Iranian Journal of Fuzzy Systems

Volume 16, Number 2, (2019), pp. 17-29

University of Sistan and Baluchestan

Stabilization of chaotic systems via fuzzy time-delayed controller approac

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Abstract

In this paper, we investigate the stabilization of unstable periodic orbits of continuous time chaotic systems using fuzzy time-delayed controllers. For this aim, we present a control method that can achieve stabilization of an unstable periodic orbit (UPO) without any knowledge of the system model. Our proposal is attained progressively. First, we combine the input-to-state linearizing controller with the fuzzy method to obtain one that achieves UPO stabilization. Then, we use reduced order sliding observer to estimate the necessary state for the controller construction. Finally, the efficiency of the proposed methods is demonstrated using numerical simulations applied to Chua's system.

Keywords: Chaos control, Time-delayed state feedback, Fuzzy adaptive systems, Sliding mode observer.

1 Introduction

In recent years, researchers have increasingly interested in controlling chaotic systems. The first chaos control method has been proposed by Ott et al. [23], nowadays known as the OGY (Ott-Grebogi-Yorke) method. This is a discrete technique that considers small perturbations applied to a crucial system parameter when the trajectory approaches the vicinity of the desired orbit when crossing a specific surface. From then onwards, numerous control techniques have been proposed for controlling chaos in different chaotic systems such as sliding-mode control [16, 37, 36, 9], H_∞ robust control [7], adaptive control algorithms [18, 40, 38], optimal control [2, 21] and heuristic methods such that genetic algorithms, practical swarm Optimization and ant colony optimization [22]. Most control cited techniques handle chaotic systems as ordinary nonlinear systems. Indeed, theses methods have been used to stabilize the unstable equilibrium points of the chaotic system or to force it to track a desired trajectory. Another method has emerged in chaos suppression and concerns the stabilization of unstable periodic orbits (UPO) embedded in the chaotic attractor [26, 25, 27, 28]. Indeed, one of the first continuous method proposed for suppression chaos was the delayed feedback control [29], which states that chaotic systems can be stabilized by a feedback gain proportional to the difference between the present and the delayed state of the system. Many alternative methods were proposed in order to overcome some limitations of the original techniques. Indeed, Pyragas [30] presented a review about improvements and applications of time-delayed feedback control. In most of the methods mentioned above, an input control signal was added and a full knowledge of the system model was required. Recently, fuzzy logic has an increasing attention in chaos control problems [4, 41, 6, 35] as well as in chaos synchronization problems [17, 33, 1, 13]. Fuzzy logic control was proposed in an attempt to control nonlinear systems [5] especially, when a complete knowledge of the plant and disturbance is not available [39]. Based on the universal approximation capability of fuzzy systems [34], many fuzzy logic schemes have been developed for unknown nonlinear systems. However, fuzzy control seems to be deficient in formal analysis and robustness aspects. To overcome this drawback, great efforts have been done in the conjunction of fuzzy techniques and conventional control approaches such as sliding mode approach [3, 19, 20, 31]. It is claimed that this union will lead to new control algorithms that exploit the advantages of both paradigms. In fact, they provide a platform for the design of nonlinear systems with the incorporation of human knowledge. In this work, we propose some methods for the stabilization of unstable periodic orbits of continuous time chaotic systems. For this purpose, we combine the time-delayed feedback controller [30], the input-to-state linearizing controller [14] and fuzzy systems to approximate unknown function of a chaotic system in order to obtain a fuzzy time delay feedback controller. Then, we designed a sliding observer to

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Received: November 2017; Revised: February 2018; Accepted: June 2018.

estimate the unmeasurable states. The efficiency of the proposed methods is finally tested by numerical simulations on Chua's system. This paper is organized as follows: In section 2, the time delay feedback controller is presented. Section 3 deals with fuzzy system. Moreover, the detailed design procedure of the fuzzy time delay feedback controller is explained in Section 4. The observer-based fuzzy time-delayed feedback controller is then investigated in section 5. Numerical simulations are carried out in Section 6 for illustration and verification of the proposed controllers. Finally some concluding remarks are given in Section 7.

Nomenclature

x	state vector in $\mathbb{R}^n$
z	transformed state vector in $\mathbb{R}^n$ with $z = \phi(x)$
$x_p(t)$	the unstable periodic orbit (UPO) in x space
$z_p(t)$	the transformed unstable periodic orbit (UPO) in z space
ρ	relative degree of the system
ε	the error $z(t) - z_p(t)$
ζ	the truncated z vector $(z_1, \ldots, z_\rho)^T$
η	the truncated error vector $(\varepsilon_1, \ldots, \varepsilon_\rho)^T$
K	the gain vector $(k_1, \ldots, k_\rho)^T$
$\xi_F(z)$	vector of fuzzy basis
θ_F	adaptive fuzzy parameter vector
θ_F^*	desired fuzzy parameter vector
Φ_F	$\theta_F^* - \theta_F$

2 Time delay feedback controller

Consider a dynamic system described by the nonlinear differential equation:

$$\begin{aligned} \dot{x}(t) &= f_n(x(t), u(t)) \\ x(t_0) &= x_0 \in \mathbb{R}^n \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}$ is the input signal. When $u(t) = 0$, system (1) behaves chaotically. Therefore, the aim of this work is to design a feedback signal in order to stabilize a predetermined unstable periodic orbit (UPO) of the chaotic system (1). An unstable periodic orbit (UPO) $x_p(t)$ of the chaotic system (1) satisfies:

$$\dot{x}_p(t) = f_n(x_p(t), 0) \tag{2}$$

which corresponds to the system output if it would move along the target UPO. For chaotic systems, this periodic signal can be reconstructed from the chaotic output, using the standard methods for extracting UPO's from chaotic time-series data [32]. The control can be achieved via adjusting the system input by a proportional feedback:

$$u(t) = K^T(x(t) - x_p(t)) \tag{3}$$

where K is the matrix gain. The experimental implementation of this method is difficult since it is not simple to extract an analytic expression $x_p(t)$ of the UPO from the experimental data and only the length of the periodic orbit to be stabilized is known, say T_p. Indeed, Pyragas has proposed a time delayed feedback controller which can be derived from (3) by replacing $x_p(t)$ with the delayed output signal $x(t - T_p)$ [29]:

$$u(t) = K^T(x(t) - x(t - T_p)) \tag{4}$$

We notice that controller (1) vanishes, and the system becomes "autonomous", when the system behaves periodically with period T_p, that is when the periodic orbit is stabilized.

3 Fuzzy system

The knowledge base for the fuzzy logic system comprises a collection of fuzzy IF-THEN rules of the form:

$$\mathbf{R_1^{(j)}} : \text{If } \Big[(x_1 \text{ is } A_1^j)...\text{and}(x_n \text{ is } A_n^j)\Big] \text{ then } \Big[(y_1 \text{ is } B_1^j)...\text{and}(y_m \text{ is } B_m^j)\Big] \tag{5}$$

for $j = 1, \ldots, H$. H is the rule number of the fuzzy logic system. The fuzzy logic system, whose rules are presented in the form of equation (5), performs a mapping from $\mathbf{V} \subset \mathbb{R}^n$ to $\mathbf{W} \subset \mathbb{R}^m$. $\mathbf{V} = V_1 \times \ldots \times V_n$ and $\mathbf{W} = W_1 \times \ldots \times W_m$ where $V_k \subset \mathbb{R}$. $k = 1, \ldots, n$ and $W_i \subset \mathbb{R}$, $i = 1, \ldots, \ldots, m$.
$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbf{V}$ and $\mathbf{y} = [y_1, y_2, \ldots, y_m]^T \in \mathbf{W}$. $x_k,\ k = 1, \ldots, n$ and $y_i,\ i = 1, \ldots, m$ denote the linguistic variables associated with the inputs and the outputs of the fuzzy logic system. A_k^j and B_i^j are labels of fuzzy sets in V and W respectively. By the use of the singleton fuzzification strategy, product inference and center-average defuzzification, the output of the fuzzy system is expressed as:

$$y_i(\mathbf{x}) \quad = \quad \frac{\sum_{j=1}^{H} y_i^j \left(\prod_{k=1}^{n} \mu_{A_k^j}(x_k) \right)}{\sum_{r=1}^{H} \left(\prod_{p=1}^{n} \mu_{A_p^r}(x_p) \right)} \tag{6}$$

where $\mu_{A_k^j}$ is the membership function of the linguistic variable A_i^j.
By introducing the concept of fuzzy basis function vector $\xi(x)$, (6) can be rewritten as:

$$y_i(\mathbf{x}) = \theta_i^T \xi(\mathbf{x}) \tag{7}$$

where

$$\theta_i = \begin{bmatrix} y_i^1 \\ : \\ y_i^H \end{bmatrix} \quad , \quad \xi(\mathbf{x}) = \begin{bmatrix} \xi^1(\mathbf{x}) \\ : \\ \xi^H(\mathbf{x}) \end{bmatrix} \tag{8}$$

and

$$\xi^j(\mathbf{x}) = \frac{\prod_{k=1}^{n} \mu_{A_k^j}(x_k)}{\sum_{r=1}^{H} \left(\prod_{p=1}^{n} \mu_{A_p^r}(x_p) \right)} \tag{9}$$

$\xi^j(\mathbf{x})$ are called fuzzy basis functions (FBF's). It is proved that these FBF's are universal approximators [34].

4 Fuzzy time delay feedback controller

In the sequel, we consider chaotic systems to which the control signal is applied in an affine way that is the controlled system can be expressed as:

$$\dot{x}(t) \quad = \quad f(x(t)) + g(x(t))u \tag{10}$$

To simplify the controller design task, we may choose a state transformation $z = \phi(x)$ such that with the state vector $z(t)$, the chaotic system (10) is described by:

$$\begin{cases} \dot{z}_i = z_{i+1}, & i = 1, ..., \rho - 1 \\ \dot{z}_\rho = F(z) + G(z)u & \\ \dot{z}_i = H_i(z), & i = \rho + 1, ..., n. \end{cases} \tag{11}$$

where F,G and H_i are differentiable functions (see ref. [14]). The control objective is to force $z(t)$ to follow $z_p(t) = \phi(x_p(t))$ such that if $\varepsilon(t) = z(t) - z_p(t)$ then:

$$\lim_{t \to \infty} \varepsilon(t) = 0 \tag{12}$$

Eventually, the problem reduces to the origin stabilization of the error system:

$$\begin{cases} \dot{\varepsilon}_i = \varepsilon_{i+1}, & i = 1, ..., \rho - 1 \\ \dot{\varepsilon}_\rho = F(z) - F(z_p) + G(z)u & \\ \dot{\varepsilon}_i = H_i(z) - H_i(z_p), & i = \rho + 1, ..., n. \end{cases} \tag{13}$$

We assume, here that if $\varepsilon_i = 0$ for $i = 1, \ldots, \rho$, then $\varepsilon_i = 0$ for all $1 \leq i \leq n$. This condition is indeed not restrictive since we can find many chaotic systems that can satisfy it with several functions ϕ, we state for instance Chua's system and the family of Lorenz systems. Should the control law be defined as

$$u_{io} \quad = \quad \frac{1}{G(z)}(-F(z) + F(z_p) - \sum_{i=1}^{\rho} k_i(\varepsilon_i(t) - \varepsilon_i(t - T_p)) \tag{14}$$

then, combining Eqs.(13) and (14) yields:

$$\begin{cases} \dot{\varepsilon}_i = \varepsilon_{i+1}, & i = 1, ..., \rho - 1 \\ \dot{\varepsilon}_\rho = -K^T(\eta(t) - \eta(t - T_p)) & \\ \dot{\varepsilon}_i = H_i(\varepsilon + z_p) - H_i(z_p), & i = \rho + 1, ..., n. \end{cases} \tag{15}$$

where $\eta = (\varepsilon_1, \ldots, \varepsilon_\rho)^T$ and $K = (k_1, \ldots, k_\rho)^T$. We can rewrite the first ρ equations in compact form as follow:

$$\dot{\eta}(t) = A\eta(t) + BK^T\eta(t - T_p) \tag{16}$$

where matrices A and B are a controllable pair expressed as follow:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ -k_1 & -k_2 & -k_3 & \ldots & -k_\rho \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{17}$$

The objective of the controller is achieved if we choose a gain K in appropriate manner such that (16) is stable and $\varepsilon_i, i = \rho+1, ..., n$ is exponentially bounded.
Let's consider the following Lyapunov function candidate [12]

$$V(\eta) = \eta^T P\eta + \int_{t-T_p}^{t} \eta^T R\eta dt \tag{18}$$

where P and R are two positive definite matrices. The time derivative of V along the trajectories of system (16) is:

$$\begin{aligned} \dot{V}(\eta) &= \dot{\eta}(t)^T P\eta(t) + \eta(t)^T P\dot{\eta}(t) + \eta(t)^T R\eta(t) - \eta(t - T_p)^T R\eta(t - T_p) = \eta(t)^T A^T P\eta(t) + \eta(t)^T PA\eta(t) + \eta(t - T_p)^T KB^T P\eta(t) \\ &+ \eta(t)^T PBK^T\eta(t - T_P) + \eta(t)^T R\eta(t) - \eta(t - T_p)^T R\eta = -(R^{\frac{1}{2}}\eta(t - T_p) - R^{-\frac{1}{2}}KB^T P\eta(t))^T \times (R^{\frac{1}{2}}\eta(t - T_p) - R^{-\frac{1}{2}}KB^T P\eta(t)) \\ &+ \eta(t)^T (A^T P + PA + PBK^T R^{-1}KB^T P + R)\eta(t) \end{aligned} \tag{19}$$

Moreover, if we choose a gain K , P and R such that

$$Q = A^T P + PA + PBK^T R^{-1}KB^T P + R < 0 \tag{20}$$

where Q is negative definite and if the error states $\varepsilon_i, i = \rho + 1, ..., n$ are exponentially bounded then the control objective is achieved. At this stage, the construction of the controller is impossible because an analytic expression of z_p is unknown. However, an approximation of z_p can be the T_p delayed state when the evolution of the chaotic system is in the vicinity of the UPO (see ref [12]), thus we may write:

$$F(z_p(t)) = F(z_p(t - T_p)) \simeq F(z(t - T_p)) \tag{21}$$

Accordingly, the control law (14) becomes, where $\zeta = (z_1, \ldots, z_\rho)^T$.

$$u_{io} = \frac{1}{G(z)}(-F(z(t)) + F(z(t - T_p)) - K^T(\zeta(t) - \zeta(t - T_p))) \tag{22}$$

Remark 4.1. *We notice from the expression of controller (22) that its construction requires the knowledge of $G(z)$ and $F(z)$. On the one hand, for most dynamical system $G(z)$ is a nonlinear function, but in the particular case of autonomous chaotic systems such as Chua system and the generalized Lorenz systems, the controller has been injected through a constant function $G(z)$, thus its knowledge is not necessary and can be circumvented by tuning a control parameter [12, 10]. In the sequel, we will consider that $G(z)$ is a constant.*

On the other hand, $F(z)$ represents the autonomous chaotic system model and needs to be known although it may be in general unknown or poorly known. To solve this problem, we will use fuzzy systems to approximate this function. Moreover, an adaptive adjusting law for parameters will be designed. In order to derive the fuzzy time delay feedback control, the fuzzy system $\hat{F}(z/\theta_F)$ is used to approximate $F(z)$ [38]. The fuzzy logic system $\hat{F}(z/\theta_F)$ is expressed by:

$$\hat{F}(z/\theta_F) = \theta_F^T \xi_F(z) \tag{23}$$

where $\xi_F(z)$ is vector of fuzzy basis supposed to be fixed, while the parameters of θ_F are parameters which will be designed by adaptive laws.

Theorem 4.2. *Consider the chaotic system described by (10). Assume that all states are available for measurement. Then the following fuzzy time delay feedback controller stabilizes unstable periodic orbits of a chaotic system, where $\hat{F}(z/\theta_F)$ is defined in equation (23).*

$$u_{fl} = \frac{1}{G(z)}(-\hat{F}(z(t)/\theta_F) + \hat{F}(z(t-T_p)/\theta_F) - K^T(\zeta(t) - \zeta(t-T_p))) \tag{24}$$

Proof. The following proof gives conditions on the choice of $\xi_F(z)$. Let θ_F^* be the desired parameter vectors of fuzzy logic system. Then, by using (24) and (21), we may express (13) as:

$$\begin{aligned}\dot{\varepsilon}_\rho &= F(z(t)) - F(z(t-T_p)) - \hat{F}(z(t)/\theta_F) + \hat{F}(z(t-T_p)/\theta_F) - K^T(\zeta(t) - \zeta(t-T_p)) \\ &= F(z(t)) - \hat{F}(z(t)/\theta_F^*) + \hat{F}(z(t)/\theta_F^*) - \hat{F}(z(t)/\theta_F) - F(z(t-T_p)) \quad + \hat{F}(z(t-T_p)/\theta_F^*) - \hat{F}(z(t-T_p)/\theta_F^*) \\ &+ \hat{F}(z(t-T_p)/\theta_F) - K^T(\zeta(t) - \zeta(t-T_p))\end{aligned} \tag{25}$$

Next, we define the minimum approximation errors as follows:

$$w_F = F(z(t)) - \hat{F}(z(t)/\theta_F^*) \tag{26}$$

and

$$w_{Fp} = F(z(t-T_p)) - \hat{F}(z(t-T_p)/\theta_F^*) \tag{27}$$

Thus, $\dot{\varepsilon}_\rho$ can be expressed by:

$$\dot{\varepsilon}_\rho = \Phi_F^T(\xi_F(z(t)) - \xi_F(z(t-T_p))) + w_F - w_{Fp} - K^T(\zeta(t) - \zeta(t-T_p)) \tag{28}$$

where Φ_F represent the fuzzy parameter errors such that:

$$\Phi_F = \theta_F^* - \theta_F \tag{29}$$

The first ρ equations of the controlled system will then have the following matrix form:

$$\dot{\eta}(t) = A\eta(t) + BK^T\eta(t-T_p) + (\Phi_F^T(\xi_F(z(t)) - \xi_F(z(t-T_p))) + w_F - w_{Fp})B \tag{30}$$

We let $W = \Phi_F^T(\xi_F(z(t)) - \xi_F(z(t-T_p))) + w_F - w_{Fp}$ and we consider the following Lyapunov function which is expressed by:

$$V(\eta, \Phi_F) = \eta^T P\eta + \int_{t-T_p}^{t} \eta^T R\eta dt + \frac{1}{2\delta}\Phi_F^T\Phi_F \tag{31}$$

where $\delta > 0$ is a positive scalar. The derivative of $V(\eta, \Phi_F)$ along the trajectories of (30) is given by:

$$\begin{aligned}\dot{V} &= \dot{\eta}(t)^T P\eta(t) + \eta(t)^T P\dot{\eta}(t) + \eta(t)^T R\eta(t) - \eta(t-T_p)^T R\eta(t-T_p) + \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F = (\eta(t)^T A^T P\eta(t) + \eta(t-T_p)^T KB^T P\eta(t) \\ &+ WB^T P\eta(t)) + \eta(t)^T PA\eta(t) + \eta(t)^T PBK^T\eta(t-T_P) + W\eta(t)^T PB + \eta(t)^T R\eta(t) - \eta(t-T_p)^T R\eta(t-T_p) + \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F \\ &= -(R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t))^T \times (R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t)) + \eta(t)^T(A^T P + PA + PBK^T R^{-1}KB^T P + R)\eta(t) \\ &+ W(B^T P\eta(t) + \eta(t)^T PB) + \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F = -(R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t))^T \times (R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t)) \\ &+ \eta(t)^T(A^T P + PA + PBK^T R^{-1}KB^T P + R)\eta(t) + (w_F - w_{Fp})(B^T P\eta(t) + \eta(t)^T PB) + \Phi_F^T(\xi_F(z(t)) \\ &- \xi_F(z(t-T_p)))(B^T P\eta(t) + \eta(t)^T PB) + \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F = -(R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t))^T \times (R^{\frac{1}{2}}\eta(t-T_p) - R^{-\frac{1}{2}}KB^T P\eta(t)) \\ &+ \eta(t)^T(A^T P + PA + PBK^T R^{-1}KB^T P + R)\eta(t) + (w_F - w_{Fp})(B^T P\eta(t) + \eta(t)^T PB) \\ &+ \Phi_F^T[(\xi_F(z(t)) - \xi_F(z(t-T_p)))(B^T P\eta(t) + \eta(t)^T PB) - \frac{1}{\delta}\dot{\theta}_F]\end{aligned} \tag{32}$$

since $\dot{\Phi}_F = -\dot{\theta}_F$. Clearly, the terms w_F and w_{Fp} are of the order of the minimum approximation error and from the Universal Approximation Theorem, Wang expected that w_F and w_{Fp} should be very small if not equal to zero in the adaptive fuzzy system [34].Therefore, we can choose the adaptive law such that:

$$\dot{\theta}_F = \delta(\xi_F(z(t)) - \xi_F(z(t-T_p)))(B^T P\eta(t) + \eta(t)^T PB) = \delta(\xi_F(z(t)) - \xi_F(z(t-T_p)))(B^T P(\zeta(t) - \zeta_p(t)) + (\zeta(t) - \zeta_p(t))^T PB) \tag{33}$$

where $\zeta_p = (z_{p1}, \ldots, z_{p\rho})^T$. Thus, the adaptation law of θ_F depends on $z_p(t)$, then it can not be constructed. To solve this problem, we again consider an approximation of z_p when the chaotic system is behaving in the vicinity of the UPO. Accordingly, we can substitute the adaptation law expression (33) by the following:

$$\dot{\theta}_F = \delta(\xi_F(z(t)) - \xi_F(z(t-T_p)))(B^T P(\zeta(t) - \zeta(t-T_p)) + (\zeta(t) - \zeta(t-T_p))^T PB)$$

(34)

If moreover we can choose P, R and K such that Q in (20) is negative definite, then we can deduce that $\eta = 0$ is a stable equilibrium of (30) which implies that $\eta(t) \in \mathbf{L}_\infty$. From (30) and by the choice (33), we obviously have $\dot{\eta}(t) \in \mathbf{L}_\infty$. Hence, $\| \eta(t) \|$ is a continuous function. Besides, we have

$$\dot{V}(\eta, \Phi_F) \leq -\lambda_{min}(Q) \| \eta(t) \|^2 \tag{35}$$

where $\lambda_{min}(Q)$ is the smallest eigenvalue of Q. Thus, by using Barbalat's lemma, the origin $\eta = 0$ of system (30) is asymptotically stable. □

5 Observer-based fuzzy time-delayed linearizing feedback controller

In order to implement the controllers proposed previously, the first ρ components of the state vector z must first be measurable. This motivates the idea of using a state observer to estimate the unmeasurable states. Besides, the observer must be designed without using the system model that we avoided in the design of the controller along the previous sections. Thus, we assume that the only measurable state is z_1 and we suggest to use a reduced order sliding mode observer described by the following equations:

$$\begin{array}{rcl} \dot{\hat{z}}_\rho & = & \varphi + w_\rho \operatorname{sign}(z_1 - \hat{z}_1) \\ \dot{\hat{z}}_i & = & \hat{z}_{i+1} + w_i \operatorname{sign}(z_1 - \hat{z}_1), \qquad i = 1, \ldots, \rho - 1 \end{array} \tag{36}$$

where φ will be determined in the sequel. The observation error $o = z - \hat{z}$ is given by:

$$\begin{array}{rcl} \dot{o}_\rho & = & F(z) + G(z)u - \varphi - w_\rho \operatorname{sign}(o_1) \\ \dot{o}_i & = & o_{i+1} - w_i \operatorname{sign}(o_1), \qquad i = 1, \ldots, \rho - 1 \end{array} \tag{37}$$

If we choose $\sigma_1 : o_1 = 0$ as a sliding surface then by considering the Lyapunov function $V_1 = \frac{1}{2} o_1^2$, the derivative is given by $\dot{V}_1 = o_1 o_2 - \omega_1 o_1 \operatorname{sign}(o_1) < 0$. Thus, if $\omega_1 > |o_2|_{\max}$ where $|o_2|_{\max} = \max_{t \geq 0} |o_2(t)|$ it follows that σ_1 is attractive. In addition, with $\omega_1 > |o_2|_{\max}$ and $\delta_1 = \omega_1 - o_2 \operatorname{sign}(o_1) > 0$ we have $\dot{o_1} = -\delta_1 \operatorname{sign}(o_1)$, which solution is given by:

$$o_1(t) = \left\{ \begin{array}{l} -\delta_1 t + o_1(0) \quad \text{if} \quad o_1(0) > 0 \\ \delta_1 t + o_1(0) \quad \text{if} \quad o_1(0) < 0 \end{array} \right. \tag{38}$$

That is we can deduce that

$$o_1(t_r) = 0 \Longleftrightarrow t_r = \frac{|o_1(0)|}{\delta_1} \tag{39}$$

Hence for $\omega_1 > |o_2|_{\max}$, $o_1(t) = 0$ for all $t > t_r$. In addition, since $o_1(t)$ is constant for all $t > t_r$ it results $\dot{o}_1 = 0$ for all $t > t_r$. Hence, we can determine the dynamics of the observation error on the sliding surface using the concept of Filipov [11]. The dynamics are determined using the invariance of the sliding surface σ_1, that is $o_1 = 0$ and $\dot{o}_1 = 0$. On the sliding surface $\overline{\operatorname{sign}}(o_1) = \frac{o_2}{\omega_1}$ where $\overline{\operatorname{sign}}(o_1)$ is the mean value of the variation of $\operatorname{sign}(o_1)$ about the sliding surface σ_1. Thus, on the sliding surface, the observation error dynamics are governed by:

$$\begin{array}{rcl} \dot{o}_\rho & = & F(z) + G(z)u - \varphi - \dfrac{\omega_\rho}{\omega_1} o_2 \\ \dot{o}_i & = & o_{i+1} - \dfrac{\omega_i}{\omega_1} o_2, i = 1, ..., \rho - 1 \end{array} \tag{40}$$

We redefine the tracking error with the estimated states as $\hat{\varepsilon} = \hat{z} - z_p$. Now, by applying the estimated state based controller, we obtain the augmented system using the tracking and the observation error vector that we define on $\mathbb{R}^{2\rho-1}$ as follows (ρ tracking errors and $\rho - 1$ estimation errors):

$$\varepsilon = \left[\begin{array}{c} \overline{\varepsilon} \\ \underline{\varepsilon} \end{array} \right] = \left[\begin{array}{c} \hat{z}_1 - z_{p_1} \\ \hat{z}_2 - z_{p_2} \\ \vdots \\ \hat{z}_\rho - z_{p_\rho} \\ z_2 - \hat{z}_2 \\ z_3 - \hat{z}_3 \\ \vdots \\ z_\rho - \hat{z}_\rho \end{array} \right] = \left[\begin{array}{c} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \vdots \\ \hat{\varepsilon}_\rho \\ o_2 \\ o_3 \\ \vdots \\ o_\rho \end{array} \right] \tag{41}$$

where $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\rho)^T$ is the tracking error and $\underline{\varepsilon} = (\varepsilon_{\rho+1}, \ldots, \varepsilon_{2\rho-1})^T$ is the observation error of the missing states only. The augmented error dynamics are given by:

$$\begin{aligned}
\dot{\varepsilon}_i &= \varepsilon_{i+1} + \frac{\omega_i}{\omega_1}\varepsilon_{\rho+1}, \quad 1 \leq i \leq \rho-1 && (1)\\
\dot{\varepsilon}_\rho &= \varphi + \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} - \dot{z}_{p_\rho} \\
\dot{\varepsilon}_j &= \varepsilon_{j+1} - \frac{\omega_{j-\rho+1}}{\omega_1}\varepsilon_{\rho+1}, \quad \rho+1 \leq j \leq 2(\rho-1) && (2)\\
\dot{\varepsilon}_{2\rho-1} &= F(z) + G(z)u - \varphi - \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} && (42)
\end{aligned}$$

Theorem 5.1. *Consider the chaotic system described by (10). Assume that only the first state z_1 is available for measurement. Then the following Observer-based fuzzy time delay feedback controller stabilizes unstable periodic orbits of a chaotic system:*

$$\hat{u}_{fl} = \frac{1}{G(z)}(-\hat{F}(\hat{z}(t)/\theta_F) + \hat{F}(\hat{z}(t-T_p)/\theta_F) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p))) \tag{43}$$

where $\hat{F}(z/\theta_F)$ and $\overline{\varepsilon}$ are defined respectively in equation (23) and (41).

Proof. Let's first define $\hat{u}_{fl}$ as:

$$\hat{u}_{fl} = \frac{1}{G(z)}(-\hat{F}(\hat{z}/\theta_F) + \hat{F}(\hat{z}_p/\theta_F) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p))) \tag{44}$$

where $\hat{F}(\hat{z}/\theta_F)$ is given by the equation (23). If moreover we choose:

$$\varphi = \hat{F}(\hat{z}_p/\theta_F) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) \tag{45}$$

then knowing that $\dot{z}_{p_\rho} = F(z_p)$, so equation (42) becomes

$$\dot{\varepsilon}_\rho = \hat{F}(\hat{z}_p/\theta_F) - F(z_p) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) + \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} \tag{46}$$

Let θ_F^* the optimal parameter vectors of fuzzy logic systems then:

$$\begin{aligned}
\dot{\varepsilon}_\rho &= \hat{F}(\hat{z}_p/\theta_F) - \hat{F}(\hat{z}_p/\theta_F^*) \quad + \hat{F}(\hat{z}_p/\theta_F^*) - F(z_p) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) + \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} \\
&= -\phi_F^T\xi(\hat{z}_p) - w_{F_p} - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) + \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} = -W_{F_p} - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) + \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1}
\end{aligned} \tag{47}$$

where ϕ_F, w_{F_p} are given respectively by (29), (27) and $W_{F_p} = \phi_F^T\xi(\hat{z}_p) + w_{F_p}$. Moreover, applying the observer based fuzzy controller to (42), yields to:

$$\dot{\varepsilon}_{2\rho-1} = F(z) + G(z)u - \varphi - \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} = F(z) - \hat{F}(\hat{z}/\theta_F) - \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} = \phi_F^T\xi(\hat{z}) + w_F - \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} = W_F - \frac{\omega_\rho}{\omega_1}\varepsilon_{\rho+1} \tag{48}$$

where ϕ_F, w_F are given respectively by (29), (26) and $W_F = \phi_F^T\xi(\hat{z}) + w_F$. Thus, the application of $\hat{u}_{fl}$ yields to the controlled system:

$$\dot{\varepsilon}(t) = \hat{A}\varepsilon(t) + \hat{B}K^T\overline{\varepsilon}(t-T_p) - W_{F_p}\hat{B} + W_F\hat{D} \tag{49}$$

where $\hat{A}$, $\hat{B}$ and $\hat{D}$ are given by:

$$\hat{A} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 & -\frac{\omega_2}{\omega_1} & 0 & \ldots & 0 \\
 & & & \ddots & & \vdots & & & \\
 & & & 0 & 1 & -\frac{\omega_{\rho-1}}{\omega_1} & & & \\
-k_1 & -k_2 & \ldots & -k_{\rho-1} & -k_\rho & -\frac{\omega_\rho}{\omega_1} & 0 & \ldots & 0 \\
0 & & \ldots & \ldots & 0 & -\frac{\omega_2}{\omega_1} & 1 & & \\
 & & & & \vdots & & & \ddots & \\
 & & & & & -\frac{\omega_{\rho-1}}{\omega_1} & & & 1 \\
0 & & \ldots & \ldots & 0 & -\frac{\omega_\rho}{\omega_1} & 0 & \ldots & 0
\end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}^T$$

$$\hat{D} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}^T \tag{50}$$

$\hat{B}$ and $\hat{D}$ are respectively ρ^{th} and $(2\rho-1)^{th}$ canonical vectors. We notice that $\hat{A}$ can be partitioned in the following form:

$$\hat{A} = \begin{bmatrix} A & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \tag{51}$$

where A is in controllable canonical form and $\hat{A}_{22}$ is in the observable canonical form. Clearly, the eigenvalues of $\hat{A}$ are the eigenvalues of A and those of $\hat{A}_{22}$. Since A is Hurwitz stable, thus, we need to choose ω_i, $i = 2, \ldots, \rho$ such that the following characteristic equation is Hurwitz stable

$$s^{\rho-1} + \frac{\omega_2}{\omega_1}s^{\rho-2} + \ldots + \frac{\omega_{\rho-1}}{\omega_1}s + \frac{\omega_\rho}{\omega_1} = 0 \tag{52}$$

We consider the following Lyapunov function which is expressed by:

$$V(\varepsilon, \Phi_F) = \varepsilon^T \hat{P}\varepsilon + \int_{t-T_p}^{t} \bar{\varepsilon}^T \hat{R}\bar{\varepsilon}dt + \frac{1}{2\delta}\Phi_F^T\Phi_F \tag{53}$$

where $\delta > 0$ is a positive scalar. The derivative of $V(\varepsilon, \Phi_F)$ along the trajectories of (49) is given by:

$$\begin{aligned}
&\dot{V}(\varepsilon) = -(\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}KB^TP\varepsilon(t))^T \times (\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}K\hat{B}^T\hat{P}\varepsilon(t)) + \varepsilon(t)^T(\hat{A}^T\hat{P} + \hat{P}\hat{A} + \hat{P}\hat{B}K^T\hat{\hat{R}}^{-1}K\hat{B}^T\hat{P} + \hat{\hat{R}})\varepsilon(t) \\
&+ \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F + (-W_{F_p}B^T + W_FD^T)\hat{P}\varepsilon + \varepsilon^T\hat{P}(-W_{F_p}B + W_FD) = -(\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}KB^TP\varepsilon(t))^T \times (\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) \\
&- \hat{\hat{R}}^{-\frac{1}{2}}K\hat{B}^T\hat{P}\varepsilon(t)) + \varepsilon(t)^T(\hat{A}^T\hat{P} + \hat{P}\hat{A} + \hat{P}\hat{B}K^T\hat{\hat{R}}^{-1}K\hat{B}^T\hat{P} + \hat{\hat{R}})\varepsilon(t) + \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F - W_{F_p}(B^TP\varepsilon + \varepsilon^TPB) + W_F(D^TP\varepsilon + \varepsilon^TPD) \\
&= -(\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}KB^TP\varepsilon(t))^T \times (\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}K\hat{B}^T\hat{P}\varepsilon(t)) + \varepsilon(t)^T(\hat{A}^T\hat{P} + \hat{P}\hat{A} + \hat{P}\hat{B}K^T\hat{\hat{R}}^{-1}K\hat{B}^T\hat{P} + \hat{\hat{R}})\varepsilon(t) \\
&+ \frac{1}{\delta}\Phi_F^T\dot{\Phi}_F - (\phi_F^T\xi(\hat{z}_p) + w_{F_p})(B^TP\varepsilon + \varepsilon^TPB) + (\phi_F^T\xi(\hat{z}) + w_F)(D^TP\varepsilon + \varepsilon^TPD) \\
&= -(\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}KB^TP\varepsilon(t))^T \times (\hat{R}^{\frac{1}{2}}\bar{\varepsilon}(t-T_p) - \hat{\hat{R}}^{-\frac{1}{2}}K\hat{B}^T\hat{P}\varepsilon(t)) + \varepsilon(t)^T(\hat{A}^T\hat{P} + \hat{P}\hat{A} + \hat{P}\hat{B}K^T\hat{\hat{R}}^{-1}K\hat{B}^T\hat{P} + \hat{\hat{R}})\varepsilon(t) \\
&\quad + \Phi_F^T((\xi(\hat{z})(D^TP\varepsilon + \varepsilon^TPD) - \xi(\hat{z}_p)(B^TP\varepsilon + \varepsilon^TPB)) - \frac{1}{\delta}\dot{\theta}_F) + w_F(D^TP\varepsilon + \varepsilon^TPD) - w_{F_p}(B^TP\varepsilon + \varepsilon^TPB)
\end{aligned} \tag{54}$$

where $\hat{\hat{R}} \in \mathbb{R}^{(2\rho-1)\times(2\rho-1)}$ and it is given by:

$$\hat{\hat{R}} = \begin{bmatrix} \hat{R} & 0 \\ 0 & 0 \end{bmatrix} \tag{55}$$

Moreover, w_F and w_{Fp} are very small if not equal to zero in the adaptive fuzzy system, then we can choose the adaptive law such that:

$$\dot{\theta}_F = \delta(\xi(\hat{z})(D^TP\varepsilon(t) + \varepsilon(t)^TPD) - \xi(\hat{z}_p)(B^TP\varepsilon(t) + \varepsilon(t)^TPB)) = \delta[(\xi(\hat{z})D^TP - \xi(\hat{z}_p)B^TP)\varepsilon(t) + \varepsilon(t)^T(\xi(\hat{z})PD - \xi(\hat{z}_p)PB)] \tag{56}$$

Thus, the adaptation law of θ_F depends on $\hat{z}_p(t)$, then it can not be constructed. To solve this problem, we again consider an approximation of z_p when the chaotic system is behaving in the vicinity of the UPO, as that used in equation (21). Accordingly, we can substitute the expression in (56) by the following:

$$\begin{aligned}
\dot{\theta}_F = \delta[(\xi(\hat{z})D^TP - \xi(\hat{z}_p)B^TP)\varepsilon(t) + \varepsilon(t)^T(\xi(\hat{z})PD - \xi(\hat{z}_p)PB)] = \delta[(\xi(\hat{z}(t))D^TP - \xi(\hat{z}(t-T_p))B^TP)\varepsilon(t) + \varepsilon(t)^T(\xi(\hat{z}(t))PD \\
- \xi(\hat{z}(t-T_p))PB)]
\end{aligned} \tag{57}$$

and if we can choose $\hat{P}$, $\hat{\hat{R}}$ and K such that

$$\hat{Q} = \hat{A}^T\hat{P} + \hat{P}\hat{A} + \hat{P}\hat{B}K^T\hat{\hat{R}}^{-1}K\hat{B}^T\hat{P} + \hat{\hat{R}} < 0 \tag{58}$$

is negative definite, then we obtain

$$\dot{V}(\varepsilon, \Phi_F) \leq -\lambda_{min}(\hat{Q}) \parallel \varepsilon(t) \parallel^2 \tag{59}$$

Eventually, we may deduce the asymptotic stability of system (49).
Once again, by using the approximation of z_p when the chaotic system is behaving in the vicinity of the UPO, the control expression becomes

$$\hat{u}_{fl} = \frac{1}{G(z)}(-\hat{F}(\hat{z}(t)/\theta_F) + \hat{F}(\hat{z}(t-T_p)/\theta_F) - K^T(\bar{\varepsilon}(t) - \bar{\varepsilon}(t-T_p))) \tag{60}$$

and

$$\varphi \quad = \quad \hat{F}(\hat{z}(t-T_p)/\theta_F) - K^T(\overline{\varepsilon}(t) - \overline{\varepsilon}(t-T_p)) \tag{61}$$

Unfortunately, the adaptation law (57) is not realizable because it uses the vector $\varepsilon(t)$ and then the unmeasurable vector $z(t)$. Thus, the adaptation law (57) requires some modifications:

$$\dot{\theta}_F = \delta[(\xi(\hat{z}(t))D^TP - \xi(\hat{z}(t-T_p))B^TP))\varepsilon(t) + \varepsilon(t)^T(\delta(\xi(\hat{z}(t))PD - \xi(\hat{z}(t-T_p))PB)] = \Gamma\varepsilon(t) + \varepsilon(t)^T\Upsilon \tag{62}$$

where $\Gamma = [\overline{\Gamma}, \underline{\Gamma}]$ and $\Upsilon = [\overline{\Upsilon}, \underline{\Upsilon}]^T$ such that $\overline{\Gamma}$ and $\overline{\Upsilon}$ are the ρ first columns of Γ and Υ respectively, thus

$$\dot{\theta}_F = \overline{\Gamma}\overline{\varepsilon}(t) + \underline{\Gamma}\ \underline{\varepsilon}(t) + \overline{\varepsilon}(t)^T\overline{\Upsilon} + \underline{\varepsilon}(t)^T\underline{\Upsilon} = (\overline{\Gamma}\overline{\varepsilon}(t) + \overline{\varepsilon}(t)^T\overline{\Upsilon}) + (\underline{\Gamma}\ \underline{\varepsilon}(t) + \underline{\varepsilon}(t)^T\underline{\Upsilon}) \tag{63}$$

If the system is behaving in the vicinity of the UPO then the first term of expression (63) can be approximated by:

$$(\overline{\Gamma}\overline{\varepsilon}(t) + \overline{\varepsilon}(t)^T\overline{\Upsilon}) = \overline{\Gamma}(\hat{\zeta}(t) - \hat{\zeta}(t-T_p)) + (\hat{\zeta}(t) - \hat{\zeta}(t-T_p))^T\overline{\Upsilon} \tag{64}$$

where $\hat{\zeta} = [\hat{z}_1, \ldots, \hat{z}_\rho]^T$. We notice here that the observer at the instant t requires the knowledge of the T_p delayed state vector $\hat{\zeta}(t-T_p)$. Thus, the observer should be initiated before the application of the controller while the chaotic system is approaching the UPO. Moreover, the convergence of the observer in T_p time, reduces to zero the observation error $\underline{\varepsilon}$ and consequently, the adaptive gain expression becomes:

$$\dot{\theta}_F \quad = \quad \overline{\Gamma}(\hat{\zeta}(t) - \hat{\zeta}(t-T_p)) + (\hat{\zeta}(t) - \hat{\zeta}(t-T_p))^T\overline{\Upsilon} \tag{65}$$

6 Simulation results and analysis

In this section, we present simulation results of stabilizing UPO's of a chaotic system using the proposed controllers. First, we notice that the controller cannot be applied unless the chaotic system behaves in the vicinity of the UPO, and this is not a very restrictive condition because chaotic systems are ergodic [12]. Subsequently, the length of the UPO (T_p) can be determined by a spectral analysis and it's value can be adjusted by minimizing recursively a cost function [8]:

$$J = \frac{1}{N}\sum_{n=0}^{N-1} \|x(t-nh) - x(t-T_p-nh)\|^2 \tag{66}$$

This cost function represents the mean square error of N points distanced by a step h on the last period and the counterpart points on the preceding period. The gradient is given by:

$$\frac{\partial J}{\partial T_p} = \frac{2}{N}\sum_{n=0}^{N-1}(x(t-nh) - x(t-T_p-nh))^T\dot{x}(t-T_p-nh) \tag{67}$$

If the system verifies

$$\|x(t) - x(t-T_p)\| \leq \varepsilon \tag{68}$$

where ε is a precision measure, then we may apply the controller and meanwhile we update T_p once each period length such that:

$$T_p(i+1) = T_p(i) - \mu\frac{\partial J}{\partial T_p(i)}, \quad \mu > 0 \tag{69}$$

with μ being an adaptation constant. When the UPO is stabilized then $J = 0$ and T_p becomes constant. For the observer based controller, we will use the same expressions calculated with the estimated state vector. To illustrate the efficiency of our proposed controlling schemes, we consider Chua's system:

$$\frac{dx_1}{dt} = \alpha x_2 - \alpha x_1 - \alpha\aleph(x_1)\ \frac{dx_2}{dt} = x_1 - x_2 + x_3\ \frac{dx_3}{dt} = -\beta x_2 + \beta u \tag{70}$$

where $\aleph(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|)$. The state variables x_1 and x_2 represent the voltages across the two capacitors, x_3 is the current through the inductor and u represents a voltage source in series with the inductor. When we use the following values of the system parameters $(\alpha, \beta, a, b) = (10, 18, -\frac{90}{60}, -\frac{44}{60})$, Chua's system behaves chaotically. The dominating frequency that we can observe when $|x_1| > 1$ can be calculated analytically by solving for the eigenvalues of the piecewise linear system and we have $\omega_0 = \sqrt{431}/6$. Hence, the length of the UPO's that we will stabilize are almost integer multiple of $T_0 = 2\pi/\omega_0$. Moreover, using the following state transformation:

$$\phi(x) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 + x_3 \\ x_1 \end{bmatrix}$$

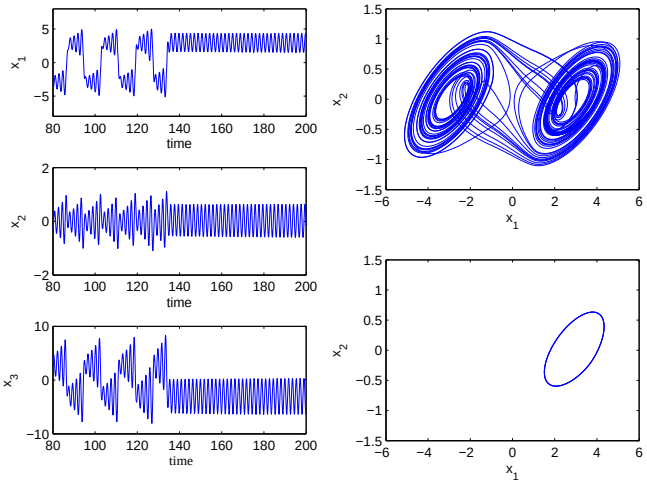


Figure 1: Stabilization of a UPO of length T_0 of the Chua's system using fuzzy time delay feedback controller.

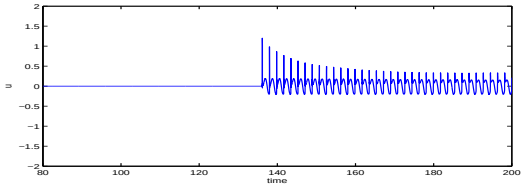


Figure 2: Representation of the fuzzy time delay feedback controller

(71)

leads to have $\rho = 2$ and $F(z) = \alpha(z_1 - z_3 - \aleph(z_3)) - z_2 - \beta z_1$ and $G(z) = \beta$.
The fuzzy time delay feedback controller is constructed using $K = [25, 10]^T$. Moreover, we consider five fuzzy levels, i.e. NB, NS, EZ, PS and PB on z_1 and z_2 to estimate $F(z)$. We use the fuzzy logic systems with center-average defuzzifier, product inference, singleton fuzzifier and Gaussian membership functions to approximate $F(z)$. To adjust the UPO length, we use $N = 20$, $h = 0.2$ and $\mu = 0.002$. Figure 1 illustrates the stabilization of the UPO of length $T_p \simeq T_0$. Indeed, we represent the time signals, the UPO alone and within the attractor. Figure 2 represents the control signal which is acceptable and doesn't exceed the limits of the applicable control ranges. To construct an observer-based fuzzy time-delayed feedback controller, we use a reduced order sliding observer with $w_1 = 50$ and $w_2 = 1000$. Similar parameters of the fuzzy time delay feedback controller are used to designed the observer-based fuzzy time-delayed feedback controller. Figure 3 shows the stabilization of several UPO's from length $T_p \simeq T_0$ to $T_p \simeq 9T_0$. Figure 4 represents the stabilization of the UPO of length $T_p \simeq T_0$. Figure 5 illustrates the evolution of the observer states and the observer-based fuzzy time-delayed feedback control signal. It is clear that the different proposed schemes achieved stabilization of different UPO's of Chua's system. The observer based controller achieves two main advantages. The first one is not using the system model and the second one is not using the unavailable system states. In [12], authors proposed an observer based adaptive time-delayed controller to stabilize the UPO's of chua's system. They used a sliding mode observer with an observer gain $w = (50, 25000)^T$. Figure 6 represents the observation error $o_2(t) = z_2(t) - \hat{z}_2(t)$ obtained with the proposed observer and the observer based adaptive time-delayed controller. Clearly, the observation error obtained with the proposed method is lower than that obtained with the observer based adaptive time-delayed controller despite the use of a high value of observer gain. Furthermore, when compared to several works in literature, our method outperforms several others such as those proposed in [15, 24] which are either not stabilizing the UPO hence not considering the chaotic characteristic of the system or not stabilizing the intrinsic UPO's of the chaotic system and hence the UPO analytic expression is considered to be known which is not always the case. Besides, the control signals obtained by their methods have relatively high level that is energy consuming whereas our method proposes low level control signals.

7 Conclusions

In this paper, we proposed two controllers to stabilize unstable periodic orbits of a chaotic system. These controllers inherit the advantages of both time-delayed feedback controller and fuzzy techniques. Thus, we firstly proposed fuzzy time delay feedback controller where we combine the time-delayed feedback controller, the input-to-state linearizing controller and fuzzy systems to

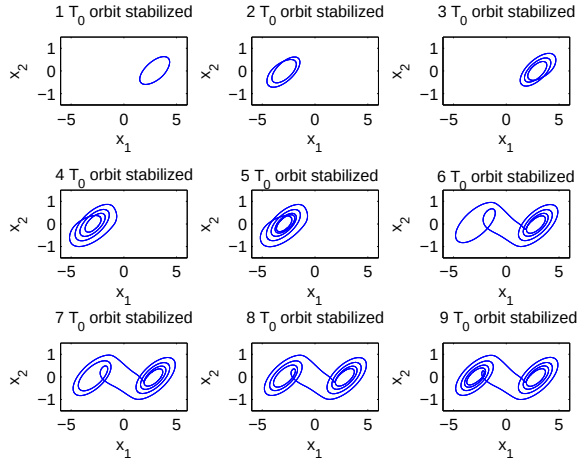


Figure 3: Stabilization of different UPO's of the Chua's system using observer-based fuzzy time delay feedback controller.

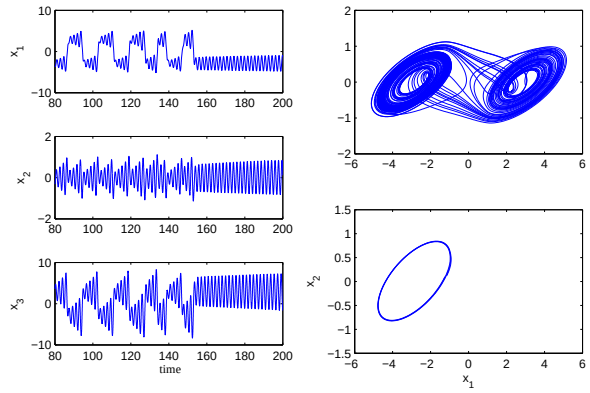


Figure 4: Stabilization of a UPO of length T_0 of the Chua's system using observer-based fuzzy time delay feedback controller.

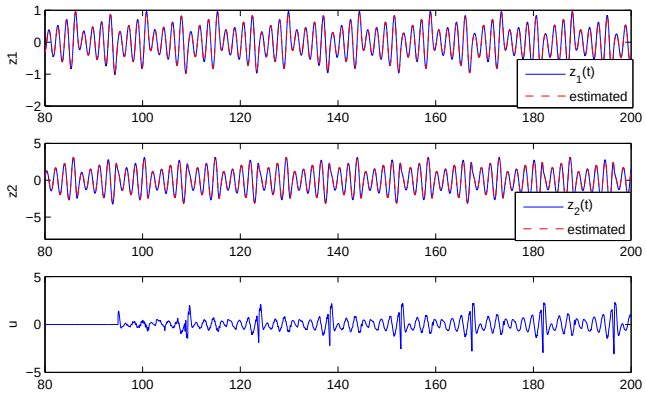


Figure 5: Representation of the observer states and the observer-based fuzzy time-delayed feedback control signal

approximate unknown function of a chaotic system. Finally, we designed a sliding observer to estimate a state vector. This observer was applied to the previous controller. The obtained controller namely the observer-based fuzzy time-delayed feedback

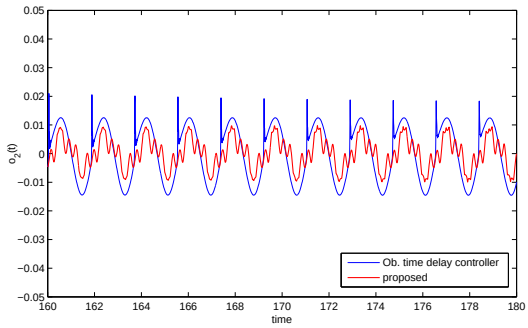


Figure 6: Representation of the observation error $o_2(t) = z_2(t) - \hat{z}_2(t)$ obtained with the proposed control and the observer based adaptive time-delayed controller.

controller has the advantage of stabilizing the different UPO's without the requirement that the state vector is measurable. Simulation results on Chua's system showed the applicability and the effectiveness of the proposed controllers.

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