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Fuzzy type theory with partial functions

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Abstract

This paper is a study of fuzzy type theory (FTT) with partial functions. Out of several possibilities we decided to introduce a special value "$*$" that represents "undefined". In the interpretation of FTT, this value lays outside of the corresponding domain. In the syntax it can be naturally represented by the description operator acting on the empty (fuzzy) set, because choosing an element from its kernel gives no result. Among many results, we will show that if, in a theory T, "$*$" is defined or provable then T is contradictory. We will also show completeness in the sense any consistent theory of FTT with partial functions has a model.

Keywords: Fuzzy type theory, EQ-algebra, partial function, description operator.

1 Introduction

In the applications of type theory to logical analysis of natural language (cf. [4, 10]) and elsewhere, e.g., in computer science, one often meets the requirement to deal with partial functions. They naturally raise, for example, when a mistake such as square root of a negative number or division by zero occurs, or when we want to express semantics of the expression "Czech president in 18^{th} century", because there was no such president before 1918. There are several ways how partial functions can be introduced. For example, W. Farmer in his paper [6] discusses 8 possibilities which appear in the literature: non-denoting expressions as non-well-formed terms, functions represented as relations, total functions with unspecified value, many-sorted language, error values, non-existent values, partial valuation for terms and formulas, partial valuation for terms but total valuation for formulas. Each of these approaches provides a certain kind of solution but has also drawbacks. It seems that fully satisfactory solution does not exist. In this paper, we introduce a fuzzy type theory with partial functions. There are two basic solutions: either we can introduce a special functional value "$*$" interpreted as "undefined", or we can modify the syntactic system to distinguish between general equality and equality of defined values. In this paper, we decided for the first option because the formal system of fuzzy type theory is fairly complicated and the second option would make it even more difficult to treat. We can also introduce "$*$" into other types. This approach has one important advantage. Tichý in [17] gave a counterexample demonstrating that introducing partial functions leads to failure of the principle of λ-conversion. As pointed by Lepage in [9], however, by introducing "$*$" as a special value, the counterexample of Tichý can be overcome and so, the λ-conversion is preserved.

Recall that in type theory, there is one specific formula which requires treatment with respect to partiality, namely the *description operator* $\iota_{\alpha(o\alpha)}$. In [11, 14], it is considered in such a way that its interpretation is just a partial function giving a value from the kernel of the corresponding fuzzy set if the latter is normal, and giving nothing otherwise. This suggests the idea that $*_o$ of type o can be defined as the formula $\iota_{o(oo)} \cdot \lambda x_o \bot$ whose interpretation requires application of the description operator to the empty set. Such value is, of course, undefined and so, we get natural interpretation of $*_o$. Using it, we can define $*_\epsilon$ as $\iota_{\epsilon(o\epsilon)} \cdot \lambda x_\epsilon *_o$, i.e., the description operator is here applied to a nowhere defined function. Similarly, we can introduce $*_\alpha$ for all types α. When dealing with "undefined" as introduced above we must be aware that it is a well formed formula. Namely, when combined with the equality it must be treated as the valid formula and so, $(A_\alpha \equiv_\alpha *_\alpha)$ means that the formula A_α is equal to the element "undefined" (which itself is also a formula). But if

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A_α is different from $*_\alpha$ then equality of these two formulas can by no means be true. Consequently, applying $*_\alpha$ to the function $g_{o\alpha} := \lambda x_\alpha \cdot (A_\alpha \equiv_\alpha x_\alpha)$ gives $g_{o\alpha} *_\alpha \equiv \bot$. We will see that this approach has several important consequences.

In this paper, we will prove that the above outlined way enables us to introduce "$*$" consistently into FTT so that partial functions can be included. We will consider the most general variant of FTT based on the EQ-algebras that are special algebras having structure of truth values that is natural for fuzzy type theory. The reason is that the basic operation in EQ-algebras is a fuzzy equality and the implication is derived from it. It can also be demonstrated that every residuated lattice is an EQ-algebra, but the converse does not hold. The paper is structured as follows: In Section 2 we introduce the EQ$_\Delta$-algebra of truth values and extend it by the special value "undefined". In Section 3 we introduce the syntax of the extended fuzzy type theory, namely introduce its axioms and prove various theorems characterizing the behavior of the formal theory. In Section 4, we will introduce semantics of the new FTT and in Section 5 we introduce the extended canonical model and prove general completeness of FTT. Finally, in Section 6 we will shortly characterize the behavior of partial functions and show that it conforms to the results of Lapierre and Lepage presented in the papers [8, 9].

2 Truth values and fuzzy equality

2.1 Truth values

The truth degrees form a linearly ordered bounded good EQ$_\Delta$-algebra (see [14, 15]) $\mathscr{E}_\Delta = \langle E, \wedge, \otimes, \sim, \mathbf{0}, \mathbf{1}, \Delta\rangle$ where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1}\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice). We put $a \le b$ iff $a \wedge b = a$, as usual. Then $\mathbf{1}$ is the top and $\mathbf{0}$ the bottom element.

(E2) $\langle E, \otimes, \mathbf{1}\rangle$ is a monoid, $\otimes$ is isotone w.r.t. $\le$.

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \le c \sim (d \wedge b)$ (substitution)

(E5) $(a \sim b) \otimes (c \sim d) \le (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \le (a \wedge b) \sim a$ (monotonicity)

(E7) $a \sim \mathbf{1} = a$ (goodness)

We define, $a \to b = (a \wedge b) \sim a$, and $\neg a = a \to \mathbf{0}$It can be proved that $a \le b$ iff $a \to b = 1$. Therefore, we can rewrite axioms (4)–(6) using $\to$ instead of $\le$. The *delta operation* $\Delta : E \to E$ has the following properties:

(ED1) $\Delta\mathbf{1} = \mathbf{1}$,

(ED2) $\Delta a \le a$,

(ED3) $\Delta a \le \Delta\Delta a$,

(ED4) $\Delta(a \sim b) \le \Delta a \sim \Delta b$,

(ED5) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.

(ED6) $\Delta(a \vee b) \le \Delta a \vee \Delta b$,

(ED7) $\Delta a \vee \neg\Delta a = \mathbf{1}$.

It should be emphasized that every residuated lattice is a good EQ-algebra with fuzzy equality being the biresiduation $a \sim b = (a \to b) \wedge (b \to a)$. An EQ-algebra $\mathscr{E}$ is *prelinear* if for all $a, b \in E$, $\mathbf{1}$ is the unique upper bound of $\{a \to b, b \to a\}$.

Lemma 2.1 ([5]). *The following is equivalent:*

(a) $\mathscr{E}$ *is prelinear.*

(b) $((a \to b) \to c) \le ((b \to a) \to c) \to c$

If $\mathscr{E}$ is prelinear then it is *lattice ordered* where the join is defined by $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$.

Convention 1. A fuzzy set is a function $A : M \to E$. We will often write $A \subsetneq M$ if we want to emphasize that A is a fuzzy set on the universe M.

The *kernel* of $A \subsetneq M$ is the set $Ker(A) = \{m \in M \mid A(m) = \mathbf{1}\}$. The fuzzy set A is *normal*, if $Ker(A) \ne \emptyset$ and *subnormal* otherwise. The set of all fuzzy sets on M is denoted by $\mathscr{F}(M)$ and the set of normal ones by $\mathscr{N}\mathscr{F}(M)$.

2.2 Extended algebra of truth values

To deal with partial functions, we will consider a special "truth value" $*$ where $* \notin E$ and interpret it as *undefined*. Then we will introduce extended EQ_Δ-algebra of truth values and denote it by $\mathscr{E}^*_\Delta$. Its support is $E^* = E \cup \{*\}$. We will extend the operations $\sim, \wedge, \otimes$ and Δ to the whole E^* as follows. The list of axioms is extended by

(EX1) $a \sim * = \mathbf{0}, \qquad a \in E,$

(EX2) $x \wedge * = *, \qquad x \in E^*,$

(EX3) $x \otimes * = *, \qquad x \in E^*.$

(EX3') $* \otimes x = *, \qquad x \in E^*.$

(EX4) $\Delta * = *, \qquad x \in E^*.$

Notice that by Axioms (EX2)–(EX4), the operations $\wedge, \otimes$ and Δ are Bochvar style [3], i.e., the $*$ collapses the result again to $*$. Taking that Axiom (E3) holds also for $*$, we obtain the following extended operations.

Corollary 2.2. *Let $a, b \in E$ and $\bigcirc \in \{\wedge, \otimes\}$. Then the following tables define the operations in the extended algebra $\mathscr{E}^*_\Delta$:*

$\sim$	b	$*$
a	$a \sim b$	$\mathbf{0}$
$*$	$\mathbf{0}$	$\mathbf{1}$

$\bigcirc$	b	$*$
a	$a \bigcirc b$	$*$
$*$	$*$	$*$

$\to$	b	$*$
a	$a \to b$	$\mathbf{0}$
$*$	$\mathbf{1}$	$\mathbf{1}$

$\vee$	b	$*$
a	$a \vee b$	$\mathbf{0}$
$*$	$\mathbf{0}$	$\mathbf{0}$

x	Δx
a	Δa
$*$	$*$

x	$\neg x$
a	$\neg a$
$*$	$\mathbf{0}$

Note that $\mathbf{0}$ remains the bottom element for all $a \in E$. We, at the same time, have $* \leq \mathbf{0}$ but $\mathbf{0} \not\leq *$.

Theorem 2.3. *Let us replace $\leq$ by $\to$ in the axioms above. Then formulas (E3)–(E6), (ED2)–(ED6) give $\mathbf{1}$ for all $x \in E^*$, (E7) and (ED7) give $\mathbf{1}$ for $a \in E$.*

Proof. This follows by straightforward verification using the operations defined above. □

In [2], a special operation $\boxast$ was introduced that we will call *star-box*[1]

x	$\boxast$
a	$\mathbf{0}$
$*$	$\mathbf{0}$
$\mathbf{1}$	$*$

This operation is characterized by the following axioms that may extend axioms of the algebra of truth values:

(ESb1) $(\boxast\mathbf{1} \sim *) = \mathbf{1},$

(ESb2) $(\boxast x \sim *) \vee \neg \boxast x = \mathbf{1}, \qquad x \in E^*.$

As we do not need this operation in this paper, we will not consider (ESb1), (ESb2) as special axioms of the extended EQ-algebra.

Finally we introduce the following derived operations on E^*:

$$?x = x \sim *, \quad !x = \neg ?x, \quad \downarrow x = x \sim \mathbf{1}, \text{ and } \uparrow x = \neg !x \vee \downarrow x.$$

Their truth tables are the following ($a \in E$):

	$?$	$!$	$\downarrow$	$\uparrow$
a	$\mathbf{0}$	$\mathbf{1}$	a	a
$*$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$

The operation "?" is a test for *undefined*, "!" is a test for *defined*, $\downarrow$ and $\uparrow$ are *star-0* and *star-1 reinterpretation*, respectively.

2.3 Fuzzy equality

A fuzzy equality $\circeq$ on M is a binary fuzzy relation on M, $\circeq\colon M \times M \to E$ that is reflexive, symmetric and $\otimes$-transitive. If $m, m' \in M$ then we usually write $[m \circeq m']$ instead of $\circeq(m, m')$. We say that $\circeq$ is *separated*, provided that $[m \circeq m'] = \mathbf{1}$ iff $m = m'$ holds for all $m, m' \in M$. A *fuzzy equality on truth values* is the operation $\sim$ from the EQ-algebra $\mathscr{E}$. This fuzzy equality is separated. Let us now extend M to $M^* = M \cup \{*\}$ where $* \notin M$. Analogously as above, $*$ represents the value "undefined" or "not known". Then we will extend the fuzzy equality $\circeq$ defined on M to $\circeq\colon M^* \times M^* \to E$ as follows[2]:

$\circeq$	y	$*$
x	$x \circeq y$	$\mathbf{0}$
$*$	$\mathbf{0}$	$\mathbf{1}$

[1]"Star box" is a name of one origami art piece.

[2]It seems unnecessary to use different symbols for fuzzy equality $\circeq$ on M and its extension to M^*.

where $x, y \in M$.

Let $M \subseteq M_\beta^{M_\alpha}$, i.e. the objects of M are *functions* $h : M_\alpha \to M_\beta$ where M_α, M_β are sets endowed with the corresponding fuzzy equalities $\doteq_\alpha, \doteq_\beta$. We will also add an element $*_{\beta\alpha} \notin M$ to M, i.e., $M^* = M \cup \{*_{\beta\alpha}\}$. Note that in this case, $*_{\beta\alpha}$ is a *nowhere defined function* on M_α. Then we introduce the fuzzy equality $\doteq: M^* \times M^* \to E$ by

$$[h \doteq h'] = \bigwedge_{m \in M_\alpha^*} [h(m) \doteq_\beta h'(m)], \qquad h, h' \in M^* \tag{1}$$

where $\doteq_\beta$ is the fuzzy equality on the set M_β. Note due to definition of $\doteq$ above, we obtain $\mathbf{0}$ in 1 whenever there is $m \in M_\alpha^*$ such that for $h, h' \in M$, either $h(m) = *$ or $h'(m) = *$, i.e., one of the functional values is undefined. Consequently, also for functions $h \in M$ we obtain $[h \doteq *_{\beta\alpha}] = \mathbf{0}$ and $[*_{\beta\alpha} \doteq *_{\beta\alpha}] = \mathbf{1}$. A function $f : M_\alpha \to M_\beta$ is *weakly extensional* if for all $m, m' \in M_\alpha$, $[m \doteq_\alpha m'] = \mathbf{1}$ implies $[f(m) \doteq_\beta f(m')] = \mathbf{1}$.

3 Syntax of partial FTT

We will consider the basic FTT whose truth degrees form a linearly ordered good EQ$_\Delta$-algebra introduced in [14]. This is the most general kind of FTT which can be extended to various specific versions including the linearly ordered IMTL-algebra originally introduced in [12].

3.1 Syntax

The basic syntactical objects of FTT are classical — see [1], namely the concepts of type and formula. The atomic types are ϵ (elements) and o (truth degrees). Complex types $(\beta\alpha)$ are formed from previously formed ones β and α. The set of all types is denoted by *Types*. The *language* of FTT denoted by J consists of variables $x_\alpha, \ldots$, special constants $\mathbf{c}_\alpha, \ldots$ ($\alpha \in$ *Types*), auxiliary symbols λ and brackets. Formulas are formed of variables, constants (each of specific type), and the symbol λ. Thus, each formula A is assigned a type (we write A_α). The set of formulas of type α is denoted by $Form_\alpha$, the set of all formulas by *Form*. Interpretation of a formula $A_{\beta\alpha}$ is a function from the set of objects of type α into the set of objects of type β. Thus, if $B \in Form_{\beta\alpha}$ and $A \in Form_\alpha$ then $(BA) \in Form_\beta$. Similarly, if $A \in Form_\beta$ and $x_\alpha \in J$, $\alpha \in$ *Types*, is a variable then $\lambda x_\alpha A_\beta \in Form_{\beta\alpha}$ is a formula whose interpretation is a function which assigns to each object of type α an object of type β represented by the formula A_β.

The set of formulas of type α, $\alpha \in$ *Types*, is denoted by $Form_\alpha$. A set of all formulas of the language J is $Form = \bigcup_{\alpha \in Types} Form_\alpha$. Specific constants always present in the language of FTT are the following.

(i) $\mathbf{E}_{(o\alpha)\alpha}$, $\quad \alpha \in \{o, \epsilon\}$, (fuzzy equality)

(ii) $\mathbf{C}_{(oo)o}$, (conjunction)

(iii) $\mathbf{S}_{(oo)o}$, (strong conjunction)

(iv) $\mathbf{D}_{oo}$, (delta)

(v) $\iota_{\alpha(o\alpha)}$, $\quad \alpha \in \{o, \epsilon\}$. (description operator)

A variable x_α is *bound* in a formula A_δ if the latter has a well formed part $(\lambda x_\alpha B_\beta)$. Otherwise x_α is *free*. A formula A is *closed* if it does not contain free variables. A closed formula A_o of type o is called a *sentence*.

Formal definitions

(i) *Basic fuzzy equality* $\equiv_{(o\alpha)\alpha} \equiv \lambda x_\alpha \lambda y_\alpha \cdot (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha$, $\alpha \in \{o, \epsilon\}$.

(ii) *Strong conjunction* $\boldsymbol{\&}_{(oo)o} \equiv \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o}\, y_o) x_o$.

(iii) *Conjunction:* $\wedge_{(oo)o} \equiv \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o}\, y_o) x_o$.

(iv) *Delta connective:* $\boldsymbol{\Delta}_{oo} \equiv \lambda x_o \mathbf{D}_{oo}\, x_o$.

(v) *Implication:* $\Rightarrow_{(oo)o} \equiv \lambda x_o \lambda y_o \cdot (x_o \wedge y_o) \equiv x_o$.

(vi) *Negation:* $\neg_{oo} \equiv \lambda x_o \cdot x_o \equiv \bot$.

(vii) *Disjunction:* $\vee_{(oo)o} \equiv \lambda x_o \lambda y_o ((x_o \Rightarrow y_o) \Rightarrow y_o) \wedge ((y_o \Rightarrow x_o) \Rightarrow x_o)$.

(viii) *General quantifier:* $(\forall x_\alpha) A_o \equiv (\lambda x_\alpha\, A_o \equiv \lambda x_\alpha\, \top)$.

(ix) Representation of *truth* and *falsity*: $\top_o \equiv (\lambda x_o x_o \equiv_{oo} \lambda x_o x_o)$, $\bot_o \equiv (\lambda x_o x_o \equiv_{oo} \lambda x_o \top)$ (we will usually omit the type at $\top$ and $\bot$). (2)

(x) *The values "undefined":* $*_o \equiv \iota_{o(oo)} \cdot \lambda x_o \bot$,

$$_\epsilon \equiv \iota_{\epsilon(o\epsilon)} \cdot \lambda x_\epsilon *_o, \text{ and } *_{\beta\alpha} \equiv \lambda x_\alpha *_\beta, \qquad \alpha, \beta \in Types.$$

Note that the element $*_{\beta\alpha}$ represents a nowhere defined function.

(xi) *Test of undefined:* $?_{o\alpha} \equiv \lambda x_\alpha \cdot x_\alpha \equiv_\alpha *_\alpha$.

(xii) *Test of defined:* $!_{o\alpha} \equiv \lambda x_\alpha \cdot \neg ?x_\alpha$.

(xiii) *Star-0 reinterpretation:* $\downarrow_{oo} \equiv \lambda x_o \cdot x_o \equiv \top$.

(xiv) *Star-1 reinterpretation:* $\uparrow_{oo} \equiv \lambda x_o \cdot \neg !x_o \vee \downarrow x_o$.

(xv) *Existential quantifier:* $(\exists x_\alpha)A_o \equiv (\forall y_o)((\forall x_\alpha)\mathbf{\Delta}(A_o \Rightarrow \uparrow y_o) \Rightarrow \uparrow y_o)$ (y_o does not occur in A_o).

Convention 1. *The fuzzy equality between formulas of the type α is denoted by $\equiv_\alpha$. We will sometimes omit the type at "$\equiv$" if it is clear from the arguments and no confusion can occur. Note also that the fuzzy equality between truth values is usually called* fuzzy equivalence.

3.2 Axioms and inference rules

Fundamental axioms.
Let $\alpha, \beta \in$ *Types*

(FT-fund1) $(\exists x_\epsilon)!x_\epsilon$.

(FT-fund2) $\mathbf{\Delta}(x_\alpha \equiv_\alpha y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv_\beta f_{\beta\alpha}\, y_\alpha)$,

(FT-fund3) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$,

(FT-fund4) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv_\beta g_{\beta\alpha}\, x_\alpha)$

(FT-fund5) $(\lambda x_\alpha B_\beta)A_\alpha \equiv_\beta C_\beta$, $\alpha, \beta \in$ *Types* , where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

(FT-fund6) $(x_\epsilon \equiv_\epsilon y_\epsilon) \equiv_\epsilon (y_\epsilon \equiv_\epsilon x_\epsilon)$,

(FT-fund7) $(x_\epsilon \equiv_\epsilon y_\epsilon)\&(y_\epsilon \equiv_\epsilon z_\epsilon) \Rightarrow (x_\epsilon \equiv_\epsilon z_\epsilon)$.

Axioms of truth degrees As usual in fuzzy logic, we have two kinds of conjunction, namely the "ordinary" conjunction $\wedge$ and the *strong conjunction* $\&$. Let $\bigcirc \in \{\wedge, \&\}$.

(FT-tval1) $\top$

(FT-tval2) $(x_o \wedge y_o) \equiv (y_o \wedge x_o)$,

(FT-tval3) $(x_o \bigcirc y_o) \bigcirc z_o \equiv x_o \bigcirc (y_o \bigcirc z_o)$,

(FT-tval4) $!A_o \Rightarrow ((A_o \equiv \top) \equiv A_o)$,

(FT-tval5) $!A_\alpha \equiv (!A_\alpha \equiv \top)$, $\alpha \in$ *Types*,

(FT-tval6) $(A_\alpha \equiv_\alpha A_\alpha) \equiv \top$, $\alpha \in$ *Types*,

(FT-tval7) $!((A_\alpha \equiv_\alpha A_\alpha) \equiv \top)$, $\alpha \in$ *Types*,

(FT-tval8) $(A_o \bigcirc \top) \equiv A_o$,

(FT-tval9) $((\top \& A_o) \equiv A_o)$,

(FT-tval10) $!A_o \Rightarrow ((A_o \wedge \bot) \equiv \bot)$,

(FT-tval11) $(A_o \wedge A_o) \equiv A_o$,

(FT-tval12) $((x_o \wedge y_o) \equiv z_o)\&(t_o \equiv x_o) \Rightarrow (z_o \equiv (t_o \wedge y_o))$,

(FT-tval13) $(x_o \equiv y_o)\&(z_o \equiv t_o) \Rightarrow (x_o \equiv z_o) \equiv (y_o \equiv t_o)$,

(FT-tval14) $(x_o \Rightarrow (y_o \wedge z_o)) \Rightarrow (x_o \Rightarrow y_o)$,

(FT-tval15) $\mathbf{\Delta}(x_o \Rightarrow y_o) \Rightarrow (x_o \& z_o \Rightarrow y_o \& z_o)$,

(FT-tval16) $\mathbf{\Delta}(x_o \Rightarrow y_o) \Rightarrow (z_o \& x_o \Rightarrow z_o \& y_o)$,

(FT-tval17) $((x_o \Rightarrow y_o) \Rightarrow z_o) \Rightarrow ((y_o \Rightarrow x_o) \Rightarrow z_o) \Rightarrow z_o$.

Axioms of delta

(FT-delta1) $(g_{oo}(\mathbf{\Delta} x_o) \wedge g_{oo}(\neg \mathbf{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\mathbf{\Delta} y_o)$,

(FT-delta2) $\mathbf{\Delta}(A_o \wedge B_o) \equiv \mathbf{\Delta} A_o \wedge \mathbf{\Delta} B_o$,

(FT-delta3) $\mathbf{\Delta}(A_o \vee B_o) \Rightarrow \mathbf{\Delta} A_o \vee \mathbf{\Delta} B_o$,

(FT-delta4) $(\mathbf{\Delta} \uparrow A_o \vee \neg \mathbf{\Delta} \uparrow A_o) \equiv \top$.

Axioms of star

(FT-B1) $(\bot \equiv *_o) \equiv \bot$,

(FT-B2) $(\top \equiv *_o) \equiv \bot$,

(FT-B3) $A_o \bigcirc *_o \equiv *_o$,

(FT-B4) $\boldsymbol{\Delta} *_o \equiv *_o$,

(FT-B5) $((A_\alpha \equiv_\alpha *_\alpha) \vee \neg (A_\alpha \equiv_\alpha *_\alpha)) \equiv \top$, $\qquad \alpha \in \{o, \epsilon\}$.

(FT-B6) $!A_o \Rightarrow (!B_o \Rightarrow !(A_o \bigcirc B_o))$,

(FT-B7) $!A_o \Rightarrow (!B_o \Rightarrow !(A_o \equiv B_o)$.

Axioms of quantifiers

(FT-quant1) $\boldsymbol{\Delta}(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$, $\quad x_\alpha$ is not free in A_o,

(FT-quant2) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha) A_o \Rightarrow B_o)$, $\quad x_\alpha$ is not free in B_o,

(FT-quant3) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha) A_o \vee B_o)$, $\quad x_\alpha$ is not free in B_o.

Axioms of descriptions

(FT-descr1) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv_\alpha y_\alpha$, $\qquad \alpha = \{o, \epsilon\}$,

(FT-descr2) $(\forall x_\alpha)(\neg \boldsymbol{\Delta}(f_{o\alpha} x_\alpha \equiv \top)) \Rightarrow (\iota_{\alpha(o\alpha)} f_{o\alpha} \equiv *_\alpha)$, $\qquad \alpha \in \{o, \epsilon\}$.

Remark 3.1. The list of axioms above may probably be redundant. To simplify them, however, will be the task of some future paper.

Remark 3.2. It follows from formal definitions (xi) and (xii) that Axioms (FT-B1), (FT-B2) and (FT-B5) can alternatively be written in the following form:

(FT-B1') $!\bot$,

(FT-B2") $\neg\neg *_o$,

(FT-B1"') $\neg *_o \equiv \bot$,

(FT-B2') $!\top$.

(FT-B5') $(?A_\alpha \vee !A_\alpha) \equiv \top$. $\qquad \alpha \in \{o, \epsilon\}$.

Recall that a formula A_o is crisp if $\vdash A_o \vee \neg A_o$. Hence, both $?A_\alpha$ as well as $?A_\alpha$ are crisp.

The inference rules remain unchanged:
Let $A_\alpha \equiv A'_\alpha \in Form_o$ and $B_o \in Form_o$ be formulas. Then we infer from them a formula B'_o which comes from B_o by replacing one occurrence of A_α by A'_α, provided that the occurrence of A_α in B_o is not an occurrence of a variable immediately preceded by λ.

(N) *Let $A_o \in Form_o$ be a formula. Then from $A_o \equiv \top$ infer $\boldsymbol{\Delta} A_o \equiv \top$.*

A theory T is a set of formulas of type o (determined by a subset of special axioms, as usual). Provability is defined as usual. If T is a theory and A_o a formula then $T \vdash A_o$ means that A_o is provable in T. A theory T is *contradictory* if $T \vdash \bot$. Otherwise it is *consistent*. The concepts of extension and conservative extension of a theory T are classical.

Theorem 3.3. *(a)* $(\exists x_\alpha)!x_\alpha$, $\alpha \in$ *Types.*

(b) $(A_o \equiv \top) \vdash !A_o$.

(c) $A_o, (A_o \equiv B_o) \vdash B_o$. *(Rule (EMP))*

(d) $\vdash A_\alpha \equiv_\alpha A_\alpha$, $\qquad \alpha \in$ *Types.*

(e) $\vdash (A_\alpha \equiv_\alpha B_\alpha) \equiv (B_\alpha \equiv_\alpha A_\alpha)$.

(f) $(A_o \equiv \top), (A_o \Rightarrow B_o) \vdash B_o$. *(Modus Ponens I)*

(g) *Let* $\vdash !A_o \equiv \top$. *Then* $\vdash A_o$ *iff* $\vdash A_o \equiv \top$.

(h) $!A_o, A_o, (A_o \Rightarrow B_o) \vdash B_o$. *(Modus Ponens II)*

(i) $!A_o \vdash (A_o \Rightarrow \bot) \Rightarrow \neg A_o$.

Proof. (a) By Axiom (FT-B2'), we have $\vdash !\top$. Therefore, $(\exists x_o)!x_o$, and so, (a) follows from this and Axiom (FT-fund1) by induction on types.

(b) follows from Axiom (FT-B2') using Rule (R).

(c) follows immediately from the assumptions using Rule (R).

(d) follows from Axioms (FT-tval6) and (FT-tval1) by Rule (EMP).

(e) is proved by induction on the complexity of types. For $\alpha \in \{o, \epsilon\}$, this follows from Axiom (FT-fund6) and the axioms of truth values (see [14, Lemma 13(g)]).

Let $\alpha = \gamma\beta$.

(L.1) $\vdash (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \equiv (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta)$ (instance of (c))

(L.2) $\vdash (f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \equiv (g_{\gamma\beta}\, x_\beta \equiv_\gamma f_{\gamma\beta}\, x_\beta)$ (inductive assumption)

(L.3) $\vdash (f_{\gamma\beta} \equiv_{\gamma\beta} g_{\gamma\beta}) \equiv (g_{\gamma\beta} \equiv_{\gamma\beta} f_{\gamma\beta})$
(L.1, L.2 using Rule (R) and formal definition (xv))

(f)

(L.1) $\vdash A_o \equiv \top$ (assumption)

(L.2) $\vdash \top \Rightarrow B_o$ (L.1, assumption, Rule (R))

(L.3) $\vdash (\top \wedge B_o) \equiv \top$ (L.2, definition of $\Rightarrow$)

(L.4) $\vdash \top \wedge B_o$ (L.3, Rule (EMP))

(L.5) $\vdash \top \wedge B_o \equiv B_o$ (Axiom (FT-tval8))

(L.6) $\vdash B_o$ (L.4, L.5, Rule (EMP))

(g) From the assumption and Axiom (FT-tval4) we obtain $\vdash (A_o \equiv \top) \equiv A_o$ using Modus Ponens I. The rest follows from this using Rule (EMP).

(h)

(L.1) $\vdash !A_o \equiv \top$ (assumption, Axiom (FT-tval5), Rule (EMP))

(L.2) $\vdash (A_o \equiv \top) \equiv A_o$ (L.1, Axiom (FT-tval4, Modus Ponens I)

(L.3) $\vdash A_o \equiv \top$ (assumption, L.2, Rule (EMP))

(L.4) $\vdash B_o$ (L.3, assumption, Modus Ponens I)

(i) Let us consider the following instance of Axiom (FT-tval12):

$$\vdash ((A_o \wedge \bot) \equiv A_o) \& (A_o \equiv A_o) \Rightarrow (A_o \equiv (A_o \wedge \bot)).$$

Realizing that $!A_o \vdash ((A_o \wedge \bot) \equiv A_o) \equiv (A_o \Rightarrow \bot)$ and $!A_o \vdash (A_o \wedge \bot) \equiv \bot$, we obtain $\vdash (A_o \Rightarrow \bot) \Rightarrow \neg A_o$ using Rule (R), Axioms (FT-tval6) and (FT-tval8). □

Remark 3.4. Note that in the proof above and most of the following ones we silently applied also symmetry of $\equiv$ due to item (c).

In the subsequent subsections we will prove a series of lemmas and theorems characterizing various properties of FTT with partial functions. Some of them are slight but necessary modifications of the properties of the standard FTT. Sometimes, however, we do not repeat the latter and refer to the properties whose proofs need not be modified by saying "using properties of FTT". We must suppose that the reader is acquainted with the papers [12, 14] because otherwise the paper would be too long.

3.3 Logical connectives and "undefined"

The following is the basic theorem characterizing definability of equivalence.

Theorem 3.5. *Let $\alpha \in Types$. Then*

(a) $\vdash !(A_\alpha \equiv A_\alpha)$,

(b) $\vdash !(*_\alpha \equiv *_\alpha)$,

(c) $!A_\alpha \vdash !(A_\alpha \equiv *_\alpha)$,

(d) $A_\alpha \equiv B_\alpha \vdash !(A_\alpha \equiv B_\alpha)$,

(e) $!A_\alpha, !B_\alpha \vdash !(A_\alpha \equiv B_\alpha)$,

(f) $!A_\alpha \vdash !!A_\alpha$,

(g) $?A_\alpha \vdash !?A_\alpha$.

Proof. (a) follows from Axioms (FT-B2') and (FT-tval6) using Rule (R).

(b) is an instance of (a).

(c) By the assumption $\vdash (A_\alpha \equiv *_\alpha) \equiv \bot$. As $\vdash !\bot$ by Axiom (FT-B1'), we obtain (c) using Rule (R).

(d) follows from (a) and the assumption using Rule (R).

(e) From the assumption, Axiom (FT-tval5) and using Rule (EMP) we obtain $\vdash !A_o \equiv \top$ and $\vdash !B_o \equiv \top$. Then (e) follows from this and Axiom (FT-B7) using Modus Ponens I.

(f) By the assumption, $\vdash (A_\alpha \equiv *_\alpha) \equiv \bot$. Then (f) follows from this and (d).

(g) follows from (b) using the assumption and Rule (R). □

Corollary 3.6. *If A_o is an axiom of FTT then $\vdash !A_o$.*

Proof. If A_o is one of Axiom (FT-B2) or (FT-tval7) then the corollary trivially follows. Otherwise, all axioms have the form of equivalence and so, the corollary follows from Theorem 3.5(d). □

Theorem 3.7. *Let $\vdash A_o$. Then $\vdash !A_o$.*

Proof. By induction on the length of the proof.

(a) If A_o is an axiom then lemma follows from Corollary 3.6.

(b) Let A_o follow from A'_o and $B_o \equiv C_o$ using Rule (R). Then $\vdash !A'_o$ by the inductive assumption. This means that $\vdash \neg(A'_o \equiv *_o)$. Then we obtain $\vdash \neg(A_o \equiv *_o)$ using Rule (R) which means that $\vdash !A_o$.

(c) Let $A_o := \mathbf{\Delta} B_o \equiv \top$ be obtained from $B_o \equiv \top$ using Rule (N). Then we have $\mathbf{\Delta} B_o \equiv \top \vdash !(\mathbf{\Delta} B_o \equiv \top)$ by Theorem 3.5(d). □

Theorem 3.8. *For all $\alpha \in$ Types, $\vdash !A_\alpha \vee ?A_\alpha$.*

Proof. For $\alpha \in \{o, \epsilon\}$, this follows from Axiom (FT-B5).

Induction step for $\alpha = \gamma\beta$. Using the properties of FTT, the inductive assumption and the definition of fuzzy equality between complex types, we can write the following sequence of provable formulas:

$$\vdash \neg(A_{\gamma\beta} x_\beta \equiv *_\beta) \Rightarrow (\exists x_\beta) \neg(A_{\gamma\beta} x_\beta \equiv *_\beta)$$
$$\vdash \neg(A_{\gamma\beta} x_\beta \equiv *_\beta) \vee (A_{\gamma\beta} x_\beta \equiv *_\beta) \Rightarrow (\exists x_\beta) \neg(A_{\gamma\beta} x_\beta \equiv *_\beta) \vee (A_{\gamma\beta} x_\beta \equiv *_\beta)$$
$$\vdash (\exists x_\beta) \neg(A_{\gamma\beta} x_\beta \equiv *_\beta) \vee (\forall x_\beta)(A_{\gamma\beta} x_\beta \equiv *_\beta)$$
$$\vdash \neg(\forall x_\beta)(A_{\gamma\beta} x_\beta \equiv *_\beta) \vee (\forall x_\beta)(A_{\gamma\beta} x_\beta \equiv *_\beta)$$
$$\vdash !A_\alpha \vee ?A_\alpha$$

Note that the previous reasoning is sound because all the formulas are defined. □

Ir follows from this theorem that both $?A_\alpha$ as well as $!A_\alpha$ are crisp and so, the following is provable (cf. [13]):

$$\vdash ?A_\alpha \equiv \mathbf{\Delta} ?A_\alpha, \quad \text{and} \quad \vdash !A_\alpha \equiv \mathbf{\Delta} !A_\alpha.$$

Lemma 3.9. *If $\vdash A_o$ then $\vdash \mathbf{\Delta} A_o$.*

Proof. (a)

(L.1) $\vdash !A_o$ (Assumption, Theorem 3.7)

(L.2) $\vdash (A_o \equiv \top) \equiv A_o$ (Assumption, L.1, Axiom (FT-tval4), MP II)

(L.3) $\vdash A_o \equiv \top$ (Assumptions, L.2, Rule (EMP))

(L.4) $\vdash \mathbf{\Delta} A_o \equiv \top$ (Rule (N))

(L.5) $\vdash \mathbf{\Delta} A_o$ (L.4, Axiom (FT-tval1), Rule (EMP))

□

Lemma 3.10. *The following holds for all $\alpha \in$ Types:*

(a) $\vdash ?*_\alpha$,

(b) $\vdash !?*_\alpha$,

(c) $\vdash \neg !*_\alpha$.

Proof. (a) follows from formal definition (xi) and Theorem 3.3(d).

(b) follows from (a) and Theorem 3.5(a).

(c) follows from (a), formal definition (xii) and the fact that $A_o \vdash \neg\neg A_o$.

□

Lemma 3.11 ([13]). *Let T be a consistent theory and $\mathbf{u}_\alpha \notin J(T)$ be a new constant. Then $T \vdash (\exists x_\alpha)\boldsymbol{\Delta} B_o$ iff $T' = T \cup \{B_o[x_\alpha/\mathbf{u}_\alpha]\}$ is a conservative extension of T.*

Theorem 3.12. *(a) $A_o \equiv \top \vdash (\forall x_\alpha)A_o$.* *(generalization I)*

(b) $!A_o, A_o \vdash (\forall x_\alpha)A_o$. *(generalization II)*

(c) $\Box A_o, A_o \equiv B_o \vdash \Box B_o$ where $\Box \in \{!, ?, \downarrow, \uparrow\}$.

Proof. (a) and (b) follow analogously as items (b) and (c) from the proof of [12, Theorem 5] using Modus Ponens I, II.

(c) follows from the assumption $\vdash A_o \equiv B_o$ using Rule (R). □

Theorem 3.13. *Let $\bigcirc \in \{\wedge, \&\}$. Then*

(a) $(A_o \equiv \top), (B_o \equiv \top) \vdash A_o \bigcirc B_o$,

(b) $!A_o, A_o, !B_o, B_o \vdash A_o \bigcirc B_o$,

(c) $\vdash (A_o \Rightarrow B_o), (B_o \Rightarrow A_o) \vdash A_o \equiv B_o$,

(d) $!A_o, !B_o \vdash !(A_o \wedge B_o)$,

(e) $\vdash !(A_o \wedge B_o) \Rightarrow (!A_o \wedge !B_o)$,

(f) $!A_o, !B_o \vdash !(A_o \Rightarrow B_o)$.

Proof. (a) can be proved from the instance $\vdash (\top \& \top) \equiv \top$ of Axiom (FT-tval8), Axiom (FT-tval1), the assumptions and Rule (R).

(b) is proved analogously using Modus Ponens II and also Axiom (FT-tval4).

(c) follows using the property $\vdash (A_o \Rightarrow B_o) \& (B_o \Rightarrow A_o) \Rightarrow (A_o \equiv B_o)$ proved in [14, Lemma 13(i)], (a) and modus ponens I.

(d) is proved in the same way as Theorem 3.5(e) from axiom (FT-B6).

(e) By the provable properties of FTT, we have $\vdash (A_o \equiv *_o) \Rightarrow ((A_o \wedge B_o) \equiv (*_o \wedge B_o))$ and $\vdash (B_o \equiv *_o) \Rightarrow ((B_o \wedge A_o) \equiv (*_o \wedge A_o))$. The we apply Axiom (FT-B3), Rule (R), the provable properties $\vdash (A_o \Rightarrow B_o) \Rightarrow (\neg B_o \Rightarrow \neg A_o)$ and $(A_o \Rightarrow B_o) \& (A_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow (B_o \wedge C_o))$ and Modus Ponens II we obtain $\vdash \neg((A_o \wedge B_o) \equiv *_o) \Rightarrow \neg(A_o \equiv *_o) \wedge \neg(B_o \equiv *_o)$ which is (e).

(f) follows from Theorem 3.5(e) and (d). □

Lemma 3.14. *(a) $!A_o, !B_o \vdash A_o \Rightarrow (A_o \vee B_o)$,*

(b) $\vdash \downarrow \top$, $\quad \vdash \uparrow \top$,

(c) $\vdash !\downarrow A_o$,

(d) $\vdash (\uparrow A_o \Rightarrow *_o) \equiv \bot$, $\quad \vdash (\downarrow A_o \Rightarrow *_o) \equiv \bot$, $\quad \vdash (*_o \Rightarrow A_o) \equiv \top$,

(e) $!A \vdash \downarrow A_o \equiv A_o$, $\quad \vdash \downarrow *_o \equiv \bot$,

(f) $!A \vdash \uparrow A_o \equiv A_o$, $\quad \vdash \uparrow *_o \equiv \top$.

Proof. (a) follows from the standard properties of FTT using Modus Ponens II.

(b) follows from $\vdash \top \equiv \top$, $\vdash !\top$, definition of $\uparrow$, (a) and the standard properties of FTT.

(c) follows from the instance of axiom (FT-tval12): $\vdash ((*_o \wedge \bot) \equiv *_o)$ and $((A \equiv \top) \equiv *_o) \Rightarrow (*_o \equiv ((A_o \equiv \top) \wedge \bot))$ axioms and Theorems 3.3 and 3.5.

(d) follows from the definition of implication and Theorems 3.3 and 3.5.

(e), (f) follow from axioms and the formal definitions of the corresponding operators.

□

Theorem 3.15. *(a) $\vdash !(*_o \vee *_o)$ and $\vdash (*_o \vee *_o) \equiv \bot$,*

(b) $\vdash !(A_o \vee *_o)$ and $\vdash (A_o \vee *_o) \equiv \bot$,

(c) $!A_o, !B_o \vdash !(A_o \vee B_o)$.

Proof. (a) Using formal definitions (v,) (vii) and Lemma 3.14(b), (d), (f) we obtain the sequence of provable formulas $\vdash (*_o \Rightarrow *_o) \equiv \top$, $\vdash (\top \Rightarrow *_o) \equiv \bot$, $\vdash (\bot \wedge \bot) \equiv \bot$ and $\vdash (*_o \vee *_o) \equiv \bot$. Then (a) follows from this and Axiom (FT-B1) using rule (R).

The second part follows from the definition of $\vee$ and the proved properties of $\Leftrightarrow$ and $\wedge$.

The proof of (b) is analogous.

(c) follows from Theorem 3.13(d) and (f). □

Lemma 3.16. $\vdash (f_{\beta\alpha} \equiv_{\beta\alpha} g_{\beta\alpha}) \equiv (\forall x_\alpha)(f_{\beta\alpha} x_\alpha \equiv_\beta g_{\beta\alpha} x_\alpha)$.

Proof. The proof is analogous to the proof of [14, Theorem 6] using Axioms (FT-fund3), Generalization II, (FT-fund4) and Theorem 3.13(c). □

Theorem 3.17. *(a) $\vdash (\forall x_\alpha)B_o \Rightarrow (B_{o,x_\alpha}[A_\alpha] \equiv \top)$,* *(substitution I)*

(b) $!B_o \vdash (\forall x_\alpha)B_o \Rightarrow B_{o,x_\alpha}[A_\alpha]$, *(substitution II)*

(c) $!A_\alpha, !B_\alpha, !C_\alpha \vdash (A_\alpha \equiv_\alpha B_\alpha) \mathbin{\&} (B_\alpha \equiv_\alpha C_\alpha) \Rightarrow (A_\alpha \equiv_\alpha C_\alpha)$.

(d) $!B_{o\alpha}A_\alpha \vdash B_{o\alpha}A_\alpha \Rightarrow (\exists x_\alpha)B_{o\alpha}x_\alpha$. *($\exists$-substitution)*

Proof. (a) The proof is based on the following instance of Axiom (FT-fund4), $\vdash (\lambda x_\alpha B_o \equiv \lambda x_\alpha \top) \Rightarrow ((\lambda x_\alpha B_o)A_\alpha \equiv (\lambda x_\alpha \top)A_\alpha)$ which reduces to $\vdash (\forall x_\alpha)B_o \Rightarrow (B_{o,x_\alpha}[A_\alpha] \equiv \top)$. As we cannot, in general, assure that $\vdash !B_{o,x_\alpha}[A_\alpha]$, we cannot apply Axiom (FT-tval4).

(b) is a corollary of (a) and Axiom (FT-tval5) using Rule (R).

(c) For $\alpha \in \{o, \epsilon\}$, this follows from Axiom (FT-fund7) and the axioms of truth values (see [14, Lemma 13(h)]).

Let $\alpha = \gamma\beta$.

(L.1) $\vdash (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \Rightarrow ((f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \equiv \top)$

(substitution axiom)

(L.2) $\vdash (\forall x_\beta)(g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta) \Rightarrow ((g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta) \equiv \top)$

(substitution axiom)

(L.3) $!A_\alpha, !B_\alpha, !C_\alpha \vdash (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \Rightarrow (f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta)$

(L.1, Theorems 3.3(g) and 3.13(f), Axiom (FT-tval4), Rule (R))

(L.4) $!A_\alpha, !B_\alpha, !C_\alpha \vdash (\forall x_\beta)(g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta) \Rightarrow (g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta)$

(L.2, Theorems 3.3(g) and 3.13(f), Axiom (FT-tval4), Rule (R))

(L.5) $\vdash (f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \mathbin{\&} (g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta) \Rightarrow (f_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta)$

(inductive assumption)

(L.6) $!A_\alpha, !B_\alpha, !C_\alpha \vdash (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma g_{\gamma\beta}\, x_\beta) \mathbin{\&} (\forall x_\beta)(g_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta) \Rightarrow (\forall x_\beta)(f_{\gamma\beta}\, x_\beta \equiv_\gamma h_{\gamma\beta}\, x_\beta)$

(L.3, L.4, L.5, Rule (N), Axiom (FT-quant1))

(L.7) $!A_\alpha, !B_\alpha, !C_\alpha \vdash (f_{\gamma\beta} \equiv_{\gamma\beta} g_{\gamma\beta}) \mathbin{\&} (g_{\gamma\beta} \equiv h_{\gamma\beta}) \Rightarrow (f_{\gamma\beta} \equiv h_{\gamma\beta})$

(L.6, using Lemma 3.16)

(d) is proved analogously as in [14], provided that the rules of substitution II and generalization II are applied. □

Lemma 3.18. *(a)* $A_o, !A_o \vdash !\boldsymbol{\Delta} A_o$,

(b) $?A_o \vdash ?\boldsymbol{\Delta} A_o$.

Proof. (a)

(L.1) $\vdash !!A_o$ (Assumption, Theorem 3.5(f))

(L.2) $\vdash (A_o \equiv \top) \equiv A_o$ (Assumptions, L.1, Axiom (FT-tval4), MP II)

(L.3) $\vdash A_o \equiv \top$ (Assumptions, L.2, (EMP))

(L.4) $\vdash \boldsymbol{\Delta} A_o \equiv \top$ (Rule (N))

(L.5) $\vdash !\boldsymbol{\Delta} A_o$ (Theorem 3.3(b))

(b)

(L.1) $\vdash (A_o \equiv *_o)$assumption

(L.2) $\vdash \boldsymbol{\Delta}(A_o \equiv *_o)$ (L.1, Rule (N))

(L.3) $\vdash \boldsymbol{\Delta}(A_o \equiv *_o) \Rightarrow (\boldsymbol{\Delta} A_o \equiv \boldsymbol{\Delta} *_o)$ (properties of FTT)

(L.4) $\vdash \boldsymbol{\Delta} A_o \equiv *_o$ (L.2, L.3, (a), Modus Ponens II, Axiom (FT-B4))

□

Lemma 3.19. *(a)* $\vdash (\exists z_o)(z_o \equiv y_o)$,

(b) $\vdash y_o \equiv (\exists z_o)(z_o \mathbin{\&} (z_o \equiv y_o))$,

(c) $\vdash (\exists z_o)z_o$,

(d) $\vdash (\forall z_o)z_o \equiv \bot$,

(e) If $\mathbf{r}_o$ *is a constant then* $\vdash (\forall x_\alpha)\mathbf{r}_o \equiv \mathbf{r}_o$ *and* $\vdash (\exists x_\alpha)\mathbf{r}_o \equiv \mathbf{r}_o$.

Proof. (a) follows from the reflexivity of $\equiv$ and $\exists$-substitution. The proof of (b) is the same as in [11] where the provable property $A_o \Rightarrow (B_o \Rightarrow C_o) \vdash A_o \mathbin{\&} B_o \Rightarrow C_o)$ must be used. (c) follows from (a) when putting $y_o := \top$. (d) is obtained from formal definition (viii) and axioms (FT-fund4) and (FT-fund5) by putting $z_o := \bot$. Similarly (e) is obtained using quantifier axioms and the properties of fuzzy equality. □

The deduction theorem holds for the defined formulas. Hence, we have the following formulation.

Theorem 3.20 (Deduction theorem). *Let T be a theory and $A_o \in Form_o$ a closed formula such that $T \vdash !A_o$. Then $T \cup \{A_o\} \vdash B_o \iff T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$, for every formula $B_o \in Form_o$ such that $T \vdash !B_o$.*

Proof. As A_o, B_o are assumed to be defined, the proof is verbatim repetition of the proof from [12, Theorem 17]. □

Theorem 3.21. *(a) If $T \vdash !*_o$ then T is contradictory.*

(b) If $T \vdash *_o$ then T is contradictory.

Proof. (a) The assumption means that $T \vdash (*_o \equiv *_o) \equiv \bot$. This together with $\vdash (*_o \equiv *_o) \equiv \top$ gives $T \vdash \bot$ which means that T is contradictory.

(b) follows from Theorem 3.7 and (a). □

Theorem 3.22. *(a) $\vdash (\forall x_\alpha) *_o \equiv \bot$,*

(b) $\vdash (\exists x_\alpha) *_o \equiv \bot$,

(c) Let $T \vdash (\exists x_\alpha) ?A_{o\alpha} x_\alpha$. Then $T \vdash (\forall x_\alpha) A_{o\alpha} x_\alpha \equiv \bot$.

(d) Let $T \vdash (\forall x_\alpha) ?A_{o\alpha} x_\alpha$. Then $T \vdash (\exists x_\alpha) A_{o\alpha} x_\alpha \equiv \bot$.

Proof. (a) By formal definition (viii), we have $\vdash (\forall x_\alpha) *_o \equiv (\lambda x_\alpha *_o \equiv \lambda x_\alpha \top)$. Then from axiom (FT-fund4), lambda conversion and substitution I we conclude that $\vdash (\forall x_\alpha) *_o \equiv (*_o \equiv \top)$. From this and Axiom (FT-B2) we obtain (a) using Rule (R).

(b) From formal definition (xv) we obtain $\vdash (\exists x_\alpha) *_o \Rightarrow ((\forall x_\alpha) \mathbf{\Delta} (*_o \Rightarrow \uparrow y_o) \Rightarrow \uparrow y_o)$. Then we apply Lemma 3.19(d), (e), generalization II and Lemma 3.14(d).

(c) Using Lemma 3.11, we can extend the language of FTT by a new constant $\mathbf{u}_\alpha$ and obtain from the assumption $T' \vdash ?A_{o\alpha} \mathbf{u}_\alpha$ which is equivalent to $T' \vdash A_{o\alpha} \mathbf{u}_\alpha \equiv *_o$. Using generalization, the quantifier properties and the fact that T' is a conservative extension of T we obtain $T \vdash (\forall x_\alpha) A_{o\alpha} x_\alpha \equiv (\forall x_\alpha) *_o$ which implies (c) using (a).

(d) From the assumption we have $T \vdash A_{o\alpha} x_\alpha \equiv *_o$. From this and $T \vdash (\exists x_\alpha) A_{o\alpha} x_\alpha \equiv (\exists x_\alpha) A_{o\alpha} x_\alpha$ we have $T \vdash (\exists x_\alpha) A_{o\alpha} x_\alpha \equiv (\exists x_\alpha) *_o$ using rule (R). This gives (d) using (b). □

It also follows from (a) that if a formula $A_{o\alpha} x_\alpha$ is undefined for some x_α then its closure is false.

Lemma 3.23. *For all $\alpha, \beta \in Types$, $\vdash *_{\beta\alpha} *_\alpha \equiv_\beta *_\beta$.*

Proof. This follows from formal definition (x) using λ-conversion. □

By this lemma, the nowhere defined function at the "undefined" point gives again the value "undefined".

3.4 Description operator and the "undefined"

The description operator $\iota_{\alpha(o\alpha)}$ for $\alpha \in \{o, \epsilon\}$ must be taken as a special constant with the proper definition of its interpretation. In fact, it can be applied to a formula $A_{o\alpha}$ provided that $\vdash (\exists x_\alpha) \mathbf{\Delta} (A_{o\alpha} x_\alpha)$. We can extend the description operator to complex types $\beta\alpha$ as follows:

$$\iota_{(\beta\alpha)(o(\beta\alpha))}\, h_{o(\beta\alpha)} := \lambda x_\alpha \cdot \iota_{\beta(o\beta)} \cdot \lambda z_\beta (\exists f_{\beta\alpha})(\mathbf{\Delta}(h_{o(\beta\alpha)} f_{\beta\alpha}) \mathbin{\&} (f_{\beta\alpha} x_\alpha \equiv z_\beta)). \tag{3}$$

The following theorem characterizes the basic properties of the description operator.

Theorem 3.24 ([11]). *(a) $\vdash \iota_{\gamma(o\gamma)}(\mathbf{E}_{(o\gamma)\gamma}\, y_\gamma) \equiv_\gamma y_\gamma$ holds for every type $\gamma \in Types$.*

(b) $\vdash \mathbf{\Delta}(p_{o\alpha} \equiv q_{o\alpha}) \Rightarrow (\iota_{\alpha(o\alpha)} p_{o\alpha} \equiv \iota_{\alpha(o\alpha)} q_{o\alpha})$,

(c) $\vdash (\exists y_\alpha)(\exists x_\alpha)(\mathbf{\Delta}(p_{o\alpha} x_\alpha \equiv (E_{(o\alpha)\alpha} y_\alpha) x_\alpha)) \Rightarrow p_{o\alpha}(\iota_{\alpha(o\alpha)} p_{o\alpha})$,

(d) $\vdash (\exists x_\alpha) \mathbf{\Delta} A_{o\alpha} x_\alpha \Rightarrow A_{o\alpha}(\iota_{\alpha(o\alpha)} A_{o\alpha})$.

By (a), the description operator can be extended to all types. The statement (d) says that if a fuzzy set interpreting the formula $A_{o\alpha}$ has a non-empty kernel then the element $\iota_{\alpha(o\alpha)} A_{o\alpha}$ belongs to it.

Theorem 3.25. *For all types γ,*

$$(\forall x_\gamma)(\neg \mathbf{\Delta}(A_{o\gamma} x_\gamma \equiv \top)) \vdash \iota_{\gamma(o\gamma)} A_{o\gamma} \equiv_\gamma *_\gamma. \tag{4}$$

Proof. By induction on the length of type. For $\gamma \in \{o, \epsilon\}$ (4) follows from axiom (FT-descr2).

Let $\gamma = \beta\alpha$. Let the inductive assumption hold.

(L.1) $\vdash \iota_{(\beta\alpha)(o(\beta\alpha))}\, A_{o(\beta\alpha)} \equiv \lambda x_\alpha \cdot \iota_{\beta(o\beta)} \cdot \lambda z_\beta (\exists f_{\beta\alpha})(\mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha})\&$ $(f_{\beta\alpha} x_\alpha \equiv z_\beta))$ (definition (3))

(L.2) $\vdash (\forall f_{\beta\alpha})(\neg\mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha} \equiv \top))$ (assumption)

(L.3) $\vdash \mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha}) \equiv \bot$ (L.2, properties of FTT)

(L.4) $\vdash \mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha})\&(f_{\beta\alpha} x_\alpha \equiv z_\beta) \equiv \bot$ (L.3, properties of FTT)

(L.5) $\vdash (\exists f_{\beta\alpha})(\mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha})\&(f_{\beta\alpha} x_\alpha \equiv z_\beta)) \equiv \bot$ (L.4, properties of FTT)

(L.6) $\vdash (\forall z_\beta)\neg\mathbf{\Delta}\,\underbrace{(\exists f_{\beta\alpha})(\mathbf{\Delta}(A_{o(\beta\alpha)} f_{\beta\alpha})\&(f_{\beta\alpha} x_\alpha \equiv z_\beta))}_{C_{o\beta} z_\beta}$

(L.5, properties of FTT)

Note that (L.6) states that some formula $\lambda z_\beta\, C_{o\beta} z_\beta$ represents a subnormal fuzzy set. Hence, using (L.1) we continue as follows:

(L.7) $\vdash \iota_{(\beta\alpha)(o(\beta\alpha))}\, A_{o(\beta\alpha)} \equiv_{\beta\alpha} \lambda x_\alpha *_\beta$

(L.1, L.6, λ-conversion, ind. assumption)

(L.8) $\vdash \iota_{(\beta\alpha)(o(\beta\alpha))}\, A_{o(\beta\alpha)} \equiv_{\beta\alpha} *_{\beta\alpha}$ (L.7, def. of $*_{\beta\alpha}$)

□

Corollary 3.26. *For all types $\alpha \in$ Types:*

(a) Let $\vdash ?A_{o\alpha}$. *Then* $\vdash \iota_{\alpha(o\alpha)} A_{o\alpha} \equiv_\alpha *_\alpha$.

(b) $\vdash \iota_{\alpha(o\alpha)} *_{o\alpha} \equiv_\alpha *_\alpha$.

Proof. (a)

(L.1) $\vdash A_{o\alpha} \equiv_{o\alpha} *_{o\alpha}$ (Assumption, definition of "?")

(L.2) $\vdash \lambda x_\alpha A_{o\alpha} x_\alpha \equiv_{o\alpha} \lambda x_\alpha *_{o\alpha}$ (L.1, properties of FTT)

(L.3) $\vdash A_{o\alpha} x_\alpha \equiv *_o$ (L.2, Axiom (FT-fund4))

(L.4) $\vdash \neg(*_o \equiv \top)$ (Axiom (FT-B2))

(L.5) $\vdash \neg(A_{o\alpha} x_\alpha \equiv \top)$ (L.3, L.4, Rule (R))

(L.6) $\vdash (\forall x_\alpha)\neg\mathbf{\Delta}(A_{o\alpha} x_\alpha \equiv \top)$ (L.5, rule (N), property of $\mathbf{\Delta}$, generalization II)

(L.7) $\vdash \iota_{\alpha(o\alpha)} A_{o\alpha} \equiv_\alpha *_\alpha$ (L.6, Theorem 3.25)

The rule of generalization (Theorem 3.12) was in line L.6 used correctly because the initial formula is defined.

(b) is a special case of (a) using Lemma 3.10(a). □

4 Semantics of partial FTT

The *extended general frame* for FTT with partial functions is the following tuple: $\mathscr{M}^* = \langle (M^*_\alpha, \doteq_\alpha)_{\alpha \in Types}, \mathscr{E}^*_\Delta, I_o, I_\epsilon\rangle$ so that the following holds:

(i) We put $M^*_o = E^*$ where the latter is the support of the extended EQ-algebra $\mathscr{E}^*_\Delta$. Furthermore, let $*_\epsilon \notin M_\epsilon$. Then we put $M^*_\epsilon = M_\epsilon \cup \{*_\epsilon\}$. For all $\alpha = \gamma\beta$, the function $*_{\gamma\beta} : M^*_\beta \to M^*_\gamma$, where $*_{\gamma\beta}(m_\beta) = *_\gamma$ for all $m_\beta \in M^*_\beta$, represents "undefined". For all types $\gamma\beta$ we put $M^*_{\gamma\beta} \subseteq (M^*_\gamma)^{M^*_\beta}$, where we require that $*_{\gamma\beta} \in M^*_{\gamma\beta}$.

(ii) The $\mathscr{E}^*_\Delta$ is the extended algebra of truth degrees (EQ$_\Delta$-algebra). We assume that the sets M^*_{oo}, $M^*_{(oo)o}$ contain all the operations discussed in Subsection 2.2.

(iii) $\doteq_\alpha$: $M^*_\alpha \times M^*_\alpha \to E$ is a fuzzy equality on M^*_α for every $\alpha \in$ *Types*. We define: $\doteq_o := \sim$ and $\doteq_\epsilon$ is the fuzzy equality on M^*_ϵ given explicitly. The fuzzy equality $\doteq_{\beta\alpha}$ for complex types is defined in (1).

(iv) $I_o : \mathscr{F}(M_o) \to M_o$, $I_\epsilon : \mathscr{F}(M_\epsilon) \to M_\epsilon$ are partial functions interpreting the basic description operators. Let $B \subsetneq M_o$ and $C \subsetneq M_\epsilon$. Then

$$I_o(B) = \begin{cases} a_B \in Ker(B) & \text{if } B \text{ is normal,} \\ *_o & \text{otherwise,} \end{cases} \quad \text{and } I_\epsilon(C) = \begin{cases} x_C \in Ker(C) & \text{if } C \text{ is normal,} \\ *_\epsilon & \text{otherwise.} \end{cases}$$

Remark 4.1. The elements a_B, x_C in the formulas above are assumed to be certain specific elements from the kernels of B and C determined on the basis of some special requirements. Note that the functions I_o, I_ϵ resemble the defuzzification operations used in the applications of fuzzy modeling.

Interpretation of formulas in a frame $\mathscr{M}$ is defined using an assignment p of elements from $\mathscr{M}$ to variables, i.e., it is a function from the set of all variables of the language J to elements from $\mathscr{M}^*$ in keeping with the corresponding types. Given an assignment p, we define a new assignment $p' = p\backslash x_\alpha$ which equals to p for all variables except for x_α. The set of all assignments over $\mathscr{M}$ is denoted by Asg($\mathscr{M}$).

Interpretation of variables and constants For arbitrary assignment $p \in \mathrm{Asg}(\mathscr{M})$ we define:

(i) If x_α is a variable then $\mathscr{M}^*_p(x_\alpha) = p(x_\alpha) \in M^*_\alpha$.

(ii) If $\mathbf{c}_\alpha$ is a constant then $\mathscr{M}_p(\mathbf{c}_\alpha) \in M^*_\alpha$ is a specific element. As special cases, we define the following:

- $\mathscr{M}_p(\mathbf{E}_{(oo)o}) :=\sim$,
- $\mathscr{M}_p(\mathbf{E}_{(o\epsilon)\epsilon} := \mathring{=}_\epsilon$,
- $\mathscr{M}_p(\mathbf{C}_{(oo)o}) := \wedge$,
- $\mathscr{M}_p(\mathbf{S}_{(oo)o}) := \otimes$,
- $\mathscr{M}_p(\mathbf{D}_{oo}) := \Delta$.

(iii) Interpretation of the description operators[3]: $\mathscr{M}_p(\iota_{\epsilon(o\epsilon)}) = I_o$ and $\mathscr{M}_p(\iota_{o(oo)}) = I_\epsilon$.

Interpretation of complex formulas

(i) Interpretation of a formula $B_{\beta\alpha}A_\alpha$, which is of type β, is $\mathscr{M}_p(B_{\beta\alpha}A_\alpha) = \mathscr{M}_p(B_{\beta\alpha})(\mathscr{M}_p(A_\alpha))$.

(ii) Interpretation of a formula $\lambda x_\alpha\, A_\beta$, which is of type $\beta\alpha$, is a function $\mathscr{M}_p(\lambda x_\alpha\, A_\beta) = F : M_\alpha \to M_\beta$, which assigns to each $m_\alpha \in M_\alpha$ the element $F(m_\alpha) = \mathscr{M}_{p'}(A_\beta)$ determined by an assignment p' such that $p' = p\backslash x_\alpha$ and $p'(x_\alpha) = m_\alpha$. It follows from this definition that F is weakly extensional w.r.t "$=_\alpha$" and "$=_\beta$".

A model of a theory T is a general frame $\mathscr{M}$ for which $\mathscr{M}_p(A_o) = \mathbf{1}$ holds for all axioms A_o of T and all assignments $p \in \mathrm{Asg}(\mathscr{M})$. A formula A_o is true in the theory T, $T \models A_o$ if it is true in the degree $\mathbf{1}$ in all its models.

By a straightforward verification we can prove that interpretation of the connectives $\equiv, \wedge, \boldsymbol{\&}, \vee, \Rightarrow, \boldsymbol{\Delta}, \equiv_\epsilon$ in $\mathscr{M}^*$ is equal to the corresponding operations $\sim, \wedge, \otimes, \vee, \to, \Delta, \mathring{=}$ introduced in Sections 2.2 and 2.3.

Lemma 4.2. *(a)* $\mathscr{M}_p((\forall x_\alpha)A_o) = \begin{cases} \bigwedge\{\mathscr{M}_{p'}(A_o) \mid p' = p\backslash x_\alpha, p'(x_\alpha) \in M^*_\alpha\}, \\ \mathbf{0}, \text{ if } \mathscr{M}_{p'}(A_o) = *_o \text{ for some } p' = p\backslash x_\alpha. \end{cases}$

(b) $\mathscr{M}_p((\exists x_\alpha)A_o) = \begin{cases} \bigvee\{\mathscr{M}_{p'}(A_o) \mid p' = p\backslash x_\alpha, p'(x_\alpha) \in M^*_\alpha, \mathscr{M}^*_{p'}(A_o) \neq *_o\}, \\ \mathbf{0}, \text{ if } \mathscr{M}_{p'}(A_o) = *_o \text{ for all } p' = p\backslash x_\alpha. \end{cases}$

Proof. Ad (a): $\mathscr{M}_p((\forall x_\alpha)A_o) = \mathscr{M}_p(\lambda x_\alpha A_o \equiv \lambda x_\alpha \top) = \bigwedge\{\mathscr{M}_{p'}(A_o) \mathring{=} \mathbf{1} \mid p' = p\backslash x_\alpha, p'(x_\alpha) \in M^*_\alpha\}$

Ad (b):

$$\mathscr{M}_p((\exists x_\alpha)A_o) = \bigwedge\{a \in E \mid \bigwedge\{\Delta(\mathscr{M}_{p'}(A_o) \to a) \mid p' = p\backslash x_\alpha, p'(x_\alpha) \in M^*_\alpha\} \to a\}$$
$$= \bigwedge\{a \in E \mid \Delta(\bigvee\{\mathscr{M}_{p'}(A_o) \mid p' = p\backslash x_\alpha, p'(x_\alpha) \in M^*_\alpha, \mathscr{M}^*_{p'}(A_o) \neq *_o\} \to a) \to a\}$$

which is (b), because if $\mathscr{M}_{p'}(A_o) = *_o$ for some $p' = p\backslash x_\alpha$ then the implication $\mathscr{M}_{p'}(A_o) \to a = \mathbf{1}$ for arbitrary $a \in E$, and so, it is ignored by the corresponding infimum operation, and if $\mathscr{M}_{p'}(A_o) = *_o$ for all assignments p' then interpretation of the existential quantifier reduces to $\bigwedge\{a \mid a \in E\} = \mathbf{0}$. □

5 Canonical model of partial FTT

The canonical model of EQ-FTT with partial functions can be obtained by extension of the canonical model for the basic FTT (see [12, 14]). We will denote it by ${}^T\mathscr{M}^*$. In the same way as in FTT, we define a special function $\mathscr{V}$, whose domain and range are equivalence classes of formulas.

5.1 Extension of theories

Let T be a theory and $\approx$ be the canonical equivalence on formulas defined in the standard way: $A_\alpha \approx B_\alpha$ iff $T \vdash A_\alpha \equiv B_\alpha$, $\alpha \in$ *Types*. The corresponding equivalence class is denoted by $|A_\alpha|$.

Definition 5.1. Let T be a theory. We say that:

(i) T is *maximally consistent* if each its extension T', $T' \supset T$ is inconsistent.

(ii) T is *linear* if for every two formulas A_o, B_o such that $T \vdash !A_o$ and $T \vdash !B$, $\vdash A_o \Rightarrow B_o$ or $T \vdash B_o \Rightarrow A_o$.

(iii) T is *extensionally complete* if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ it follows that there is a closed formula C_α such that $T \vdash !C_\alpha$, $T \not\vdash A_{\beta\alpha}C_\alpha \equiv B_{\beta\alpha}C_\alpha$.

Theorem 5.2 ([14])**.** *Every consistent theory T can be extended to a maximally consistent linear theory.*

[3] Recall that the description operator represents, in fact, a defuzzification operation (cf. [16], Chapter 3).

Proof. The proof proceeds in the same way as the proof of item 3 of Lemma 2 from [7] because the prelinearity property is provable in FTT. □

Theorem 5.3. *Every consistent theory T can be extended to an extensionally complete consistent theory $\overline{T}$.*

Proof. The proof is obtained by small modification of the proof of [14, Theorem 12]. Namely, we have to confine to defined formulas. Let us denote $\pi = \mathrm{Card}(J(T))$. Let K_α be a well ordered set of new constants of type $\alpha \in \mathit{Types}$, $\mathrm{Card}(K_\alpha) \leq \pi$ and put $K = \bigcup_{\alpha \in \mathit{Types}} K_\alpha$ and $J^+(T) = J(T) \cup K$. Therefore, we will consider closed formulas A_α, B_α of the language $J(T)$ such that $T \vdash !A_\alpha, !B_\alpha$ and enumerate them by ordinal numbers. Theorem 3.5(e) then assures us that $T \vdash !(A_\alpha \equiv B_\alpha)$.

In the proof, we construct a sequence of theories T_μ, $\mu \leq \pi$ and a sequence of special sets of formulas Ψ_μ such that:

(a) $T_0 = T$ and $T_\nu \subset T_\mu$ for $\nu < \mu$,

(b) $\Psi_\nu \subset \Psi_\mu$ and $T_\mu \not\vdash \Psi_\mu$.

The construction proceeds by transfinite recursion. We suppose that $T_{<\mu} = \bigcup_{\nu \in \mu} T_\nu$ and $\Psi_{<\mu} = \bigcup_{\nu \in \mu} \Psi_\nu$ are already constructed.

Let $D \in \Psi_{<\mu}$. Furthermore, let $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be the first not yet processed formula of this form, and let $\mathbf{c}_\alpha \in K$ be the first not yet used constant (both in the given well ordering). We distinguish two cases:

(i) Let $T_{<\mu} \vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$. Then we put $\Psi_\mu = \Psi_{<\mu}$ and $T_\mu = T_{<\mu} \cup \{A_{\beta\alpha} \equiv B_{\beta\alpha}\}$.

(ii) Let $T_{<\mu} \not\vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$. Then we put $\Psi_\mu = \Psi_{<\mu} \cup \{D \vee (A_{\beta\alpha} c_\alpha \equiv B_{\beta\alpha} c_\alpha) \mid D \in \Psi_{<\mu}\}$ and $T_\mu = T_{<\mu}$.

Now we must show that in both cases $T_\mu \not\vdash \Psi_\mu$. This is done analogously as in [7]. In case (ii), we must show that the assumption $T_\mu \vdash D \vee (A_{\beta\alpha} c_\alpha \equiv B_{\beta\alpha} c_\alpha)$ leads to contradiction; indeed, replacing c_α by a variable not occurring in D and $A_{\beta\alpha}, B_{\beta\alpha}$ and further, using (FT-quant3) and Lemma (3.16) we obtain $T_\mu \vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$.

To show, finally, that T_π is extensionally complete, let $A_{\beta\alpha} \equiv B_{\beta\alpha}$ be a formula processed in step μ and $T_\pi \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$. Then it had to be processed as the case (ii), i.e. $T_\mu \not\vdash D \vee (A_{\beta\alpha} \equiv B_{\beta\alpha})$, otherwise we get a contradiction. But the previous reasoning, together with Theorem 3.17(b), leads to $T_\pi \not\vdash (A_{\beta\alpha} c_\alpha \equiv B_{\beta\alpha} c_\alpha)$ for some constant $\mathbf{c}_\alpha$. □

5.2 Extended canonical algebra of truth values

Now we will put ${}^T E = \{|A| \mid A \in \mathit{Form}_o, A \text{ closed}, T \vdash !A\}$, and ${}^T E^* = {}^T E \cup \{| *_o |\}$, where $*_o$ is defined in (2). It follows from the definition of "$\approx$" that $| *_o | = \{A \mid A \in \mathit{Form}_o, T \vdash ?A\}$.

Theorem 5.4. *Let T be a consistent linear extensionally complete theory. Then the algebra ${}^T\mathscr{E}^* = \langle {}^T E^*, {}^T\wedge, {}^T\otimes, {}^T\!\sim, {}^T\Delta, {}^T\boldsymbol{1}, {}^T\boldsymbol{0}\rangle$ is an extended linearly ordered good EQ_Δ-algebra.*

Proof. We must prove that the operations behave in accordance with the extended operations due to Subsection 2.2. The proof of the standard operations on ${}^T E$ and linearity of the algebra ${}^T\mathscr{E}$ is the same as in [14].

Let us now check the extended operations due to Axioms (EX1)–(EX4). First note the following: if $T \vdash ?A_o$ then $A_o \in | *_o |$. On the other hand, if $T \vdash !A_o$ then $|A_o| \in {}^T E$. Indeed, otherwise $A_o \in |*_o|$ which would mean that $T \vdash A_o \equiv *_o$ and, consequently, T would be contradictory. We thus conclude that ${}^T\mathscr{E}^*$ is not degenerated.

The extended operations in ${}^T\mathscr{E}^*$ due to Corollary 2.2 are obtained as follows:

(i) Let $|A_o|, |B_o| \in {}^T E$. Then $T \vdash !A_o$ and $T \vdash !B_o$ and so $T \vdash !(A_o \equiv B_o)$ by Theorem 3.5(e). Therefore, the operation $|A_o| \,{}^T\!\sim |B_o| = |A_o \equiv B_o|$ is closed on ${}^T E$. Furthermore, $| *_o | \,{}^T\!\sim | *_o | = |\top|$ by Axiom (FT-tval6). Finally, $T \vdash !A_o$ implies $T \vdash (A_o \equiv *_o) \equiv \bot$ which gives $|A_o| \,{}^T\!\sim | *_o | = |\bot|$ for all $|A_o| \in {}^T E$.

(ii) $|A_o| \,{}^T\bigcirc | *_o | = |A_o \bigcirc *_o| = | *_o |$ by Axiom (FT-B3), where $\bigcirc \in \{\wedge, \&\}$.

(iii) ${}^T\Delta | *_o | = |\Delta *_o | = | *_o |$ by Axiom (FT-B4).

□

5.3 Canonical frame and completeness

Now we define the canonical frame.

Definition 5.5. Let T be a consistent, linear and extensionally complete theory. Then the canonical frame is $\mathscr{M}^* = \langle (M^*_\alpha, \doteq_\alpha)_{\alpha \in \mathit{Types}}, {}^T\mathscr{E}^*_\Delta, I_o, I_\epsilon\rangle$ where:

(1) $M^*_\alpha = \{\mathscr{V}(A_\alpha) \mid A_\alpha \in \mathit{Form}_\alpha, A_\alpha \text{ closed}\}$, $\alpha \in \mathit{Types}$. The construction of $\mathscr{V}$ proceeds inductively:

   (i) If $\alpha = o$ then $\mathscr{V}(A_o) = |A_o|$, and $\mathscr{V}(*_o) = | *_o | = |\iota_{o(oo)} \cdot \lambda x_o \bot|$.

   (ii) If $\alpha = \epsilon$ then $\mathscr{V}(A_\epsilon) = |A_\epsilon|$, $\mathscr{V}(A_\epsilon) = |A_\epsilon|$, $\mathscr{V}(*_\epsilon) = | *_{\epsilon|=} |\iota_{\epsilon(o\epsilon)} \cdot \lambda x_\epsilon *_o |$.

   (iii) If $\alpha = \gamma\beta$ then we put $\mathscr{V}(A_{\gamma\beta}) \subseteq M^*_\beta \times M^*_\gamma$ which is a relation consisting of couples $\langle \mathscr{V}(B_\beta), \mathscr{V}(A_{\gamma\beta} B_\beta)\rangle$ for all closed $B_\beta \in \mathit{Form}_\beta$ and $A_{\gamma\beta} \in \mathit{Form}_{\gamma\beta}$. As a special case, $\mathscr{V}(*_{\gamma\beta}) = \{\langle \mathscr{V}(A_\beta), \text{ and } \mathscr{V}(*_\gamma)\rangle \mid \mathscr{V}(A_\beta) \in M^*_\beta\}$.

The fuzzy equality on M^*_α is defined by $[\mathscr{V}(A_\alpha) \doteq_\alpha \mathscr{V}(B_\alpha)] = |A_\alpha \equiv B_\alpha|$.

(2) The ${}^{T}\mathscr{E}^{e*}_{\Delta}$ is the extended EQ_Δ-algebra from Theorem 5.4.

(3) Let $T \vdash !A_{oo}$ and $\mathscr{V}(A_{oo}) = \{\langle \mathscr{V}(B_o) \text{ and } \mathscr{V}(A_{oo}B_o)\rangle \mid \mathscr{V}(B_o) \in M_o^*\}$ be a fuzzy set on M_o^*. Then we put

$$I(\mathscr{V}(A_{oo})) = \begin{cases} |\iota_{o(oo)}A_{oo}|, \text{ if } \vdash (\exists x_o)\boldsymbol{\Delta}(A_{oo}x_o), \\ \mathscr{V}(*_o), \text{ if } \vdash (\forall x_o)(\neg\boldsymbol{\Delta}(A_{oo}x_o \equiv \top)). \end{cases}$$

Similarly for a fuzzy set $\mathscr{V}(A_{o\epsilon})$ on M_ϵ^* we put

$$I(\mathscr{V}(A_{o\epsilon})) = \begin{cases} |\iota_{\epsilon(o\epsilon)}A_{o\epsilon}|, \text{ if } \vdash (\exists x_\epsilon)\boldsymbol{\Delta}(A_{o\epsilon}x_\epsilon), \\ \mathscr{V}(*_\epsilon), \text{ if } \vdash (\forall x_\epsilon)(\neg\boldsymbol{\Delta}(A_{o\epsilon}x_\epsilon \equiv \top)). \end{cases}$$

Finally we will prove the theorem saying that every consistent theory of FTT with partial functions has a model.

Theorem 5.6 (Completeness of FTT with partial functions)**.** *Let T be a theory, the special axioms of which have the form $A_o \equiv \top$. Then T is consistent iff it has a general model $\mathscr{M}$.*

Proof. All axioms of FTT are defined by Corollary 3.6. Moreover, using Axiom (FT-B2') and Rule (R) we can prove that also $T \vdash !A_o$ holds for all special axioms A_o of T. Using Theorems 5.2 and 5.3 we extend T into a linear and extensionally complete theory. Then this theorem follows from the construction of the canonical frame above using the same arguments as in [14]. □

6 Partial functions

In this section we will briefly touch the results introduced and analyzed by Lapierre and Lepage in [8, 9].

The following definitions characterize the basic concepts introduced in these papers:

(i) *Total function:* $\mathrm{TotF}_{o(\beta\alpha)} \equiv \lambda f_{\beta\alpha} \cdot (\forall x_\alpha)(!x \Rightarrow !(f_{\beta\alpha}x_\alpha))$,

(ii) *Partial function:* $\mathrm{PartF}_{o(\beta\alpha)} \equiv \lambda f_{\beta\alpha} \cdot (\exists x_\alpha)(!x_\alpha \boldsymbol{\&} ?(f_{\beta\alpha}x_\alpha))$,

(iii) *Strict function:* $\mathrm{StrictF}_{o(\beta\alpha)} \equiv \lambda f_{\beta\alpha} \cdot (\forall x_\alpha)(?x \Rightarrow ?(f_{\beta\alpha}x_\alpha))$,

(iv) *Non-strict function:* $\mathrm{NStrictF}_{o(\beta\alpha)} \equiv \lambda f_{\beta\alpha} \cdot (\exists x_\alpha)(?x_\alpha \boldsymbol{\&} !(f_{\beta\alpha}x_\alpha))$.

Note that all these predicates are crisp, i.e., any function is either total or partial, or either strict or non-strict.

We will also introduce a special ordering: $\lhd_{(o\alpha)\alpha} \equiv \lambda x_\alpha \lambda y_\alpha \cdot ?x_\alpha \vee \boldsymbol{\Delta}(x_\alpha \equiv y_\alpha)$. It can be proved that this relation is crisp and is indeed an ordering. On the basis of it, we can define *monotonous function:*

$$\mathrm{MonF}_{o(\beta\alpha)} \equiv \lambda f_{\beta\alpha} \cdot (\forall x_\alpha)(\forall y_\alpha)((x_\alpha \lhd y_\alpha) \equiv (f_{\beta\alpha}x_\alpha \lhd f_{\beta\alpha}y_\alpha)). \tag{5}$$

Lapierre and Lepage gave many arguments in favor of the idea to consider all the functions to be monotonous in the sense of (5).

Lemma 6.1. *Let T be a consistent theory, $f_{\beta\alpha} \in Form_{\beta\alpha}$ be a formula such that $T \vdash TotF\, f_{\beta\alpha} \wedge NStrictF\, f_{\beta\alpha}$ and, moreover, the following is provable: $T \vdash (\exists x_\alpha)\boldsymbol{\Delta}(!x_\alpha \text{ and } \neg\boldsymbol{\Delta}(f_{\beta\alpha}x_\alpha \equiv f_{\beta\alpha}*_\alpha))$. Then $T \not\vdash MonF\, f_{\beta\alpha}$.*

According to this lemma, monotonicity excludes also improper functions having the properties from Lemma 6.1. This result is in accordance with the results of [9].

To obtain a special case of FTT in accordance with the papers [8, 9], we might extend the list of its axioms by the following one:

(FT-fund6) $(\forall f_{\beta\alpha})\, \mathrm{MonF}\, f_{\beta\alpha}$, $\alpha, \beta \in$ *Types.*

It seems, however, that this axiom should not belong among logical axioms of FTT and should be added only to special theories. The reason is that it would restrict FTT too much. For example, the operation $\circledast$ considered in Subsection 2.2 and in [2] could not be included because it is not monotonous.

7 Conclusions

In this paper, we studied possibilities how partial functions can be introduced into fuzzy type theory. We have chosen the most general kind of FTT based on EQ-algebra of truth degrees, see [14].

From several options how a special value "undefined" (denoted by $*$) can be introduced, we decided to represent it by an element laying outside of the given domain (for each specific type). In our construction, we used the fact that the description operator $\iota_{\alpha(o\alpha)}$ gives no result when applied to formulas representing subnormal fuzzy sets. Hence, we defined $*_o$ as the formula $\iota_{o(oo)} \cdot \lambda x_o \bot$ and accordingly the other values "undefined". This made it possible to have "undefined" inside the set *Form* of all the formulas without necessity to extend the language by new constant with special properties. Further research will be focused on formalization and deeper elaboration of the theory of partial functions using the means of the fuzzy type theory.

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