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Quantale-valued fuzzy Scott topology

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Abstract

The aim of this paper is to extend the truth value table of lattice-valued convergence spaces to a more general case and then to use it to introduce and study the quantale-valued fuzzy Scott topology in fuzzy domain theory. Let $(L, *, \varepsilon)$ be a commutative unital quantale and let $\otimes$ be a binary operation on L which is distributive over nonempty subsets. The quadruple $(L, *, \otimes, \varepsilon)$ is called a generalized GL-monoid if $(L, *, \varepsilon)$ is a commutative unital quantale and the operation $*$ is $\otimes$-semi-distributive. For generalized GL-monoid L as the truth value table, we systematically propose the stratified L-generalized convergence spaces based on stratified L-filters, which makes various existing lattice-valued convergence spaces as special cases. For L being a commutative unital quantale, we define a fuzzy Scott convergence structure on L-fuzzy dcpos and use it to induce a stratified L-topology. This is the inducing way to the definition of quantale-valued fuzzy Scott topology, which seems an appropriate way by some results.

Keywords: Commutative unital quantale, Generalized GL-monoid, Stratified L-filter, Stratified L-generalized convergence space, Stratified L-topology; L-fuzzy dcpo, Fuzzy Scott topology.

1 Introduction

Domain theory, a formal basis for the semantics of programming languages, originated in work by Dana Scott [42, 43]. Domain models for various types of programming languages, including imperative, functional, nondeterministic and probabilistic languages, have been studied extensively.

Quantitative Domain Theory (QDT for short) has undergone active research in the past four decades which models concurrent systems and in the hope of arriving at semantics that allows not only qualitative results but also taking in account complexity, runtime, etc. Generalized (ultra)metric spaces [41], continuity spaces [10] and Ω-categories [44] are the well-known frameworks of quantitative domain theory.

In 2001 and 2005, with a complete Heyting algebra L as the truth value table, the concept of L-fuzzy dcpos by using the approach of fuzzy sets firstly appeared in [6] and [57]. Then in 2007, with a commutative unital quantal Ω as the truth value table, under the terminology of category theory, Lai and Zhang [30] defined a new version of Ω-fuzzy dcpos (called $\mathcal{I}$-cocomplete Ω-categories in that paper) in the framework of Ω-category theory. These topics and research contents are now called fuzzy domain theory or fuzzy set approach to quantitative domain theory.

From the viewpoint of pure mathematics, domain theory can be considered as a combination of order theory and topology, where the main ordered structure is a complete directed poset or a dcpo for short (i.e., a poset such that every directed subset possesses a join), the main mapping is the Scott continuous mapping (i.e., a mapping preserving suprema of all directed subsets) and the main topology is the Scott topology on dcpos (i.e., the topology consists of Scott open sets—upper sets which are inaccessible by directed joins). Scott-continuous mappings show up in the study

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of models for lambda calculi and the denotational semantics of computer programs [43]; and the Scott topology is a topological model such that Scott continuity and topological continuity coincide [11].

In 2010, by restricting the truth value table back to a complete Heyting algebra L, Yao and Shi [54] defined and studied fuzzy Scott topology on Lai-Zhang's L-fuzzy dcpos. Like the classical case, the category of fuzzy dcpos is cartesian-closed. In classical domain theory, the Scott topology on a dcpo can be defined by two equivalent ways [5, 11, 35]: the first one is the direct way, that is, we can define a Scott open set as an upper set which is join-inaccessible by ideals (i.e., $\bigvee I \in U$ implies $U \cap I \neq \emptyset$ for every ideal I); the second one is the inducing way, that is, the Scott topology can be induced by Scott convergence of filters or nets. In [54], these two ways were generalized to define fuzzy Scott topology on L-fuzzy dcpos. They are indeed equivalent for L being a complete Heyting algebra.

As we have seen, different lattices can be selected as the truth value table for the study of fuzzy domain. A complete Heyting algebra is relatively simpler than a commutative unital quantale since it has an idempotent tensor product (i.e., the meet operation). It is still an open question, as follows:

How can we define the fuzzy Scott topology on L-fuzzy dcpos for L being a commutative unital quantale? Are these two ways still equivalent in this setting?

The aim of this paper is to answer these questions.

2 The brief history of lattice-valued convergence spaces

The convergence of nets and filters is always an important topic in topology and related areas. In some aspects, filters are relatively more suitable or easier to serve as a tool of convergence in topology, since they completely rely on the background set itself, not on directed sets as that nets should firstly have. The theory of filter convergence, as a relatively isolated structure from topology, can be dated back to Kent [29]. More recently, this theory has been systematically investigated in [4].

Since fuzzy topology was introduced by Chang [3] in 1968, the related Moore-Smith convergence theory was opened for more than ten years. It was Pu and Liu [38] who firstly developed a theory of convergence of nets of fuzzy points paralleling the classical theory of net convergence as that in [28]. Almost at the same time, Lowen [34] developed a theory of convergence in stratified [0,1]-topological spaces based on the concept of prefilters.

Following a suggestion by Höhle and fundamental work on lattice-valued convergence based on lattice-valued filters [14, 15, 17], Jäger [18] introduced the so-called stratified L-generalized convergence spaces by means of stratified L-filters. In [17], the truth value table L for stratified L-filters is cl-quasi-monoid, while in [18] it is a complete Heyting algebra. This condition makes the category of stratified L-generalized convergence spaces to be a cartesian-closed supercategory of the category of stratified L-topological spaces. To be precise, the concept of (lattice-valued fuzzy) convergence structures is more convenient to be captured than (lattice-valued fuzzy) topological structures from the categorical aspects mainly since it has suitable function structures.

We now recall some basic definitions of stratified L-filters and stratified L-generalized convergence spaces [18] for L being a complete Heyting algebra. The definitions of different lattices will be introduced in the next section.

For a nonempty set X and an element λ in a lattice L, we use λ_X to denote the constant L-subset with the value a.

Definition 2.1. *Let L be a complete lattice and let X be a nonempty set. A mapping $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter if*

(LF1) $\mathcal{F}(0_X) = 0,\ \mathcal{F}(1_X) = 1$;
(LF2) $\forall A, B \in L^X,\ A \leq B$ *implies* $\mathcal{F}(A) \leq \mathcal{F}(B)$;
(LF3^) $\forall A, B \in L^X,\ \mathcal{F}(A) \wedge \mathcal{F}(B) \leq \mathcal{F}(A \wedge B)$;
(LFS^) $\forall \lambda \in L,\ A \in L^X,\ \lambda \wedge \mathcal{F}(A) \leq \mathcal{F}(\lambda_X \wedge A)$.

Let $\mathbf{F}_L^s(X)$ be the set of all stratified L-filters on X. For every $x \in X$, $[x]$ is the stratified L-filter sending every $A \in L^X$ to $A(x)$, called the *pointed L-filter* of x.

Definition 2.2. *Let L be a complete lattice and let X be a nonempty set. A stratified L-generalized convergence structure on X is a mapping* $\lim : \mathbf{F}_L^s(X) \longrightarrow L^X$ *satisfying that*

(LC0) $\lim[x](x) = 1$;
(LC2) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}_L^s(X),\ \lim \mathcal{F} \leq \lim \mathcal{G}$.

The pair $(X, \lim)$ is called a stratified L-generalized convergence space.

After Jäger's pioneer work [18], the theory of lattice-valued convergence spaces is subsequently developed further. One important topic is to deeply investigate the theory of frame-valued L-generalized convergence spaces. Jäger [19, 20, 21, 22, 23, 24, 25, 26] has done a lot of work in this direction. Another topic is to extend the truth value table to some more general lattices. In [45], Yao used a complete residuated lattice $(L,*)$ as the truth value table to study lattice-valued convergence spaces based on stratified L-filters. Although there are some mistakes in [45] and seem to have not been completely solved in [52], but the idea therein is consequently used by [8, 31, 32, 33].

For details in [45], for a complete residuated lattice $(L,*)$, the conditions (LF3^) and (LFS^) are respectively replaced by

(LF3*) $\forall A, B \in L^X$, $\mathcal{F}(A) * \mathcal{F}(B) \leq \mathcal{F}(A * B)$;

(LFS*) $\forall \lambda \in L,\ A \in L^X$, $\lambda * \mathcal{F}(A) \leq \mathcal{F}(\lambda_X * A)$.

Li et al. [31, 32, 33] and Fang [7] studied stratified L-generalized convergence spaces by using the idea of fuzzy order and replacing (LFS^) by (LFS*) and replacing (LC2) by

(LC2*) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}_L^s(X)$, $\mathrm{sub}(\mathcal{F}, \mathcal{G}) \leq \mathrm{sub}(\lim \mathcal{F}, \lim \mathcal{G})$.

Therein, *sub* is the fuzzy inclusion operator between L-subsets [12].

In [8], Fang used a commutative unital quantale $(L, *, \varepsilon)$ (here ε is the unit of the semigroup operation $*$) as the truth value table and replaced (LC0) by

(LC1) $\forall x \in X,\ \lim[x](x) \geq \varepsilon$.

In [36], Orpen and Jäger used an ecl-premonoid $(L, \leq, *, \otimes)$ (see Section 2 for details) as a more general truth value table to study stratified L-generalized convergence spaces. For stratified L-filters, they used the axioms (LF1), (LF2), (LFS*) and

(LF3$^{\otimes}$) $\forall A, B \in L^X$, $\mathcal{F}(A) \otimes \mathcal{F}(B) \leq \mathcal{F}(A \otimes B)$.

The readers should notice that there are some lattice-valued convergence structures using different tools other than stratified L-filters, for example, L-filters of crisp sets [46], $\top$-filters in [9, 55], classical nets [48], (L, M)-filters [27, 37] and (L, M)-nets [49].

In Section 4, we will firstly use a more general lattice as the truth value table to study lattice-valued convergence spaces based on stratified L-filters such that this kind of structures can be supplied as a common framework for the stratified L-generalized convergence spaces in [7, 8, 18, 31, 32, 33, 36]. Then in Section 5, in case of L being a commutative unital quantale, we will define an L-fuzzy Scott convergence on L-fuzzy dcpos as a special case of lattice-valued convergence structures and further, investigate the induced L-fuzzy Scott topology associated with the approach, called the quantale-valued fuzzy Scott topology.

3 Necessary materials: preparations for lattice-valued convergence spaces

3.1 The truth value tables for lattice-valued convergence spaces

Generally speaking, it was Jäger [18] who firstly studied lattice-valued convergence spaces for a general lattice L as the truth value table rather than the unit interval [0,1]. In that paper, Jäger chose L a complete Heyting algebra (or called a frame) since its operations possess rich and powerful logical meaning which can help us easily translate definitions and results from the crisp case to the lattice-valued case, and importantly it also makes the category of stratified L-generalized convergence spaces to be a cartesian-closed supercategory of the category of stratified L-topological spaces.

Definition 3.1. [40] *Let L be a complete lattice and let $*$ be a semigroup operation on L. The pair $(L,*)$ is called a quantale if the operation $*$ is distributive over joins, that is, for all $a \in L$, $a * (\bigvee S) = \bigvee_{s \in S}(a * s)$ and $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ and $S \subseteq L$. A quantale $(L,*)$ is called* commutative *(resp., unital) if the operation $*$ is commutative (resp., has a unit element ε).*

For a commutative quantale $(L,*)$, the operation $*$ has a right adjoint $\rightarrow: L \times L \longrightarrow L$ given by, for all $a, b \in L$, $a \rightarrow b = \bigvee\{c \in L \mid a * c \leq b\}$. A complete lattice L is called a *complete Heyting algebra* or a *frame* if $(L, \wedge)$ is a quantale.

A quantale $(L,*)$ is called a *complete residuated lattice* if it is commutative and strictly two-sided (i.e., the top element 1 is the unit of $*$). A quantale $(L,*)$ is called a *GL-monoid* [13] if it is a complete residuated lattice and divisible (i.e., for all $a, b \in L$, $a \leq b$ implies that there exists $c \in L$ such that $a = b * c$).

Example 3.2. (1) *Let $L = [0, \infty]$ denote the extended interval of all non-negative real numbers with the same ordering as the real numbers. Let $\times$ be the usual multiplication on real numbers extended to cope with the infinity such that $x \times \infty = \infty$ for every $x \in [0, +\infty]$ and $0 \times \infty = 0$. Then $(L, \times, 1)$ is a commutative unital quantale.*

(2) *Let $L = [0, \infty]^{op}$ denote the extended interval of all non-negative real numbers with the opposite ordering as real numbers (so 0 is the greatest element). Let $+$ be the usual addition on real numbers extended to cope with infinity, such that $x + \infty = \infty$ for every $x \in L$. Then $(L, +, 0)$ is a commutative unital quantale.*

(3) *The unit interval $[0, 1]$ equipped with a left continuous t-norm (e.g., the usual $\times$, $\wedge$ and $\min$) becomes a complete residuated lattice.*

(4) *Heyting algebras and complete MV-algebras are GL-monoids.*

Proposition 3.3. [17] *Suppose that* $(L, *, \varepsilon)$ *is a commutative unital quantale. Then*
(Q1) $\varepsilon \leq p \to q \Longleftrightarrow p \leq q$;
(Q2) $\varepsilon \to p = p$;
(Q3) $p * (p \to q) \leq q$;
(Q4) $q \leq a \to (p * q)$;
(Q5) $(p \to q) * (q \to r) \leq p \to q$;
(Q6) $(\bigvee_i p_i) \to q = \bigwedge_i (p_i \to q)$;
(Q7) $p \to (\bigwedge_i q_i) = \bigwedge_i (p \to q_i)$;
(Q8) $(r \to p) \to (r \to q) \geq p \to q$;
(Q9) $(p \to r) \to (q \to r) \geq q \to p$;
(Q10) $p \to (q \to r) = (p * q) \to r$.

Let $\otimes$ be an operation on a poset L. The triple $(L, \otimes)$ is called a *partially ordered groupoid* or a *po-groupoid* [2] if
(Iso) $a_1 \leq b_1$ and $a_2 \leq b_2$ imply $a_1 \otimes a_2 \leq b_1 \otimes b_2$ for all $a_1, a_2, b_1, b_2 \in L$.

Definition 3.4. *Let* $(L, *)$ *be a commutative quantale and let* $(L, \otimes)$ *be a po-groupoid. We say that* $(L, *)$ *is* $\otimes$*-semi-distributive if*
(SD1) $\forall a, b, c \in L,\ a * (b \otimes c) \leq (a * b) \otimes (a * c)$;
(SD2) $\forall a, b, c \in L,\ (a \to b) \otimes (a \to c) \leq a \to (b \otimes c)$.

Proposition 3.5. (SD1) *and* (SD2) *are equivalent.*

Proof. (SD1)$\Longrightarrow$(SD2). For all $a, b, c \in L$, $a * [(a \to b) \otimes (a \to c)] \leq [a * (a \to b)] \otimes [a * (a \to c)] \leq b \otimes c$ and then $(a \to b) \otimes (a \to c) \leq a \to (b \otimes c)$.

(SD2)$\Longrightarrow$(SD1). For all $a, b, c \in L$, $a \to [(a * b) \otimes (a * c)] \geq [a \to (a * b)] \otimes [a \to (a * c)] \geq b \otimes c$ and then $a * (b \otimes c) \leq (a * b) \otimes (a * c)$. ☐

Clearly, every commutative quantale $(L, *)$ is $\wedge$-semi-distributive.

Definition 3.6. *We call the quadruple* $(L, *, \otimes, \varepsilon)$ *an sd-quantale if* $(L, *, \varepsilon)$ *is a* $\otimes$*-semi-distributive commutative unital quantale. We call an sd-quantale a generalized GL-monoid if*
(CL) $(L, \otimes)$ *is a cl-groupoid* [2], *i.e.,* $\otimes$ *is distributive over nonempty joins.*

Example 3.7. (1) *It is shown by Lemma 2.7 in* [36] *that every GL-monoid is a complete Heyting algebra. Hence, for every GL-monoid* $(L, *)$, *the quadruple* $(L, *, \wedge, 1)$ *is a generalized GL-monoid and then every complete MV-algebra is a generalized GL-monoid.*

(2) *For every commutative unital quantale* $(L, *, \varepsilon)$, *the quadruple* $(L, *, \vee, \varepsilon)$ *is a generalized GL-monoid. Then for every complete Heyting algebra* L, *the quadruple* $(L, \wedge, \vee, 1)$ *is a generalized GL-monoid.*

(3) *If* $(L, *, \varepsilon)$ *is an idempotent commutative unital quantale, then* $(L, *, *, \varepsilon)$ *is a generalized GL-monoid. Then for every complete Heyting algebra* L, $(L, \wedge, \wedge, 1)$ *is a generalized GL-monoid.*

(4) *The quadruple* $([0, +\infty], \times, +, 1)$ *is a generalized GL-monoid, where* $[0, +\infty]$ *is the set of all non-negative real numbers including* $+\infty$.

Remark 3.8. *For some results in the next section, we only need to assume that* L *is an sd-quantale. It is easy to see that for every commutative unital quantale* $(L, *, \varepsilon)$, *consequently complete residuated lattice, the structure* $(L, *, \wedge, \varepsilon)$ *is an sd-quantale.*

A *cl-premonoid* $(L, \otimes)$ [17] is a cl-groupoid with the condition that $\alpha \leq (\alpha \otimes 1) \wedge (1 \otimes \alpha)$ for every $\alpha \in L$. An *enriched cl-premonoid* or an *ecl-premonoid* [17] is a quadruple $(L, \leq, *, \otimes)$ such that $(L, *)$ is a GL-monoid and $(L, \otimes)$ is a cl-premonoid which together satisfy the *domination condition*
(Dom) $(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \leq (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2)$ for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$.
In [36], Orpen and Jäger used an ecl-premonoid $(L, \leq, *, \otimes)$ as the truth value table to study stratified L-generalized convergence spaces. In that paper, a further condition was always used, as follows:
(M) $\alpha \leq \alpha \otimes \alpha$ for every $\alpha \in L$.
Let us now study the relation among the conditions (SD), (Dom) and (M).

Proposition 3.9. *Let* $(L, *)$ *be a commutative quantale and let* $(L, \otimes)$ *be a po-groupoid. Then* (SD) *follows from* (Dom) *and* (M).

Proof. For all $\alpha, \beta, \gamma \in L$, $\alpha * (\beta \otimes \gamma) \leq (\alpha \otimes \alpha) * (\beta \otimes \gamma) \leq (\alpha * \beta) \otimes (\alpha * \gamma)$. ☐

Proposition 3.9 indicates that every ecl-premonoid is a generalized GL-monoid. While not every generalized GL-monoid is an ecl-premonoid since (SD) can not imply either (M) or (Dom). Here we have an example.

Example 3.10. *Let $L = \{0, a, b, 1\}$ be the diamond lattice, that is, $0 \leq a, b \leq 1$ and $a \not\leq b, b \not\leq a$. Define two operations $*, \otimes : L \times L \longrightarrow L$ by Table 1 and Table 2, respectively.*

$*$	0	a	b	1
0	0	0	0	0
a	0	a	b	1
b	0	b	b	b
1	0	1	b	1

Table 1: The operation $*$ on L

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	1
1	0	a	1	1

Table 2: The operation $\otimes$ on L

It is easy to see that both $(L, *, a)$ and $(L, \otimes, b)$ are commutative unital quantales.

(SD1) *For all $\alpha, \beta, \gamma \in L$, $\alpha * (\beta \otimes \gamma) \leq (\alpha * \beta) \otimes (\alpha * \gamma)$.*

Case 1. If one of α, β, γ is 0, then the left side is 0 and so that the inequality holds.

Case 2. If $\alpha = a$, then $\alpha * (\beta \otimes \gamma) = \beta \otimes \gamma = (\alpha * \beta) \otimes (\alpha * \gamma)$, and the inequality holds.

Case 3. If $\alpha = b$ and $\beta, \gamma \neq 0$, then $\alpha * (\beta \otimes \gamma) \leq b$ and $(\alpha * \beta) \otimes (\alpha * \gamma) = (b * \beta) \otimes (b * \gamma) = b * b = b$, and the inequality holds.

Case 4. If $\alpha = 1$ and $\beta, \gamma \neq 0$,

Subcase 1. If $\beta = b$, then $\alpha * (\beta \otimes \gamma) = \alpha * \gamma$ and $(\alpha * \beta) \otimes (\alpha * \gamma) = b \otimes (\alpha * \gamma) = \alpha * \gamma$, and the inequality holds.

Subcase 2. If $\beta \neq b$, that is $\beta \in \{a, 1\}$, then $(\alpha * \beta) \otimes (\alpha * \gamma) = (1 * \beta) \otimes (1 * \gamma) \geq 1 \otimes b = 1$, and the inequality holds.

While, Condition (M) *does not hold for a. Condition (Dom) does not hold for $\alpha_1 = \beta_2 = 1,\ \alpha_2 = \beta_1 = b$. Indeed, $(1 \otimes b) * (b \otimes 1) = 1 * 1 = 1$, but $(1 * b) \otimes (b * 1) = b * b = b$.*

Proposition 3.9 and Example 3.10 indicate that a generalized GL-monoid is much more general than an ecl-premonoid used in [36].

3.2 Stratified L-filters and stratified L-topology

In this subsection, we assume that $(L, *, \otimes, \varepsilon)$ is an sd-quantale.

For a nonempty set X, the notation L^X denotes the set of all mappings from X to L and every member of L^X is called an *L-subset* of X. The constant mapping with value $\lambda \in L$ will be denoted as λ_X. For the operation $\& \in \{*, \otimes, \wedge, \vee\}$ and for all $A, B \in L^X$, $A\&B$ is the L-subset defined by $(A\&B)(x) = A(x)\&B(x)$ $(\forall x \in X)$. Define $\mathrm{sub} : L^X \times L^X \longrightarrow L$ by $\mathrm{sub}(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$, called the *subset degree operation* of A in B.

Let $f : X \longrightarrow Y$ be an ordinary mapping, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called the L-valued Zadeh function or the L-forward powerset operator [39]) and $f^{\leftarrow} : L^Y \longrightarrow L^X$ (called the L-backward powerset operator [39]) by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f_L^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively.

Definition 3.11. *[17]Let X be a nonempty set. A mapping $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies*

(LF1) $\mathcal{F}(0_X) = 0,\ \mathcal{F}(1_X) = 1$*;*

(LF2) *for all $A, B \in L^X$, $A \leq B$ implies $\mathcal{F}(A) \leq \mathcal{F}(B)$;*

($\mathrm{LF3}^{\otimes}$) *for all $A, B \in L^X$, $\mathcal{F}(A) \otimes \mathcal{F}(B) \leq \mathcal{F}(A \otimes B)$;*

(LFS^{*}) $\lambda * \mathcal{F}(A) \leq \mathcal{F}(\lambda_X * A)$ *for all $\lambda \in L,\ A \in L^X$.*

Let $\mathbf{F}_L^s(X)$ be the set of all stratified L-filters on X. For every $x \in X$, the notation $[x]$ means the stratified L-filter sending $A \in L^X$ to $A(x)$, called the *pointed L-filter* at x.

Suppose that $f : X \longrightarrow Y$ is a mapping and $\mathcal{F} \in \mathbf{F}_L^s(X)$. It is easy to verify that $f(\mathcal{F}) : L^Y \longrightarrow L$ $(B \mapsto \mathcal{F}(f_L^{\leftarrow}(B)))$ is a stratified L-filter on Y.

Remark 3.12. (1) *Let L be a frame. For the generalized GL-monoid $L = (L, \wedge, \wedge, 1)$, the related stratified L-filters were used in* [18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

(2) *Let $(L, *)$ be a complete residuated lattice. For the sd-quantale $(L, *, \wedge, 1)$, the related stratified L-filters were used in* [7] *and* [31, 32, 33].

(3) *Let $(L, *, \varepsilon)$ be a commutative unital quantale. For the sd-quantale $(L, *, \wedge, \varepsilon)$, the related stratified L-filters were used in* [8].

Definition 3.13. [17] *Let X be a nonempty set. A subfamily $\delta \subseteq L^X$ is called a stratified L-topology on X if it satisfies*

(O1) $0_X, 1_X \in \delta$;

(O2) $A, B \in \delta$ *implies* $A \otimes B \in \delta$;

(O3) $\{A_i |\ i \in I\} \subseteq \delta$ *implies* $\bigvee_i A_i \in \delta$;

(OS) $A \in \delta$ *implies* $\lambda_X * A \in \delta$ *for every* $\lambda \in L$.

The pair (X, δ) is called a stratified L-topological space.

A mapping $f : X \longrightarrow Y$ is called *continuous* with respect to two given stratified L-topological spaces (X, δ_1) and (Y, δ_2) if $f_L^{\leftarrow}(B) \in \delta_1$ for every $B \in \delta_2$. The category of stratified L-topological spaces with continuous mappings as morphisms is denoted by **SL-Top**. For concepts from category theory, we refer to [1].

By Section 6.1 in [17] we know that, for L a generalized GL-monoid, a stratified L-topology is uniquely determined by the L-interior operator $\mathcal{K} : L^X \longrightarrow L^X$ satisfying the following axioms:

(K1) $\mathcal{K}(1_X) = 1_X$;

(K2) $\forall A \in L^X$, $\mathcal{K}(A) \leq A$;

(K3) $\forall A, B \in L^X$, $\mathcal{K}(A) \leq \mathcal{K}(B)$;

(K4) $\forall A, B \in L^X$, $\mathcal{K}(A) \otimes \mathcal{K}(B) \leq \mathcal{K}(A \otimes B)$;

(K5) $\forall A \in L^X$, $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}(A)$.

(K6) $\forall \lambda \in L, A \in L^X$, $\lambda * \mathcal{K}(A) \leq \mathcal{K}(\lambda_X * A)$.

In fact, the lattice L in Section 6.1 in *[17]* is a bit different from a generalized GL-monoid, but the proofs are similar.

Let X be a nonempty set. For a stratified L-topology δ on X. Its associate L-interior operator $\mathcal{K}_\delta : L^X \longrightarrow L^X$ is given by $\mathcal{K}_\delta(A) = \bigvee\{B \in \delta |\ B \leq A\}$, and its neighborhood filter $\mathcal{U}_\delta^x : L^X \longrightarrow L$ at x is given by $\mathcal{U}_\delta^x(A) = \mathcal{K}_\delta(A)(x)$. The neighborhood filter $\mathcal{U}_\delta^x$ is indeed a stratified L-filter which is less than or equal to $[x]$. Conversely, for an L-interior operator $\mathcal{K} : L^X \longrightarrow L^X$, its associate stratified L-topology is given by $\delta_{\mathcal{K}} = \{A \in L^X |\ \mathcal{K}(A) = A\}$.

4 Stratified L-generalized convergence structures

In this section, we will use a generalized GL-monoid $(L, *, \otimes, \varepsilon)$ as the truth value table (unless otherwise stated) to define and study stratified L-generalized convergence spaces and their relation with stratified L-topology. These kinds of structures will make the stratified L-generalized convergence spaces in the sense of [7, 8, 18, 31, 32, 33, 36] as special cases of our approach.

Definition 4.1. *(see [18] for L being a frame and* [8] *for L being a commutative unital quantale) A* stratified L-generalized convergence structure *on a nonempty set X is a mapping* $\lim : \mathbf{F}_L^s(X) \longrightarrow L^X$ *satisfying that*

(LC1) $\forall x \in X,\ \lim[x](x) \geq \varepsilon$;

(LC2) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}_L^s(X)$, $\mathcal{F} \leq \mathcal{G}$ *implies* $\lim \mathcal{F} \leq \lim \mathcal{G}$.

The pair $(X, \lim)$ is called a stratified L-generalized convergence space.

A stratified L-generalized convergence space $(X, \lim)$ is called topological or a stratified L-topological convergence space if

(Lp) $\forall (\mathcal{F}, x) \in \mathbb{F}_L^s(X) \times X,\ \lim \mathcal{F}(x) = \mathrm{sub}(\mathcal{U}_{\lim}^x, \mathcal{F})$;

(Lt) $\forall x \in X$, $\mathcal{U}_{\lim}^x \leq \mathcal{U}_{\lim}^x(\mathcal{U}_{\lim}^{(\cdot)})$.

Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two stratified L-generalized convergence spaces. A mapping $f : X \longrightarrow Y$ is called *continuous* if $\lim^X \mathcal{F} \leq f_L^{\leftarrow}(\lim^Y f(\mathcal{F}))$ for every $\mathcal{F} \in \mathbf{F}_L^s(X)$. The resulting category is denoted by **SL-GCS**. Clearly, every discrete structure and every indiscrete structure are respectively discrete object and indiscrete object in **SL-GCS**. Thus every mapping from a discrete space to another space is continuous, and every mapping from a space to an indiscrete space is continuous. We use **SL-TCS** to denote the full subcategory of **SL-GCS** consisting of stratified L-topological convergence spaces.

Remark 4.2. (1) *Let L be a complete Heyting algebra. For the generalized GL-monoid $L = (L, \wedge, \wedge, 1)$, a stratified L-generalized convergence structure in Definition 4.1 turns to be the one in* [18].

(2) *Let $(L, *)$ be a complete residuated lattice. For the sd-quantale $(L, *, \wedge, 1)$, Fang [7] and Li et al* [31, 32, 33] *studied stratified L-generalized convergence spaces by changing the axiom* (LC2) *to*

(LC2*) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}^s_L(X)$, $\mathrm{sub}(\mathcal{F}, \mathcal{G}) \leq \mathrm{sub}(\lim \mathcal{F}, \lim \mathcal{G})$.

(3) *Let $(L, *, \varepsilon)$ be a commutative unital quantale. For the sd-quantale $(L, *, \wedge, \varepsilon)$, the stratified L-generalized convergence structure in Definition 4.1 turns to be the one in* [8] *based on L-ordered filters.*

(4) *Let $(L, *, \otimes)$ be an enriched cl-premonoid. For the generalized GL-monoid $(L, *, \otimes, 1)$, a stratified L-generalized convergence structure in Definition 4.1 turns to be the one in* [36].

From now on, we will establish a categorical reflectivity of stratified L-topological spaces in stratified L-generalized convergence spaces and the induced categorical isomorphism between stratified L-topological spaces and stratified L-topological convergence spaces.

Remark 4.3. *We will not use any new technique in the rest of this section. All the definitions, approaches and results are similar and imitated from those in* [36] *and other literatures. Our one aim is to introduce a more general lattice which can be considered as the truth value table for lattice-valued convergence spaces such that we can use it to define L-fuzzy Scott topology in the next section.*

Proposition 4.4. *Suppose $(X, \lim)$ is a stratified L-generalized convergence space.*
(1) *For every $x \in X$, define $\mathcal{U}^x_{\lim} : L^X \longrightarrow L$ by $\mathcal{U}^x_{\lim}(A) = \bigwedge_{\mathcal{F} \in \mathbf{F}^s_L(X)} \lim \mathcal{F}(x) \to \mathcal{F}(A)$. Then every $\mathcal{U}^x_{\lim}$ is a stratified L-filter on X, which is less than or equal to $[x]$.*
(2) *The family $\delta_{\lim} = \{A \in L^X \mid A(x) \leq \mathcal{U}^x_{\lim}(A),\ \forall x \in X\}$ is a stratified L-topology on X.*

Proof. We here only prove (LF3$^{\otimes}$) in (1), which is related to (SD2).

(LF3$^{\otimes}$) For all $A, B \in L^X$,

$$\begin{aligned}
\mathcal{U}^x_{\lim}(A) \otimes \mathcal{U}^x_{\lim}(B) &\leq \bigwedge_{\mathcal{F} \in \mathbf{F}^s_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(A)) \otimes (\lim \mathcal{F}(x) \to \mathcal{F}(B)) \\
&\leq \bigwedge_{\mathcal{F} \in \mathbf{F}^s_L(X)} \lim \mathcal{F}(x) \to (\mathcal{F}(A) \otimes \mathcal{F}(B)) \quad \text{(the condition (SD2) is used)} \\
&\leq \bigwedge_{\mathcal{F} \in \mathbf{F}^s_L(X)} \lim \mathcal{F}(x) \to \mathcal{F}(A \otimes B) = \mathcal{U}^x_{\lim}(A \otimes B).
\end{aligned}$$

The other conclusions are similar to those in [36]. □

Remark 4.5. *By Proposition 4.4, if we only focus on the induced L-topology of a stratified L-generalized convergence space, then we only need the condition of $(L, *, \otimes, \varepsilon)$ being an sd-quantale. We know that every commutative unital quantale is an sd-quantale with $\otimes = \wedge$. Then for a commutative unital quantale $(L, *, \varepsilon)$, every stratified L-generalized convergence space induces a stratified L-topology in the sense of Definition 3.13 with $\otimes = \wedge$.*

Proposition 4.6. *Suppose that (X, δ) is a stratified L-topological space. Define $\lim_\delta : \mathbf{F}^s_L(X) \longrightarrow L^X$ by*

$$\lim{}_\delta \mathcal{F}(x) = \mathrm{sub}(\mathcal{U}^x_\delta, \mathcal{F}) \ (\forall \mathcal{F} \in \mathbf{F}^s_L(X), \forall x \in X).$$

Then $\lim_\delta$ is a stratified L-topological convergence structure on X and $\mathcal{U}^x_{\lim_\delta} = \mathcal{U}^x_\delta$ for every $x \in X$.

Proof. Similar to those in [36]. □

Proposition 4.7. (1) *Suppose that δ is a stratified L-topology on X. Then $\delta_{\lim_\delta} = \delta$.*

(2) *Suppose that $(X, \lim)$ is a stratified L-generalized convergence space. Then* $\lim$ *is topological iff* $\lim_{\delta_{\lim}} = \lim$.

By using similar methods in [7, 8, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33], we can show that the transformations between **SL-Top** and **SL-GCS** are functorial. Here we have,

Theorem 4.8. **SL-Top** *is reflective in* **SL-GCS**, *and* **SL-Top** *is isomorphic to* **SL-TCS**.

Remark 4.9. *For a stratified L-topological convergence space, Condition (LC2) can imply Condition (LC2*). In fact, if* $\lim$ *is a stratified L-topological convergence space, then there exists a stratified L-topology δ on X such that* $\lim = \lim_\delta$. *Then for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}^s_L(X)$ and all $x \in X$, we have*

$$\lim \mathcal{F}(x) * \mathrm{sub}(\mathcal{F}, \mathcal{G}) = \lim{}_\delta \mathcal{F}(x) * \mathrm{sub}(\mathcal{F}, \mathcal{G}) = \mathrm{sub}(\mathcal{U}^x_\delta, \mathcal{F}) * \mathrm{sub}(\mathcal{F}, \mathcal{G}) \leq \mathrm{sub}(\mathcal{U}^x_\delta, \mathcal{G}) = \lim{}_\delta \mathcal{G}(x).$$

5 A quantale-valued fuzzy Scott topology

In this section, we assume that $(L,*,\varepsilon)$ is a commutative unital quantale. We would like to define and study a quantale-valued fuzzy Scott topology on L-fuzzy dcpos.

In classical domain theory, the Scott topology on dcpo can be defined via two equivalent ways [5, 11, 35]: the first one is the direct way, that is, a Scott open set is defined as an upper set which is join-inaccessible by ideals (i.e., $\bigvee I \in U$ implies $U \cap I \neq \emptyset$ for every ideal I); the second one is the inducing way, that is, the Scott topology can be induced by Scott convergence of filters or nets. In [54], for L being a complete Heyting algebra, these two ways were generalized to define fuzzy Scott topology on L-fuzzy dcpos.

We first would like to recall some basic definitions and results on quantale-valued fuzzy dcpos. Notice that we will consider a commutative unital quantale $(L,*,\varepsilon)$ as an sd-quantale $(L,*,\wedge,\varepsilon)$ defined in Section 3.

Definition 5.1. *An L-fuzzy poset is a pair (X,e) such that X is a nonempty set and $e: X\times X \longrightarrow L$ is a mapping, called an L-fuzzy order, satisfying that for all $x,y,z \in X$,*

(E1) *$e(x,x) \geq \varepsilon$;*

(E2) *$e(x,y) * e(y,z) \leq e(x,z)$;*

(E3) *if $e(x,y) \geq \varepsilon$ and $e(y,x) \geq \varepsilon$, then $x = y$.*

An L-fuzzy poset in Definition 5.1 is equivalent to a skeletal L-category in [30].

Example 5.2. (1) *Define $e_L : L\times L \longrightarrow L$ by $e_L(x,y) = x \to y$ for all $x,y \in L$. Then e_L is an L-fuzzy order on L.*

(2) *Let X be a nonempty set. The subset degree operation* sub *on X is an L-fuzzy order on L^X.*

Definition 5.3. *Suppose that (X,e) is an L-fuzzy poset. An L-subset $S \in L^X$ is called an upper (resp., a lower) set if $S(x) * e(x,y) \leq S(y)$ (resp., $S(x) * e(y,x) \leq S(y)$) for all $x,y \in X$.*

Let (X,e) be an L-fuzzy poset and $x \in X$. Define $\uparrow x$, $\downarrow x \in L^X$ respectively by $\uparrow x(y) = e(x,y)$ and $\downarrow x = e(y,x)$ for every $y \in X$. Then $\uparrow x$ is an upper set and $\downarrow x$ is a lower set.

For a stratified L-filter $\mathcal{F}$ on X (with the conditions (LF1), (LF2), (LF3^) and (LFS*)), define $\mathcal{F}^l \in L^X$ by

$$\mathcal{F}^l(x) = \bigvee_{A\in L^X} \mathcal{F}(A) * \mathrm{sub}(A, \uparrow x)\ (\forall x \in X),$$

where $\mathcal{F}^l(x)$ can be interpreted as the degree for x to be a lower bound of some $A \in \mathcal{F}$. Clearly, $\mathcal{F}^l$ is a lower set.

Proposition 5.4. (1) *For two stratified L-filters $\mathcal{F},\mathcal{G}$, if $\mathcal{F} \leq \mathcal{G}$ then $\mathcal{F}^l \leq \mathcal{G}^l$;*

(2) *for any $x \in X$, $[x]^l = \downarrow x$.*

Proof. (1) is obvious.

(2) For every $y \in X$, on one hand, $[x]^l(y) = \bigvee\limits_{A\in L^X} [x](A) * \mathrm{sub}(A,\uparrow y) \geq \uparrow y(x) * \mathrm{sub}(\uparrow y, \uparrow y) = \downarrow x(y)$. On the other hand, for all $A \in L^X$, we have $\mathrm{sub}(A,\uparrow y) \leq A(x) \to e(y,x)$ and $A(x) * \mathrm{sub}(A,\uparrow y) \leq e(y,x) = \downarrow x(y)$. By the arbitrariness of $A \in L^X$, we get that $[x]^l(y) \leq \downarrow x(y)$. Hence, $[x]^l = \downarrow x$. □

Definition 5.5. *Suppose (X,e) is an L-fuzzy poset. An L-subset $I \in L^X$ is called a fuzzy ideal if*

(I1) *I is a lower set;*

(I2) *I is nonempty, that is, $\bigvee_{x\in X} I(x) \geq \varepsilon$;*

(I3) *$I(x) * I(y) \leq \bigvee_{z\in X} I(z) * e(x,z) * e(y,z)$ for all $x,y \in X$.*

The set of all fuzzy ideals of an L-fuzzy poset (X,e) is denoted by $\mathcal{I}_L(X)$.

Definition 5.6. *Suppose that (X,e) is an L-fuzzy poset and $S \in L^X$. An element x_0 is called a join of S, in symbols $x_0 = \sqcup S$, if $e(x_0,x) = \bigwedge\limits_{y\in X} I(y) \to e(y,x)$ for every $x \in X$. An L-fuzzy poset (X,e) is called an L-fuzzy dcpo if every fuzzy ideal has a join.*

Clearly, for every element x in an L-fuzzy poset (X,e), we have $\downarrow x \in \mathcal{I}_L(X)$ and $x = \sqcup\downarrow x$. Let (X,e) be an L-fuzzy dcpo and define a mapping $\mathcal{S} : \mathbf{F}^s_L(X)\times X \longrightarrow L$ by $\mathcal{S}(\mathcal{F},x) = \bigvee\limits_{I\in\mathcal{I}_L(X)} \mathrm{sub}(I,\mathcal{F}^l) * e(x,\sqcup I)$, for all $(\mathcal{F},x) \in \mathbf{F}^s_L(X)\times X$. The value $\mathcal{S}(\mathcal{F},x)$ can be interpreted as the degree of $\mathcal{F}$ being Scott convergent to x.

Proposition 5.7. *For all $x,y \in L$, we have $\mathcal{S}([y],x) = e(x,y)$.*

Proof. For $x, y \in X$, $\mathcal{S}([y], x) = \bigvee_{I \in \mathcal{I}_L(X)} \mathrm{sub}(I, [y]^l) * e(x, \sqcup I) = \bigvee_{I \in \mathcal{I}_L(X)} \mathrm{sub}(I, \downarrow y) * e(x, \sqcup I)$. Firstly,

$$\mathcal{S}([y], x) \geq \mathrm{sub}(\downarrow x, \downarrow y) * e(x, \sqcup \downarrow x) = e(x, y) * e(x, x) = e(x, y).$$

Secondly, $\mathcal{S}([y], x) = \bigvee_{I \in \mathcal{I}_L(X)} e(\sqcup I, y) * e(x, \sqcup I) \leq e(x, y)$. □

Theorem 5.8. *For every L-fuzzy dcpo (X, e), $\mathcal{S}$ is a stratified L-generalized convergence structure on X.*

Proof. (LC1) is obvious from Proposition 5.7. (LC2) is also followed from Proposition 5.4(2). □

By Proposition 4. 4, the stratified L-generalized convergence structure $\mathcal{S}$ induces a stratified L-topology (in the sense of Definition 3.1 3 with $\otimes = \wedge$) $\sigma_L(X)$ on X given by

$$\sigma_L(X) = \{A \in L^X \mid \forall (\mathcal{F}, x) \in \mathbf{F}_L^s(X) \times X,\ A(x) * \mathcal{S}(\mathcal{F}, x) \leq \mathcal{F}(A)\},$$

called the *fuzzy Scott topology* on X. The members of $\sigma_L(X)$ are called *fuzzy Scott open sets*. The corresponding L-topological space is denoted by $\Sigma_L(X)$. This is the inducing way of the fuzzy Scott topology on L-fuzzy dcpos for L being a commutative unital quantale.

In [54], it is shown that for L being a complete Heyting algebra, fuzzy Scott topology also can be described directly as follows:

Proposition 5.9. *Suppose that L is a complete Heyting algebra and (X, e) is an L-fuzzy dcpo. Then for every $A \in L^X$, $A \in \sigma_L(X)$ iff $A(\sqcup I) = \bigvee_{x \in X} I(x) \wedge A(x)$ for all $I \in \mathcal{I}_L(X)$.*

Proposition 5.10. *Every fuzzy Scott open set is an upper set.*

Proof. Suppose A is a fuzzy Scott open set. For all $x, y \in X$, we have $A(x) * \mathcal{S}([y], x) \leq [y](A)$. This is exactly to say, $A(x) * e(x, y) \leq A(y)$. Hence A is an upper set. □

For a commutative unital quantale $(L, *, \varepsilon)$, the condition "$A(\sqcup I) = \bigvee_{x \in X} I(x) \wedge A(x)$ for all $I \in \mathcal{I}_L(X)$" should be rewritten as

(DW*) $A(\sqcup I) = \bigvee_{x \in X} I(x) * A(x)$ for all $I \in \mathcal{I}_L(X)$.

Proposition 5.11. *Let (X, e) be an L-fuzzy dcpo and $A \in L^X$. The following statements are equivalent:*

(1) *$A \in L^X$ satisfies (DW*);*

(2) *A is an upper set and $A(\sqcup I) \leq \bigvee_{x \in X} I(x) * A(x)$ for all $I \in \mathcal{I}_L(X)$.*

Proof. We only need to show that A is an upper set iff $I(x) * A(x) \leq A(\sqcup I)$ for all $x \in X,\ I \in \mathcal{I}_L(X)$. In fact, if A is an upper set, then $I(x) * A(x) \leq e(x, \sqcup I) * A(x) \leq A(\sqcup I)$. Conversely, for all $x, y \in X$, we have $\downarrow y \in \mathcal{I}_L(X)$ and $A(x) * e(x, y) = A(x) * \downarrow y(x) \leq A(\sqcup \downarrow y) = A(y)$. Then A is an upper set. □

We now call an L-subset with the condition (DW*) a DW*-open set and denote the set of all DW*-open sets by $\mathrm{DW}_L^*(X)$.

Definition 5.12. *[54] Let (X, e) be an L-fuzzy dcpo. For $x \in X$, define a map $\Downarrow x \in L^X$ by*

$$\Downarrow x(y) = \bigwedge_{I \in \mathcal{I}_L(X)} e(x, \sqcup I) \to I(y)\ (\forall y \in X).$$

A fuzzy dcpo (X, e) is called continuous or a fuzzy domain if $\Downarrow x \in \mathcal{I}_L(X)$ for each $x \in X$.

The value $\Downarrow x(y)$ can be interpreted as the degree for y to be way below x, and $\Uparrow y$ is defined by $\Uparrow y(x) = \Downarrow x(y)$.

In classical domain theory, for a continuous dcpo X, the family $\{\Uparrow x \mid x \in X\}$ is a base of Scott topology. This result is very important for establishing a categorical duality between domains and completely distributive lattices and a categorical isomorphism between injective T_0 spaces and continuous lattices. In [47], we get a counterpart of this result in fuzzy domain theory, and was applied to the duality between fuzzy domains and strongly fuzzy completely distributive lattices [53] and the isomorphism between fuzzy injective T_0 spaces and fuzzy continuous lattices [50]. We here will prove that, in a quantale-valued fuzzy domain, every $\Uparrow x$ is fuzzy Scott open. Firstly we need a proposition.

Proposition 5.13. *Let (X, e) be an L-fuzzy dcpo. Then*
(1) $\forall x, u, v, y \in X,\ e(u, x) * \Downarrow y(x) * e(y, v) \leq \Downarrow v(u)$.
(2) $\forall I \in \mathcal{I}_L(X)$, $\forall x \in X$, $\Downarrow(\sqcup I)(x) \leq I(x)$.

Proof. (1) It follows from [47]. (2) It is obvious from Definition 5.12. □

Proposition 5.14. *Let (X, e) be a fuzzy domain. Then for every $x \in X$, $\Uparrow x$ is a DW*-open set.*

Proof. We need to show that $\Uparrow x(\sqcup I) = \bigvee_{y \in X} I(y) * \Uparrow x(y)$ for all $I \in \mathcal{I}_L(X)$. Firstly, for every $y \in X$,

$$I(y) * \Uparrow x(y) \leq e(y, \sqcup I) * \Uparrow x(y) \leq \Uparrow x(\sqcup I).$$

Secondly,
$$\Uparrow x(\sqcup I) = \Downarrow(\sqcup I)(x) = \bigvee_{y \in X} \Downarrow(\sqcup I)(y) * \Downarrow y(x) \leq \bigvee_{y \in X} I(y) * \Uparrow x(y).$$ □

We now have a question in our mind that

For L a commutative unital quantale, can the fuzzy Scott topology on L-fuzzy dcpos also be equivalently described as a direct way?

If the answer of this question is positive, then $\mathrm{DW}_L^*(X) = \sigma_L(X)$. But unfortunately, it is not right. We only have $\mathrm{DW}_L^*(X) \subseteq \sigma_L(X)$ (see Proposition 5.16 below) and even worse $\mathrm{DW}_L^*(X)$ is not necessarily an L-topology.

Example 5.15. *Let $L = \{0, a, b, 1\}$ be the diamond lattice in Example 3.10. The quantale operation $* : L \times L \longrightarrow L$ is given by Table 1 and the related implication operation $\rightarrow: L \times L \longrightarrow L$ is given by the following Table 3.*

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	a	b	1
b	0	0	1	1
1	0	0	b	1

Table 3: The implication $\rightarrow$ on L

As has been stated in Example 3.10, $(L, *, a)$ is an idempotent commutative unital quantale. Let $X = \{x, y\}$ and define $e(x, x) = a,\ e(y, x) = 0,\ e(x, y) = e(y, y) = 1$. Then (X, e) is an L-fuzzy poset.

(1) We firstly determine $\mathcal{I}_L(X)$ for (X, e). Suppose that I is a fuzzy ideal of (X, e). Then

(I1) I is nonempty, that is, $I(x) \vee I(y) \geq a$.

(I2) I is directed, that is, $I(u) * I(v) \leq (I(x) * e(u, x) * e(v, x)) \vee (I(y) * e(u, y) * e(v, y))$ for all $u, v \in X$. This condition can be equivalently written as: $I(y) \leq I(y) * 1$.

(I3) I is a lower set, that is, $I(u) * e(v, u) \leq I(v)$ for all $u, v \in X$. This condition can be equivalently written as: $I(y) * 1 \leq I(x),\ I(y) * 1 \leq I(y)$.

By (I1-I3), we have, $I \in \mathcal{I}_L(X)$ iff $I(x) \vee I(y) \geq a$ and $I(y) * 1 = I(y) \leq I(x)$.

Hence, we have $\mathcal{I}_L(X) = \{I_1 = (a, 0),\ I_2 = (1, 0),\ I_3 = (1, b),\ I_4 = (1, 1)\}$.

(2) We now show that if (X, e) is an L-fuzzy dcpo. Let $I \in \mathcal{I}_L(X)$. Then $u = \sqcup I$ iff

$$e(u, x) = (I(x) \rightarrow e(x, x)) \wedge (I(y) \rightarrow e(y, x)),\ e(u, y) = (I(x) \rightarrow e(x, y)) \wedge (I(y) \rightarrow e(y, y)).$$

After checking, we find that these two conditions can be equivalently written as, $e(u, x) = (I(x) \rightarrow a) \wedge (I(y) \rightarrow 0)$. Then we have $\sqcup I_1 = x,\ \sqcup I_2 = \sqcup I_3 = \sqcup I_4 = y$. Hence, (X, e) is an L-fuzzy dcpo.

(3) We now determine the family $\mathrm{DW}_L^*(X)$. For every $A \in \mathrm{DW}_L^*(X)$, we have
$A(x) = A(\sqcup I_1) = (I_1(x) * A(x)) \vee (I_1(y) * A(y)) = A(x)$,
$A(y) = A(\sqcup I_2) = (I_2(x) * A(x)) \vee (I_2(y) * A(y)) = 1 * A(x)$,
$A(y) = A(\sqcup I_3) = (I_3(x) * A(x)) \vee (I_3(y) * A(y)) = (1 * A(x)) \vee (b * A(y))$,
$A(y) = A(\sqcup I_4) = (I_4(x) * A(x)) \vee (I_4(y) * A(y)) = (1 * A(x)) \vee (1 * A(y))$.

These conditions can be equivalently replaced by, $1 * A(y) \leq A(y) = 1 * A(x)$. Hence, we have four such kind of L-subsets, $A_1 = (0, 0),\ A_2 = (a, 1), A_3 = (b, b),\ A_4 = (1, 1)$. Clearly, $A_2 \wedge A_3 = (0, b)$ does not satisfies (DW). That is, $\mathrm{DW}_L^*(X) = \{A_1, A_2, A_3, A_4\}$ is not an L-topology.*

Proposition 5.16. $\mathrm{DW}_L^*(X) \subseteq \sigma_L(X)$.

Proof. Suppose that $A \in \mathrm{DW}_L^*(X)$,

$$\begin{aligned} A(x) * \mathcal{S}(\mathcal{F}, x) &= \bigvee_{I \in \mathcal{I}_L(X)} A(x) * \mathrm{sub}(I, \mathcal{F}^l) * e(x, \sqcup I) \leq \bigvee_{I \in \mathcal{I}_L(X)} \mathrm{sub}(I, \mathcal{F}^l) * A(\sqcup I) \\ &= \bigvee_{I \in \mathcal{I}_L(X)} \bigvee_{y \in X} \mathrm{sub}(I, \mathcal{F}^l) * A(y) * I(y) \leq \bigvee_{y \in X} \mathcal{F}^l(y) * A(y) \\ &= \bigvee_{y \in X} \bigvee_{B \in L^X} \mathcal{F}(B) * \mathrm{sub}(B, \uparrow y) * A(y) = \bigvee_{y \in X} \bigvee_{B \in L^X} \mathcal{F}(B) * \mathrm{sub}(B, \uparrow y) * \mathrm{sub}(\uparrow y, A) \\ &\leq \bigvee_{B \in L^X} \mathcal{F}(B) * \mathrm{sub}(B, A) \leq \mathcal{F}(A). \end{aligned}$$

Then $A \in \sigma_L(X)$. □

Corollary 5.17. *Let X be a fuzzy domain, then $\Uparrow x \in \sigma_L(X)$ for every $x \in X$.*

From these redult, if L is a complete Heyting algebra, then for an L-fuzzy dcpo (X, e), the two ways to the fuzzy Scott open sets are equivalent. But when L is a commutative unital quantale, the two ways are not equivalent any more, and even more unfortunately, the fuzzy Scott open sets defined by the direct way *do not necessarily form an L-topology* any more.

For an L-topological space (X, δ), instead of the axiom (O2), some authors (e.g. in [16, 45]) used the axiom (O2*): $A * B \in \delta$ for all $A, B \in \delta$.

We now have a further question,

Whether the family of L-subsets $A \in L^X$ with Condition (DW) satisfies the axiom (O2*)?*

Again unfortunately, this is not right either.

Proposition 5.18. *On the L-fuzzy poset (L, e_L), the followings are equivalent:*

(1) *$A * B$ is an upper set for all upper sets A, B on (L, e_L);*

(2) *$a \leq a * a$ for every $a \in L$, that is the condition* (M) *holds for the operation $*$.*

Proof. (1)$\Longrightarrow$(2). Let $A = \mathrm{id}_L$ be the identity mapping on L. It is clear that A is an upper set. Then for every $a \in L$, $(A * A)(\varepsilon) * (\varepsilon \to a) \leq (A * A)(a)$ and hence $a \leq a * a$.

(2)$\Longrightarrow$(1). Suppose that A, B are two upper sets. Then for all $a, b \in L$, $a \to b \leq A(a) \to A(b)$ and $a \to b \leq B(a) \to B(b)$ and then

$$a \to b \leq (a \to b) * (a \to b) \leq (A(a) \to A(b)) * (B(a) \to B(b)) \leq (A * B)(a) \to (A * B)(b).$$

Hence, $A * B$ is an upper set. □

If the operation $*$ satisfies the condition (M), then a commutative unital quantale will have a lot of same or similar properties as a complete Heyting algebra, and techniques in fuzzy domains are also similar for these two different truth value tables (see [51]).

For a summary, what should be a fuzzy Scott topology in the case of for L being a commutative unital quantale? It could be $\sigma_L(X)$ induced by the fuzzy Scott convergence structure $\mathcal{S}$, which seems a right way to define the quantale-valued fuzzy Scott topology on L-fuzzy dcpos.

6 Conclusions

In this paper, we have proposed the concepts of sd-quantales and generalized GL-monoids and used them as the truth value table for stratified L-generalized convergence spaces. A generalized GL-monoid makes a GL-monoid and a Heyting algebra as special cases. We systematically studied some relations between stratified L-generalized convergence spaces and stratified L-topological spaces. As desired, we showed that the category of stratified L-topological spaces can be reflectively embedded in the category of stratified L-generalized convergence spaces, and stratified L-topologies and stratified L-topological convergence structures are categorically isomorphic. As a matter of fact, we have not used any new technique in this part. All the definitions, approaches and results are similar and imitated from those in the literatures. One aim of this paper is to introduce a more general lattice which can be considered as the truth value table for lattice-valued convergence spaces in the future study, but more importantly, we can apply the theory of stratified L-generalized convergence spaces to define of quantale-valued fuzzy Scott topology on L-fuzzy dcpos. It seems that the inducing way is an appropriate way to quantale-valued fuzzy Scott topology.

There are some further potential works as follows:

1. To study the quantale-valued fuzzy Scott topology.

It has been explained clearly that, for a commutative unital quantale L, the inducing way is a right way to fuzzy Scott topology on L-fuzzy dcpos. It is a question worth further studying that,

Do the results related to fuzzy Scott topology in [54] still hold in quantale-valued setting, for example, the fuzzy sobriety and the convergence theory of stratified L-filters?

2. To study properties of category of stratified L-generalized convergence spaces and its subcategories.

A main aim of [18] to introduce stratified frame-valued generalized convergence spaces is that these structures form a cartesian-closed topological category which embeds the category of topological spaces as a reflective subcategory. In [19], Jäger studied in detail the properties of several subcategories of the category of stratified L-generalized convergence spaces. For L being a generalized GL-monoid, we also should study the topologicalness and cartesian-closedness for the category of stratified L-generalized convergence spaces and its subcategories as well as their interrelations.

3. To study stratified L-generalized convergence spaces in topology sense.

In [21, 22, 23, 24, 25, 26], Jäger studied the regularity, compactness and compactification of stratified L-generalized convergence spaces. For L being a generalized GL-monoid, we also could study separation axioms, compactness, compactifications and other topology-like properties.

4. To study some relations between stratified L-ordered convergence spaces and strong stratified L-topological spaces

In [8], for L a commutative unital quantale, Fang defined stratified L-ordered filters and used them to define the concept of stratified L-ordered convergence structures, and then studied the categorical relations between stratified L-ordered convergence spaces and strong stratified L-topological spaces.

A stratified L-filter $\mathcal{F} : L^X \longrightarrow L$ is called a *stratified L-ordered filter* [8] if

(FO) $a \to \mathcal{F}(A) \le \mathcal{F}(a_X \to A)$ for all $a \in L$ and $A \in \delta$.

A stratified L-topological space (X, δ) is called *strong* [17, 56] if

(OS) $a_X \to A \in \delta$ for all $a \in L$ and $A \in \delta$.

Let $\mathbf{OF}^s_L(X)$ be the set of all stratified L-ordered filters on the set X. A *stratified L-ordered convergence structure* on a nonempty set X [8] is a mapping $\lim : \mathbf{OF}^s_L(X) \longrightarrow L^X$ with Condition (LC1) and

(OLC2) $\mathcal{F} \le \mathcal{G}$ implies $\lim \mathcal{F} \le \lim \mathcal{G}$ $(\forall \mathcal{F}, \mathcal{G} \in \mathbf{OF}^s_L(X))$.

We can replace the commutative unital quantale in [8, 56] by a generalized GL-monoid and then study the similar contents in Section 3 and that in [8].

5. By using different kinds of fuzzy filters, we can get different types of fuzzy generalized convergence structures.

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