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Exponential membership function and duality gaps for I-fuzzy linear programming problems

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Abstract

Fuzziness is ever presented in real life decision making problems. In this paper, we adapt the pessimistic approach to study a pair of linear primal-dual problem under intuitionistic fuzzy (I-fuzzy) environment and prove certain duality results. We generate the duality results using exponential membership and non-membership functions to represent the decision maker's satisfaction and dissatisfaction level. Further, two numerical examples have been given. In each of these illustrations, varying the values of the shape variables in the exponential membership functions, various nonlinear optimization problems have been constructed, analyzed and solved. The duality gaps for all these optimization problems have been computed and compared with the duality gap under the linear membership function. We found that these gaps for the I-fuzzy linear primal-dual pair under the exponential membership functions are smaller as compared with the linear membership functions. The main advantage of using the exponential membership function is that it has the flexibility of altering the values of the shape parameters (as per the decision-maker's satisfaction). Finally, with the help of a suitable ranking defuzzification function, we have extended our approach to the I-fuzzy linear problem with fuzzy parameters and fuzzy constraints.

Keywords: Fuzzy optimization, Intuitionistic fuzzy set (I-fuzzy set), Primal-dual problems, Fuzzy duality theorems, Duality gaps.

1 Introduction

Linear programming (LP) is the classical tool for optimal allocation of scarce resources, among a number of competitive activities. Since their introduction [15], it has been intensively studied by many researchers and applied to solve real world problems such as shortest path, network flow, matching, assignment, transportation, blending petroleum products etc. In traditional form, to optimize a linear objective function under some linear conditions, we take the help of LP problem along with the assumption that all the input parameters are completely accurate or exactly determined. However, in existing LP problem, the input data or parameters such as resources, cost and objective function are characterized by imprecise/fuzzy due to incomplete or unobtainable information. Fuzzy set theory [26], is one of the extensively used tool to model such kind of uncertainty in decision making problems. Zimmermann [27] was the first who introduced the theory of fuzzy sets in LP problem. Following the Bellman-Zadeh's fuzzy decision algorithm [10], Zimmermann represented the goal of objective and the constraints by fuzzy sets and then aggregated them in order to derive the maximizing decision. Several papers dealing with the optimal solutions of fuzzy LP problems appeared in [20]-[22]. Duality theory plays an important role in LP problem. In duality theory, we establish relationship between the original LP problem (primal) and the LP problem derived from original problem by taking transpose of rows and columns (dual). It does also provide deep insight to decision makers in taking alternative course of action and their relative values. Rodder and Zimmermann [19] first studied the duality concept in fuzzy LP. Indicating the inherent difficulties in the formulation of primal and dual in [19], Bector and Chandra [6] created a pair of primal and dual LP

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problems based on the aspiration level approach [27] and acquired certain duality relations taking the linear membership function. Later on, Gupta and Mehlawat [13] considered a pair of fuzzy LP problem and established duality relations under exponential membership function.

In the mean time, the theory of fuzzy sets has been extensively developed and enriched with diverse modifications and generalizations of the core concept. Among those fuzzy sets, intuitionistic fuzzy set (I-fuzzy set) is the more generalized one. The concept of I-fuzzy sets was introduced by Atanassov [4] assuming that each element in the set must be characterized by membership and non-membership function, which can take value between 0 and 1 with sum of degree not exceeding 1. Many researchers like [5, 11, 16, 23, 25] worked in this direction and proved that the approach using I-fuzzy sets are much more useful in comparison with ordinary fuzzy sets. With the concept of degrees of acceptance of objectives and constraints along with their degrees of rejection, Angelov [3] was the first to make an effort to use I-fuzzy set in optimization problems. By following the idea of maximizing the degree of acceptance of I-fuzzy objective(s) and constraints and minimizing the degree of rejection of I-fuzzy objective(s) and constraints, Angelov [3] constructed an I-fuzzy optimization model. This model can be further converted into equivalent crisp optimization problems using extension principle [10] and the I-fuzzy aggregation operators. Following the approach discussed in [3], Aggarwal et al. [2] extended the concept of fuzzy LP problem [6] to the I-fuzzy scenario. To its generalization, in this paper, we have developed the duality relations for the same problem considered in Aggarwal et al. [2], under the exponential membership function. Based on the aspiration levels of the players as fuzzy numbers in two person zero sum matrix games with fuzzy goals and fuzzy payoffs, a new solution concept has been proposed by Aggarwal et al. [1] in the I-fuzzy scenario. Recently, Ramik and Vlach [18] proposed a level set approach for the I-fuzzy linear optimization problems and proved weak and strong duality results in the I-fuzzy sense.

In this paper, we follow the concept of pessimistic approach for studying a pair of primal-dual LP problem with exponential membership and non-membership functions under I-fuzzy environment. In Section 2, we define some terminologies and concepts which shall be using throughout the paper. Section 3 introduces I-fuzzy linear primal-dual LP problem and exponential membership and non-membership functions to indicate the decision maker's vague aspiration level and proved suitable duality relations. Changing the shape parameters involved in the exponential membership function, in accordance with the decision maker's acceptance, several non-linear problems are obtained, which after solving, leads us to evaluate the duality gaps. In the Section 4, we have considered two examples. At first, taking the example 1 in [2], we have calculated the duality gaps with the exponential membership function and obtained lesser gaps when compared with the linear membership function [2]. In the same section, we have constructed another non-trivial example based on the real life. Using the exponential and the linear membership function, the I-fuzzy linear problem is converted to the corresponding two different optimization problems-one involves the shape parameters, and the other does not. In order to discuss the duality gaps, the duals to the problems are formulated. Eventually, the duality gaps are calculated for the two sets of primal-dual pairs, which when compared, again results in decrease of the gaps for the problem obtained using exponential function unlike the one obtained under linear membership function. Further, using a suitable linear ranking defuzzification function, we have defined the exponential membership and non-membership functions. Using these membership functions, we have converted the I-fuzzy pair to the crisp primal-dual pair, and hence extended our approach to the I-fuzzy LP problems with fuzzy parameters and fuzzy constraints in Section 5. Finally some concluding note and future directions in the last section.

2 Preliminaries

2.1 Basic Definitions

Definition 2.1. [28] *Let X be a universal set. The set $\tilde{A}$ is called a* **fuzzy set** *in X if $\tilde{A}$ is represented by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ and $\mu_{\tilde{A}}(x)$ is the membership function of x in $\tilde{A}$. The definition reduces to the classical set if $\mu_{\tilde{A}} : X \to \{0,1\}$.*

Definition 2.2. [4] *For a universal set X, an* **I-fuzzy set** *is described by a set A^* where $A^* = \{(x, \mu_{A^*}(x), \nu_{A^*}(x)) \mid x \in X\}$, where $\mu_{A^*} : X \to [0,1]$ and $\nu_{A^*} : X \to [0,1]$ such that $\mu_{A^*}(x)$ and $\nu_{A^*}(x)$ are defined as the degree of belongingness (membership grade) and degree of non-belongingness (non-membership grade) of an element $x \in X$ to the set A^* having the property that $0 \leq \mu_{A^*}(x) + \nu_{A^*}(x) \leq 1$. If $\mu_{A^*}(x) + \nu_{A^*}(x) = 1$, for all $x \in X$, then A^* reduces to the Definition 2.1.*

Definition 2.3. [4] *Let A be an I-fuzzy set in X. The function $\Pi_A : X \to [0,1]$ defined as $\Pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the* **degree of indeterminacy** *of the element x in A. It is very obvious that for the case of ordinary fuzzy sets $\Pi_A(x) = 0$, for every $x \in X$.*

2.2 Meaning of $(x \gtrsim_{IF} a)$ in I-fuzzy sense

Considering '$x \geq a$' in crisp environment, means the inequality holds only if 'x is greater than or equal to a'. That is, the decision maker does not have any freedom with the system constraints. However, if '$x \gtrsim a$', in fuzzy sense, then the decision maker has flexibility with the constraints. Basically, '$\gtrsim$' is associated with a fuzzy set whose membership function is formulated considering the degree of satisfaction level of decision maker. Therefore for an element x which satisfies $(x \gtrsim a)$ is certainly feasible (hence considered) whereas for an element x which do not satisfy $(x \gtrsim a)$, i.e, $x < a - p,\ p > 0$, is surely infeasible, where p is the limit of the admissible violation of the inequality. However, if $a - p \leq x < a$, then the belongingness of x will be defined by a membership function which will be decided by a decision maker (see [2]). In the sense of Zimmermann [28], '$\gtrsim$' means 'x is essentially greater than or equal to a'. The statement '$\gtrsim_{IF}$' denotes '$\gtrsim$' in I-fuzzy sense. To understand this inequality, the pessimistic approach [12] is described below:

2.2.1 Pessimistic approach

In the pessimistic sense for I-fuzzy set, we interpret any element x satisfying $a - p + q \leq x < a$, $0 < q < p$, as those points for which the decision maker is not willing to accept totally although their degree of rejection is zero. Assuming the tolerances $p,\ q$, such that $0 < q < p$, the linear membership $\mu_0(x)$ and non-membership $\nu_0(x)$ functions are defined as follows:

$$\mu_0(x) = \begin{cases} 1 & \text{if } x \geq a, \\ 1 - \dfrac{a - x}{p} & \text{if } a - p \leq x < a, \\ 0 & \text{if } x < a - p, \end{cases} \text{ and } \nu_0(x) = \begin{cases} 1 & \text{if } x \leq a - p, \\ 1 - \dfrac{x - a + p}{q} & \text{if } a - p < x \leq a - p + q, \\ 0 & \text{if } x > a - p + q, \end{cases}$$

The pessimistic approach under exponential membership and non-membership function for the degree of acceptance $\mu_0(x)$ and the degree of rejection $\nu_0(x)$ are defined as:

$$\mu_0(x) = \begin{cases} 1 & \text{if } x \geq a, \\ \dfrac{e^{-\alpha_0((a-x)/p)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } a - p \leq x < a, \\ 0 & \text{if } x < a - p, \end{cases} \text{ and } \nu_0(x) = \begin{cases} 1 & \text{if } x \leq a - p, \\ \dfrac{e^{-\alpha_0((x-a+p)/q)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } a - p < x \leq a - p + q, \\ 0 & \text{if } x > a - p + q, \end{cases}$$

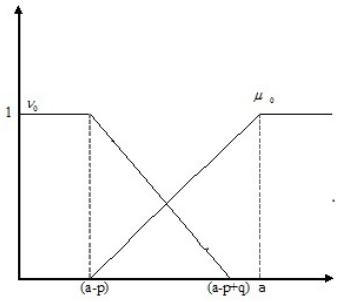


Figure 1: Pessimistic linear membership and non-membership functions

Remark 2.4. *The notation '$x \gtrsim_{IF(p)} a$' which is used to accentuate the tolerance p associated with the I-fuzzy statement '$x \gtrsim_{IF} a$' (discussed in Section 2.2). The I-fuzzy inequality '$x \lesssim_{IF(q)} a - p$' is equivalent to '$(-x) \gtrsim_{IF(q)} p - a$'. Hence, in the pessimistic approach, the I-fuzzy statement '$x \gtrsim_{IF(p,q)} a$' is defined by if $x \gtrsim_{IF(p,q)} a$, then* $\begin{cases} x \gtrsim_{IF(p)} a, \\ x \lesssim_{IF(q)} a - p, \end{cases}$ *with $a - p \leq a - p + q \leq a$. This means that the expression '$x \gtrsim_{IF(p,q)} a$' includes membership function, governed by the statement '$x \gtrsim_{IF(p)} a$' and the non-membership function, defined by the inequality '$x \lesssim_{IF(q)} a - p$'.*

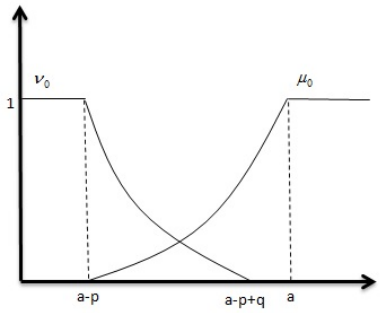


Figure 2: Pessimistic exponential membership and non-membership functions

2.3 Solving I-fuzzy optimization problems

Let X be the universal set. Considering the sets of r objectives $G_i,\ i = 1, 2, ..., r$, the sets of m constraints C_j, $j = 1, 2, .., m$, Atanassov [4] defined the I-fuzzy set as: $D = \{(x, \mu_D(x), \nu_D(x)),\ x \in X\}$, where the I-fuzzy decision, $D = (G_1 \cap G_2 \cap, ..., \cap G_r) \cap (C_1 \cap C_2 \cap, ..., \cap C_m)$ and $\mu_D(x) = \min\limits_{i,j}\{\mu_{G_i}(x), \mu_{C_j}(x)\}$, and $\nu_D(x) = \max\limits_{i,j}\{\nu_{G_i}(x), \nu_{C_j}(x)\}$. Let $\min\limits_{i,j}\{\mu_{G_i}(x), \mu_{C_j}(x)\} = \alpha$ and $\max\limits_{i,j}\{\nu_{G_i}(x), \nu_{C_j}(x)\} = \beta$. Let the score function of the I-fuzzy decision D be defined by $s(x) = \mu_D(x) - \nu_D(x),\ (x \in X) = \alpha - \beta$. Therefore, an optimal decision in I-fuzzy sense is given by $\max_{x \in X} s(x)$. Hence, the I-fuzzy optimization problem can be converted to the following crisp optimization problem [3]: max $\alpha - \beta$ subject to $\alpha \leq \mu_{G_i}(x),\ \beta \geq \nu_{G_i}(x),\ \alpha \leq \mu_{C_j}(x), \beta \geq \nu_{C_j}(x), i = 1, 2, ..., r,\ j = 1, 2, ..., m,\ \alpha \geq \beta \geq 0,\ \alpha + \beta \leq 1,\ x \in X$, where α, β denote the minimal degree of acceptance and the maximal degree of rejection of the sets of m constraints $C_j,\ j = 1, 2, .., m$, respectively.

3 Duality in LP problem with I-fuzzy sets

Linear programming and duality theory belongs to the best understood fields in mathematical programming. The duality theory has been significantly developed by various researchers, like, Hamacher et al. [14], Rodder and Zimmermann [19], Liu et al. [17], Bector et al. [8, 9] etc. In this section, formulating an I-fuzzy primal-dual pair, we have obtained their corresponding duality relations in I-fuzzy scenario under exponential membership and non-membership function using pessimistic approach. Consider the classical primal-dual linear problems under crisp environment:

$$\begin{array}{llll} \text{(LP)} & \text{Maximize} & c^T x & \\ & \text{subject to} & Ax \leq b, & \\ & & x \geq 0. & \end{array} \qquad \begin{array}{llll} \text{(LD)} & \text{Minimize} & b^T w & \\ & \text{subject to} & A^T w \geq c, & \\ & & w \geq 0. & \end{array}$$

Let $\mathbb{R}^n$ represents the n-dimensional Euclidean space. Let $c,\ x \in \mathbb{R}^n$, $b,\ w \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. The I-fuzzy versions of the problems (LP) and (LD), defined in Aggarwal et al. [2] are as follows:

$$\begin{array}{ll} \text{(IFLP)} & \text{Find } x \in \mathbb{R}^n, \text{ such that} \\ & c^T x \gtrsim_{IF} z_0, \\ & Ax \lesssim_{IF} b, \\ & x \geq 0, \end{array} \qquad \begin{array}{ll} \text{(IFLD)} & \text{Find } w \in \mathbb{R}^m, \text{ such that} \\ & b^T w \lesssim_{IF} w_0, \\ & A^T w \gtrsim_{IF} c, \\ & w \geq 0. \end{array}$$

Here "$\gtrsim_{IF}$" and "$\lesssim_{IF}$" are the I-fuzzy version of the inequalities "$\geq$" and "$\leq$", respectively. These inequalities are characterized by the I-fuzzy sets whose membership function describes decision maker's degree of satisfaction and non-membership function represents the decision maker's degree of dissatisfaction, respectively. The constants z_0 and w_0 indicate the aspiration levels of the two objectives $c^T x$ and $b^T w$ chosen by the decision maker.

Let $\mu_0(.)$ and $v_0(.)$ represent respectively the membership and the non-membership functions for the objective I-fuzzy linear inequality, and $\mu_i(.)$ and $v_i(.)$, $i = 1, ..., m$, denote the membership and non-membership functions for the I-fuzzy linear constraints, respectively. Also, $p_i,\ q_i,\ 0 < q_i < p_i,\ i = 0, 1, ..., m$, represents the tolerance associated with the acceptance and the rejection of $m + 1$ constraints in (IFLP), respectively and $s_j,\ t_j,\ 0 < t_j < s_j,\ (j = 0, 1, ..., n)$

denotes the tolerances associated with respect to the acceptance and the rejection of the $n+1$ constraints in (IFLD), respectively.

Let α, β denote the minimal degree of acceptance and maximal degree of rejection, respectively, of $m+1$ constraints of the problem (IFLP) and δ, η represents the minimal degree of acceptance and maximal degree of rejection of $n+1$ constraints of (IFLD), respectively. Under the linear membership and non-membership functions (as discussed in Section 2.2.1) and following Angelov [3], the crisp equivalent of the problems (IFLP) and (IFLD) under pessimistic scenario are given as:

$$\text{(IFL)} \quad \text{Maximize} \quad \alpha - \beta \quad \text{subject to}$$

$$(1-\alpha)p_0 + c^T x - z_0 \geq 0, (1-\alpha)p_i - A_i x + b_i \geq 0,\ i = 1, ..., m$$

$$(1-\beta)q_0 - c^T x + (z_0 - p_0) \leq 0, (1-\beta)q_i + A_i x - (b_i + p_i) \leq 0,\ i = 1, ..., m$$

$$0 \leq \beta \leq \alpha \leq 1,\ \alpha + \beta \leq 1,\ x \geq 0.$$

$$\text{(IFM)} \quad \text{Maximize} \quad \delta - \eta \quad \text{subject to}$$

$$(1-\delta)s_0 - b^T w + w_0 \geq 0,\ (1-\delta)s_j + A_j^T w - c_j \geq 0,\ j = 1, ..., n$$

$$(1-\eta)t_0 + b^T w - (w_0 + s_0) \leq 0,\ (1-\eta)t_j - A_j^T w + (c_j - s_j) \leq 0,\ j = 1, ..., n$$

$$0 \leq \eta \leq \delta \leq 1,\ \delta + \eta \leq 1,\ w \geq 0.$$

Now, to indicate the decision maker's level of satisfaction and dissatisfaction in the pessimistic approach for the objective function and the system of constraints, consider the following form of exponential membership and non-membership function:

$$\mu_0(c^T x) = \begin{cases} 1 & \text{if } c^T x \geq z_0, \\ \dfrac{e^{\alpha_0((c^T x - z_0)/p_0)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } z_0 - p_0 \leq c^T x < z_0, \\ 0 & \text{if } c^T x < z_0 - p_0, \end{cases}$$

$$\nu_0(c^T x) = \begin{cases} 1 & \text{if } c^T x \leq z_0 - p_0, \\ \dfrac{e^{\alpha_0((z_0 - p_0 - c^T x)/q_0)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } z_0 - p_0 < c^T x \leq z_0 - p_0 + q_0, \\ 0 & \text{if } c^T x > z_0 - p_0 + q_0, \end{cases}$$

$$\mu_i(A_i x) = \begin{cases} 1 & \text{if } A_i x \leq b_i, \\ \dfrac{e^{\alpha_i((b_i - A_i x)/p_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} & \text{if } b_i < A_i x \leq b_i + p_i, \\ 0 & \text{if } A_i x > b_i + p_i, \end{cases}$$

$$\nu_i(A_i x) = \begin{cases} 1 & \text{if } A_i x \geq b_i + p_i, \\ \dfrac{e^{\alpha_i((A_i x - b_i - p_i)/q_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} & \text{if } b_i + p_i - q_i \leq A_i x < b_i + p_i, \\ 0 & \text{if } A_i x < b_i + p_i - q_i. \end{cases}$$

where $i = 1, 2, ..., m$, α_0, α_i, $0 < \alpha_0,\ \alpha_i < \infty$ are called shape parameters that measure the degree of vagueness and A_i, b_i denote the ith row of the matrix A and ith component of the vector b $(i = 1, ..., m)$, respectively. Now, let λ, γ denote the minimal degree of acceptance and maximal degree of rejection, respectively, of $m+1$ constraints of the problem (IFLP). Following Angelov [3] and using exponential membership and non-membership functions, the (IFLP) problem is analogous to the following crisp optimization problem:

$$\text{(CP)} \quad \text{Maximize} \quad \lambda - \gamma \quad \text{subject to}$$

$$\lambda \leq \frac{e^{\alpha_0((c^T x - z_0)/p_0)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}}, \gamma \geq \frac{e^{\alpha_0((z_0 - p_0 - c^T x)/q_0)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}},$$

$$\lambda \leq \frac{e^{\alpha_i((b_i - A_i x)/p_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}}, \gamma \geq \frac{e^{\alpha_i((A_i x - b_i - p_i)/q_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}},\ i = 1, ..., m$$

$$0 \leq \gamma \leq \lambda \leq 1,\ \lambda + \gamma \leq 1,\ x \geq 0.$$

The problem (CP) can be rewritten as:

$$\text{(CP-1)} \quad \text{Maximize} \qquad \lambda - \gamma \text{subject to}$$

$$p_0 \ln(\lambda(1-e^{-\alpha_0}) + e^{-\alpha_0}) \leq \alpha_0(c^T x - z_0), \tag{1}$$

$$q_0 \ln(\gamma(1-e^{-\alpha_0}) + e^{-\alpha_0}) \geq \alpha_0(z_0 - p_0 - c^T x), \tag{2}$$

$$p_i \ln(\lambda(1-e^{-\alpha_i}) + e^{-\alpha_i}) \leq \alpha_i(b_i - A_i x),\ (i = 1, ..., m) \tag{3}$$

$$q_i \ln(\gamma(1-e^{-\alpha_i}) + e^{-\alpha_i}) \geq \alpha_i(A_i x - b_i - p_i),\ (i = 1, ..., m) \tag{4}$$

$$0 \leq \gamma \leq \lambda \leq 1,\ \lambda + \gamma \leq 1,\ x \geq 0.$$

Let δ, η denote the minimal degree of acceptance and maximal degree of rejection of $n+1$ constraints of the problem (IFLD). Similarly, the crisp optimization problem corresponding to (IFLD) is as follows:

$$\text{(CD)} \quad \text{Maximize} \quad \delta - \eta \text{subject to}$$

$$\delta \leq \frac{e^{\beta_0((w_0 - b^T w)/s_0)} - e^{-\beta_0}}{1 - e^{-\beta_0}}, \eta \geq \frac{e^{\beta_0((b^T w - w_0 - s_0)/t_0)} - e^{-\beta_0}}{1 - e^{-\beta_0}},$$

$$\delta \leq \frac{e^{\beta_j((A_j^T w - c_j)/s_j)} - e^{-\beta_j}}{1 - e^{-\beta_j}}, \eta \geq \frac{e^{\beta_j((c_j - s_j - A_j^T w)/t_j)} - e^{-\beta_j}}{1 - e^{-\beta_j}},\ j = 1, ..., n$$

$$0 \leq \eta \leq \delta \leq 1,\ \delta + \eta \leq 1,\ w \geq 0,$$

where A_j^T represents the jth row of the matrix A^T, c_j denotes the jth component of $c\ (j = 1, ..., n)$ and β_0, β_j, $0 < \beta_0, \beta_j < \infty\ (j = 0, 1, ..., n)$ are the shape parameters. Again, the problem (CD) can be expressed as

$$\text{(CD-1)} \quad \text{Maximize} \qquad \delta - \eta \text{subject to}$$

$$s_0 \ln(\delta(1-e^{-\beta_0}) + e^{-\beta_0}) \leq \beta_0(w_0 - b^T w), \tag{5}$$

$$t_0 \ln(\eta(1-e^{-\beta_0}) + e^{-\beta_0}) \geq \beta_0(b^T w - w_0 - s_0), \tag{6}$$

$$s_j \ln(\delta(1-e^{-\beta_j}) + e^{-\beta_j}) \leq \beta_j(A_j^T w - c_j), \quad j = 1, 2, ..., n \tag{7}$$

$$t_j \ln(\eta(1-e^{-\beta_j}) + e^{-\beta_j}) \geq \beta_j(c_j - s_j - A_j^T w), \quad j = 1, 2, ..., n \tag{8}$$

$$0 \leq \eta \leq \delta \leq 1,\ \delta + \eta \leq 1,\ w \geq 0.$$

The crisp equivalent pair (CP-1) and (CD-1) of the I-fuzzy linear primal-dual pair (IFLP) and (IFLD) are nonlinear programming problems. Medium or large-sized problems of this type can be solved computationally with many software programs used for solving nonlinear programming problems, like, MAPLE 12, MATHEMATICA, LINGO etc.

Remark 3.1. (*i*) *Taking* $\alpha_0 \to 0$, $\alpha_i \to 0$, $\forall i = 1, 2, ..., m$ *and* $\beta_0 \to 0$, $\beta_j \to 0$, $\forall j = 1, 2, ..., n$, *the problems (CP) and (CD) reduces to (IFPC) and (IFDC), respectively studied in* [2]. *Hence, the problems and results considered in* [2] *are the particular cases of the problems discussed in the paper.*

(*ii*) *If all the membership grades* $\mu_0(x)$, $\mu_i(x)$, $i = 1, 2, ..., m$ *are set equal to unity, then the problems (CP-1) and (CD-1) reduces to the problems studied in Gupta and Mehlawat* [13].

Now, we establish the usual duality relations for the pair (CP-1) and (CD-1).

Theorem 3.2. *Let (x, λ, γ) and (w, δ, η) be the feasible solutions for (CP-1) and (CD-1), respectively. Then,*

$$\sum_{i=1}^{m}\left(\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)p_i w_i + \sum_{j=1}^{n}\left(\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)s_j x_j \leq b^T w - c^T x, \tag{9}$$

$$\sum_{i=1}^{m}\left(\frac{\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)q_i w_i + \sum_{j=1}^{n}\left(\frac{\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)t_j x_j \geq (c-s)^T x - (b+p)^T w. \tag{10}$$

Proof. Multiplying (3) by $w_i \geq 0$ for $i = 1, ..., m$ and adding all the resultant m inequalities, we obtain

$$\sum_{i=1}^{m}\left(\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)p_i w_i \leq b^T w - x^T A^T w. \tag{11}$$

Again, multiplying (7) by $x_j \geq 0$ for $j = 1, ..., n$ and adding them, we get

$$\sum_{j=1}^{n}\left(\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)s_j x_j \leq w^T A x - c^T x. \tag{12}$$

Adding (11) and (12), we have

$$\sum_{i=1}^{m}\left(\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)p_i w_i + \sum_{j=1}^{n}\left(\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)s_j x_j \leq b^T w - c^T x.$$

Similarly, from (4) and (8), we obtain

$$\sum_{i=1}^{m}\left(\frac{\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)q_i w_i + \sum_{j=1}^{n}\left(\frac{\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)t_j x_j \geq (c-s)^T x - (b+p)^T w.$$

Hence the result. □

Remark 3.3. *When $\lambda = 1,\ \delta = 1,\ p_i = q_i\ (i = 0, 1, ..., m)$ and $s_j = t_j\ (j = 0, 1, ..., n)$, the inequalities (9) and (10) reduces to $b^T w \geq c^T x$, which is the weak duality result for the crisp LP problem. For other values of $\lambda,\ \delta \in (0, 1)$, the situation remains in I-fuzzy sense.*

Remark 3.4. *The I-fuzzy pair (IFLP) and (IFLD) gives us the crisp equivalents (CP-1) and (CD-1). However, (CP-1) and (CD-1) are not duals in strict sense. So, we may not be able to find a direct or converse duality relation between them. Moreover, if there exists optimal solution of (CP-1) and (CD-1), then the optimal values of the two objective functions may not be equal in general.*

Remark 3.5. *It follows from (1) and (5) that*

$$\frac{\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0 + \frac{\ln(\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0 \leq (c^T x - b^T w) + (w_0 - z_0). \tag{13}$$

The above inequality represents the relative difference of the aspiration levels z_0 of $c^T x$ and w_0 of $b^T w$ in terms of the degree of satisfaction and tolerance levels. Also, adding (2) and (6), we get

$$\frac{\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0 + \frac{\ln(\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0 \geq (b^T w - c^T x) + (z_0 - p_0) - (w_0 + s_0). \tag{14}$$

The inequality (14) denotes the relative difference between the minimum aspiration level $(z_0 - p_0)$ of $c^T x$ and maximum aspiration level $w_0 + s_0$ of $b^T w$ in terms of their degree of dissatisfaction. Moreover, if $\lambda = 1,\ \delta = 1,\ p_i = q_i\ (i = 0, 1, 2, ..., m),\ s_j = t_j\ (j = 0, 1, 2, ..., n)$ and $w_0 = z_0$, then (9) and (13) implies $c^T x = b^T w$, which is the condition for attaining optimal values for the problems (LP) and (LD). With the same values of the parameters, this result will also be obtained from (10) and (14).

Theorem 3.6. *Let $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ be feasible solution for (CP-1) and let $(\bar{w}, \bar{\delta}, \bar{\eta})$ be the feasible solution for (CD-1) such that*

(i) $\displaystyle\sum_{i=1}^{m}\left(\frac{\ln(\bar{\lambda}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)p_i\bar{w}_i + \sum_{j=1}^{n}\left(\frac{\ln(\bar{\delta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)s_j\bar{x_j} = b^T\bar{w} - c^T\bar{x},$

(ii) $\displaystyle\sum_{i=1}^{m}\left(\frac{\ln(\bar{\gamma}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\right)q_i\bar{w}_i + \sum_{j=1}^{n}\left(\frac{\ln(\bar{\eta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\right)t_j\bar{x_j} = (c-s)^T\bar{x} - (b+p)^T\bar{w},$

(iii) $\displaystyle\frac{\ln(\bar{\lambda}(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0 + \frac{\ln(\bar{\delta}(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0 = (c^T\bar{x} - b^T\bar{w}) + (w_0 - z_0),$

(iv) $\displaystyle\frac{\ln(\bar{\gamma}(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0 + \frac{\ln(\bar{\eta}(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0 = (b^T\bar{w} - c^T\bar{x}) + (z_0 - w_0) - (s_0 + p_0),$

(v) the aspiration levels z_0 and w_0 satisfy $z_0 - w_0 \le 0$,

then $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ is optimal to (CP-1) and $(\bar{w}, \bar{\delta}, \bar{\eta})$ is optimal to (CD-1).

Proof. Let (x, λ, γ) be feasible for (CP-1) and let (w, δ, η) be feasible for (CD-1). Using Theorem 3.2, we have

$$\sum_{i=1}^{m}\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_iw_i + \sum_{j=1}^{n}\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_jx_j - (b^Tw - c^Tx) \le 0,$$

$$\text{and}\quad (c-s)^Tx - (b+p)^Tw - \sum_{i=1}^{m}\frac{\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_iw_i - \sum_{j=1}^{n}\frac{\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_jx_j \le 0.$$

From (i) and (ii), we have

$$\sum_{i=1}^{m}\frac{\ln(\bar{\lambda}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_i\bar{w}_i + \sum_{j=1}^{n}\frac{\ln(\bar{\delta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_j\bar{x_j} - (b^T\bar{w} - c^T\bar{x}) = 0 \tag{15}$$

$$(c-s)^T\bar{x} - (b+p)^T\bar{w} - \sum_{i=1}^{m}\frac{\ln(\bar{\gamma}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_i\bar{w}_i - \sum_{j=1}^{n}\frac{\ln(\bar{\eta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_j\bar{x_j} = 0. \tag{16}$$

These above relations follow that if we have any feasible solution (x, λ, γ) for (CP-1) and any feasible solution (w, δ, η) for (CD-1), then

$$\sum_{i=1}^{m}\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_iw_i + \sum_{j=1}^{n}\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_jx_j - \sum_{i=1}^{m}\frac{\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_iw_i$$

$$-\sum_{j=1}^{n}\frac{\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_jx_j - (b^Tw - c^Tx) + (c-s)^Tx - (b+p)^Tw \le \sum_{i=1}^{m}\frac{\ln(\bar{\lambda}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_i\bar{w}_i$$

$$+\sum_{j=1}^{n}\frac{\ln(\bar{\delta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_j\bar{x_j} - \sum_{i=1}^{m}\frac{\ln(\bar{\gamma}(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_i\bar{w}_i - \sum_{j=1}^{n}\frac{\ln(\bar{\eta}(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_j\bar{x_j}$$

$$-(b^T\bar{w} - c^T\bar{x}) + (c-s)^T\bar{x} - (b+p)^T\bar{w}.$$

It follows that $(\bar{x}, \bar{\lambda}, \bar{\gamma}, \bar{w}, \bar{\delta}, \bar{\eta})$ is optimal solution to the following optimization problem whose maximum value is zero:

$$\text{Maximize}\quad \sum_{i=1}^{m}\frac{\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_iw_i + \sum_{j=1}^{n}\frac{\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_jx_j -$$

$$\sum_{i=1}^{m}\frac{\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_iw_i-\sum_{j=1}^{n}\frac{\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_jx_j-(b^Tw-c^Tx)+(c-s)^Tx-(b+p)^Tw$$

$$\text{subject to}\quad \frac{p_0\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}\le c^Tx-z_0,\quad \frac{q_0\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}\ge z_0-p_0-c^Tx,$$

$$\frac{p_i\ln(\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\le b_i-A_ix,\quad (i=1,...,m)\text{ and }\frac{q_i\ln(\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}\ge A_ix-b_i-p_i,\quad (i=1,...,m)$$

$$\frac{s_0\ln(\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}\le w_0-b^Tw,\text{ and }\frac{t_0\ln(\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}\ge b^Tw-w_0-s_0,$$

$$\frac{s_j\ln(\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\le A_j^Tw-c_j,\quad (j=1,...,n)\text{ and }\frac{t_j\ln(\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}\ge c_j-s_j-A_j^Tw,\quad (j=1,...,n)$$

$$\lambda+\gamma\le 1,\ \delta+\eta\le 1,\ \text{and } 0\le\gamma\le\lambda\le 1,\ 0\le\eta\le\delta\le 1,\ x\ge 0,\ \ w\ge 0.$$

Now, from the given conditions (iii) and (iv), we have

$$\frac{\ln(\bar\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0+\frac{\ln(\bar\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0-(c^T\bar x-b^T\bar w)-(w_0-z_0)=0, \tag{17}$$

$$(b^T\bar w-c^T\bar x)+(z_0-w_0)-(s_0+p_0)-\frac{\ln(\bar\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0-\frac{\ln(\bar\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0=0. \tag{18}$$

Adding (15)-(18), we obtain

$$\sum_{i=1}^{m}\left(\frac{\ln(\bar\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_i-\frac{\ln(\bar\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_i-p_i\right)\bar w_i+\sum_{j=1}^{n}\left(\frac{\ln(\bar\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_j-\frac{\ln(\bar\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_j-s_j\right)\bar x_j+\left(\frac{\ln(\bar\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0-\frac{\ln(\bar\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0-p_0\right)+\left(\frac{\ln(\bar\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0-\frac{\ln(\bar\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0-s_0\right)+2(z_0-w_0)=0.$$

In the above equation, each term in the left hand side is non-positive (as $\bar\lambda,\bar\gamma,\ \bar\delta,\ \bar\eta\le 1$ and $p_i>q_i, s_j>t_j$) and it follows that each of the terms should individually equal to zero, i.e.

$$\sum_{i=1}^{m}\left(\frac{\ln(\bar\lambda(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}p_i-\frac{\ln(\bar\gamma(1-e^{-\alpha_i})+e^{-\alpha_i})}{\alpha_i}q_i-p_i\right)\bar w_i=0,$$

$$\sum_{j=1}^{n}\left(\frac{\ln(\bar\delta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}s_j-\frac{\ln(\bar\eta(1-e^{-\beta_j})+e^{-\beta_j})}{\beta_j}t_j-s_j\right)\bar x_j=0,$$

$$\frac{\ln(\bar\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0+\left(-\frac{\ln(\bar\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0-p_0\right)=0, \tag{19}$$

$$\frac{\ln(\bar\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0+\left(-\frac{\ln(\bar\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0-s_0\right)+2(z_0-w_0)=0. \tag{20}$$

Again by same arguments as above, from (19) and (20), we have

$$\frac{\ln(\bar\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0=0,\ \frac{\ln(\bar\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0+p_0=0,\ \frac{\ln(\bar\delta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}s_0=0,\ \frac{\ln(\bar\eta(1-e^{-\beta_0})+e^{-\beta_0})}{\beta_0}t_0+s_0=0.$$

Since $(\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})/\alpha_0)p_0\le 0$ and $(\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})/\alpha_0)q_0+p_0\ge 0$ (as $\lambda,\ \gamma\le 1$ and $p_0>q_0$), implies that

$$\frac{\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0\le\frac{\ln(\bar\lambda(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}p_0\text{ and }\frac{\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0+p_0\ge\frac{\ln(\bar\gamma(1-e^{-\alpha_0})+e^{-\alpha_0})}{\alpha_0}q_0+p_0.$$

As $p_0,\ q_0,\ \alpha_0>0$ and "ln" is an increasing function in $(0,\infty)$, we have $\lambda\le\bar\lambda$ and $\gamma\ge\bar\gamma$. This implies that $\lambda-\gamma\le\bar\lambda-\bar\gamma$. Similarly, $\delta-\eta\le\bar\delta-\bar\eta$. Hence the result. □

Remark 3.7. *In the crisp scenario, i.e.* $\bar\lambda=1,\ \bar\delta=1,\ w_0=z_0,\ p_i=q_i\ (i=0,1,...,m)$ *and* $s_j=t_j\ (j=0,1,...,n)$*, the conditions (i)-(v) of Theorem 3.6 implies that* $c^T\bar x=b^T\bar w$ *which is the strong duality result in the crisp LP problem.*

4 Duality Gaps

4.1 Comparison with linear membership function [2]

The study of duality relations for a fuzzy nonlinear optimization problem is incomplete without discussing the duality gap. Unlike crisp optimization problems, there may be a huge duality gap in fuzzy nonlinear models. The practical applications of such problems are relevant in various real world problems only if the duality gap becomes smaller or tends to zero. Smaller the duality gap is, closer will be the primal and dual feasible regions and hence will be more significant to the practical situations.

Remark 4.1. *In the problem discussed in Section 3, there are two duality gaps viz., duality gap 1 and 2 under exponential membership function. These are defined as the absolute difference of LHS and RHS of inequalities* (9) *and* (10)*, respectively. Aggarwal et al. [2] have studied the models (IFLP) and (IFLD), under linear membership functions. The duality gaps 1 and 2 under the linear membership function are:*

$$\left|[(\alpha-1)p^Tw+(\delta-1)s^Tx]-[b^Tw-c^Tx]\right| \text{ and } \left|[(\beta-1)q^Tw+(\eta-1)t^Tx]-[(c-s)^Tx-(b+p)^Tw]\right|,$$

respectively, given in the Theorem 1 in [2] for the primal-dual pair (IFL) and (IFM).

In the succeeding discussion, we have shown that the duality gaps 1 and 2 are significantly reduced under exponential membership function unlike with linear membership function. This is mainly due to flexibility of changing the shape parameters associated with the exponential membership function.

Consider the example 1 in [2], which is as follows:
(IFLP-E) Find $x \in R^2$ such that $x_1+x_2 \gtrsim_{IF} 7, 4x_1-x_2 \lesssim_{IF} 10,\ \ x_1+2x_2 \lesssim_{IF} 8, 5x_1+2x_2 \lesssim_{IF} 20,\ x_1,x_2 \geq 0.$

(IFLD-E) Find $w \in R^3$ such that $10w_1+8w_2+20w_3 \lesssim_{IF} 6,\ 4w_1+w_2+5w_3 \gtrsim_{IF} 1, -w_1+2w_2+2w_3 \gtrsim_{IF} 1, w_1,w_2,w_3 \geq 0.$

Comparing the above problems with (IFLP) and (IFLD) given in Section 3, we have

$$z_0=7,\ w_0=6,\ c=\begin{bmatrix}1\\1\end{bmatrix},\ A=\begin{bmatrix}4&-1\\1&2\\5&2\end{bmatrix}, b=\begin{bmatrix}10\\8\\20\end{bmatrix}, x=\begin{bmatrix}x_1\\x_2\end{bmatrix},\ w=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

4.1.1 Under linear membership function [2]

For $p_0=3,\ p_1=3,\ p_2=4,\ p_3=5$ and $q_0=2.5,\ q_1=2,\ q_2=3,\ q_3=4$, and using substituting the data of (IFLP-E) above in (IFL), the following problem (EL) is obtained:

$$\text{(EL) Maximize } \alpha-\beta \text{ subject to } x_1+x_2-3\alpha \geq 4, -4x_1+x_2-3\alpha \geq -13,$$

$$-x_1-2x_2-4\alpha \geq -12, -5x_1-2x_2-5\alpha \geq -25, -x_1-x_2-2.5\beta \leq -6.5,$$

$$4x_1-x_2-2\beta \leq 11, x_1+2x_2-3\beta \leq 9, 5x_1+2x_2-4\beta \leq 21,$$

$$0 \leq \beta \leq \alpha \leq 1,\ \alpha+\beta \leq 1,\ x_1,x_2 \geq 0.$$

Similarly, for $s_0=4,\ s_1=4,\ s_2=3$ and $t_0=2.5,\ t_1=1.5,\ t_2=0.5$, the problem (IFM) becomes:

$$\text{(ED) Maximize } \delta-\eta \text{ subject to } -10w_1+8w_2+20w_3-4\delta \geq -10,$$

$$4w_1+w_2+5w_3-4\delta \geq -3, -w_1+2w_2+2w_3-3\delta \geq -2, 10w_1+8w_2+20w_3-2.5\eta \leq 7.5,$$

$$-4w_1-w_2-5w_3-1.5\eta \leq 1.5, w_1-2w_2-2w_3-0.5\eta \leq 1.5,$$

$$0 \leq \eta \leq \delta \leq 1,\ \delta+\eta \leq 1,\ w_1,w_2,w_3 \geq 0.$$

The optimal solution of (EL) is $\bar{x}_1=3.07$ and $\bar{x}_2=3.048$ and the optimal solution of (ED) is $\bar{w}_1=0,\ \bar{w}_2=0,$ and $\bar{w}_3=0.375$. Therefore, the duality gaps 1 and 2 under linear membership function are 15.3208 and 17.2770, respectively.

4.1.2 Under exponential membership function

For the problems (IFLP-E) and (IFLD-E) and taking the same values of $p_i,\ q_i,\ i = 0, 1, 2, 3$ and $s_j,\ t_j,\ j = 0, 1, 2$ (as in 4.1.1), the models (CP-1) and (CD-1) reduce to the following problems (CP-E) and (CD-E):

$$\text{(CP-E) Maximize} \quad \lambda - \gamma \text{ subject to}$$

$$3\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0}) \le \alpha_0(x_1+x_2-7),\quad 2.5\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0}) \ge \alpha_0(4-x_1-x_2),$$
$$3\ln(\lambda(1-e^{-\alpha_1})+e^{-\alpha_1}) \le \alpha_1(10-4x_1+x_2),\quad 2\ln(\gamma(1-e^{-\alpha_1})+e^{-\alpha_1}) \ge \alpha_1(4x_1-x_2-7),$$
$$4\ln(\lambda(1-e^{-\alpha_2})+e^{-\alpha_2}) \le \alpha_2(8-x_1-2x_2),\quad 3\ln(\gamma(1-e^{-\alpha_2})+e^{-\alpha_2}) \ge \alpha_2(x_1+2x_2-4),$$
$$5\ln(\lambda(1-e^{-\alpha_3})+e^{-\alpha_3}) \le \alpha_3(20-5x_1-2x_2),\quad 4\ln(\gamma(1-e^{-\alpha_3})+e^{-\alpha_3}) \ge \alpha_3(5x_1+2x_2-15),$$
$$0 \le \gamma \le \lambda \le 1,\ \lambda+\gamma \le 1,\ x \ge 0.$$

$$\text{(CD-E) Maximize} \quad \delta - \eta \text{ subject to}$$

$$4\ln(\delta(1-e^{-\beta_0})+e^{-\beta_0}) \le \beta_0(6-10w_1-8w_2-20w_3),\quad 2.5\ln(\eta(1-e^{-\beta_0})+e^{-\beta_0}) \ge \beta_0(10w_1+8w_2+20w_3-2),$$
$$4\ln(\delta(1-e^{-\beta_1})+e^{-\beta_1}) \le \beta_1(4w_1+w_2+5w_3-1),\quad 1.5\ln(\eta(1-e^{-\beta_1})+e^{-\beta_1}) \ge \beta_1(-3-4w_1-w_2-5w_3),$$
$$3\ln(\delta(1-e^{-\beta_2})+e^{-\beta_2}) \le \beta_2(-w_1+2w_2+2w_3-1),\quad 0.5\ln(\eta(1-e^{-\beta_2})+e^{-\beta_2}) \ge \beta_2(-2+w_1-2w_2-2w_3),$$
$$0 \le \eta \le \delta \le 1,\ \delta+\eta \le 1,\ w \ge 0.$$

Taking the shape parameters $\alpha_0 = 1,\ \alpha_1 = 1.5,\ \alpha_2 = 2,\ \alpha_3 = 2.5$ (corresponding to the objective function and the system of constraints of (IFLP)) and $\beta_0 = 1,\ \beta_1 = 2,\ \beta_2 = 3$ (with respect to the objective function and the system of constraints of (IFLD)), the problems (CP-E) and (CD-E) becomes:

$$\text{(ECP) Maximize } \lambda - \gamma \text{ subject to}$$

$$3\ln(0.632121\lambda+0.367879) \le x_1+x_2-7,\quad 2.5\ln(0.632121\gamma+0.367879) \ge 4-x_1-x_2,$$
$$2\ln(0.776870\lambda+0.223130) \le 10-4x_1+x_2,\quad 1.333333\ln(0.776870\gamma+0.223130) \ge 4x_1-x_2-13,$$
$$2\ln(0.864665\lambda+0.135335) \le 8-x_1-2x_2,\quad 1.5\ln(0.864665\gamma+0.135335) \ge x_1+2x_2-12,$$
$$2\ln(0.917915\lambda+0.082085) \le 20-5x_1-2x_2,\quad 1.6\ln(0.917915\gamma+0.082085) \ge 5x_1+2x_2-25,$$
$$0 \le \gamma \le \lambda \le 1,\ \lambda+\gamma \le 1,\ x_1, x_2 \ge 0.$$

$$\text{(ECD) Maximize } \delta - \eta \text{ subject to}$$

$$4\ln(0.632121\delta+0.367879) \le 6-10w_1-8w_2-20w_3,\quad 2.5\ln(0.632121\eta+0.367879) \ge 10w_1+8w_2+20w_3-10,$$
$$2\ln(0.864665\delta+0.135335) \le 4w_1+w_2+5w_3-1,\quad 0.75\ln(0.864665\eta+0.135335) \ge -3-4w_1-w_2-5w_3,$$
$$\ln(0.950213\delta+0.0497871) \le -w_1+2w_2+2w_3-1,\quad 0.167\ln(0.950213\eta+0.0497871) \ge -2+w_1-2w_2-2w_3,$$
$$0 \le \eta \le \delta \le 1,\ \delta+\eta \le 1,\ w_1, w_2, w_3 \ge 0.$$

Using the software MATHEMATICA, the optimal solutions of (ECP) are $x_1 = 3.01988,\ x_2 = 2.98048,\ \lambda = 0.551697$ and $\gamma = 0.128749$ and of (ECD) is $w_1 = 0.112283,\ w_2 = 0.557976,\ w_3 = 0.0077498,\ \delta = 1$ and $\eta = 0$, respectively. Hence, the duality gaps 1 and 2 for the problems (ECP) and (ECD) are 0.392922 and 15.9891, respectively, which are smaller than the gaps obtained under linear membership function.

Remark 4.2. *Fixing the values of $p_i,\ q_i,\ i = 0, 1, 2, 3$ and $s_j,\ t_j,\ j = 0, 1, 2$ (as in 4.1.1) in the problems (CP-E) and (CD-E), if we perform sensitivity analysis on the shape parameters $\alpha_i's$ and $\beta_j's$, $(i = 0, 1, 2, 3,\ j = 0, 1, 2)$ involved in the constraint functions, we obtain different optimal solutions of the problems (CP-E) and (CD-E). It has been found that small perturbations in the parameters, results in change in the optimal values of the problems, which eventually, provides different duality gaps. These gaps reduces to a larger extent with the use of exponential membership function when compared with the linear membership function.*

Altering the values of the shape parameters $\alpha_i,\ i = 0, 1, 2, 3$ and $\beta_j,\ j = 0, 1, 2$, in the problems (CP-E) and (CD-E), number of optimization problems are solved using MATHEMATICA, and the duality gaps 1 and 2 under exponential membership function are computed as shown in Table 1 below: Since the duality gaps 1 and 2 calculated under linear membership function are 15.3208 and 17.277 (see 4.1.1), respectively, therefore, from the table, we can conclude that these gaps are smaller than linear membership function.

Primal Problem							Dual Problem						Gap 1	Gap 2
α_0	α_1	α_2	α_3	λ	γ	$c^T x$	β_0	β_1	β_2	δ	η	$b^T w$		
1	1.5	2	2.5	0.55169	0.12875	6.00036	1	2	3	1	0	5.74163	0.39292	15.9891
1	0.5	2	3	0.54753	0.13189	5.98932	1	1.5	3	1	0	5.8374	0.48649	16.0341
1	0.6	2	3	0.54753	0.13189	5.98932	1	1.6	3	1	0	5.85684	0.48508	16.0412
1	0.7	2	3	0.54753	0.13189	5.98932	1	1.7	3	1	0	5.83172	0.45542	16.011
1	0.8	2	3	0.54753	0.13189	5.98932	1	1.8	3	1	0	5.8209	0.44488	16.0053
1	0.9	2	3	0.54753	0.13189	5.98932	1	1.9	3	1	0	5.8291	0.46538	16.0275
1	1	2	3	0.54753	0.13189	5.98932	1	2	3	1	0	5.82444	0.43766	16.0031
1	1.1	2	3	0.54753	0.13189	5.98932	1	2.1	3	1	0	5.94548	0.65972	16.2275
1	1.2	2	3	0.54753	0.13189	5.98932	1	2.2	3	1	0	5.82528	0.43657	16.0072
1	1.3	2	3	0.54753	0.13189	5.98932	1	2.3	3	1	0	5.83406	0.47558	16.0495
1	1.4	2	3	0.54753	0.13189	5.98932	1	2.4	3	1	0	5.7493	0.43207	16.0086
1	1.5	2	3	0.54753	0.13189	5.98932	1	2.5	3	1	0	5.88259	0.52678	16.1171
1	1.5	2	3.5	0.54408	0.13452	5.98015	1	3	3	1	0	5.8413	0.44424	16.0279
1	1.5	2	4	0.54122	0.13671	5.97252	1	3.5	3	1	0	5.9051	0.55419	16.1383
1	1.5	2	4.5	0.53883	0.13855	5.96613	1	4	3	1	0	5.84756	0.47486	16.0641
1	1.5	2	5	0.53683	0.14009	5.96076	1	4.5	3	1	0	5.7911	0.52162	16.0354
1	1.5	2	5.5	0.53513	0.14141	5.95621	1	5	3	1	0	5.76756	0.49786	16.0107
1	1.5	2	6	0.53369	0.14254	5.95232	1	5.5	3	1	0	5.80074	0.55128	16.0502
1	1.5	2	6.5	0.53244	0.14351	5.94897	1	6	3	1	0	9.97598	0.46817	16.0722
1	1.5	2	7	0.53136	0.14436	5.94605	1	6.5	3	1	0	5.85443	0.62808	16.1281
1	1.5	2	7.5	0.53041	0.14509	5.9435	1	7	3	1	0	5.94331	0.72357	16.2152
1	1.5	2.5	2.5	0.53927	0.13821	5.9673	1	2	3.5	1	0	5.99446	1.41632	16.4222
1	1.5	3	2.5	0.52864	0.14649	5.93871	1	2	4	1	0	5.86482	0.44764	16.1929
1	1.5	3.5	2.5	0.51959	0.15368	5.91418	1	2	4.5	1	0	5.84071	0.41095	16.2188
1	1.5	4	2.5	0.51191	0.15988	5.89317	1	2	5	1	0	5.82095	0.38038	16.2373
1	1.5	4.5	2.5	0.0804	0.05036	6.29254	1	2	5.5	1	0	5.57065	0.56049	16.0397
1	1.5	4.6	2.5	0.07889	0.0457	6.311	1	2	5.6	1	0	5.7643	0.77951	16.2821
1	1.5	4.7	2.5	0.0725	0.04555	6.31161	1	2	5.7	1	0	5.7643	0.79989	16.2938
1	1.5	4.8	2.5	0.09519	0.02656	6.38842	1	2	5.8	1	0	5.7643	0.59352	16.2823
1	1.5	4.9	2.5	0.06328	0.04303	6.32165	1	2	5.9	1	0	5.7643	0.81623	16.316
1	1.5	5	2.5	0.08411	0.02485	6.39547	1	2	6	1	0	5.76429	0.61312	16.3003
1	1.5	5.1	2.5	0.49876	0.17073	5.85689	1	2	6.1	1	0	5.7643	0.28539	16.0774
1	1.5	5.2	2.5	0.49777	0.17156	5.85415	1	2	6.2	1	0	5.7643	0.28526	16.0821
1	1.5	5.3	2.5	0.05969	0.02832	6.38122	1	2	6.3	1	0	5.7643	0.72445	16.3434
1	1.5	5.4	2.5	0.49588	0.17314	5.84889	1	2	6.4	1	0	5.7643	0.28503	16.091
1	1.5	5.5	2.5	0.04735	0.03036	6.37285	1	2	6.5	1	0	5.76429	0.79093	16.3692
1	1.5	5.6	2.5	0.49409	0.17465	5.84392	1	2	6.6	1	0	5.7643	0.28481	16.0992
1	1.5	5.7	2.5	0.03835	0.03152	6.36816	1	2	6.7	1	0	5.7643	0.84127	16.3922
1	1.5	5.8	2.5	0.05116	0.01925	6.41866	1	2	6.8	1	0	5.7643	0.67973	16.3669
1	1.5	5.9	2.5	0.03221	0.03102	6.37019	1	2	6.9	1	0	5.7643	0.86917	16.4116
1	1.5	6	2.5	0.06226	0.03114	6.36971	1	2	7	1	0	5.76429	0.63581	16.4215
1	1.5	6.1	2.5	0.05216	0.02571	6.39194	1	2	7.1	1	0	5.7643	0.6645	16.4207
1	1.5	6.2	2.5	0.48931	0.1787	5.83056	1	2	7.2	1	0	5.7643	0.28421	16.1204
1	1.5	6.3	2.5	0.05216	0.02933	6.37711	1	2	7.3	1	0	5.7643	0.65728	16.4472
1	1.5	6.4	2.5	0.02876	0.01914	6.41909	1	2	7.4	1	0	5.7643	0.79172	16.4274
1	1.5	6.5	2.5	0.03943	0.02207	6.40694	1	2	7.5	1	0	5.76429	0.69641	16.4484
1	1.5	6.6	2.5	0.04834	0.03156	6.36797	1	2	7.6	1	0	5.7643	0.66005	16.4767
1	1.5	6.7	2.5	0.04309	0.02888	6.37891	1	2	7.7	1	0	5.7643	0.67539	16.4819
1	1.5	6.8	2.5	0.02633	0.01287	6.44532	1	2	7.8	1	0	5.7643	0.74409	16.4277
1	1.5	6.9	2.5	0.02465	0.01237	6.4474	1	2	7.9	1	0	5.91378	0.98174	16.7168

Figure 3: Various solutions and duality gaps for different set of shape parameters

4.2 Numerical illustration

The manager of a dairy milk decides that each cow should get daily nutrients of A, B and C. Two varieties of feed are available, variety I and variety II. The contents of nutrients A, B and C in units per kg in variety I and II are given in the following table:

Variety	A (units/kg)	B (units/kg)	C (units/kg)
I	3	1	3
II	1	2	2

The requirement of nutrient A in variety I cannot exceed that of variety II by 9 units/kg per day, possibly extensible up to 12 units/kg in a day. Further, the total requirements of nutrients B and C in the diet are 7 and 16 units/kg per day which may extend up to 11 and 21 units/kg/day, respectively. Suppose the varieties I and II earns a profit of \$ 1 per kg sale. The manager wants to earn a profit at least \$ 6, and will stop producing these varieties if the profit earned is less than or equal to \$ 3. Formulating the problem in the I-fuzzy sense, determine how much of feed of each variety should be made to feed a cow daily to enhance the overall profit?

4.2.1 Formulation

Let x_1 kg and x_2 kg be the amounts of variety I and variety II to be produced/day, respectively. From the problem, we have $z_0 = 6$, $p_0 = 3$. Therefore, I-fuzzy linear problem is formulated as follows:

$$\text{(IFP) Find } x \in R^2 \text{ such that } x_1 + x_2 \gtrsim_{IF} 6, 3x_1 - x_2 \lesssim_{IF} 9, \quad x_1 + 2x_2 \lesssim_{IF} 7, 3x_1 + 2x_2 \lesssim_{IF} 16, x_1, x_2 \geq 0,$$

where $z_0 = 6,\ c = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 3 & 2 \end{bmatrix}, b = \begin{bmatrix} 9 \\ 7 \\ 16 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. From the problem, $p_0 = 3,\ p_1 = 3,\ p_2 = 4$ & $p_3 = 5$. Now, $0 < q_0 < 3,\ 0 < q_1 < 3,\ 0 < q_2 < 4,\ 0 < q_4 < 5$. If we choose $q_0 = 2.5,\ q_1 = 2,\ q_2 = 3$ & $q_3 = 4$, the problem

(CP-1) becomes:

(CP-2) Maximize $\lambda - \gamma$ subject to

$$3\ln(\lambda(1-e^{-\alpha_0})+e^{-\alpha_0}) \le \alpha_0(x_1+x_2-6),\quad 2.5\ln(\gamma(1-e^{-\alpha_0})+e^{-\alpha_0}) \ge \alpha_0(3-x_1-x_2),$$
$$3\ln(\lambda(1-e^{-\alpha_1})+e^{-\alpha_1}) \le \alpha_1(9-3x_1+x_2),\quad 2\ln(\gamma(1-e^{-\alpha_1})+e^{-\alpha_1}) \ge \alpha_1(3x_1-x_2-12),$$
$$4\ln(\lambda(1-e^{-\alpha_2})+e^{-\alpha_2}) \le \alpha_2(7-x_1-2x_2),\quad 3\ln(\gamma(1-e^{-\alpha_2})+e^{-\alpha_2}) \ge \alpha_2(x_1+2x_2-11),$$
$$5\ln(\lambda(1-e^{-\alpha_3})+e^{-\alpha_3}) \le \alpha_3(16-3x_1-2x_2),\quad 4\ln(\gamma(1-e^{-\alpha_3})+e^{-\alpha_3}) \ge \alpha_3(3x_1+2x_2-21),$$
$$0 \le \gamma \le \lambda \le 1,\ \lambda+\gamma \le 1,\ x_1, x_2 \ge 0.$$

Taking the shape parameters as $\alpha_0 = 3,\ \alpha_1 = 5,\ \alpha_2 = 1$ and $\alpha_3 = 1$, the optimal solution of (CP-2) is $x_1 = 3.72407,\ x_2 = 1.99439,\ \lambda = 0.741765$ and $\gamma = 0$. The optimal value of (CP-2) is $\bar{\lambda} - \bar{\gamma} = 0.741765$ and the corresponding objective function value is $x_1 + x_2 = 5.71846$. Now, let us consider the problem (IFP) with linear membership function. Substituting the values of $p_i,\ i = 0, 1, 2, 3$, in the problem (IFL), we get (EP) max$\alpha - \beta$ subject to $x_1 + x_2 - 3\alpha \ge 3, -3x_1 + x_2 - 3\alpha \ge -12, -x_1 - 2x_2 - 4\alpha \ge -11, -3x_1 - 2x_2 - 5\alpha \ge -21, -x_1 - x_2 - q_0\beta \le -3 - q_0, 3x_1 - x_2 - q_1\beta \le 12 - q_1, x_1 + 2x_2 - q_2\beta \le 11 - q_2, 3x_1 + 2x_2 - q_3\beta \le 21 - q_3, \alpha + \beta \le 1,\ \alpha \ge \beta \ge 0,\ x_1,\ x_2 \ge 0$. Taking $q_0 = 2.5,\ q_1 = 2,\ q_2 = 3$ & $q_3 = 4$, the I-fuzzy primal optimal solution of (EP) is $x_1 = 3.75,\ x_2 = 1.875,\ \alpha = 0.875$ and $\beta = 0$. The optimal value of (EP) is $\alpha - \beta = 0.875$ and the corresponding objective function value is $x_1 + x_2 = 5.625$. It may be remarked that the overall profit for the manager, in case of the exponential membership function is higher than that of the linear membership function.

4.2.2 Duality gaps

To study the duality gaps, we write the intuitionistic duals of the problem (IFP) under exponential and linear membership function letting $w_0 = 5$. The intuitionistic dual of problem (IFP) is formulated as follows:

(IFD) Find $w \in R^3$ such that $9w_1 + 7w_2 + 16w_3 \lesssim_{IF} 5,\ 3w_1 + w_2 + 3w_3 \gtrsim_{IF} 1, -w_1 + 2w_2 + 2w_3 \gtrsim_{IF} 1, w_1, w_2, w_3 \ge 0$.

Taking tolerances $s_0 = 4,\ s_1 = 4$ & $s_2 = 3$. Now, $0 < t_0 < 4,\ 0 < t_1 < 4,\ 0 < t_2 < 3$. If we choose $t_0 = 2.5,\ t_1 = 1.5$ & $t_2 = 0.5$, the problem (CD-1) (for the problem (IFD)) reduces to

(CD-2) Maximize $\delta - \eta$ subject to

$$4\ln(\delta(1-e^{-\beta_0})+e^{-\beta_0}) \le \beta_0(5-9w_1-7w_2-16w_3),\quad 2.5\ln(\eta(1-e^{-\beta_0})+e^{-\beta_0}) \ge \beta_0(9w_1+7w_2+16w_3-9),$$
$$4\ln(\delta(1-e^{-\beta_1})+e^{-\beta_1}) \le \beta_1(3w_1+w_2+3w_3-1),\quad 1.5\ln(\eta(1-e^{-\beta_1})+e^{-\beta_1}) \ge \beta_1(-3-3w_1-w_2-3w_3),$$
$$3\ln(\delta(1-e^{-\beta_2})+e^{-\beta_2}) \le \beta_2(-w_1+2w_2+2w_3-1),\quad 0.5\ln(\eta(1-e^{-\beta_2})+e^{-\beta_2}) \ge \beta_2(-2+w_1-2w_2-2w_3),$$
$$0 \le \eta \le \delta \le 1,\ \delta+\eta \le 1,\ w_1, w_2, w_3 \ge 0,$$

and again substituting the problem (IFD) in (IFM) for the same values of $s_i,\ i = 0, 1, 2$, the following problem (ED) is obtained: (ED) max $\delta - \eta$ subject to $-9w_1 - 7w_2 - 16w_3 - 4\delta \ge -9, 3w_1 + w_2 + 3w_3 - 4\delta \ge -3, -w_1 + 2w_2 + 2w_3 - 3\delta \ge -2, 9w_1 + 7w_2 + 16w_3 - t_0\eta \le 9 - t_0, -3w_1 - w_2 - 3w_3 - t_1\eta \le 3 - t_1, w_1 - 2w_2 - 2w_3 - t_2\eta \le 2 - t_2, \delta + \eta \le 1,\ \delta \ge \eta \ge 0,\ w_1,\ w_2,\ w_3 \ge 0$. For the same values of $t_i,\ i = 0, 1, 2$, the duality gaps 1 and 2 under linear membership function for the problems (EP) and (ED) are 0 and 20.6517, respectively. In particular, taking the values of the parameters $\alpha_0 = 1,\ \alpha_1 = 0.5,\ \alpha_2 = 2,\ \alpha_3 = 2.5$ for the problem (CP-2) and $\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 3$ for the problem (CD-2), the corresponding objective function values are 5.58199 and 5.05476, respectively. Consequently, the duality gaps 1 and 2 are 0 and 14.33370, respectively. The duality gap 2 is smaller than the duality difference obtained under the linear membership function.

The same result has been verified for the number of problems by varying the shape parameters $\alpha_i,\ i = 0, 1, 2, 3$ and $\beta_j,\ j = 0, 1, 2$ in (CP-2) and (CD-2). The problems are solved using MATHEMATICA and are shown in Table 2 above. This table demonstrates that under exponential membership function the duality gap 1 remains zero while the duality gap 2 decreases when compared with the duality gap 2, which equals 20.6517 under linear membership case.

Primal Problem							Dual Problem						Gap 1	Gap 2
α_0	α_1	α_2	α_3	λ	γ	c^Tx	β_1	β_2	β_3	δ	η	b^Tw		
1	0.5	2	2.5	0.79424	0	5.58199	1	1	3	0.97849	0	5.05476	0	14.3337
1	0.6	2	3	0.79349	0	5.58036	1	1.1	3	0.97789	0	5.05632	0	14.1161
1	0.7	2	3.5	0.79278	0	5.57881	1	1.2	3	0.97729	0	5.05788	0	14.123
1	0.8	2	4	0.7921	0	5.57732	1	1.3	3	0.97669	0	5.05935	0	14.1293
1	0.9	2	4.5	0.79144	0	5.57589	1	1.4	3	0.97609	0	5.06082	0	14.135
1	1	2	5	0.79082	0	5.57452	1	1.5	3	0.97551	0	5.06245	0	14.1401
1	1.1	2	5.5	0.79021	0	5.5732	1	1.6	3	0.97493	0	5.0639	0	14.1447
1	1.2	2	6	0.78964	0	5.57194	1	1.7	3	0.97435	0	5.06537	0	14.1489
1	1.3	2	6.5	0.78909	0	5.57074	1	1.8	3	0.97378	0	5.06682	0	14.1527
1	1.4	2	7	0.78856	0	5.56958	1	1.9	3	0.97322	0	5.06827	0	14.1561
6	1.5	2	2.5	0.65122	0	5.78621	6	2	3	0.96677	0	5.02247	0	14.3819
5.5	1.5	2	2.5	0.65869	0	5.77342	5.5	2	3	0.96699	0	5.02433	0	14.3648
5	1.5	2	2.5	0.66712	0	5.75914	5	2	3	0.96726	0	5.0264	0	14.3458
4.5	1.5	2	2.5	0.67666	0	5.74313	4.5	2	3	0.96758	0	5.02895	0	14.3246
4	1.5	2	2.5	0.68749	0	5.72519	4	2	3	0.96795	0	5.03203	0	14.3012
3.5	1.5	2	2.5	0.69979	0	5.70505	3.5	2	3	0.9684	0	5.03557	0	14.275
3	1.5	2	2.5	0.71376	0	5.68255	3	2	3	0.96894	0	5.03989	0	14.2461
2.5	1.5	2	2.5	0.72953	0	5.65755	2.5	2	3	0.96961	0	5.04529	0	14.2147
2	1.5	2	2.5	0.74722	0	5.63005	2	2	3	0.97043	0	5.05184	0	14.1809
1.5	1.5	2	2.5	0.76678	0	5.60022	1.5	2	3	0.97143	0	5.05984	0	14.1452
1	1.5	2	2.5	0.78805	0	5.56847	1	2	3	0.97266	0	5.06971	0	14.3337
1	1.5	2	3	0.78805	0	5.56847	1	2.5	3	0.97	0	5.07656	0	14.1161
1	1.5	2	3.5	0.78805	0	5.56847	1	3	3	0.96755	0	5.08288	0	14.123
1	1.5	2	4	0.78805	0	5.56847	1	3.5	3	0.96532	0	5.08865	0	14.1293
1	1.5	2	4.5	0.78805	0	5.56847	1	4	3	0.96331	0	5.09387	0	14.135
1	1.5	2	5	0.78805	0	5.56847	1	4.5	3	0.96149	0	5.09857	0	14.1401
1	1.5	2	5.5	0.78805	0	5.56847	1	5	3	0.95986	0	5.1028	0	14.1447
1	1.5	2	6	0.78805	0	5.56847	1	5.5	3	0.95839	0	5.10663	0	14.1489
1	1.5	2	6.5	0.78805	0	5.56847	1	6	3	0.95706	0	5.11007	0	14.1527
1	1.5	2	7	0.78805	0	5.56847	1	6.5	3	0.95586	0	5.1132	0	14.1561
1	1.5	2	7.5	0.78805	0	5.56847	1	7	3	0.95477	0	5.11603	0	14.1591
1	1.5	2.5	2.5	0.7785	0	5.54749	1	2	3.5	0.97212	0	5.07112	0	14.079
1	1.5	3	2.5	0.77034	0	5.52944	1	2	4	0.97167	0	5.07228	0	14.0534
1	1.5	3.5	2.5	0.76339	0	5.51398	1	2	4.5	0.97129	0	5.07325	0	14.0315
1	1.5	4	2.5	0.75747	0	5.50075	1	2	5	0.97098	0	5.07407	0	14.0127
1	1.5	4.5	2.5	0.75241	0.00248	5.48939	1	2	5.5	0.97071	0	5.07476	0	14.072
1	1.5	5	2.5	0.74806	0.00477	5.47961	1	2	6	0.97047	0	5.07536	0	14.1636
1	1.5	5.5	2.5	0.74431	0.00676	5.47112	1	2	6.5	0.97027	0	5.07588	0	14.271
1	1.5	6	2.5	0.74105	0.00851	5.46373	1	2	7	0.97009	0	5.07633	0	14.3802
1	1.5	6.5	2.5	0.73819	0.01004	5.45724	1	2	7.5	0.96995	0	5.07673	0	14.482
1	1.5	7	2.5	0.73567	0.01139	5.45152	1	2	8	0.96981	0	5.07708	0	14.5726
1	1.5	7.1	2.5	0.7352	0.01165	5.45045	1	2	8.1	0.96978	0	5.07708	0	14.5893
1	1.5	7.2	2.5	0.73475	0.01189	5.44941	1	2	8.2	0.96976	0	5.07716	0	14.6054
1	1.5	7.3	2.5	0.73429	0.01214	5.4484	1	2	8.3	0.96973	0	5.07724	0	14.6214
1	1.5	7.4	2.5	0.73386	0.01237	5.4474	1	2	8.4	0.96971	0	5.07732	0	14.6366
1	1.5	7.5	2.5	0.73344	0.01261	5.44643	1	2	8.5	0.96969	0	5.07733	0	14.6516
1	1.5	7.6	2.5	0.73302	0.01283	5.44548	1	2	8.6	0.96967	0	5.07741	0	14.666
1	1.5	7.7	2.5	0.73261	0.01305	5.44455	1	2	8.7	0.96964	0	5.07751	0	14.6801
1	1.5	7.8	2.5	0.73222	0.01327	5.44364	1	2	8.8	0.96962	0	5.07759	0	14.6939
1	1.5	7.9	2.5	0.73183	0.01348	5.44276	1	2	8.9	0.9696	0	5.0776	0	14.7071
1	1.5	8	2.5	0.73145	0.01369	5.44189	1	2	9	0.96958	0	5.07761	0	14.72

Figure 4: Various solutions and duality gaps for different set of shape parameters

5 Extension in I-fuzzy sets

Consider the I-fuzzy versions of (LP) and (LD) as (FP) and (FD) given below:

$$\begin{array}{ll} \text{(FP)} & \text{Find } x \in \mathbb{R}^n, \text{ such that} \\ & c^T x \gtrsim_{IF(p_0,q_0)} z_0, \\ & Ax \lesssim_{IF(p,q)} b, \\ & x \geq 0. \end{array} \qquad \begin{array}{ll} \text{(FD)} & \text{Find } w \in \mathbb{R}^m, \text{ such that} \\ & b^T w \lesssim_{IF(s_0,t_0)} w_0, \\ & A^T w \gtrsim_{IF(s,t)} c, \\ & w \geq 0, \end{array}$$

where

(*i*) the fuzzy quantities z_0 and w_0, represents the aspiration levels of the objective functions in (FP) and (FD), respectively,

(*ii*) the adequacy between the objective function $c^T x$ and the aspiration level z_0 is measured by the tolerance associated with their acceptance and rejection, viz, p_0 and q_0, respectively. In the same way, for the objective function $b^T w$ and the aspiration level w_0, s_0 (tolerance associated with the acceptance) and t_0 (tolerance associated with the rejection) measures the adequacy between them, and

(*iii*) for the constraints of (FP), the tolerance associated with the acceptance, p (fuzzy vector) and the tolerance associated with the rejection, q (fuzzy vector), measures the adequacy between the fuzzy numbers (fuzzy vectors) $A^T x$ and b. Similarly, the adequacy between the fuzzy numbers (fuzzy vectors) $A^T w$ and c are measured by s and t, respectively, which are also the tolerance associated with the acceptance and the rejection of the constraints for the problem (FD).

Further, in the I-fuzzy sense, $Ax \lesssim_{IF(p,q)} b$, can be explicitly written as: $Ax(\leq)b + p(1-\lambda)$, and $Ax(\leq)(b+p) - q(1-\gamma)$, $\lambda,\ \gamma \in [0,1]$, where the notation $(\leq)$ (or $(\geq)$) represents an ordering relation between the fuzzy numbers that preserve the ranking whenever they are multiplied by non-negative scalars. Suppose $N(R)$ represents the set of all fuzzy numbers in R. Let us consider a suitable linear ranking (defuzzification) function $F : N(R) \to R$, which is treated in such a way that $a\ (\leq)\ b$ implies $F(a) \leq F(b)$, for more details for such type of function, refer to [7, 24]. To define the level of desirability of the decision maker for the objective function and the system constraints, we choose the following membership functions:

$$\mu_0(x) = \begin{cases} 1 & \text{if } C \geq Z, \\ \dfrac{e^{\alpha_0((C-Z)/P)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } Z - P \leq C < Z, \\ 0 & \text{if } C < Z - P, \end{cases} \text{ and } \nu_0(x) = \begin{cases} 1 & \text{if } C \leq Z - P, \\ \dfrac{e^{\alpha_0((Z-P-C)/Q)} - e^{-\alpha_0}}{1 - e^{-\alpha_0}} & \text{if } Z - P < C \leq Z - P + Q, \\ 0 & \text{if } C > Z - P + Q, \end{cases}$$

$$\mu_i(x) = \begin{cases} 1 & \text{if } M_i \leq B_i, \\ \dfrac{e^{\alpha_i((B_i-M_i)/P_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} & \text{if } B_i < M_i \leq B_i + P_i, \\ 0 & \text{if } M_i > B_i + P_i, \end{cases} \text{ and } \nu_i(x) = \begin{cases} 1 & \text{if } M_i \geq B_i + P_i, \\ \dfrac{e^{\alpha_i((M_i-B_i-P_i)/Q_i)} - e^{-\alpha_i}}{1 - e^{-\alpha_i}} & \text{if } B_i + P_i - Q_i \leq M_i < B_i + P_i, \\ 0 & \text{if } M_i < B_i + P_i - Q_i. \end{cases}$$

where $F(c^T x) = C,\ F(z_0) = Z,\ F(p_0) = P,\ F(q_0) = Q$ and for $i = 1, 2, ..., m,\ F(b_i) = B_i,\ F(A_i x) = M_i,\ F(p_i) = P_i,\ F(q_i) = Q_i$.

Using the membership and non-membership functions defined above, the crisp equivalent of (FP) for $i = 1, 2, ..., m$ is formulated as follows:

$$\text{(CLP) Maximize } \lambda - \gamma \text{ subject to}$$
$$P\ln(\lambda(1-e^{-\alpha_0}) + e^{-\alpha_0}) \leq \alpha_0(C - Z), \quad Q\ln(\gamma(1-e^{-\alpha_0}) + e^{-\alpha_0}) \geq \alpha_0(Z - P - C),$$
$$P_i\ln(\lambda(1-e^{-\alpha_i}) + e^{-\alpha_i}) \leq \alpha_i(B_i - M_i), \quad Q_i\ln(\gamma(1-e^{-\alpha_i}) + e^{-\alpha_i}) \geq \alpha_i(M_i - B_i - P_i),$$
$$0 \leq \gamma \leq \lambda \leq 1,\ \lambda + \gamma \leq 1,\ x \geq 0.$$
$$\text{(CLD) Maximize } \delta - \eta \text{ subject to}$$
$$S_0\ln(\delta(1-e^{-\beta_0}) + e^{-\beta_0}) \leq \beta_0(W - B), \quad T\ln(\eta(1-e^{-\beta_0}) + e^{-\beta_0}) \geq \beta_0(B - W - S),$$
$$S_j\ln(\delta(1-e^{-\beta_j}) + e^{-\beta_j}) \leq \beta_j(N_j - C_j), \quad T_j\ln(\eta(1-e^{-\beta_j}) + e^{-\beta_j}) \geq \beta_j(C_j - S_j - N_j),$$
$$0 \leq \eta \leq \delta \leq 1,\ \delta + \eta \leq 1,\ w \geq 0.$$

where $F(b^T w) = B,\ F(w_0) = W,\ F(s_0) = S,\ F(t_0) = T$ and for $j = 1, 2, ..., n,\ F(c_j) = C_j,\ F(A_j^T w) = N_j,\ F(s_j) = S_j,\ F(t_j) = T_j$. Now, we will state some relations for the pair (CLP) and (CLD), whose proofs can be obtained on the lines of Theorem 3.2 and Theorem 3.6.

Theorem 5.1. *Let (x, λ, γ) and (w, δ, η) be the feasible solutions for (CLP) and (CLD), respectively. Then*

(i) $\displaystyle\sum_{i=1}^{m} \frac{\ln(\lambda(1-e^{-\alpha_i}) + e^{-\alpha_i})}{\alpha_i} F(p_i w_i) + \sum_{j=1}^{n} \frac{\ln(\delta(1-e^{-\beta_j}) + e^{-\beta_j})}{\beta_j} F(s_j x_j \leq F(b^T w) - F(c^T x).$

(ii) $\displaystyle\sum_{i=1}^{m} \frac{\ln(\gamma(1-e^{-\alpha_i}) + e^{-\alpha_i})}{\alpha_i} F(q_i w_i) + \sum_{j=1}^{n} \frac{\ln(\eta(1-e^{-\beta_j}) + e^{-\beta_j})}{\beta_j} F(t_j x_j) \geq (F(c^T x) - F(s^T x)) - (F(b^T w) + F(p^T w)).$

(iii) $\displaystyle\frac{\ln(\lambda(1-e^{-\alpha_0}) + e^{-\alpha_0})}{\alpha_0} F(p_0) + \frac{\ln(\delta(1-e^{-\beta_0}) + e^{-\beta_0})}{\beta_0} F(s_0) \leq (F(c^T x) - F(b^T w)) + (F(w_0) - F(z_0)).$

(iv) $\displaystyle\frac{\ln(\gamma(1-e^{-\alpha_0}) + e^{-\alpha_0})}{\alpha_0} F(q_0) + \frac{\ln(\eta(1-e^{-\beta_0}) + e^{-\beta_0})}{\beta_0} F(t_0) \geq (F(b^T w) - F(c^T x)) + (F(z_0) - F(w_0)) - (F(s_0) + F(p_0)).$

Theorem 5.2. *Let $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ be feasible for (CLP) and let $(\bar{w}, \bar{\delta}, \bar{\eta})$ be feasible for (CLD) such that*

(i) $\displaystyle\sum_{i=1}^{m} \frac{\ln(\bar{\lambda}(1-e^{-\alpha_i}) + e^{-\alpha_i})}{\alpha_i} F(p_i \bar{w}_i) + \sum_{j=1}^{n} \frac{\ln(\bar{\delta}(1-e^{-\beta_j}) + e^{-\beta_j})}{\beta_j} F(s_j \bar{x_j}) = F(b^T \bar{w}) - F(c^T \bar{x}),$

(ii) $\displaystyle\sum_{i=1}^{m} \frac{\ln(\bar{\gamma}(1-e^{-\alpha_i}) + e^{-\alpha_i})}{\alpha_i} F(q_i \bar{w}_i) + \sum_{j=1}^{n} \frac{\ln(\bar{\eta}(1-e^{-\beta_j}) + e^{-\beta_j})}{\beta_j} F(t_j \bar{x_j}) = (F(c^T \bar{x}) - F(s^T \bar{x})) - (F(b^T \bar{w}) + F(p^T \bar{w})),$

(iii) $\displaystyle\frac{\ln(\bar{\lambda}(1-e^{-\alpha_0}) + e^{-\alpha_0})}{\alpha_0} F(p_0) + \frac{\ln(\bar{\delta}(1-e^{-\beta_0}) + e^{-\beta_0})}{\beta_0} F(s_0) = (F(c^T \bar{x}) - F(b^T \bar{w})) + (F(w_0) - F(z_0)),$

(iv) $\displaystyle\frac{\ln(\bar{\gamma}(1-e^{-\alpha_0}) + e^{-\alpha_0})}{\alpha_0} F(q_0) + \frac{\ln(\bar{\eta}(1-e^{-\beta_0}) + e^{-\beta_0})}{\beta_0} F(t_0) = (F(b^T \bar{w}) - F(c^T \bar{x})) + (F(z_0) - F(w_0)) - (F(s_0) + F(p_0)),$

(v) the aspiration level $F(z_0)$ and $F(w_0)$ satisfy $F(z_0) - F(w_0) \leq 0$,

then $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ is optimal to (CLP) and $(\bar{w}, \bar{\delta}, \bar{\eta})$ is optimal to (CLD).

6 Conclusions

In this paper, we have presented an I-fuzzy pair of primal-dual linear programming problem. Representing the decision maker's aspiration level by exponential membership and non-membership function, we have obtained the duality results in I-fuzzy scenario. Based on the exponential membership approach, we have compared the duality gaps with linear membership approach [2] and proved that the gaps reduces to a larger extent under exponential membership. This is mainly due to the presence of the shape variables corresponding to the objective function and the system of constraints. Varying the values of these shape parameters, different nonlinear programming problems are framed and solved, using the software MATHEMATICA. We have also constructed a non-trivial example in I-fuzzy environment. This new I-fuzzy problem is further converted to two different forms of crisp optimization problems using the linear and the exponential membership functions. For these pairs, the duality gaps are calculated, by changing the shape parameters, which eventually, shows that the gaps under the exponential membership function are much smaller than that under the linear membership function. This shows the practical significance of such problems in real world situations. Finally, using a linear ranking defuzzification function, we have studied some extensions of I-fuzzy linear programming problem considering fuzzy parameters and constraints. The techniques developed in this paper can be used to discuss quadratic, nonlinear and fractional I-fuzzy optimization problems. Moreover, the results discussed in the manuscript also encourage to explore the study of duality gaps and its advantages for the I-fuzzy multiobjective optimization problems under linear/exponential membership and non-membership functions. In future, some of these issues will be addressed.

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