \text{and}$$

$$\mu^*(q, xb, p) = \bigvee \{\mu(q, x, r) \wedge \mu^*(r, b, p) : r \in Q\}, \forall b \in X^*, x \in X.$$

An ffsm $\mathcal{M}$ is said to be trivial machine which does not accept any strings, i.e. $\mu(q, \beta, q) = 1, \forall q \in Q$. Let $\mathcal{M} = (Q, X, \mu)$ be a fuzzy finite state machine. $\mathcal{M}$ is called complete [25] if for all $q \in Q, \beta \in X$, there exists $p \in Q$ such that $\mu(q, \beta, p) > 0$. Consider $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ as fuzzy finite state machines. Let $\eta : Q_2 \to Q_1$ be a surjective partial function and let $\xi : X_1 \to X_2$ be a function. Then the pair $(\eta, \xi)$ is called a covering [15] of $\mathcal{M}_1$ by $\mathcal{M}_2$ which is written as $\mathcal{M}_1 \leq \mathcal{M}_2$, if $\mu_1^+(\eta(p_2), b, \eta(q_2)) = \bigvee\{\mu_2^+(p_2, \xi^+(b), q_2)\}$, $\forall b \in X_1^+, p_2, q_2$ belong to the domain of $\eta$ where $\xi^+ : X_1^+ \to X_2^+$ is defined by $\xi^+(b_1 b_2 \dots b_n) = \xi(b_1)\xi(b_2)\dots\xi(b_n)$ for $b_1 b_2 \dots b_n \in X_1$. For all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, $\mathcal{M}_1 \times \mathcal{M}_2 = (Q_1 \times Q_2, X_1 \times X_2, \mu_1 \times \mu_2)$ are called the full direct product [24] of $\mathcal{M}_1$ and $\mathcal{M}_2$ if and only if $(\forall (b_1, b_2) \in X_1 \times X_2)$ $\mu_1 \times \mu_2\big((q_1, q_2), (b_1, b_2), (p_1, p_2)\big) = \mu_1(q_1, b_1, p_1) \wedge \mu_2(q_2, b_2, p_2)$. $\mathcal{M}_1 \wedge \mathcal{M}_2 = (Q_1 \times Q_2, X, \mu_1 \wedge \mu_2)$ is called the restricted direct product [24] of $\mathcal{M}_1$ and $\mathcal{M}_2$ if and only if, $\mu_1 \wedge \mu_2\big((q_1, q_2), b, (p_1, p_2)\big) = \mu_1(q_1, b, p_1) \wedge \mu_2(q_2, b, p_2)$, $\forall b \in X$ where $X_1 = X_2$. Let $\omega$ be a function of $Q_2 \times X_2$ into $X_1$. Let $Q = Q_1 \times Q_2$ and $\mu^\omega : Q \times X_2 \times Q \to [0, 1]$ be as follows: $\forall\big((q_1, q_2), b, (p_1, p_2)\big) \in Q \times X_2 \times Q$, $\mu^\omega\big((q_1, q_2), b, (p_1, p_2)\big) = \mu_1(q_1, \omega(q_2, b), p_1) \wedge \mu_2(q_2, b, p_2)$. Then $\mathcal{M} = (Q, X_2, \mu^\omega)$ is called the cascade product [24] of $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\mathcal{M} = \mathcal{M}_1 \omega \mathcal{M}_2$. Let $f$ be a function of $Q_2$ into $X_1$, $Q = Q_1 \times Q_2$ and $\mu^0 : Q \times (X_1^{Q_2} \times X_2) \times Q \to [0, 1]$ be as follows: $\forall\big((q_1, q_2), (f, b), (p_1, p_2)\big) \in Q \times (X_1^{Q_2} \times X_2) \times Q$, $\mu^0\big((q_1, q_2), (f, b), (p_1, p_2)\big) = \mu_1(q_1, f(q_2), p_1) \wedge \mu_2(q_2, b, p_2)$. Then $\mathcal{M} = (Q, X_1^{Q_2} \times X_2, \mu^0)$ is a ffsm. Then $\mathcal{M} = \mathcal{M}_1 \circ \mathcal{M}_2$ is called the wreath product [24] of $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $X_1 \cap X_2 = \emptyset, a \in X_1 \cup X_2$. $\mathcal{M}_1 \otimes \mathcal{M}_2 = (Q_1 \times Q_2, X_1 \cup X_2, \mu_1 \otimes \mu_2)$ is called the Cartesian composition of $\mathcal{M}_1$ and

$\mathcal{M}_2$ if

$$(\mu_1 \otimes \mu_2)((q_1, q_2), a, (p_1, p_2)) = \begin{cases} \mu_1(q_1, a, p_1) & if\ a \in X_1\ and\ q_2 = p_2 \\ \mu_2(q_2, a, p_2) & if\ a \in X_2\ and\ q_1 = p_1 \\ 0\ otherwise \end{cases}$$

**Definition 2.1.** [34] *Let $\mathcal{M} = (Q, X, \mu)$ be a fuzzy finite state machine. $\mathcal{M}$ is called switching if and only if $\mu(q, a, p) = \mu(p, a, q)$ $\forall q, p \in Q$ and $\forall a \in X$. $\mathcal{M}$ is called commutative if and only if $\mu(q, ab, p) = \mu(q, ba, p)$ $\forall q, p \in Q$ and $\forall a, b \in X$. If $\mathcal{M}$ is switching and commutative, then $\mathcal{M}$ is called a fuzzy finite switchboard state machine (ffssm).*

**Definition 2.2.** [34] *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ be mappings. A pair $(\alpha, \beta)$ is called a switching homomorphism if $\mu_2(\alpha(q), \beta(\sigma), \alpha(p)) = \bigvee\{\mu_1(s, \sigma, t) \mid s, t \in Q_1, \alpha(t) = \alpha(p), \alpha(s) = \alpha(q)\}$ for $\forall q, p \in Q_1$ and $\forall \sigma \in X_1$.*

# 3 Covering and Product of Fuzzy Finite Switchboard State Machine

We give the following simple algorithm which generates the switchboard of a given fuzzy finite state machine.
Let $\mathcal{M} = (Q, X, \mu)$ is *ffsm*, when the number of input elements is 1, obviously $\mathcal{M}$ is commutative and switching. So, set $X = \{x_1, x_2, \cdots, x_n\}$.

**Algorithm 3.1.** *Construction of the Fuzzy Finite Switchboard State Machine.*

*Input : the set $Q$ of ($n$ states) states of ffsm $\mathcal{M} = (Q, X, \mu)$;*

*the set $X$ of input alphabets of $\mathcal{M}$;*

*the transition table $\mu$ of $\mathcal{M}$.*

*Output : YES, if $\mathcal{M}$ is ffssm with $n$ states, or NO, if $\mathcal{M}$ is not ffssm.*

*Procedure :*

**Step 1** *Enter the state transition matrices $\mu_{x_1}, \mu_{x_2}, \cdots, \mu_{x_n}$.*

**Step 2** *Set $i$ be the initial value, $i = 1$ and $n \geq 2$.*

**Step 3** *for $i \leq n - 1$, calculate $\mu_{x_i}\mu_{x_{i+1}}(q, p)$ and $\mu_{x_{i+1}}\mu_{x_i}(q, p)$ $\forall q, p \in Q$*

- *if $\mu_{x_i}\mu_{x_{i+1}}(q, p) \neq \mu_{x_{i+1}}\mu_{x_i}(q, p)$ then STOP, the output $\mathcal{M}$ is not commutative;*
- *if $\mu_{x_i}\mu_{x_{i+1}}(q, p) = \mu_{x_{i+1}}\mu_{x_i}(q, p)$, recalculate*

$$\mu_{x_i}\mu_{x_{i+2}}(q, p)\ and\ \mu_{x_{i+2}}\mu_{x_i}(q, p);$$

- *if both are not equal then STOP, the output $\mathcal{M}$ is not commutative, NO;*
- *otherwise, recalculate $\mu_{x_i}\mu_{x_{i+3}}(q, p)$ and $\mu_{x_{i+3}}\mu_{x_i}(q, p)$ and so on;*
- *if necessary, calculate until $\mu_{x_i}\mu_{x_n}(q, p)$ and $\mu_{x_n}\mu_{x_i}(q, p)$;*
- *if both are not equal, the output $\mathcal{M}$ is not commutative, NO;*
- *if both are equal Go to* **Step 4**.

**Step 4** *$i = i + 1$ repeat* **Step 3**.

**Step 5** *$i = n$, STOP, the output $\mathcal{M}$ is commutative, YES.*

**Step 6** *for $i \leq n$, calculate $\mu_{x_i}(q, p)$ and $\mu_{x_i}(p.q)$ $\forall p, q \in Q$.*

- *if $\mu_{x_i}(q, p) \neq \mu_{x_i}(p, q)$ then STOP, the output $\mathcal{M}$ is not switching;*
- *if $\mu_{x_i}(p, q) = \mu_{x_i}(p, q)$, recalculate $\mu_{x_{i+1}}(p, q)$ and $\mu_{x_{i+1}}(p, q)$;*
- *if both are not equal then STOP, the output $\mathcal{M}$ is not switching, NO;*
- *otherwise, recalculate $\mu_{x_{i+2}}(p, q)$ and $\mu_{x_{i+2}}(p, q)$ and so on;*

- *if both are not equal, the output $\mathcal{M}$ is not switching, NO;*
- *if both are equal Go to* **Step 7**.

**Step 7** $i = i + 1$ *repeat* **Step 6**.

**Step 8** $i = n$, *STOP, the output is switching, YES.*

We are now interested in obtaining the covering and some product results involving the cascade and wreath product of ffssm.

**Definition 3.2.** *Let $\mathcal{M} = (Q, X, \mu)$ be a ffssm. If $\sigma \in X$ exists such that $\mu(q, \sigma, p) = \mu(p, \sigma, q)$ for $q, p \in Q$, $q$ and $p$ are in a switching relation and denote a switching class by $[[q]_\sigma = \{p \mid \mu(q, \sigma, p) = \mu(p, \sigma, q), \sigma \in X\}$*

**Proposition 3.3.** *Let $\mathcal{M} = (Q, X, \mu)$ be a ffssm. Define $Q_\sigma = \{[q_\sigma] \mid q \in Q\}$ and $X_\sigma = X \backslash \{\sigma\}$ for $\sigma \in X$. Take $\alpha = [q]_\sigma, q \in Q$. For all $\tau \in X_\sigma$, define the fuzzy subset $\mu_\sigma$ of $Q_\sigma \times X_\sigma \times Q_\sigma$ by $\mu_\sigma(\alpha, \tau, \beta) = \mu([q]_\sigma, \tau, [p]_\sigma) = \bigvee\{\mu(r, \tau, s) : r \in [q]_\sigma, s \in [p]_\sigma\}$ $\forall \alpha, \beta \in Q_\sigma$, then $\mathcal{M}_\sigma = (Q_\sigma, X_\sigma, \mu_\sigma)$ is a ffssm.*

*Proof.* Let $\alpha, \beta \in Q_\sigma$ and take $\alpha = [q]_\sigma, \beta = [p]_\sigma, \forall p, q \in Q$. Suppose that $\alpha = \beta$. Then $\mu(q, \sigma, p) = \mu(p, \sigma, q)$ or $q = p$. Now, assume that $\tau = \rho, \forall \tau, \rho \in X_\sigma$.

**Case 1(a)** if $q = p$, then

$$\mu_\sigma(\alpha, \tau, \beta) = \mu([q]_\sigma, \tau, [p]_\sigma) = \mu([q]_\sigma, \rho, [p]_\sigma) = \mu([p]_\sigma, \rho, [q]_\sigma) = \mu_\sigma(\beta, \tau, \alpha).$$

**Case 1(b)** If $\mu(q, \sigma, p) = \mu(p, \sigma, q)$, then

$$\begin{aligned}\mu_\sigma(\alpha, \tau, \beta) &= \mu([q]_\sigma, \tau, [p]_\sigma) = \mu([q]_\sigma, \rho, [p]_\sigma) = \vee\{\mu(r, \rho, s) \mid r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{[\mu(s, \rho, r) \mid r \in [q]_\sigma, s \in [p]_\sigma\} = \mu([p]_\sigma, \rho, [q]_\sigma) = \mu_\sigma(\beta, \rho, \alpha).\end{aligned}$$

Since, $\mu(r, \sigma, s) \in [0, 1], \forall r \in Q$ and $[r]_\sigma$ is a set of states. Thus $\mu_\sigma$ is well-defined. Let $\alpha, \beta \in Q_\sigma, r \in X_\sigma$ and take $\alpha = [q]_\sigma, \beta = [p]_\sigma$. Suppose $\alpha = \beta$. Then $\mu(q, \tau, p) = \mu(p, \tau, q)$ or $\mu(p, \tau\sigma, q) = \mu(q, \tau\sigma, p)$.

**Case 2(a)** If $\mu(q, \tau, p) = \mu(p, \tau, q)$, then

$$\begin{aligned}\mu_\sigma(\alpha, \tau, \beta) &= \mu([q]_\sigma, \tau, [p]_\sigma) = \vee\{\mu(r, \tau, s) | r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{[\mu(s, \tau, r) | r \in [q]_\sigma, s \in [p]_\sigma\} = \mu([p]_\sigma, \tau, [q]_\sigma) = \mu_\sigma(\beta, \tau, \alpha).\end{aligned}$$

**Case 2(b)** If $\mu(p, \tau\sigma, q) = \mu(q, \tau\sigma, p)$, then

$$\begin{aligned}\mu_\sigma(\alpha, \tau\sigma, \beta) &= \mu([q]_\sigma, \tau\sigma, [p]_\sigma) = \vee\{\mu(r, \tau\sigma, s) | r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{\mu(r, \tau, k) \wedge \mu(k, \sigma, s) | k \in [t]_\sigma, r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{[\mu(k, \tau, r) \wedge \mu(s, \sigma, k) | k \in [t]_\sigma, r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{[\mu(s, \sigma, k) \wedge \mu(k, \tau, r) | k \in [t]_\sigma, r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \mu([p]_\sigma, \sigma\tau, [q]_\sigma) = \mu([p]_\sigma, \tau\sigma, [q]_\sigma) \quad \text{(by Definition 2.1)} \\ &= \mu_\sigma(\beta, \tau\sigma, \alpha).\end{aligned}$$

Hence $\mathcal{M}_\sigma$ is switching.

Let $\alpha, \beta \in Q_\sigma, \tau, \rho \in X_\sigma$ and take $\alpha = [q]_\sigma, \beta = [p]_\sigma \forall q, p \in Q$. Since $\mathcal{M}$ is commutative, therefore

$$\begin{aligned}\mu_\sigma(\alpha, \tau\rho, \beta) &= \mu([q]_\sigma, \tau\rho, [p]_\sigma) = \vee\{\mu([q]_\sigma, \tau, [t]_\sigma) \wedge \mu([t]_\sigma, \rho, [p]_\sigma) | t \in Q\} \\ &= \vee\{(\vee\{\mu(r, \tau, t) | r \in [q]_\sigma, t \in [t]_\sigma\}) \wedge (\vee\{\mu(t, \rho, s) | t \in [t]_\sigma, s \in [q]_\sigma\}) | t \in Q\} \\ &= \vee\{\vee\{\mu(r, \tau, t) \wedge \mu(t, \rho, s) | t \in [t]_\sigma, t \in Q\} | r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{\mu(r, \tau\rho, s) | r \in [q]_\sigma, s \in [p]_\sigma\} = \vee\{\mu(r, \rho\tau, s) | t \in [q]_\sigma, k \in [p]_\sigma\} \\ &= \vee\{\vee\{\mu(r, \rho, t) \wedge \mu(t, \tau, s) | t \in [t]_\sigma, t \in Q\} | r \in [q]_\sigma, s \in [p]_\sigma\} \\ &= \vee\{(\vee\{\mu(r, \rho, t) | r \in [q]_\sigma, t \in [t]_\sigma\}) \wedge (\vee\{\mu(t, \tau, s) | t \in [t]_\sigma, s \in [q]_\sigma\}) | r \in Q\} \\ &= \vee\{\mu([q]_\sigma, \rho, [t]_\sigma) \wedge \mu([t]_\sigma, \tau, [p]_\sigma) | t \in Q\} = \mu([q]_\sigma, \rho\tau, [p]_\sigma) = \mu_\sigma(\alpha, \rho\tau, \beta).\end{aligned}$$

Hence $\mathcal{M}$ is commutative. ∎

**Remark 3.4.** *Now, $\mathcal{M}_\sigma = (Q_\sigma, X_\sigma, \mu_\sigma)$ can called as an quotient ffssm for $\sigma \in X$.*

**Theorem 3.5.** *If $\mathcal{M}_\sigma = (Q_\sigma, X_\sigma, \mu_\sigma)$, $\sigma \in X$, and $\mathcal{M} = (Q, X, \mu)$ are ffssm, then $\mathcal{M}_\sigma \leq \mathcal{M}$.*

*Proof.* Let $\eta$ be a surjective partial function of $Q$ to $Q_\sigma$ as $\eta(q) = [q]_\sigma, \forall q \in Q$ and define a function $\xi : X_\sigma \to X$ by $\xi(\sigma') = \sigma', \forall \sigma' \in X_\sigma$. Then $\xi$ is clearly a one-one mapping. Let $q, p \in Q, \tau \in X_\sigma$. Then

$$\begin{aligned}\mu_\sigma(\eta(q), \tau, \eta(p)) &= \mu([q]_\sigma, \tau, [p]_\sigma) = \vee\{\mu(r, \tau, s) | r \in [q]_\sigma, s \in [p]_\sigma, \eta(s) = \eta(p)\} \\ &= \vee\{\mu(r, \xi(\tau), s) | r \in [q]_\sigma, s \in [p]_\sigma, \eta(s) = \eta(p)\} = \mu([q]_\sigma, \xi(\tau), [p]_\sigma).\end{aligned}$$

Thus $\mathcal{M}_\sigma \leq \mathcal{M}$. □

**Theorem 3.6.** *Let $\mathcal{M}_i = (Q_i, X_i, \mu_i)$ be ffssms, $i = 1, 2$ and $\mathcal{M}_\sigma = (Q_\sigma, X_\sigma, \mu_\sigma)$ be a ffssm over $\sigma \in X_i$. If $\mathcal{M}_\sigma \leq \mathcal{M}_1$ and $\mathcal{M}_1 \leq \mathcal{M}_2$, then $\mathcal{M}_\sigma \leq \mathcal{M}_2$.*

*Proof.* Straightforward □

A ffsm $\mathcal{M} = (Q, \{1_Q\}, \mu)$ is known as a trivial machine. Clearly the identical machines are fuzzy finite switchboard state machines. For example, the machine $\mathcal{M}$ has $Q = \{q_0, q_1, q_2\}$ states and $\sigma \in X$ input is a trivial machine such as:

$$\mu(q_0, \sigma, q_0) = 1, \quad \mu(q_1, \sigma, q_1) = 1 \quad \mu(q_2, \sigma, q_2) = 1.$$

**Proposition 3.7.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ be a ffsm and $\mathcal{M}_2 = (Q_2, \{1_{Q_2}\}, \mu_2)$ be a trivial machine, $|X_1| \leq |Q_2|$, then $\mathcal{M}_1 \circ \mathcal{M}_2$ is switching if and only if $\mathcal{M}_1$ is switching.*

*Proof.* Let $p_1, q_1 \in Q_1, p_2, q_2 \in Q_2$ and $f$ is a function from $Q_2$ to $X_1$. Since $\mathcal{M}_1 \circ \mathcal{M}_2$ is switching, therefore

$$\mu^0((p_1, p_2), (f, 1_{Q_2}), (q_1, q_2)) = \mu^0((q_1, q_2), (f, 1_{Q_2}), (p_1, p_2))$$

Now,

$$\begin{aligned}\mu_1(p_1, f(p_2), q_1) &= \mu_1(p_1, f(p_2), q_1) \wedge 1 = \mu_1(p_1, f(p_2), q_1) \wedge \mu_2(p_2, 1_{Q_2}, q_2) \\ &= \mu^0((p_1, p_2), (f, 1_{Q_2}), (q_1, q_2)) = \mu^0((q_1, q_2), (f, 1_{Q_2}), (p_1, p_2)) \\ &= \mu_1(q_1, f(p_2), p_1) \wedge \mu_2(q_2, 1_{Q_2}, p_2) = \mu_1(q_1, f(p_2), p_1) \wedge 1 = \mu_1(q_1, f(p_2), p_1)\end{aligned}$$

Since $|X_1| \leq |Q_2|$, therefore $\exists f \in X_1^{Q_2}$ such that $f(Q_2) = X_1$. Thus $\mathcal{M}_1$ is switching. Conversely, let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $(f, 1_{Q_2}) \in (X_1^{Q_2} \times X_2)$. Since $\mathcal{M}_1$ is switching, $\mu_1(p_1, f(p_2), q_1) = \mu_1(q_1, f(p_2), p_1)$ and $\mu_2(p_2, 1_{Q_2}, q_2) = \mu_2(q_2, 1_{Q_2}, p_2)$. Now,

$$\begin{aligned}\mu^0((p_1, p_2), (f, 1_{Q_2}), (q_1, q_2)) &= \mu_1(p_1, f(p_2), q_1) \wedge \mu_2(p_2, 1_{Q_2}, q_2) \\ &= \mu_1(q_1, f(p_2), p_1) \wedge \mu_2(q_2, 1_{Q_2}, p_2) = \mu^0((q_1, q_2), (f, 1_{Q_2}), (p_1, p_2)).\end{aligned}$$

Hence, $\mathcal{M}_1 \circ \mathcal{M}_2$ is switching. □

**Proposition 3.8.** *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssm. Then $\mathcal{M}_1 \circ \mathcal{M}_2$ is a ffssm if and only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssms.*

*Proof.* Let $(p_1, p_2), (q_1, q_2) \in Q$ and $(\alpha, \sigma), (\beta, \tau) \in (X_1^{Q_2} \times X_2)$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching, it follows from Proposition 3.7 that $\mathcal{M}_1 \circ \mathcal{M}_2$ is switching. Also, $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative, $\mu_1(q_1, ab, p_1) = \mu_1(q_1, ba, p_1)$ and $\mu_2(q_2, ab, p_2) = \mu_2(q_2, ba, p_2)$. Thus, for some $p_2^{(1)} \in Q_2$,

$$\begin{aligned}\mu^0((p_1, p_2), (\alpha, \sigma)(\beta, \tau), (q_1, q_2)) &= \mu_1(p_1, \alpha(p_2)\beta(p_2^{(1)}), q_1) \wedge \mu_2(p_2, \sigma\tau, q_2) \\ &= \mu_1(p_1, \beta(p_2^{(1)})\alpha(p_2), q_1) \wedge \mu_2(p_2, \tau\sigma, q_2) = \mu^0((p_1, p_2), (\beta, \tau)(\alpha, \sigma), (q_1, q_2)).\end{aligned}$$

Hence, $\mathcal{M}_1 \circ \mathcal{M}_2$ is commutative. Therefore, $\mathcal{M}_1 \circ \mathcal{M}_2$ is a fuzzy switchboard wreath product of $\mathcal{M}_1$ and $\mathcal{M}_2$. Conversely, let $q_1, p_1 \in Q_1, q_2, p_2 \in Q_2$ and $(\alpha, \sigma), (\beta, \tau) \in (X_1^{Q_2} \times X_2)$. Since $\mathcal{M}_1 \circ \mathcal{M}_2$ is switching, when $q_2 = p_2$ and $q_1 = p_1$, it follows from Proposition 3.7 that $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching. Now, we have to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative. Since $\mathcal{M}_1 \circ \mathcal{M}$ is commutative. Suppose $p_2 = q_2$. Then $\mu(p_2, \sigma\tau, q_2) = 1 = \mu(p_2, \tau\sigma, q_2)$. Hence

$$\begin{aligned}\mu(p_1, \alpha(p_2)\beta(p_2^{(1)}), q_1) &= \mu(p_1, \alpha(p_2)\beta(p_2^{(1)}), q_1) \wedge 1 \\ &= \mu(p_1, \alpha(p_2)\beta(p_2^{(1)}), q_1) \wedge \mu(p_2, \sigma\tau, q_2) = \mu^0((p_1, p_2), (\alpha, \sigma)(\beta, \tau), (q_1, q_2)) \\ &= \mu^0((p_1, p_2), (\beta, \tau)(\alpha, \sigma), (q_1, q_2)) = \mu(p_1, \beta(p_2^{(1)})\alpha(p_2), q_1) \wedge \mu(p_2, \tau\sigma, q_2) \\ &= \mu(p_1, \beta(p_2^{(1)})\alpha(p_2), q_1) \wedge 1 = \mu(p_1, \beta(p_2^{(1)})\alpha(p_2), q_1)\end{aligned}$$

Similarly, when $p_1 = q_1$, we have $\mu(p_2, \sigma\tau, q_2) = \mu(p_2, \tau\sigma, q_2)$. Hence $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative. Therefore $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssm. □

Now, we introduce the concept of restricted cascade product of ffsms.

**Definition 3.9.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be complete ffsms. Define the restricted cascade product $\mathcal{M}_1 \varpi \mathcal{M}_2 = (Q_1 \times Q_2, X_2, \mu^{\varpi})$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to mapping $\varpi : X_2 \to X_1$ as,*

$$\mu^{\varpi}((p_1, p_2), \sigma_2, (q_1, q_2)) = \mu_1(p_1, \varpi(\sigma_2), q_1) \wedge \mu_2(q_2, \sigma_2, p_2),$$

*where $\mu^{\varpi} : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \to [0, 1]$, $\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $\sigma_2 \in X_2$.*

**Example 3.10.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be ffsm's, where $Q_1 = \{p_1, p_2\}, X_1 = \{\sigma, \tau, \rho\}, Q_2 = \{q_1, q_2\}, X_2 = \{\sigma', \tau'\}$, and $\mu_1$ and $\mu_2$ are defined as follows:*

$$\begin{array}{lcllcl}
\mu_1(p_1, \sigma, p_1) & = & 0.5 & \mu_1(p_2, \rho, p_1) & = & 0.7 \\
\mu_1(p_2, \sigma, p_1) & = & 0.3 & \mu_2(q_1, \sigma', q_1) & = & 0.6 \\
\mu_1(p_1, \tau, p_2) & = & 0.3 & \mu_2(q_2, \sigma', q_1) & = & 0.3 \\
\mu_1(p_2, \tau, p_2) & = & 0.6 & \mu_2(q_1, \tau', q_2) & = & 0.5 \\
\mu_1(p_1, \rho, p_2) & = & 0.3 & \mu_2(q_2, \tau', q_1) & = & 0.25.
\end{array}$$

*Now define the function $\varpi : X_2 \to X_1$ as $\varpi(\sigma') = \sigma$, $\quad \varpi(\tau') = \tau$. Also, the partial function $\mu^{\varpi} : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \to [0, 1]$ is defined as:*

$$\begin{array}{lclcl}
\mu^{\varpi}((p_1, q_1), \sigma', (p_1, q_1)) & = & \mu_1(p_1, \varpi(\sigma'), p_1) \wedge \mu_2(q_1, \sigma', q_1) & = & 0.5 \\
\mu^{\varpi}((p_1, q_2), \sigma', (p_1, q_1)) & = & \mu_1(p_1, \varpi(\sigma'), p_1) \wedge \mu_2(q_2, \sigma', q_1) & = & 0.3 \\
\mu^{\varpi}((p_2, q_1), \sigma', (p_1, q_1)) & = & \mu_1(p_2, \varpi(\sigma'), p_1) \wedge \mu_2(q_1, \sigma', q_1) & = & 0.3 \\
\mu^{\varpi}((p_2, q_2), \sigma', (p_1, q_1)) & = & \mu_1(p_2, \varpi(\sigma'), p_1) \wedge \mu_2(q_2, \sigma', q_1) & = & 0.3 \\
\mu^{\varpi}((p_1, q_1), \tau', (p_2, q_2)) & = & \mu_1(p_1, \varpi(\tau'), p_2) \wedge \mu_2(q_1, \tau', q_2) & = & 0.3 \\
\mu^{\varpi}((p_1, q_2), \tau', (p_2, q_1)) & = & \mu_1(p_1, \varpi(\tau'), p_2) \wedge \mu_2(q_2, \tau', q_1) & = & 0.25 \\
\mu^{\varpi}((p_2, q_1), \tau', (p_2, q_2)) & = & \mu_1(p_2, \varpi(\tau'), p_2) \wedge \mu_2(q_1, \tau', q_2) & = & 0.5 \\
\mu^{\varpi}((p_2, q_2), \tau', (p_2, q_1)) & = & \mu_1(p_2, \varpi(\tau'), p_2) \wedge \mu_2(q_2, \tau', q_1) & = & 0.25
\end{array}$$

*and $\mu^{\varpi}$ is 0 elsewhere. It follows that $\mathcal{M}_1 \varpi \mathcal{M}_2$ is a restricted cascade product.*

**Proposition 3.11.** *Let $\mathcal{M}_i = (Q_i, X_i, \mu_i)$ be ffsms,$i = 1, 2$. Then there exists $\omega : Q_2 \times X_2 \to X_1$ for all $\varpi : X_2 \to X_1$ such that $\mathcal{M}_1 \varpi \mathcal{M}_2 \cong \mathcal{M}_1 \omega \mathcal{M}_2$.*

*Proof.* Let $\omega$ be defined by $\omega(p_2, \sigma') = \varpi(\alpha(p_2, \sigma'))$ $\forall (p_2, \sigma') \in Q_2 \times X_2$ where $\alpha : Q_2 \times X_2 \to X_2$ is a projection mapping and by definition, it is well-defined. Let $\xi$ be an identity map on $X_2$ and $\eta$ be an identity map on $Q_1 \times Q_2$. Then

$$\begin{aligned}
&\mu^{\varpi}(\eta(p_1, p_2), \sigma', \eta(q_1, q_2)) = \mu^{\varpi}((p_1, p_2), \sigma', (q_1, q_2)) \\
&\quad = \mu_1(p_1, \varpi(\sigma'), q_1) \wedge \mu_2(p_2, \sigma', q_2)) = \mu_1(p_1, \omega(p_2, \sigma'), q_1) \wedge \mu_2(p_2, \sigma', q_2)) \\
&\quad = \mu^{\omega}((p_1, p_2), \sigma', (q_1, q_2)) = \mu^{\omega}(\eta(p_1, p_2), \xi(\sigma'), \eta(q_1 . q_2)).
\end{aligned}$$

Hence, $\mathcal{M}_1 \varpi \mathcal{M}_2 \cong \mathcal{M}_1 \omega \mathcal{M}_2$. □

**Proposition 3.12.** *Let $\mathcal{M}_i = (Q_i, X_i, \mu_i)$ be ffsms, $i = 1, 2$ and $\varpi : S(\mathcal{M}_2) \to S(\mathcal{M}_1)$ be a semigroup homomorphism. Then $\mathcal{M}_1 \varpi \mathcal{M}_2$ is a ffssm if and only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssms.*

*Proof.* Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssms. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative and $\varpi$ is homomorphism, therefore for all $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2, \sigma', \tau' \in X_2$.

$$\begin{aligned}
&\mu^{\varpi}((p_1, p_2), \sigma'\tau', (q_1, q_2)) = \mu_1(p_1, \varpi(\sigma'\tau'), q_1) \wedge \mu_2(p_2, \sigma'\tau', q_2) \\
&= \mu_1(p_1, \varpi(\sigma')\varpi(\tau'), q_1) \wedge \mu_2(p_2, \sigma'\tau', q_2) = \mu_1(p_1, \varpi(\tau')\varpi(\sigma'), q_1) \wedge \mu_2(p_2, \tau'\sigma', q_2) \\
&= \mu_1(p_1, \varpi(\tau'\sigma'), q_1) \wedge \mu_2(p_2, \tau'\sigma', q_2) = \mu^{\varpi}((p_1, p_2), \tau'\sigma', (q_1, q_2)).
\end{aligned}$$

Thus $\mathcal{M}_1 \varpi \mathcal{M}_2$ is commutative. Let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$, $\sigma' \in X_2$ and

$$\mu^{\varpi}((p_1, p_2), \sigma', (q_1, q_2)) = \mu_1(p_1, \varpi(\sigma'), q_1) \wedge \mu_2(p_2, \sigma', q_2).$$

Then as $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching, we have $\mu_1(p_1, \varpi(\sigma'), q_1) = \mu_1(q_1, \varpi(\sigma'), p_1)$ and $\mu_2(p_2, \sigma', q_2) = \mu_2(q_2, \sigma', p_2)$. Thus $\mu^{\varpi}((p_1, p_2), \sigma', (q_1, q_2)) = \mu_1(q_1, \varpi(\sigma'), p_1) \wedge \mu_2(q_2, \sigma', p_2) = \mu^{\varpi}((q_1, q_2), \sigma', (p_1, p_2))$. Therefore $\mathcal{M}_1\varpi\mathcal{M}_2$ is switching. Conversely, we assume that $\mathcal{M}_1\varpi\mathcal{M}_2$ is a ffssm. Let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2, \sigma', \tau' \in X_2$. Since $\mathcal{M}_1\varpi\mathcal{M}_2$ is commutative,

$$\begin{aligned}&\mu_1(p_1, \varpi(\sigma')\varpi(\tau'), q_1) \wedge \mu_2(p_2, \sigma'\tau', q_2) = \mu_1(p_1, \varpi(\sigma'\tau'), q_1) \wedge \mu_2(p_2, \sigma'\tau', q_2)\\ &\quad= \mu^{\varpi}((p_1, p_2), \sigma'\tau', (q_1, q_2)) = \mu^{\varpi}((p_1, p_2), \tau'\sigma', (q_1, q_2))\\ &\quad= \mu_1(p_1, \varpi(\tau'\sigma'), q_1) \wedge \mu_2(p_2, \tau'\sigma', q_2) = \mu_1(p_1, \varpi(\tau')\varpi(\sigma'), q_1) \wedge \mu_2(p_2, \tau'\sigma', q_2),\end{aligned}$$

which gives $\mu_1(p_1, \varpi(\sigma'\tau'), q_1) = \mu_1(p_1, \varpi(\tau'\sigma'), q_1)$ $\mu_2(p_2, \sigma'\tau', q_2) = \mu_2(p_2, \tau'\sigma', q_2)$, whereby $\mathcal{M}_2$ is commutative. Now, consider $\sigma, \tau \in X_1$. Since the function $\varpi$ is surjective, there exists $\sigma'' \in X_2$ such that $\varpi(\sigma'') = \sigma$ and $\tau'' \in X_2$ such that $\varpi(\tau'') = \tau$. Also, $\mu_1(p_1, \sigma\tau, q_1) = \mu_1(p_1, \varpi(\sigma'')\varpi(\tau''), q_1) = \mu_1(p_1, \varpi(\tau'')\varpi(\sigma''), q_1)$ $= \mu_1(p_1, \tau\sigma, q_1)$. Thus $\mathcal{M}_1$ is commutative. Now, let $p_1, q_1 \in Q_1, p_2, q_2 \in Q_2, \sigma' \in X_2$. Since $\mathcal{M}_1\varpi\mathcal{M}_2$ is switching , we have $\mu_1(p_1, \varpi(\sigma'), q_1) \wedge \mu_2(p_2, \sigma', q_2) = \mu^{\varpi}((p_1, p_2), \sigma', (q_1, q_2)) = \mu^{\varpi}((q_1, q_2), \sigma', (p_1, p_2))= \mu_1(q_1, \varpi(\sigma'), p_1) \wedge \mu_2(q_2, \sigma', p_2)$. Therefore, $\mu_1(p_1, \varpi(\sigma'), q_1) = \mu_1(q_1, \varpi(\sigma'), p_1)$ and $\mu_2(p_2, \sigma', q_2) = \mu_2(q_2, \sigma', p_2)$. Hence, $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching. □

Finally, we examine some results related to the direct products and Cartesian composition of ffssms.

**Proposition 3.13.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be ffsm. Then $\mathcal{M}_1 \times \mathcal{M}_2$ is a ffssm if and only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssm.*

*Proof.* For all, $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $\forall (a_1, a_2) \in X_1 \times X_2$, we have

$$\mu_1(q_1, a_1, p_1) \wedge \mu_2(q_2, a_2, p_2) = \mu_1 \times \mu_2 \left((q_1, q_2), (a_1, a_2), (p_1, p_2)\right).$$

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are *ffssms*, we have $\mu_1(q_1, a_1, p_1) = \mu_1(p_1, a_1, q_1)$, $\mu_2(q_2, a_2, p_2) = \mu_2(p_2, a_2, q_2)$ $\forall q_1, p_1 \in Q_1, q_2, p_2 \in Q_2$ and $a_1 \in X_1, a_2 \in X_2$. Now, we have to prove that $\mathcal{M}_1 \times \mathcal{M}_2$ is switching. So,

$$\begin{aligned}\mu_1 \times \mu_2 \left((p_1, p_2), (a_1, a_2), (q_1, q_2)\right) &= \mu_1(p_1, a_1, q_1) \wedge \mu_2(p_2, a_2, q_2)\\ &= \mu_1(q_1, a_1, p_1) \wedge \mu_2(q_2, a_2, p_2)\\ &= \mu_1 \times \mu_2 \left((q_1, q_2), (a_1, a_2), (p_1, p_2)\right)\end{aligned}$$

Hence, $\mathcal{M}_1 \times \mathcal{M}_2$ is switching. Let $(q_1, q_2) \in Q_1 \times Q_2$, $(a_1, a_2), (b_1, b_2) \in X_1 \times X_2$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative, we have $\mu_1(q_1, a_1b_1, p_1) = \mu_1(q_1, b_1a_1, p_1)$, $\mu_2(q_2, a_2b_2, p_2) = \mu_2(q_2, b_2a_2, p_2)$. Then $\mu_1 \times \mu_2 \left((q_1, q_2), (a_1, a_2)(b_1, b_2), (p_1, p_2)\right)=$ $\mu_1 \times \mu_2 \left((q_1, q_2), (a_1b_1, a_2b_2), (p_1, p_2)\right) = \mu_1(q_1, a_1b_1, p_1) \wedge \mu_2(q_2, a_2b_2, p_2)$ $= \mu_1(q_1, b_1a_1, p_1) \wedge \mu_2(q_2, b_2a_2, p_2) = \mu_1 \times \mu_2 \left((q_1, q_2), (b_1a_1, b_2a_2), (p_1, p_2)\right)$ $= \mu_1 \times \mu_2 \left((q_1, q_2), (b_1, b_2)(a_1, a_2), (p_1, p_2)\right)$ which means $\mathcal{M}_1 \times \mathcal{M}_2$ is commutative. Consequently, $\mathcal{M}_1 \times \mathcal{M}_2$ is an ffssm. Conversely, since $\mathcal{M}_1 \times \mathcal{M}_2$ is switching, for all $q_1, p_1 \in Q_1$, $q_2, p_2 \in Q_2$, $a \in X_1, b \in X_2$, we have $\mu_1(q_1, a, p_1) \wedge \mu_2(q_2, b, p_2) = \mu_1 \times \mu_2 \left((q_1, q_2), (a, b), (p_1, p_2)\right) = \mu_1 \times \mu_2 \left((p_1, p_2), (a, b), (q_1, q_2)\right) = \mu_1(p_1, a, q_1) \wedge \mu_2(p_2, b, q_2)$. Therefore, we get $\mu_1(q_1, a, p_1) = \mu_1(p_1, a_1, q_1)$ and $\mu_2(q_2, b, p_2) = \mu_2(p_2, b, q_2)$ which imply that $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching. Clearly, $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative, since $\mathcal{M}_1 \times \mathcal{M}_2$ is commutative, $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssms. □

**Proposition 3.14.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be ffsm. Then $\mathcal{M}_1 \wedge \mathcal{M}_2$ is a ffssm if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssm.*

*Proof.* Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $u, v \in X_1 = X_2, r \in Q_1, s \in Q_2$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative, then $\mu_1 \wedge \mu_2((q_1, q_2), uv, (p_1, p_2)) = \mu_1(q_1, uv, p_1) \wedge \mu_2(q_2, uv, p_2) = \mu_1(q_1, u, r) \wedge \mu_1(r, v, p_1) \wedge \mu_2(q_2, u, s) \wedge \mu_2(s, v, p_2) = \mu_1(r, v, p_1) \wedge \mu_1(q_1, u, r) \wedge \mu_2(s, v, p_2) \wedge \mu_2(q_2, u, s) = \mu_1(q_1, uv, p_1) \wedge \mu_2(q_2, uv, p_2)$. Similarly $\mu_1 \wedge \mu_2((q_1, q_2), vu, (p_1, p_2)) = \mu_1(q_1, uv, p_1) \wedge \mu_2(q_2, uv, p_2)$. Therefore, $\mathcal{M}_1 \wedge \mathcal{M}_2$ are commutative. Suppose $\forall b \in X_1 = X_2$, $\mu_1 \wedge \mu_2((q_1, q_2), b, (p_1, p_2)) = \mu_1(q_1, b, p_1) \wedge \mu_2(q_2, b, p_2)$. As, $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching, $\mu_1(q_1, b, p_1) = \mu_1(p_1, b, q_1)$, $\mu_1(q_2, b, p_2) = \mu_2(p_2, b, q_2)$. Thus, $\mu_1 \wedge \mu_2((q_1, q_2), b, (p_1, p_2))$ $= \mu_1(q_1, b, p_1) \wedge \mu_2(q_2, b, p_2)= \mu_1(p_1, b, q_1) \wedge \mu_1(p_2, b, q_2)= \mu_1 \wedge \mu_2((p_1, p_2), b, (q_1, q_2))$.
Hence, $\mathcal{M}_1 \wedge \mathcal{M}_2$ is switching. Therefore, $\mathcal{M}_1 \wedge \mathcal{M}_2$ is the fuzzy switchboard restricted direct product of $\mathcal{M}_1$ and $\mathcal{M}_2$. The proof of the converse is simple and straightforward. □

**Proposition 3.15.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be ffsm. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is an ffssm if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssm.*

*Proof.* For all, $(q_1, q_2) \in Q_1 \times Q_2$ and $x, y \in X_1$. Since $\mathcal{M}_1$ is commutative, we have

$$\begin{aligned}
&(\mu_1 \otimes \mu_2)((q_1, q_2), xy, (p_1, p_2)) \\
&= \vee_{(r_1,r_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), x, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), y, (p_1, p_2))\} \\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1, x, r_1) \wedge (\mu_1 \otimes \mu_2)((r_1, q_2), y, (p_1, p_2))\} \\
&= \begin{cases} \vee_{r_1\in Q_1}\{\mu_1(q_1, x, r_1) \wedge \mu_1(r_1, y, p_1)\} \text{ if } q_2 = p_2 \\ 0 \quad \text{otherwise} \end{cases} = \begin{cases} \mu_1(q_1, xy, r_1) \text{ if } q_2 = p_2 \\ 0 \quad \text{otherwise} \end{cases} \\
&= \begin{cases} \mu_1(q_1, yx, r_1) \text{ if } q_2 = p_2 \\ 0 \quad \text{otherwise} \end{cases} = \begin{cases} \vee_{r_1\in Q}\{\mu_1(q_1, y, r_1) \wedge \mu_1(r_1, x, p_1)\} \text{ if } q_2 = p_2 \\ 0 \quad \text{otherwise} \end{cases} \\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1, y, r_1) \wedge (\mu_1 \otimes \mu_2)((r_1, q_2), x, (p_1, p_2))\} \\
&= \vee_{(r_1,r_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), y, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), x, (p_1, p_2))\} \\
&= (\mu_1 \otimes \mu_2)((q_1, q_2), yx, (p_1, p_2)).
\end{aligned}$$

Similarly, let $(q_1, q_2) \in Q_1 \times Q_2$ and $u, v \in X_2$. Since $\mathcal{M}_2$ is commutative, we have

$$\begin{aligned}
&(\mu_1 \otimes \mu_2)((q_1, q_2), uv, (p_1, p_2)) \\
&= \vee_{(s_1,s_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), u, (s_1, s_2)) \wedge (\mu_1 \otimes \mu_2)((s_1, s_2), v, (p_1, p_2))\} \\
&= \vee_{s_2\in Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), u, (p_1, s_2)) \wedge \mu_2(s_2, v, p_2)\} \\
&= \begin{cases} \vee_{s_2\in Q_2}\{\mu_2(q_2, u, s_2) \wedge \mu_2(s_2, v, p_2)\} \text{ if } q_1 = p_1 \\ 0 \quad \text{otherwise} \end{cases} = \begin{cases} \mu_2(q_2, uv, s_2) \text{ if } q_1 = p_1 \\ 0 \quad \text{otherwise} \end{cases} \\
&= \begin{cases} \mu_2(q_2, vu, s_2) \text{ if } q_1 = p_1 \\ 0 \quad \text{otherwise} \end{cases} = \begin{cases} \vee_{r_2\in Q_2}\{\mu_2(q_2, v, s_2) \wedge \mu_2(s_2, u, p_2)\} \text{ if } q_1 = p_1 \\ 0 \quad \text{otherwise} \end{cases} \\
&= \vee_{s_2\in Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), v, (p_1, s_2)) \wedge \mu_2(s_2, u, p_2)\} \\
&= \vee_{(s_1,s_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), v, (s_1, s_2)) \wedge (\mu_1 \otimes \mu_2)((s_1, s_2), u, (p_1, p_2))\} \\
&= (\mu_1 \otimes \mu_2)((q_1, q_2), vu, (p_1, p_2)).
\end{aligned}$$

Let $x \in X_1, u \in X_2$. Then

$$\begin{aligned}
&(\mu_1 \otimes \mu_2)((q_1, q_2), xu, (p_1, p_2)) \\
&= \vee_{(r_1,r_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), x, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), u, (p_1, p_2))\} \\
&= \vee_{r_1\in Q_1}\{\vee_{r_2\in Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), x, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), u, (p_1, p_2))\}\} \\
&= \vee_{r_1\in Q_1}\{(\mu_1 \otimes \mu_2)((q_1, q_2), x, (r_1, p_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, p_2), u, (p_1, p_2))\} \\
&= \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, u, p_2).
\end{aligned}$$

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative and by Theorem 6.13.3 in [26],

$$(\mu_1 \otimes \mu_2)((q_1, q_2), xu, (p_1, p_2)) = (\mu_1 \otimes \mu_2)((q_1, q_2), ux, (p_1, p_2)).$$

Similarly, let $y \in X_1, v \in X_2$, we get $(\mu_1 \otimes \mu_2)((q_1, q_2), vy, (p_1, p_2)) = (\mu_1 \otimes \mu_2)((q_1, q_2), yv, (p_1, p_2))$. Hence $\mathcal{M}_1 \otimes \mathcal{M}_2$ is commutative. Now, we have to show that $\mathcal{M}_1 \otimes \mathcal{M}_2$ is switching. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $x \in X$. From the definition, we have $(\mu_1 \otimes \mu_2)((q_1, q_2), x, (p_1, p_2)) = \mu_1(q_1, x, p_1)$ if $q_2 = p_2$. Since $\mathcal{M}_1$ is switching, we get $(\mu_1 \otimes \mu_2)((q_1, q_2), x, (p_1, p_2) = \mu_1(q_1, x, p_1) = \mu_1(p_1, x, q_1) = (\mu_1 \otimes \mu_2)((p_1, p_2), x, (q_1, q_2))$. Similary, let $u \in X_2$ and we obtain $(\mu_1 \otimes \mu_2)((q_1, q_2), y, (p_1, p_2)) = (\mu_1 \otimes \mu_2)((p_1, p_2), y, (q_1, q_2))$. Hence $\mathcal{M}_1 \otimes \mathcal{M}_2$ is switching. Conversely, let $q_1, p_1 \in Q_1$, $q_2, p_2 \in Q_2$ and $x, y \in X_1$. Since $\mathcal{M}_1 \otimes \mathcal{M}_2$ is commutative, when $q_2 = p_2$,

$$\begin{aligned}
\mu_1(q_1, xy, p_1) &= \vee_{r_1\in Q_1}\{\mu_1(q_1, x, r_1) \wedge \mu_1(r_1, y, p_1)\} \\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1, x, r_1) \wedge (\mu_1 \otimes \mu_2)((r_1, q_2), y, (p_1, p_2))\} \\
&= \vee_{(r_1,r_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), x, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), y, (p_1, p_2))\} \\
&= (\mu_1 \otimes \mu_2)((q_1, q_2), xy, (p_1, p_2)) = (\mu_1 \otimes \mu_2)((q_1, q_2), yx, (p_1, p_2)) \\
&= \vee_{(r_1,r_2)\in Q_1\times Q_2}\{(\mu_1 \otimes \mu_2)((q_1, q_2), y, (r_1, r_2)) \wedge (\mu_1 \otimes \mu_2)((r_1, r_2), x, (p_1, p_2))\} \\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1, y, r_1) \wedge (\mu_1 \otimes \mu_2)((r_1, q_2), x, (p_1, p_2))\} \\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1, y, r_1) \wedge \mu_1(r_1, x, p_1)\} = \mu_1(q_1, yx, p_1).
\end{aligned}$$

Therefore, $\mathcal{M}_1$ is commutative. In a similar way, we can prove that $\mathcal{M}_2$ is commutative. Now, if $x \in X_1$ and $p_2 = q_2$, suppose

$$\begin{aligned}
\mu_1(p_1,x,q_1) &= \mu_1(p_1,x\lambda,q_1) = \vee_{r_1\in Q_1}\{\mu_1(p_1,x,r_1)\wedge\mu_1(r_1,\lambda,q_1)\}\\
&= \vee_{r_1\in Q_1}\mu_1(p_1,x,r_1)\wedge(\mu_1\otimes\mu_2)((r_1,q_2),\lambda,(q_1,q_2))\}\\
&= \vee_{r_1\in Q_1}\{(\mu_1\otimes\mu_2)((p_1,q_2),x,(r_1,q_2)\wedge(\mu_1\otimes\mu_2)((r_1,q_2),\lambda,(q_1,q_2))\}\\
&= (\mu_1\otimes\mu_2)((p_1,q_2),x\lambda,(q_1,q_2)) = (\mu_1\otimes\mu_2)((p_1,q_2),x,(q_1,q_2))\\
&= (\mu_1\otimes\mu_2)((q_1,q_2),x,(p_1,q_2)) \quad \text{Since } \mathcal{M}_1\otimes\mathcal{M}_2 \text{ is switching}\\
&= (\mu_1\otimes\mu_2)((q_1,q_2),x\lambda,(p_1,q_2))\\
&= \vee_{r_1\in Q_1}\{(\mu_1\otimes\mu_2)((q_1,q_2),x,(r_1,q_2)\wedge(\mu_1\otimes\mu_2)((r_1,q_2),\lambda,(p_1,q_2))\}\\
&= \vee_{r_1\in Q_1}\mu_1(q_1,x,r_1)\wedge(\mu_1\otimes\mu_2)((r_1,q_2),\lambda,(p_1,q_2))\}\\
&= \vee_{r_1\in Q_1}\{\mu_1(q_1,x,r_1)\wedge\mu_1(r_1,\lambda,q_1)\} = \mu_1(q_1,x\lambda,p_1) = \mu_1(q_1,x,p_1).
\end{aligned}$$

Hence, $\mathcal{M}_1$ is switching. Similarly, we can prove that $\mathcal{M}_2$ is switching. Therefore $\mathcal{M}_1$ and $\mathcal{M}_2$ are ffssms. □

The following example shows that the relationship between the Cartesian composition and the direct product which does not hold each other.

**Example 3.16.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be ffssm, where $Q_1 = \{p_1, p_2\}, X = \{\sigma, \tau\}, Q_2 = \{q_1, q_2\}, X_2 = \{\sigma'\}$, and $\mu_1$ and $\mu_2$ are defined as follows:*

$$\begin{array}{ll}
\mu_1(p_1,\sigma,p_1)=0 & \mu_1(p_2,\tau,p_1)=0\\
\mu_1(p_1,\sigma,p_2)=c_1>0 & \mu_1(p_2,\tau,p_2)=c_4>0\\
\mu_1(p_1,\tau,p_1)=c_2>0 & \mu_2(q_1,\sigma',q_1)=0\\
\mu_1(p_2,\sigma,p_1)=c_3>0 & \mu_2(q_2,\sigma',q_1)=d_2>0\\
\mu_1(p_2,\sigma,p_2)=0 & \mu_2(q_2,\sigma',q_2)=0.
\end{array}$$

*From $\mu_1$ and $\mu_2$, the defined Cartesian composition $\mathcal{M}_1 \otimes \mathcal{M}_2$ is presented as follows:*

$$\begin{array}{ll}
(\mu_1\otimes\mu_2)((p_1,q_1),\sigma,(p_2,q_1))=c_1 & (\mu_1\otimes\mu_2)((p_2,q_1),\tau,(p_2,q_1))=c_4\\
(\mu_1\otimes\mu_2)((p_1,q_2),\sigma,(p_2,q_2))=c_1 & (\mu_1\otimes\mu_2)((p_2,q_2),\tau,(p_2,q_2))=c_4\\
(\mu_1\otimes\mu_2)((p_2,q_1),\sigma,(p_1,q_1))=c_3 & (\mu_1\otimes\mu_2)((p_1,q_1),\sigma',(p_1,q_2))=d_1\\
(\mu_1\otimes\mu_2)((p_2,q_2),\sigma,(p_1,q_2))=c_3 & (\mu_1\otimes\mu_2)((p_1,q_2),\sigma',(p_1,q_1))=d_2\\
(\mu_1\otimes\mu_2)((p_1,q_1),\tau,(p_1,q_1))=c_2 & (\mu_1\otimes\mu_2)((p_2,q_1),\sigma',(p_2,q_2))=d_1\\
(\mu_1\otimes\mu_2)((p_1,q_2),\tau,(p_1,q_2))=c_2 & (\mu_1\otimes\mu_2)((p_2,q_2),\sigma',(p_2,q_1))=d_2,
\end{array}$$

*and the defined direct product $\mathcal{M}_1 \times \mathcal{M}_2$ is presented as follows:*

$$\begin{array}{l}
(\mu_1\times\mu_2)((p_1,q_1),(\sigma,\sigma'),(p_2,q_2))=c_1\wedge d_1\\
(\mu_1\times\mu_2)((p_1,q_2),(\sigma,\sigma'),(p_2,q_1))=c_1\wedge d_2\\
(\mu_1\times\mu_2)((p_2,q_1),(\sigma,\sigma'),(p_1,q_2))=c_3\wedge d_1\\
(\mu_1\times\mu_2)((p_2,q_2),(\sigma,\sigma'),(p_1,q_1))=c_3\wedge d_2\\
(\mu_1\times\mu_2)((p_1,q_1),(\tau,\sigma'),(p_1,q_2))=c_2\wedge d_1\\
(\mu_1\times\mu_2)((p_1,q_2),(\tau,\sigma'),(p_1,q_1))=c_2\wedge d_2\\
(\mu_1\times\mu_2)((p_2,q_1),(\tau,\sigma'),(p_2,q_2))=c_4\wedge d_1\\
(\mu_1\times\mu_2)((p_2,q_2),(\tau,\sigma'),(p_2,q_1))=c_4\wedge d_2.
\end{array}$$

*Now, we consider the relation between the Cartesian composition ($\mathcal{M}_1 \otimes \mathcal{M}_2$) and direct product ($\mathcal{M}_1 \times \mathcal{M}_2$). Now define $\eta : Q_1 \times Q_2 \to Q_1 \times Q_2$ by $\eta((q, q')) = (q, q')$ and $\xi : X_1 \cup X_2 \to X_1 \times X_2$ by $\xi(\tau) = (\tau, \tau')$ be any function and let $\forall (p, p'), (q, q'), (q_1, q_1') \in Q_1 \times Q_2, \forall \tau \in X_1 \cup X_2$. Suppose $c_i > d_j$ for $i = 1, \ldots, 4$ and $j = 1, 2$. Then $(\mu_1 \otimes \mu_2)(\eta((p, p')), \tau, \eta((q, q'))) = (\mu_1 \otimes \mu_2)((p, p'), \tau, (q, q')) = \mu_1(p, \tau, q)$ if $p' = q'$, and $(\mu_1 \times \mu_2)(\eta((p, p')), \xi(\tau), \eta((q_1, q_1')))$ $= (\mu_1 \times \mu_2)((p, p')), \xi(\tau), (q_1, q_1')) = \mu_1(p, \tau, q_1) \wedge \mu_2(p', \tau', q_1')$. It is clear that $(q, q') \neq (q_1, q_1')$. Since $\eta$ is bijective, then $\eta((q, q')) \neq \eta((q_1, q_1'))$. Therefore $\mathcal{M}_1 \otimes \mathcal{M}_2 \leq \mathcal{M}_1 \times \mathcal{M}_2$ does not hold.*

**Definition 3.17.** *Let $\mathcal{M} = (Q, X, \mu)$ be a ffssm. If $\mathcal{M}$ is strongly connected, then $\mathcal{M}$ is said to be perfect switchboard machine.*

**Proposition 3.18.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be a ffssm and strongly connected. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is a perfect switchboard machine if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are perfect switchboard machine.*

*Proof.* Similar to the proof of Proposition 3.15. □

**Theorem 3.19.** *Let $\mathcal{M}_i = (Q_i, X_i, \mu_i)$ be ffssms, $i = 1, 2$. Then the following properties hold:*

*(a)* $\mathcal{M}_1 \varpi \mathcal{M}_2 \leq \mathcal{M}_1 \circ \mathcal{M}_2$,

*(b)* $\mathcal{M}_1 \circ \mathcal{M}_2 \leq \mathcal{M}_1 \times \mathcal{M}_2$,

*(c)* $\mathcal{M}_1 \varpi \mathcal{M}_2 \leq \mathcal{M}_1 \times \mathcal{M}_2$,

*(d)* $\mathcal{M}_1 \wedge \mathcal{M}_2 \leq \mathcal{M}_1 \times \mathcal{M}_2$.

*Proof.* The proof of the theorem is a direct consequence of the relationship between products of fuzzy finite switchboard state machine. The proofs follow Proposition 3.8-3.14 and Theorem 6.17.2(1) of [26]. □

# 4 Asynchronous Fuzzy Finite Switchboard State Machine

In this section we shall introduce and study asynchronous fuzzy finite switchboard state machine. The transition within the states of an asynchronous fuzzy finite switchboard state machine is controlled by the event input and no need to wait for a clock signal input. For more information asynchronous automata we refer to [9, 36].

**Definition 4.1.** *Let $\mathcal{M} = (Q, X, \mu)$ be a ffssm. $\mathcal{M}$ is an asynchronous fuzzy finite switchboard state machine (affssm) if $\mu(q, \sigma\sigma, p) = [q]_\sigma$ $\forall p, q \in Q$ and $x \in X$.*

**Lemma 4.2.** *Let $\mathcal{M} = (Q, X, \mu)$ be an affssm. Then $\mu(q, \sigma\sigma, p) = \mu(p, \sigma, q)$ holds for any $p, q \in Q$ and any $\sigma \in X$.*

*Proof.* The proof of the lemma is similar to Lemma 1 in [36]. □

**Proposition 4.3.** *Let $\mathcal{M} = (Q, X, \mu)$ be an affssm and let $p, q$ be a pair of states in $Q$. If $\mu(q, \sigma, p) = \mu(p, \sigma, q)$ and $\mu(p, \tau, q) = \mu(q, \tau, p)$ for some $\sigma, \tau \in X$, then $q = p$.*

*Proof.* Let $\mathcal{M} = (Q, X, \mu)$ be an affssm and let $q, p \in Q$. Suppose that $\mu(q, \sigma, p) = \mu(p, \sigma, q)$ and $\mu(p, \tau, q) = \mu(q, \tau, p)$ for some $\sigma, \tau \in X$. Then, by Lemma 4.2 we have $\mu(q, \tau, q) = \mu(q, \tau\tau, q) = \bigvee\{\mu(q, \tau, p) \wedge \mu(p, \tau, q) | p \in Q\} = \bigvee\{\mu(p, \tau, q) \wedge \mu(p, \tau, q) | p \in Q\} = \mu(p, \tau, q)$. Since $\mathcal{M}$ is switching, $\mu(p, \tau, q) = \bigvee\{\mu(q, \sigma, p) \wedge \mu(p, \tau, q)\} = \mu(q, \sigma\tau, q) = \mu(q, \tau\sigma, q) = \bigvee\{\mu(q, \tau, p) \wedge \mu(p, \sigma, q)\} = \bigvee\{\mu(q, \tau, q) \wedge \mu(p, \sigma, q)\} = \bigvee\{1 \wedge (q, \sigma, p)\} = \mu(q, \sigma, p)$. □

**Proposition 4.4.** *Let $\mathcal{M}_1 = (Q_1, X_1, \mu_1)$ be an affssm and $\mathcal{M}_2 = (Q_2, X_2, \mu_2)$ be a ffsm. Let $\mathcal{M}_2$ is an onto-switching homomorphic image of $\mathcal{M}_1$. Then $\mathcal{M}_2$ is an affssm.*

*Proof.* Let $\mathcal{M}_1$ be an affssm and for all $q_2, p_2 \in Q_2$. Since $\alpha : Q_1 \to Q_2$ is onto mapping, $\exists$ $q_1, p_1 \in Q_1$ such that $\alpha(q_1) = q_2$ and $\alpha(p_1) = p_2$. Let $\forall \sigma_2, \tau_2 \in X_2$. Since $\beta : X_1 \to X_2$ is onto mapping, $\exists$ $\sigma_1, \tau_1 \in X_1$ such that $\beta(\sigma_1) = \sigma_2$ and $\beta(\tau_1) = \tau_2$. Clearly $\mathcal{M}_2$ is commutative, since $\mathcal{M}_1$ is commutative. So, Now we have to prove that $\mathcal{M}_2$ is switching. Since $\mathcal{M}_1$ is switching, then

$$
\begin{aligned}
\mu_2(q_2, \sigma_2, p_2) &= \mu(q_2, \sigma_2\sigma_2, p_2) \\
&= \mu_2(\alpha(q_1), \beta(\sigma_1)\beta(\sigma_1), \alpha(p_1)) = \mu_2(\alpha(q_1), \beta(\sigma_1\sigma_1), \alpha(p_1)) \\
&= \bigvee\{\mu_1(s_1, \sigma_1\sigma_1, t_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\
&= \bigvee\{\mu_1(s_1, \sigma_1, t_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\
&= \bigvee\{\mu_1(t_1, \sigma_1, s_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\
&= \mu_2(\alpha(p_1), \beta(\sigma_1), \alpha(q_1)) = \mu_2(p_2, \sigma_2, q_2).
\end{aligned}
$$

Hence $\mathcal{M}_2$ is switching. Thus $\mathcal{M}_2$ is an affssm. □

# 5 Application

**Single Pattern: The one-minute microwave**[33]

A simple one-minute microwave system with a single control button is a useful application for fuzzy finite state machines. This system has two subsystems such as the Oven ffsm that turns the power-tube on/off, turns the light on/off and etc. On the other hand, the Timer ffsm keeps track of the time remaining and counts down with every clock tick. Each process has three states: The Timer includes the Sleep state, Idle/Waiting state and Active state while the Oven includes the OpenDoor state, Idle/Waiting state, and Active state. Separating the Timer from the Oven enables the modification of both components to be performed independently. This formation indicates that the use of a State Pattern with two controllers, each holding the transition table for the corresponding ffsm. However, these two subsystems and an object inside one of the subsystems need to communicate with the object inside the subsytem. To maintain the communication between these separate components, an additional functionality called the switchboard is needed. Through the switchboard, the Timer subsystem generates events corresponding to all the information that the Timer has to send to the Oven. The switchboard reference is available to the Oven and the related Oven states can then register themselves to receive to those events. By using the restricted product of ffssm, one can reduce the number of states and evaluate the subsystem state space without knowing the whole system.

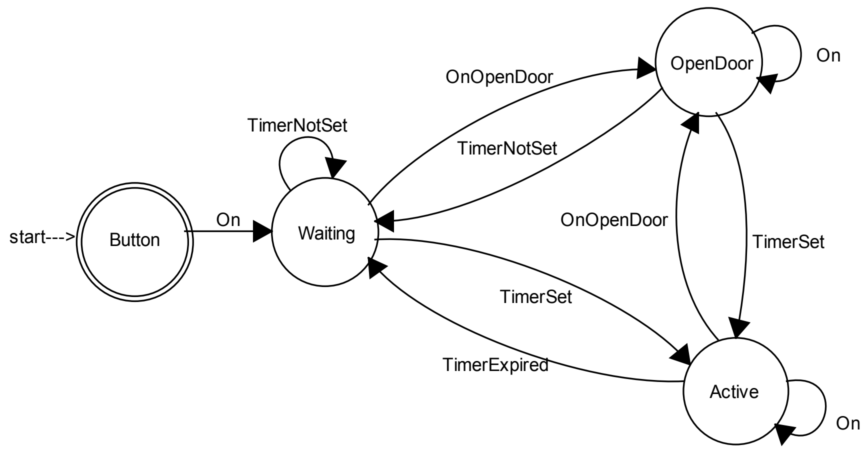


Figure 1: One-minute Microwave

## 6 Conclusions

One of the basic operations in automata theory is the product. In a nutshell, inspired by [34], we have considered the idea of switchboards in the concept of products and the covering of fuzzy finite state machine for the study of algebraic automata. We have presented an example for the definition of the restricted product of fuzzy finite switchboard state machine. We have given an algorithm for constructing fuzzy finite switchboard state machine and a single pattern one-minute microwave as an example to represent the definition of the restricted product of fuzzy finite switchboard state machine. The notion of asynchronous fuzzy finite switchboard state machine is introduced and studied their onto-switching homomorphic image of it. For future research, it will be of interest to study other types of products [24] in a general framework of nondeterministic (fuzzy) automata [2] and for the possible application on the work, the fuzzy automata and the idea of switchboard could be used to describe the regular safety properties of possibilistic linear temporal properties of an uncertain system based on the methods from Li *et al.*[19, 20].

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