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Properties of fuzzy relations and aggregation process in decision making

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Abstract

In this contribution connections between input fuzzy relations $R_1, \dots, R_n$ on a set X and the output fuzzy relation $R_F = F(R_1, \dots, R_n)$ are studied. F is a function of the form $F : [0,1]^n \to [0,1]$ and R_F is called an aggregated fuzzy relation. In the literature the problem of preservation, by a function F, diverse types of properties of fuzzy relations $R_1, \dots, R_n$ is examined. Here, it is considered the converse approach. Namely, fuzzy relation $R_F = F(R_1, \dots, R_n)$ is assumed to have a given property and then it is checked if fuzzy relations $R_1, \dots, R_n$ have this property. Moreover, a discussion on the mentioned two approaches is provided. The properties, which are examined in this paper, depend on their notions on binary operations $B : [0,1]^2 \to [0,1]$. By incorporating operation B these properties are generalized versions of known properties of fuzzy relations.

Keywords: Fuzzy relation, Aggregation function, Decision making.

1 Introduction

There exist diverse types of fuzzy relation properties [39, 40], for example some of the classes are presented in [20, 25, 33]. Fuzzy relations have many applications. For example in decision making problems, in the context of aggregation process (cf. [4, 3, 26]), there are considered various properties of fuzzy relations [6, 25, 33]. In multicriteria decision making, a finite set of alternatives $X = \{x_1, \dots, x_m\}$, $m \in \mathbb{N}$ and a finite set of criteria $K = \{k_1, \dots, k_n\}$, on the base of which the alternatives are evaluated, may be considered. Fuzzy relations $R_1, \dots, R_n$ on the set X corresponding to each criterion may be provided. These relations may be represented by matrices, where $R_k : X \times X \to [0,1]$, $k = 1, \dots, n$, $n \in \mathbb{N}$, $R_k(x_i, x_j) = r^k_{ij}$, $1 \leqslant i, j \leqslant m$. Fuzzy relations in such setting represent the preferences of decision makers over given alternatives. With the use of a function $F : [0,1]^n \to [0,1]$, usually an aggregation function, the aggregated fuzzy relation $R_F = F(R_1, \dots, R_n)$ is obtained. On the base of this relation the final result is determined, i.e. a solution alternative. It is useful to know which properties of fuzzy relations $R_1, \dots, R_n$ are preserved by aggregation function, i.e. it is considered a problem if R_F has the same properties as fuzzy relations $R_1, \dots, R_n$ have (cf. [25]). There exist several works contributed to the problem of preservation of fuzzy relation properties [9, 33, 35].

In this paper we consider the "converse" approach. Namely, a fuzzy relation $R_F = F(R_1, \dots, R_n)$ is assumed to have a given property and the properties of fuzzy relations $R_1, \dots, R_n$ are examined under suitable assumptions on F. This approach in studies of connections between input fuzzy relations $R_1, \dots, R_n$ and the output one R_F has not been studied in the literature for the case of B dependent properties of fuzzy relations, where $B : [0,1]^2 \to [0,1]$. Similar problem was considered in [6] but for other type of fuzzy relation properties. Precisely speaking, we assume that R_F has a given property and ask if there exist at least one relation R_i for $i \in \{1, ..., n\}$ such that it has the same property. Sometimes it appears that all relations $R_1, \dots, R_n$ have the same property as R_F has. Some issues of these problems were presented at FUZZ IEEE 2017 conference (cf. [5]). Furthermore, in this contribution a discussion on the two ways of checking connections between input relations and the output one is provided, namely the approach of checking preservation of properties and the "converse" problem. In this paper, the properties depending on binary operations $B : [0,1]^2 \to [0,1]$, i.e. generalized versions of properties given in [25] are considered. These are the following properties: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, transitivity, negative transitivity, Ferrers

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property, semitransitivity. They were introduced in [9] and considered along with the problem of preservation of these properties in aggregation process. Here appropriate assumptions on F to fulfill the required dependence are considered. Namely, assuming that $R_F \in \mathcal{FR}(X)$ has a given property we ask under what assumptions on F the input fuzzy relations $R_1,\ldots,R_n \in \mathcal{FR}(X)$ have this property. There are presented the results with the minimal assumptions on F acting as an "aggregation function". However, practically not all aggregation operators make sense in such context [23]. Averaging functions $F:[0,1]^n \to [0,1]$ (compensative functions or in other words functions fulfilling the Pareto principle) are the ones which fulfil the property $\min(x_1,\ldots,x_n) \leqslant F(x_1,\ldots,x_n) \leqslant \max(x_1,\ldots,x_n)$ for $x_1,\ldots,x_n \in [0,1]$. Taking into account the meaning of the process of aggregation, averaging functions are appropriate for the presented considerations. Examples of such functions are quasi-linear means, minimum and maximum. Note that quasi-linear means are one of the most often appearing aggregation functions in decision making real-life situations [41]. In this paper, after giving the result with the weakest assumptions on F, we put examples of averaging functions fulfilling the considered property. Moreover, for some of the properties (i.e. reflexivity, irreflexivity, connectedness, asymmetry) we present also other type than known in the literature (cf. [9]) sufficient conditions for a function F to preserve these properties of fuzzy relations. The new results and results existing in the literature [9] are presented and discussed. As a result, this paper provides some useful information for aggregation process in decision making, i.e. which aggregation functions F and which operations B are more suitable in applications if the output fuzzy relation R_F, and all or at least one inputs $R_1,\ldots,R_n$, are supposed to have the same B dependent property.

The paper is organized as follows. In Section 2, useful definitions and concepts are collected. In Section 3, notions and properties of fuzzy relations are recalled. In Section 4, the main problem of this paper is studied, i.e. the connection between the properties of aggregated fuzzy relation $R_F = F(R_1,\ldots,R_n) \in \mathcal{FR}(X)$ and the input fuzzy relations $R_1,\ldots,R_n$ as well as a discussion on the obtained results and the ones on preservation of the properties are provided. To sum up, in Section 5 there are given algorithms using the presented in the paper considerations along with the illustrating examples.

2 Basic notions

We recall the notion of an aggregation function, some properties and classes of aggregation functions, and concepts of dominance as well as commuting property.

2.1 Aggregation functions and their properties

We present here some classes and properties of aggregation functions.

Definition 2.1. [14] *Let $n \geqslant 2$. A function $F:[0,1]^n \to [0,1]$ is called an aggregation function, if it is increasing with respect to any variable*

$$\underset{s_1,\ldots,s_n,t_1,\ldots,t_n \in [0,1]}{\forall} \left(\underset{1\leqslant i\leqslant n}{\forall} s_i \leqslant t_i\right) \Rightarrow F(s_1,\ldots,s_n) \leqslant F(t_1,\ldots,t_n) \tag{1}$$

and $F(0,\ldots,0)=0$, $F(1,\ldots,1)=1$.

Example 2.2. *Let $\varphi:[0,1]\to\mathbb{R}$ be a continuous, strictly monotonic function. A quasi-linear mean (cf. [25], p. 112) is an aggregation function, where*

$$F(t_1,\ldots,t_n) = \varphi^{-1}\left(\sum_{i=1}^{n} w_i\varphi(t_i)\right), \qquad \sum_{i=1}^{n} w_i = 1, w_i \in [0,1]. \tag{2}$$

If in (2) weights $w_i = \frac{1}{n}$ for any $i=1,\ldots,n$, we get a quasi-arithmetic mean. Particular cases of quasi-linear means are weighted arithmetic means $F(t_1,\ldots,t_n) = \sum_{i=1}^{n} w_i t_i$, or weighted geometric means $F(t_1,\ldots,t_n) = \prod_{i=1}^{n} t_i^{w_i}$ for $w_i > 0$.

Other than (1) type of monotonicity (increasingness) for functions $F:[0,1]^n \to [0,1]$ (also aggregation functions) is given below.

Definition 2.3. [14] *A function $F:[0,1]^n \to [0,1]$ is called jointly strictly monotone, if for all $s_1,\ldots,s_n,t_1,\ldots,t_n \in [0,1]$*

$$\left(\underset{1\leqslant i\leqslant n}{\forall} s_i < t_i\right) \Rightarrow F(s_1,\ldots,s_n) < F(t_1,\ldots,t_n). \tag{3}$$

Example 2.4. [14] *Aggregation functions which are jointly strictly monotone are: quasi-linear means,* min*,* max*, product* $F(t_1, \ldots, t_n) = \prod_{i=1}^{n} t_i$.

Other useful in our consideration properties of functions (aggregation functions) in $[0, 1]$ are p–boundary condition and possessing of zero element.

Definition 2.5. *Let* $n \in \mathbb{N}$. *A function* $F : [0, 1]^n \to [0, 1]$*:*
- *has a zero element* $z \in [0, 1]$ *(cf.* [14]*, Definition 10), if*

$$\forall_{1\leqslant k\leqslant n} \quad \forall_{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n\in[0,1]} \quad F(x_1, ..., x_{k-1}, z, x_{k+1}, ..., x_n) = z,$$

- *is without zero divisors (cf. [14], Definition 10), if*

$$\forall_{x_1,\ldots,x_n\in[0,1]} \quad (F(x_1, ..., x_n) = z \;\Rightarrow\; (\exists_{1\leqslant k\leqslant n} x_k = z)),$$

- *fulfils strong* p*–boundary condition (cf. [8]), if for* $p \in [0, 1]$

$$\forall_{t_1,\ldots,t_n\in[0,1]} \quad (F(t_1, ..., t_n) = p \;\Leftrightarrow\; (\forall_{1\leqslant i\leqslant n} t_i = p)).$$

In the literature (cf. [34]), strong p–boundary condition for $p = 1$ is also called "one strict" condition. Important classes of binary aggregation functions consist of fuzzy conjunctions and disjunctions.

Definition 2.6. cf. [17], [22] $F : [0, 1]^2 \to [0, 1]$ *is called a fuzzy conjunction if it is increasing and* $F(1, 1) = 1$, $F(0, 0) = F(0, 1) = F(1, 0) = 0$. $F : [0, 1]^2 \to [0, 1]$ *is called a fuzzy disjunction if it is increasing and* $F(0, 0) = 0$, $F(1, 1) = F(0, 1) = F(1, 0) = 1$.

In our further considerations we will need the concept of a dual function.

Definition 2.7. (cf. [14], p. 31) *Let* $F : [0, 1]^n \to [0, 1]$. *A function* F^d *is called a dual function to* F*, if for all* $x_1, \ldots, x_n \in [0, 1]$ $F^d(x_1, \ldots, x_n) = 1 - F(1 - x_1, \ldots, 1 - x_n)$. F *is called a self-dual function, if it holds* $F = F^d$.

As a result we see that a fuzzy disjunction is a dual notion to a fuzzy conjunction. By Definition 2.6 it follows that a fuzzy conjunction has a zero element $z = 0$ (dually, fuzzy disjunction has a zero element $z = 1$). Below, there are presented some subclasses of fuzzy conjunctions and disjunctions.

Definition 2.8. *An operation* $F\colon [0, 1]^2 \to [0, 1]$ *is called:*

- *t-seminorm* [38] *(a semicopula* [2]*, a conjunctor* [15]*) if it is a fuzzy conjunction which has a neutral element* 1*,*
- *a t-semiconorm if it is a fuzzy disjunction which has a neutral element* 0*.*

Definition 2.9. *An operation* $F\colon [0, 1]^2 \to [0, 1]$ *is called:*

- *an overlap function* [12] *if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition* $F(s, t) = 1$ *if and only if* $s = t = 1$*,*
- *a grouping function* [13] *if it is a commutative, continuous fuzzy disjunction without zero divisors, fulfilling condition* $F(s, t) = 0$ *if and only if* $s = t = 0$*.*

Triangular norms and conorms are examples of fuzzy conjunctions and disjunctions, respectively.

Definition 2.10 ([29])**.** *A triangular norm* $T : [0, 1]^2 \to [0, 1]$ *(respectively a triangular conorm* $S : [0, 1]^2 \to [0, 1]$*) is an arbitrary associative, commutative, increasing in both variables function having a neutral element* $e = 1$ *(respectively* $e = 0$*).*

Basic triangular norms and conorms are presented below along with their well-known notions.

Example 2.11 ([29], p. 6)**.** *For arbitrary* $s, t \in [0, 1]$ *we have the following t-norms and t-conorms:*
- *lattice,* $T_M(s, t) = \min(s, t)$, $S_M(s, t) = \max(s, t)$,
- *Łukasiewicz,* $T_L(s, t) = \max(s + t - 1, 0)$, $S_L(s, t) = \min(s + t, 1)$,
- *product,* $T_P(s, t) = st$, $S_P(s, t) = s + t - st$,
- *drastic,* $T_D(s,t) = \begin{cases} 0, & s,t < 1 \\ s, & t = 1 \\ t, & s = 1 \end{cases}$, $S_D(s,t) = \begin{cases} 1, & s,t > 0 \\ s & t = 0 \\ t, & s = 0 \end{cases}$.

Example 2.12. *Triangular norms, t–seminorms, overlap functions and quasi–linear means (including as a special case the weighted arithmetic means) have strong* 1*–boundary property. Triangular conorms, t–semiconorms, grouping functions and quasi–linear means (including as a special case the weighted arithmetic means) have strong* 0*–boundary property.*

Special case of aggregation functions are the ones which are idempotent and they form the class of so called *averaging functions* (functions that are bounded by minimum and maximum), i.e. averaging functions fulfil the following condition

$$\underset{t_1,\ldots,t_n\in[0,1]}{\forall} \min(t_1,\ldots,t_n) \leqslant F(t_1,\ldots,t_n) \leqslant \max(t_1,\ldots,t_n).$$

Proposition 2.13 ([25], Proposition 5.1)**.** *Every function* $F:[0,1]^n \to [0,1]$ *increasing in each variable and idempotent, i.e.* $F(t,\ldots,t)=t$ *for any* $t\in[0,1]$*, is an averaging function.*

Examples of averaging aggregation functions are given in Example 2.2. It is worth to mention that an averaging function need not to be an aggregation function. For example, Lehmer means (cf. [11]) are averaging functions but they do not fulfil the monotonicity condition (1) (as a result Lehmer means are not aggregation functions).

Aggregation functions such that $F \leqslant \min$ are called *conjunctive aggregation functions* and aggregation functions such that $F \geqslant \max$ are called *disjunctive aggregation functions* (cf. [34]). Below, we give sufficient conditions for a function F to fulfil $F \leqslant \min$ or $F \geqslant \max$.

Proposition 2.14 (cf. [9])**.** *If a function* $F:[0,1]^n \to [0,1]$ *is increasing in each variable and has a neutral element* $e=1$*, i.e.*

$$\underset{t\in[0,1]}{\forall}\ \underset{1\leqslant k\leqslant n}{\forall} F(1,\ldots,1,t,1,\ldots,1)=t,$$

where t *is at the* k*-th position, then* $F \leqslant \min$*.*
If a function $F:[0,1]^n \to [0,1]$ *is increasing in each variable and has a neutral element* $e=0$*, i.e.*

$$\underset{t\in[0,1]}{\forall}\ \underset{1\leqslant k\leqslant n}{\forall} F(0,\ldots,0,t,0,\ldots,0)=t,$$

where t *is at the* k*-th position, then* $F \geqslant \max$*.*

Triangular norms or t-seminorms are examples od conjunctive aggregation functions and t-conorms or t-semiconorms are examples of disjunctive aggregation functions.

2.2 Dominance and commuting property

There exist diverse connections between functions. For example, we may consider dominance of one function over another or a commuting property.

Definition 2.15. cf. [37], [35]*Let* $m,n\in\mathbb{N}$*. A function* $F:[0,1]^m \to [0,1]$ *dominates a function* $G:[0,1]^n \to [0,1]$ *(*$F \gg G$*), if for arbitrary matrix* $[a_{ik}] = A \in [0,1]^{m\times n}$ *we have*

$$F(G(a_{11},\ldots,a_{1n}),\ldots,G(a_{m1},\ldots,a_{mn})) \geqslant G(F(a_{11},\ldots,a_{m1}),\ldots,F(a_{1n},\ldots,a_{mn})). \tag{4}$$

Proposition 2.16. *Let* $G:[0,1]^n \to [0,1]$ *be increasing,* $m=2$ *in (4). Thus* $\min \gg G$ *([35], p. 16) and* $G \gg \max$ *(cf.[21], Theorem 2), so we have respectively*

$$\underset{s_1,\ldots,s_n,t_1,\ldots,t_n\in[0,1]}{\forall} \min(G(s_1,...,s_n),G(t_1,...,t_n)) \geqslant G(\min(s_1,t_1),...,\min(s_n,t_n))$$

and

$$\underset{s_1,\ldots,s_n,t_1,\ldots,t_n\in[0,1]}{\forall} G(\max(s_1,t_1),...,\max(s_n,t_n)) \geqslant \max(G(s_1,...,s_n),G(t_1,...,t_n)).$$

Example 2.17 (cf. [35])**.** *A weighted arithmetic mean dominates* T_L*, a weighted geometric mean dominates* T_P*.*

Special case of dominance is a commuting property.

Definition 2.18. [36] *Let* $m,n\in\mathbb{N}$*. A function* $F:[0,1]^m \to [0,1]$ *commutes with a function* $G:[0,1]^n \to [0,1]$ *(or* F *and* G *are commuting), if for arbitrary matrix* $[a_{ik}] = A \in [0,1]^{m\times n}$ *we have*

$$F(G(a_{11},\ldots,a_{1n}),\ldots,G(a_{m1},\ldots,a_{mn})) = G(F(a_{11},\ldots,a_{m1}),\ldots,F(a_{1n},\ldots,a_{mn})).$$

Corollary 2.19. [36] *Two operations commute if and only if they dominate each other.*

The class of all functions commuting with the minimum is the same as the class of all functions dominating the minimum (cf. [36]).

Theorem 2.20 (cf. [35], Proposition 5.1)**.** *An increasing function* $F\colon [0,1]^n \to [0,1]$ *dominates minimum if and only if for each* $t_1,\dots,t_n \in [0,1]$ $F(t_1,\dots,t_n) = \min(f_1(t_1),\dots,f_n(t_n))$*, where* $f_k\colon [0,1] \to [0,1]$ *is increasing with* $k = 1,\dots,n$*.*

Dually, we may state that the class of all functions commuting with the maximum is the same as the class of all functions dominated by the maximum.

Theorem 2.21. *An increasing function* $F\colon [0,1]^n \to [0,1]$ *is dominated by maximum if and only if for each* $t_1,\dots,t_n \in [0,1]$*,* $F(t_1,\dots,t_n) = \max(f_1(t_1),\dots,f_n(t_n))$*, where* $f_k\colon [0,1] \to [0,1]$ *is increasing with* $k = 1,\dots,n$*.*

Example 2.22. *A weighted arithmetic mean commutes with the operation* $+$*(cf.* [7]*). A weighted geometric mean commutes with the product* T_P*. Moreover, if* F *is a dual function to a weighted geometric mean (thus* F *is an aggregation function), then* F *commutes with* S_P *(cf.* [36]*, Corollary 5).*

3 Properties of fuzzy relations

We recall suitable concepts for fuzzy relations.

Definition 3.1. [39] *A fuzzy relation on* $X \neq \emptyset$ *is a function* $R : X \times X \to [0,1]$*. The family of all fuzzy relations on* X *is denoted by* $\mathcal{FR}(X)$*.*

Remark 3.2. *If* $card(X) = n$*,* $X = \{x_1,\dots,x_n\}$*, then a relation* $R \in \mathcal{FR}(X)$ *may be presented by a matrix* $R = [r_{ik}]$*, where* $r_{ik} = R(x_i,x_k)$*,* $i,k = 1,\dots,n$*.*

We recall the notions of composition and dual composition. In the notion of dual composition we follow the notational convention of Bandler and Kohout (cf. [1]).

Definition 3.3 (cf. [39])**.** *Let* $B\colon [0,1]^2 \to [0,1]$*. A* sup*–*$B$*–composition of relations* $R,W \in \mathcal{FR}(X)$ *is the relation* $(R \circ_B W) \in \mathcal{FR}(X)$ *such that for any* $(x,z) \in X \times X$ *it holds*

$$(R \circ_B W)(x,z) = \sup_{y\in X} B(R(x,y),W(y,z)). \tag{5}$$

Dually, an inf*–*B*–composition of fuzzy relations* $R,W \in \mathcal{FR}(X)$ *is the relation* $(R \lhd_B W) \in \mathcal{FR}(X)$ *such that for any* $(x,z) \in X \times X$ *it holds*

$$(R \lhd_B W)(x,z) = \inf_{y\in X} B(R(x,y),W(y,z)). \tag{6}$$

Now, we give the list of the fuzzy relation properties considered in this paper.

Definition 3.4. *Let* $B, B_1, B_2 : [0,1]^2 \to [0,1]$ *be binary operations. Relation* $R \in \mathcal{FR}(X)$ *is:*

- *reflexive, if* $\forall_{x\in X}\ R(x,x) = 1$,
- *irreflexive, if* $\forall_{x\in X}\ R(x,x) = 0$,
- *symmetric, if* $\forall_{x,y\in X}\ R(x,y) = R(y,x)$,
- *totally* B*-connected, if* $\forall_{x,y\in X}\ B(R(x,y),R(y,x)) = 1$,
- B*-connected, if* $\forall_{x,y\in X, x\neq y}\ B(R(x,y),R(y,x)) = 1$,
- B*-asymmetric, if* $\forall_{x,y\in X}\ B(R(x,y),R(y,x)) = 0$,
- B*-antisymmetric, if* $\forall_{x,y\in X, x\neq y}\ B(R(x,y),R(y,x)) = 0$,
- B*-transitive, if* $\forall_{x,y,z\in X}\ B(R(x,y),R(y,z)) \leqslant R(x,z)$,
- *negatively* B*-transitive, if* $\forall_{x,y,z\in X}\ B(R(x,y),R(y,z)) \geqslant R(x,z)$,
- B_1*-*B_2*-Ferrers, if* $\forall_{x,y,z,w\in X}\ B_1(R(x,y),R(z,w)) \leqslant B_2(R(x,w),R(z,y))$,
- B_1*-*B_2*-semitransitive, if* $\forall_{x,y,z,w\in X}\ B_1(R(x,w),R(w,y)) \leqslant B_2(R(x,z),R(z,y))$.

In Definition 3.4, an operation B (respectively B_1) takes place of a t-norm or an operation B (respectively B_2) takes place of a t-conorm in the properties considered by Fodor and Roubens [25]. Particularly, minimum or maximum in the standard properties introduced for fuzzy relations by Zadeh [40] are replaced with B (B_1, B_2, respectively). This is why these properties may be considered as a generalization of classically used properties. This is also justified by connections between the classically used "min"/"max" depending properties and the ones given in Definition 3.4 (cf. Propositions 3.10 - 3.12 and Proposition 2.14). It is natural to consider fuzzy conjunctions, for example in definition of transitivity, and fuzzy disjunctions for example in definition of connectedness properties. This is a generalization of the standard properties given by Zadeh, and then by Fodor and Roubens (which is also a generalization of the meaning of the adequate properties for crisp relations). For example in [19], [16], [18] in the context of preference relations, transitivity with respect to a conjunction (instead of a t-norm) was considered as a suitable type of B-transitivity. However, not all conjunctions are adequate for defining transitivity. For reflexive fuzzy relations (such as large preference relations) there should be considered conjunctions upper bounded by the minimum only (cf. [17]). Examples of such conjunctions are the ones with a neutral element 1, like t-norms or more generally t-seminorms. As a result we see that the choice of an operation B is important from practical point of view. Let us see also the following two statements.

Proposition 3.5. *Let B have a zero element 0 and have no zero divisors. Then B-asymmetry (B-antisymmetry) is equivalent to* min-*asymmetry (*min-*antisymmetry).*

Proof. Let $R \in \mathcal{FR}(X)$, B have a zero element 0 and have no zero divisors. B-asymmetry is equivalent to

$$B(R(x,y), R(y,x)) = 0 \Leftrightarrow R(x,y) = 0 \vee R(y,x) = 0 \Leftrightarrow \min(R(x,y), R(y,x)) = 0,$$

which is equivalent to the fact that R is min-asymmetric. The proof for antisymmetry is analogous. □

Dually, we may prove

Proposition 3.6. *Let B have a zero element 1 and have no zero divisors. Then total B-connectedness (B-connectedness) is equivalent to total* max-*connectedness (*max-*connectedness).*

As a result B-asymmetry properties and B-connectedness properties with respect to operations B without zero divisors (with adequate zero element of B) are rather strong properties to be fulfilled in practice, for example for preference relations obtained on the basis of decision makers judgements.

It is worth to mention that some properties may be characterized with the use of composition. This may be useful in practice for the purpose of checking if the adequate properties are fulfilled (cf. Section 5).

Corollary 3.7. *Let $B_1, B_2 : [0,1]^2 \to [0,1]$. A relation $R \in \mathcal{FR}(X)$ is B_1–B_2–semitransitive if and only if $R \circ_{B_1} R \leqslant R \triangleleft_{B_2} R$.*

Proof. Let $R \in \mathcal{FR}(X)$. By (5), (6) and definition of B_1–B_2–semitransitivity we get

$$R \circ_{B_1} R \leqslant R \triangleleft_{B_2} R \Leftrightarrow \underset{x,y \in X}{\forall} \sup_{w \in X} B_1(R(x,w), R(w,y)) \leqslant \inf_{z \in X} B_2(R(x,z), R(z,y))$$

$$\Leftrightarrow \underset{x,y,z,w \in X}{\forall} B_1(R(x,w), R(w,y)) \leqslant B_2(R(x,z), R(z,y)),$$

which means that R is B_1–B_2–semitransitive. □

As a result we see that semitransitivity is a weaker property than transitivity. Analogously we may prove that

Corollary 3.8. *(cf. [17]) Let $B : [0,1]^2 \to [0,1]$. A relation $R \in \mathcal{FR}(X)$ is B–transitive if and only if $R \circ_B R \leqslant R$. A relation $R \in \mathcal{FR}(X)$ is negatively B–transitive if and only if $R \triangleleft_B R \geqslant R$.*

Thorough study on transitive and negatively transitive fuzzy relations, in the context of preference relations, one may find in [18]. The property of transitivity is studied with respect to a fuzzy conjunction. The authors consider almost all the possible generators and therefore all the possible strict preference relations obtained from the reflexive relation. They also provide a general expression for the negative transitivity that those relations satisfy.

Proposition 3.9. *Let $B_1, B_2 : [0,1]^2 \to [0,1]$. If $R \in \mathcal{FR}(X)$ is B_1-transitive (respectively negatively B_2-transitive), then R is B_1–B_2–semitransitive for $B_2 = (B_1)^d$ (respectively $B_1 = (B_2)^d$).*

Proof. If $R \in \mathcal{FR}(X)$ is B_1-transitive, then $R \circ_{B_1} R \leqslant R$, and also by duality principle, R is negatively B_2-transitive for $B_2 = (B_1)^d$, so $R \leqslant R \triangleleft_{B_2} R$. As a result we get B_1–B_2–semitransitivity of R. □

There are many connections between properties of fuzzy relations. These are dependencies between diverse classes of properties (as for example transitivity and semitransitivity), which were above presented, and dependencies inside one class, for example semitransitivity with respect to diverse operations B, which will be given below.

By the definition of properties and the fact that values of fuzzy relations and operation B belong to the unit interval $[0,1]$ we get the monotonicity in the family of fuzzy relations with respect to a given property and operation B.

Proposition 3.10. *Let $B, B^* : [0,1]^2 \to [0,1]$, $B \leqslant B^*$. Thus if $R \in \mathcal{FR}(X)$ is totally B-connected (respectively B-connected, negatively B-transitive), then it is also totally B^*-connected (respectively B^*-connected, negatively B^*-transitive).*

Proof. Let $x, y \in X$. By assumption $B(R(x,y), R(y,x)) = 1$. Since $B, B^* : [0,1]^2 \to [0,1]$, $B \leqslant B^*$, we get $B^*(R(x,y), R(y,x)) = 1$. What proves the dependence for total B-connectedness. The remaining properties may be justified in a similar way. □

Analogously we may prove that

Proposition 3.11. *(cf. [17]) Let $B, B^* : [0,1]^2 \to [0,1]$, $B^* \leqslant B$. Thus if $R \in \mathcal{FR}(X)$ is B-asymmetric (respectively B-antisymmetric, B-transitive), then it is also B^*-asymmetric (respectively B^*-antisymmetric, B^*-transitive).*

As a result, if it comes for example to the transitivity property with respect to the well known t-norms, since $T_D \leqslant T_L \leqslant T_P \leqslant T_M$, T_L-transitivity turns out to be the most useful from practical point of view property (one of the easiest to be fulfilled in practice, cf. [10]).

Proposition 3.12. *Let $B_i, B_i^* : [0,1]^2 \to [0,1]$, $i = 1,2$, $B_1^* \leqslant B_1$, $B_2 \leqslant B_2^*$. Thus if $R \in \mathcal{FR}(X)$ is B_1-B_2-Ferrers (respectively B_1-B_2-semitransitive), then it is also B_1^*-B_2^*-Ferrers (respectively B_1^*-B_2^*-semitransitive).*

Proof. Let $x, y, z, w \in X$. We will prove the dependence for the Ferrers property. Thus by assumptions

$$B_1^*(R(x,y), R(z,w)) \leqslant B_1(R(x,y), R(z,w)) \leqslant B_2(R(x,w), R(z,y)) \leqslant B_2^*(R(x,w), R(z,y)),$$

which means that relation R has B_1^*-B_2^*-Ferrers property. The proof for B_1^*-B_2^*-semitransitivity is analogous. □

4 Aggregation of fuzzy relations

Now, dependencies between relations $R_1, \ldots, R_n$ on a set X and the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$ in the context of aggregation process will be investigated. These properties are related to B-properties listed in Definition 3.4. Our aim is to ask under what assumptions on $F : [0,1]^n \to [0,1]$, if $R_F \in \mathcal{FR}(X)$ has a given property, there exist at least one $R_i \in \mathcal{FR}(X)$ for $i \in \{1, ..., n\}$ such that it has the same property possessed by the relation R_F. This is the converse problem to the one considered in [9], where preservation of these type of properties was discussed, i.e. it was studied if for every relation $R_1, \ldots, R_n \in \mathcal{FR}(X)$ having a given property, R_F also has this property [25].

Firstly, we will consider reflexivity and irreflexivity. It is easy to observe the following result.

Proposition 4.1. *Let $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ be reflexive. If $F \leqslant \min$, then all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are reflexive.*

We may give also the following dependence which follows straightforwardly from definition of strong 1–boundary condition.

Proposition 4.2. *Let F fulfil strong 1–boundary condition. $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is reflexive if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are reflexive.*

The next example shows that the condition presented in Proposition 4.1 is only a sufficient one.

Example 4.3. *Let $card(X) = 2$. We consider fuzzy relations with matrices:*

$$R_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, R_F = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix},$$

where F is the arithmetic mean. Relation R_F is reflexive and relations R_1, R_2 also have this property, but F is an averaging function.

Remark 4.4. *Conditions on function F given in Propositions 4.1, 4.2 are not equivalent. However, if $F \leqslant \min$, then F fulfils strong 1–boundary condition. Namely*

$$\min(t_1, \ldots, t_n) \geqslant F(t_1, \ldots, t_n) = 1 \Leftrightarrow \min(t_1, \ldots, t_n) = 1 \Leftrightarrow t_1 = \cdots = t_n = 1.$$

The converse is not true, since the weighted arithmetic mean fulfils strong 1–boundary condition but it attains values that are strictly greater than the minimum (cf. Proposition 2.13).

In virtue of Example 2.12 and Propositions 4.1, 4.2 we get

Corollary 4.5. *Let F be an overlap function, a t-seminorm, a t-norm or a quasi–linear mean (a weighted arithmetic mean). Relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is reflexive if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are reflexive.*

Dually to reflexivity we obtain the following results for irreflexivity.

Proposition 4.6. *Let $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ be irreflexive. If $F \geqslant \max$, then all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are irreflexive.*

Proposition 4.7. *Let F fulfil strong 0–boundary condition. $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is irreflexive if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are irreflexive.*

Example 4.8. *The condition on function F presented in Proposition 4.6 is only the sufficient one. Let $card(X) = 2$. We consider fuzzy relations with matrices:*

$$R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, R_F = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix},$$

where F is the arithmetic mean. Relation R_F is irreflexive and relations R_1, R_2 also have this property, but F is an averaging function.

Remark 4.9. *Conditions on F given in Propositions 4.6, 4.7 are not equivalent. However, if $F \geqslant \max$, then F fulfils strong 0–boundary condition. The converse is not true, since the weighted arithmetic mean fulfils strong 0–boundary condition but it attains values that are strictly smaller than the maximum (cf. Proposition 2.13 and Remark 4.4).*

In virtue of Example 2.12 and Propositions 4.6, 4.7 we get

Corollary 4.10. *Let F be a grouping function, a t-semiconorm, a t-conorm or a quasi–linear mean (a weighted arithmetic mean). If a fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is irreflexive, then all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are also irreflexive.*

Let us notice, that Proposition 4.2 and Proposition 4.7 also present sufficient conditions for preservation of reflexivity and irreflexivity, respectively. In [9] preservation of reflexivity (respectively irreflexivity) by a function $F : [0,1]^n \to [0,1]$ was characterized by having an idempotent element 1 (respectively 0) by F. Certainly, strong 1–boundary condition (respectively strong 0–boundary condition) for F is a stronger assumption than having an idempotent element 1 (respectively having an idempotent element 0) by F.

If it comes to the ,,converse problem" for symmetry we did not obtain satisfactory results. If we would assume that F is injective with respect to all arguments, then if a fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is symmetric, then also all relations $R_1, \ldots, R_n$ are symmetric. However, injectivity with respect to all arguments, as a property itself, is not so easy to be fulfilled (for example arithmetic mean, minimum, maximum, geometric mean, t-norms and t-conorms, are not injective with respect to all arguments). Assuming injectivity with respect to a fixed variable, for example to the first variable, $F(x_1, y) = F(x_2, y) \Rightarrow x_1 = x_2$ for all $y \in [0,1]$ (in such case we should also assume that F is without zero element) we get a condition which is not enough to obtain the required result. This is shown by the following counter-example. Let $card(X) = 2$. We consider fuzzy relations with matrices:

$$R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; R_F = \frac{R_1 + R_2}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

Relations R_1, R_2 are not symmetric while $R_F = \frac{R_1+R_2}{2}$ is symmetric and the arithmetic mean is injective with a fixed variable (both the first and the second one). However, for decision making problems if we consider the problem of preservation of fuzzy relation properties, we get simple result that each operation $F : [0,1]^n \to [0,1]$ preserves the property of symmetry [9]. As a result, the approach of checking symmetry of each individual relations $R_1, \ldots, R_n$ is a better way than applying the "converse" method.

Now, we consider asymmetry (antisymmetry) and connectedness (total connectedness) properties.

Proposition 4.11. *Let $B : [0,1]^2 \to [0,1]$ and $F : [0,1]^n \to [0,1]$ commute, F fulfil strong 0–boundary condition. A fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B-asymmetric (respectively B-antisymmetric) if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are B-asymmetric (respectively B-antisymmetric).*

Proof. Let $x, y \in X$. If R_F is B-asymmetric, i.e. $B(F(R_1(x,y), \ldots, R_n(x,y)), F(R_1(y,x), \ldots, R_n(y,x))) = 0$ then by commuting property of F and B we get $F(B(R_1(x,y), R_1(y,x)), \ldots, B(R_n(x,y), R_n(y,x))) = 0$ and since F fulfils strong 0–boundary condition we obtain $B(R_i(x,y), R_i(y,x)) = 0$ which means that R_i are B-asymmetric for $i = 1, \ldots, n$. Proof for B-antisymmetry is analogous. □

By Proposition 4.11 and Example 2.22 we obtain

Corollary 4.12. *Let $B = T_P$, F be a weighted geometric mean. A fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B-asymmetric (respectively B-antisymmetric) if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are B-asymmetric (respectively B-antisymmetric).*

Example 4.13. *The conditions on F and B given in Proposition 4.11 are only sufficient for simultaneous B-asymmetry (B-antisymmetry) of R_F and $R_1, \ldots, R_n$. If $R_1 = R_2 \equiv 0$, then $R_F \equiv 0$, where F is the arithmetic mean and $B = T_P$. However, the arithmetic mean and product do not commute, i.e. it is not true that $\frac{ab+cd}{2} = \frac{a+c}{2} \cdot \frac{b+d}{2}$ for any $a, b, c, d \in [0,1]$ (it is enough to take $a = 0$, $c = 1$, $b = 0.2$, $d = 0.3$).*

The obtained results for total B-connectedness and B-connectedness, are dual to the ones for B-asymmetry and B-antisymmetry, respectively.

Proposition 4.14. *Let $B : [0,1]^2 \to [0,1]$ and $F : [0,1]^n \to [0,1]$ commute, F fulfil strong 1–boundary condition. A fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B-connected (respectively totally B-connected) if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are B-connected (respectively totally B-connected).*

By Proposition 4.14 and Example 2.22 we obtain

Corollary 4.15. *Let $B = S_P$, F be a dual aggregation function to a weighted geometric mean. A fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B-connected (respectively totally B-connected) if and only if all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ are B-connected (respectively totally B-connected).*

Example 4.16. *The conditions given in Proposition 4.14 are only sufficient for simultaneous B-connectedness (total B-connectedness) of R_F and $R_1, \ldots, R_n$. If $R_1 = R_2 \equiv 1$, then $R_F \equiv 1$, where F is the arithmetic mean and $B = T_P$. However, as we noticed in Example 4.13, the arithmetic mean and product do not commute.*

Remark 4.17. *In Proposition 4.11 and Proposition 4.14 we have a sufficient condition for simultaneous B-asymmetry (B-antisymmetry) and B-connectedness (total B-connectedness) of input fuzzy relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ and the output $R_F \in \mathcal{FR}(X)$. As a result in these propositions we get other sufficient conditions for preservation of adequate properties by aggregating functions.*

In the following two statements we recall the results connected with preservation of B-asymmetry (B-antisymmetry) and similarly B-connectedness (total B-connectedness). These results characterize functions F that preserve the mentioned B dependent properties (let us notice that Proposition 4.11 and Proposition 4.14 present assumptions on F and B for simultaneous possessing a given property by inputs $R_1, \ldots, R_n \in \mathcal{FR}(X)$ and the output $R_F \in \mathcal{FR}(X)$).

Theorem 4.18. [9] *Let $card(X) \geqslant 2$, B have a zero element 0 and has no zero divisors. A function F preserves B-asymmetry (B-antisymmetry) if and only if it satisfies the following condition for all $s, t \in [0,1]^n$*

$$\underset{1 \leqslant k \leqslant n}{\forall} \min(s_k, t_k) = 0 \Rightarrow \min(F(s), F(t)) = 0.$$

Theorem 4.19. [9] *Let $card(X) \geqslant 2$, B have a zero element 1 and has no zero divisors. A function F preserves total B-connectedness (B-connectedness) if and only if it satisfies the following condition for all $s, t \in [0,1]^n$*

$$\underset{1 \leqslant k \leqslant n}{\forall} \max(s_k, t_k) = 1 \Rightarrow \max(F(s), F(t)) = 1. \tag{7}$$

As we see, Theorem 4.18 and Theorem 4.19 deal with operations B without zero divisors. This is a strong requirement since by Propositions 3.5 and 3.6, B-asymmetry (B-antisymmetry) and total B-connectedness (B-connectedness) for B without zero divisors (with adequate zero element) are equivalent to min-asymmetry (min-antisymmetry) and total B-connectedness (B-connectedness), respectively. As a result, such B-properties are strong to be fulfilled in practice.

It is worth to mention that in Proposition 4.11 and Proposition 4.14 we obtained sufficient conditions on F for preservation of B-asymmetry (B-antisymmetry) and B-connectedness (total B-connectedness) - not necessarily for operations B without zero divisors. Moreover, the presented conditions on F and B in Proposition 4.11 (respectively Proposition 4.14) and Theorem 4.18 (respectively Theorem 4.19) are independent. This is shown in the following example for the case of connectedness properties. For asymmetry and antisymmetry we may consider examples obtained dually.

Example 4.20. *The following set of conditions are independent:*
1) $B : [0,1]^2 \to [0,1]$, $F : [0,1]^n \to [0,1]$, B and F commute, F fulfils strong 1-boundary condition,
2) $B : [0,1]^2 \to [0,1]$, B has a zero element 1 and is without zero divisors, $F : [0,1]^n \to [0,1]$ fulfils (7).
Indeed, let us consider $B \equiv 0$ or $B \equiv 1$ and F the arithmetic mean. Thus B and F fulfil the set of conditions 1) but they do not fulfil the set of conditions 2) (namely, $B \equiv 0$ does not have a zero element 1 and $B \equiv 1$ has a zero element 1 but it has zero divisors). Now, let $B = S_P$ and $F = \max$. Thus B and F fulfil the set of conditions 2) but they do not fulfil the set of conditions 1), since $\max$ and S_P do not commute. If they would commute, then for any $a, b, c, d \in [0,1]$

$$\max(a + b - ab, c + d - cd) = \max(a, c) + \max(b, d) - \max(a, c)\max(b, d).$$

However, for $a = d = 0$, $b = 0.9$, $c = 0.8$ we have $0.9 = \max(0.9, 0.8) < 1.7 - 0.72 = 0.98$.

Now we will turn to the remaining properties. For transitivity, negative transitivity, semitransitivity and Ferrers property we obtained only existence of an input relation $R_i \in \mathcal{FR}(X)$, $i \in \{1, ..., n\}$, that has the same property as the aggregated fuzzy relation $R_F \in \mathcal{FR}(X)$.

Theorem 4.21. *If $F : [0,1]^n \to [0,1]$ is jointly strictly monotone and $B \gg F$, then if a fuzzy relation $R_F = F(R_1, \dots, R_n) \in \mathcal{FR}(X)$ is B-transitive, then there exists a relation $R_i \in \mathcal{FR}(X)$ for $i = \{1, \dots, n\}$, which is B-transitive.*

Proof. Let $x, y, z \in X$. By assumption that R_F is B-transitive we get $B(F(R_1(x,y), ..., R_n(x,y)), F(R_1(y,z), ..., R_n(y,z))) \leqslant F(R_1(x,z), ..., R_n(x,z))$ and since $B \gg F$, $F(B(R_1(x,y), R_1(y,z)), ..., B(R_n(x,y), R_n(y,z))) \leqslant F(R_1(x,z), ..., R_n(x,z))$. Condition of joint strict monotonicity (3) is equivalent to

$$F(t_1, \dots, t_n) \leqslant F(s_1, \dots, s_n) \Rightarrow (\underset{1 \leqslant i \leqslant n}{\exists} t_i \leqslant s_i), \tag{8}$$

for any $t_1, \dots, t_n, s_1, \dots, s_n \in [0,1]$. By (8) we get B-transitivity for at least one relation $R_i \in \mathcal{FR}(X)$. □

Corollary 4.22. *By Theorem 4.21 and Examples 2.4, 2.17 we see that a weighted geometric mean F commutes with $B = T_P$, as a result T_P dominates F, so if a fuzzy relation $R_F = F(R_1, \dots, R_n) \in \mathcal{FR}(X)$ is B-transitive, then there exist at least one relation $R_1, \dots, R_n \in \mathcal{FR}(X)$ which is B-transitive.*

Remark 4.23. *Using similar condition to (8) but with universal quantifier, in Theorem 4.21 we cannot obtain the result for all relations $R_1, \dots, R_n \in \mathcal{FR}(X)$. Namely, let us consider two-argument version of it $F(a,b) \leqslant F(c,d) \Rightarrow (a \leqslant c$ and $b \leqslant d)$, $a, b \in [0,1]$. This condition is equivalent to $(a > c$ or $b > d) \Rightarrow F(a,b) > F(c,d)$, $a, b \in [0,1]$. Such function $F : [0,1]^2 \to [0,1]$ does not exist. If it would exist, then $F(0,1) > F(1,0) > F(0,1)$.*

Dually to B-transitivity we get the result for negative B-transitivity.

Theorem 4.24. *If $F : [0,1]^n \to [0,1]$ is jointly strictly monotone and $F \gg B$, then if a fuzzy relation $R_F = F(R_1, \dots, R_n) \in \mathcal{FR}(X)$ is negatively B-transitive, then there exists a relation $R_i \in \mathcal{FR}(X)$ for $i = \{1, \dots, n\}$, which is negatively B-transitive.*

By Theorem 4.24 and Example 2.17 we obtain

Corollary 4.25. *Let $B = S_P$, F be a dual aggregation function to a weighted geometric mean. If a fuzzy relation $R_F = F(R_1, \dots, R_n) \in \mathcal{FR}(X)$ is negatively B-transitive, then there exist at least one relation $R_1, \dots, R_n \in \mathcal{FR}(X)$ which is negatively B-transitive.*

Let us notice that in [9] preservation of B-transitivity and negative B-transitivity were characterized. Namely, we have the following results which characterize functions F that preserve a given B-transitivity (respectively negative B-transitivity).

Theorem 4.26. *Let card $X \geqslant 3$, B have a zero element $z = 0$. An increasing function $F\colon [0,1]^n \to [0,1]$ preserves B-transitivity if and only if $F \gg B$.*

Theorem 4.27. *Let card $X \geqslant 3$, B have a zero element $z = 1$. An increasing function $F\colon [0,1]^n \to [0,1]$ preserves negative B-transitivity if and only if $B \gg F$.*

We see that in the case of transitivity and negative transitivity checking the preservation of fuzzy relation properties is much more fruitful approach than considering the converse approach (while assuming the property of R_F, only the existence of some input fuzzy relation $R_1, \ldots, R_n$ may be obtained with the same property, cf. Theorem 4.21 and Theorem 4.24). It turns out that assumption of dominance is a useful one in both approaches, i.e. checking preservation of properties of input fuzzy relations and the "converse" approach.

Now we will consider Ferrers property and semitransitivity.

Theorem 4.28. *If $F : [0,1]^n \to [0,1]$ is jointly strictly monotone $B_1 \gg F$, $F \gg B_2$, then if a fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B_1-B_2-Ferrers (respectively B_1-B_2-semitransitive), then there exist a relation $R_i \in \mathcal{FR}(X)$ for $i = \{1, \ldots, n\}$, which is B_1-B_2-Ferrers (respectively B_1-B_2-semitransitive).*

Proof. Let $x, y, z, w \in X$. We will prove the Ferrers property. By assumption R_F is B_1-B_2-Ferrers, so

$$B_1(F(R_1(x,y),...,R_n(x,y)),F(R_1(z,w),...,R_n(z,w))) \leqslant B_2(F(R_1(x,w),...,R_n(x,w)),F(R_1(z,y),...,R_n(z,y)))$$

and since $B_1 \gg F$, $F \gg B_2$, we get

$$F(B_1(R_1(x,y),R_1(z,w)),...,B_1(R_n(x,y),R_n(z,w)))) \leqslant F(B_2(R_1(x,w),R_1(z,y)),...,B_2(R_n(x,w),R_n(z,y))).$$

By (8) we get B_1-B_2-Ferrers property for at least one relation R_i. Proof for semitransitivity is analogous. □

A weighted geometric mean (which is a jointly strictly monotone function, cf. Example 2.4) is increasing (cf. (1)), then it dominates the maximum (cf. Proposition 2.16) and T_P dominates a weighted geometric mean (cf. Example 2.22). As a result by Theorem 4.28 we get the following result

Corollary 4.29. *Let $B_1 = T_P$ and $B_2 = \max$, F be a weighted geometric mean. If a fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ is B_1-B_2-Ferrers (respectively B_1-B_2-semitransitive), then there exist a relation $R_i \in \mathcal{FR}(X) \in \mathcal{FR}(X)$ for $i = \{1, \ldots, n\}$, which is B_1-B_2-Ferrers (respectively B_1-B_2-semitransitive).*

In [9] sufficient conditions for functions $F : [0,1]^n \to [0,1]$ to preserve B_1-B_2-Ferrers property and B_1-B_2-semitransitivity were provided. Namely, we have the following results.

Theorem 4.30. *If an increasing function $F\colon [0,1]^n \to [0,1]$ fulfils $F \gg B_1$ and $B_2 \gg F$, then it preserves B_1-B_2-Ferrers property (B_1-B_2-semitransitivity).*

We see that in Theorem 4.28 we get in some sense similar assumptions relating to function F and B, i.e. some monotonicity condition for F and dominance between F and B. However, for the converse problem with the given assumptions, we have only a guarantee that one of the input fuzzy relations have the assumed property of R_F.

5 Algorithms for decision making

In this section the idea of multicriteria (or similarly multiagent) decision making is recalled. The presented problem is related to considerations provided in this paper. Fuzzy relations in such setting represent the preferences of decision makers over given alternatives. Let $card(X) = m$, $m \in \mathbb{N}$, $X = \{x_1, \ldots, x_m\}$ be a set of alternatives. In multicriteria decision making a decision maker has to choose among the alternatives with respect to a set of criteria. Let $K = \{k_1, \ldots, k_n\}$ be the set of criteria on the base of which the alternatives are evaluated. $R_1, \ldots, R_n$ be fuzzy relations corresponding to each criterion represented by matrices, where $R_k : X \times X \to [0,1]$, $k = 1, \ldots, n$, $n \in \mathbb{N}$, $R_k(x_i, x_j) = r^k_{ij}$, $1 \leqslant i, j \leqslant m$. Similarly, if we consider multiagent decision making problems, relations $R_1, \ldots, R_n$ represent the preferences of each agent and there is no criteria (certainly, we can combine these two situations, i.e. many criteria and many agents). With the use of a function $F : [0,1]^n \to [0,1]$, we aggregate given fuzzy relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$, where $n \in \mathbb{N}$. We obtain an aggregated fuzzy relation $R_F \in \mathcal{FR}(X)$ of the form

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)), \quad x, y \in X.$$

We may ask, for the need of decision making problems, if a function $F : [0,1]^n \to [0,1]$ preserves properties of fuzzy relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$, i.e. if for every relation $R_1, \ldots, R_n \in \mathcal{FR}(X)$ having this property, R_F also has this property [25]. For example, projections $P_k(t_1, \ldots, t_n) = t_k$, $k \in \{1, \ldots, n\}$ preserve each property of fuzzy relations since for $F = P_k$, we get $R_F = R_k$. In this contribution we also consider other approach, namely we assume that

$R_F \in \mathcal{FR}(X)$ has a given property and then we ask if at least one relation $R_1, \dots, R_n \in \mathcal{FR}(X)$ has this property, where $F : [0,1]^n \to [0,1]$. Properties of fuzzy relations, if they are fulfilled by fuzzy relations, may have a form of measure of consistency of choices or may provide the interpretation of choices. This is why it is important to check the connections between input fuzzy relations and the output one. From practical point of view, for multicriteria or multiagent decision making problems, it means that we check if the particular choices of decision makers are of the same type (or they are consistent) as the aggregated fuzzy relation presents.

We put here description of two algorithms (cf. Procedure 1 and Procedure 2) to obtain the final solution from a given set of alternatives for decision making problems where fuzzy relations represent preferences of decision makers over this set of alternatives. The aggregated fuzzy relation R_F is supposed to be a basis for deriving the ordering of alternatives. Ways of obtaining the final ordering of alternatives are discussed for example in [27, 28], where one of the presented methods is a 'voting' one. To find a solution alternative in decision making problems we may apply also nondominance method [30] or methods considered in [33], where also a numerical example dealing with a real-life multicriteria decision making problem (concerning water supplies in China) is given.

For Procedure 1, we will have the following inputs:
$X = \{x_1, ..., x_m\}$, $R_1, ..., R_n \in \mathcal{FR}(X)$, $\mathcal{F} = \{F|F : [0,1]^n \to [0,1]\}$ - a finite set of aggregation functions preserving a B-property, where $B \in \mathcal{B}$, $\mathcal{B} = \{B|B : [0,1]^2 \to [0,1]\}$ - a finite set of given comparable operations including constant operations B^0 and B^1, where $B^0(s,t) = 0$ and $B^1(s,t) = 1$ for each $s, t \in [0,1]$, respectively.

Procedure 1

1. Check the type of a given B-property for each $R_1, \dots, R_n \in \mathcal{FR}(X)$ and each $B \in \mathcal{B}$
2. Establish the common type of B-property (with respect to B) for relations $R_1, \dots, R_n \in \mathcal{FR}(X)$
3. Determine the relation R_F with the use of an aggregation function F

Output: the aggregated fuzzy relation $R_F \in \mathcal{FR}(X)$ with the B–property.

In the first step we check the type of the B-property for all relations $R_1, \dots, R_n \in \mathcal{FR}(X)$ and each $B \in \mathcal{B}$. We may use dependencies presented in Propositions 3.10 - 3.12. As a result in the second step of Procedure 1 we may establish the common type of the B-property for all relations $R_1, \dots, R_n \in \mathcal{FR}(X)$, i.e. we may find the strongest type of the given B-property (depending on operation B) and attribute it to each $R_1, \dots, R_n$. At this step again we may apply results from Propositions 3.10 - 3.12. Since we assume that B^0 and B^1 belong to the class $\mathcal{B}$, we always may establish the type of B-property depending on operation B. Namely, B^0-transitivity, B^0-asymmetry, B^0-antisymmetry conditions are fulfilled trivially, and similarly negative B^1-transitivity, total B^1-connectedness, B^1-connectedness, B^0-B^1-semitransitivity and B^0-B^1-Ferrers property are fulfilled trivially by definition. As a result the algorithm may be run to the end (however not giving an interesting result - not always guaranteeing a possibility to choose an alternative). In the third step of the algorithm we may apply adequate function F, preserving the B-property, to aggregate input relations $R_1, \dots, R_n \in \mathcal{FR}(X)$ (cf. [9]).

Now, we will present a numerical example illustrating Procedure 1. Let $X = \{x_1, x_2, x_3, x_4\}$ and be given fuzzy relations $R_1, R_2 \in \mathcal{FR}(X)$ which present preferences over alternatives x_1, x_2, x_3, x_4, where

$$R_1 = \begin{bmatrix} 1 & 0.2 & 0.3 & 0.5 \\ 1 & 1 & 0.3 & 0.5 \\ 0.7 & 0.6 & 1 & 0.6 \\ 0.6 & 0.6 & 0.5 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and $\mathcal{B} = \{B^0, T_D, T_L, T_P, \min, G, B^1\}$, where G is the geometric mean (so it as an overlap function) and $B^0 \leqslant T_D \leqslant T_L \leqslant T_P \leqslant \min \leqslant G \leqslant B^1$. Moreover, the set $\mathcal{F}$ consists of some aggregation functions that preserve B-transitivity for $B \in \mathcal{B}$ (for this set $\mathcal{B}$ it may be a family of quasi-linear means, cf. [9]). We see that R_1 is T_L-transitive ($R \circ_{T_L} R = R$, cf. Corollary 3.8) so it is also T_D-transitive and B^0-transitive but R_1 is not B-transitive for $B \in \{T_P, \min, G, B^1\}$. Whereas R_2 is G-transitive so it is also B-transitive for $B \in \{B^0, T_D, T_L, T_P, \min\}$. As a result both R_1 and R_2 are T_L-transitive (cf. Proposition 3.11). One of the aggregation functions preserving this type of property is a weighted arithmetic mean (cf. [9]), so fuzzy relations R_1, R_2 may be aggregated with the use of a weighted arithmetic mean, here for example $F(s,t) = 0.6s + 0.4t$. As a result, $R_F = F(R_1, R_2)$ is also T_L-transitive, where

$$R_F = \begin{bmatrix} 1 & 0.12 & 0.18 & 0.3 \\ 1 & 1 & 0.18 & 0.3 \\ 0.82 & 0.76 & 1 & 0.36 \\ 0.76 & 0.76 & 0.7 & 1 \end{bmatrix}.$$

Now, we will go to Procedure 2, where we will have the following inputs:
$X = \{x_1, ..., x_m\}$, $R_1, ..., R_n \in \mathcal{FR}(X)$, $\mathcal{F} = \{F | F : [0,1]^n \to [0,1]\}$ - a finite set of aggregation functions including quasi-linear means, $\mathcal{B} = \{B | B : [0,1]^2 \to [0,1]\}$ - a finite set of given comparable operations including constant operations B^0 and B^1.

Procedure 2

1. Determine the relation R_F with the use of an aggregation function F
2. Check if R_F has the B–property for each $B \in \mathcal{B}$

Output: the aggregated fuzzy relation $R_F \in \mathcal{FR}(X)$ with the B–property (and guarantee that at least one/all relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ have the B–property).

In Procedure 2 we apply aggregation functions that are suitable for the 'converse' problem to preservation of properties. Such functions F were described in this contribution (cf. Propositions 4.1 - 4.14 and Theorems 4.21 - 4.28). Quasi-linear means are useful examples of aggregation functions both from practical point of view and the assumptions presented in the mentioned above results. This is why it is worth to include them in the set $\mathcal{F}$.

Below we give a numerical example illustrating Procedure 2. Let $X = \{x_1, x_2, x_3\}$ and be given fuzzy relations $R_1, R_2 \in \mathcal{FR}(X)$ which present preferences over alternatives x_1, x_2, x_3, where R_F was obtained with a use of the geometric mean, where

$$R_1 = \begin{bmatrix} 1 & 0 & 0.8 \\ 0.7 & 1 & 0.75 \\ 0.2 & 0.25 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1 & 0.3 & 0.8 \\ 0.7 & 1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix},$$

$\mathcal{B} = \{B^0, T_D, T_L, T_P, \min, B^1\}$, $B^0 \leqslant T_D \leqslant T_L \leqslant T_P \leqslant \min \leqslant B^1$ and

$$R_F = \begin{bmatrix} 1 & 0 & 0.8 \\ 0.7 & 1 & 0.6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Relation $R_F = F(R_1, R_2)$ is min-transitive and also T_P-transitive (cf. Proposition 3.11). As a result by Theorem 4.21 and Corollary 4.21 at least one relation R_1 or R_2 should be T_P-transitive. In this example, it turns out that R_1 is not T_P-transitive while R_2 fulfils T_P-transitivity condition (cf. Corollary 3.8).

Now we will turn to the the problem of determining complexity of the presented algorithms. Fuzzy relations $R_1, \ldots, R_n$ are defined on a set X consisting of m elements, so complexity of algorithms will depend on the variable m (the size of a matrix representing $R_1, \ldots, R_n$). Taking into account that the remaining operations in the presented algorithms should be performed at maximum n times (n is the finite number of fuzzy relations) we see that with respect to the complexity, both algorithms described in Procedure 1 and Procedure 2 (however with another number of steps) are of the same time profitability.

Corollary 5.1. *Algorithms described in Procedure 1 and Procedure 2 take the same computational time complexity for each property. This is polynomial time complexity $O(m^k)$, where $m, k \in \mathbb{N}$. The degree of polynomial depends on a given property: reflexivity/irreflexivity $O(m)$, connectedness/asymmetry $O(m^2)$, transitivity/semitransitivity $O(m^3)$, Ferrers property $O(m^4)$.*

However, in the converse approach to checking preservation of properties for fuzzy relations we may not always have guarantee that all input fuzzy relations have a given property. Namely, this is the case of transitivity, negative transitivity, Ferrers property and semitransitivity.

6 Conclusions

In this contribution the study on connections between properties of fuzzy relation $R_F = F(R_1, \ldots, R_n) \in \mathcal{FR}(X)$ and properties of input fuzzy relations $R_1, \ldots, R_n \in \mathcal{FR}(X)$ was conveyed. Conditions for functions $F : [0,1]^n \to [0,1]$ to fulfill each given dependence were presented. Moreover, a discussion on these results and the results on preservation of properties of fuzzy relations was provided. It is worth to mention that in some cases new sufficient conditions for the preservation of properties of $R_1, \ldots, R_n$ by a function F in aggregation process were obtained. For many properties a weighted geometric mean turned out to be a useful tool as an aggregation function to fulfill the required dependencies. Finally, suitable decision making algorithms along with illustrating examples were provided. There were analyzed the time complexities of the given algorithms.

The possible future generalization of the presented research may be related to other fields such as for example intuitionistic fuzzy relations or interval-valued fuzzy relations (cf. [24], [31], [32]).

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