

INTERVAL-VALUED FUZZY B -ALGEBRAS

A. BORUMAND SAEID

ABSTRACT. In this note the notion of interval-valued fuzzy B -algebras (briefly, i-v fuzzy B -algebras), the level and strong level B -subalgebra is introduced. Then we state and prove some theorems which determine the relationship between these notions and B -subalgebras. The images and inverse images of i-v fuzzy B -subalgebras are defined, and how the homomorphic images and inverse images of i-v fuzzy B -subalgebra becomes i-v fuzzy B -algebras are studied.

1. Introduction

Y. Imai and K. Iseki [4] introduced two classes of abstract algebras: BCK -algebras and BCI -algebras. It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. In [8], J. Neggers and H. S. Kim introduced the notion of d -algebras, which is generalization of BCK -algebras and investigated relation between d -algebras and BCK -algebras. Also they introduced the notion of B -algebras [7]. Y. B. Jun et. al. applied the fuzzy notions to B -algebras and introduced the notions of fuzzy B -algebras [5]. The concept of a fuzzy set, which was introduced in [10].

In [11], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set, also he constructed a method of approximate inference using his i-v fuzzy sets. Biswas [1], defined interval-valued fuzzy sub groups and S. M. Hong et. al. applied the notion of interval-valued fuzzy to BCI -algebras [3].

In the present paper, we using the notion of interval-valued fuzzy set by Zadeh and introduced the concept of interval-valued fuzzy B -subalgebras (briefly i-v fuzzy B -subalgebras) of a B -algebra, and study some of their properties. We prove that every B -subalgebra of a B -algebra X can be realized as an i-v level B -subalgebra of an i-v fuzzy B -subalgebra of X , then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. [7] A B -algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

$$(I) \quad x * x = 0,$$

Received: April 2005; Accepted: February 2006

Key words and phrases: B -algebra, Fuzzy B -subalgebra, Interval-valued fuzzy set, Interval-valued fuzzy B -subalgebra.

- (II) $x * 0 = x$,
 (III) $(x * y) * z = x * (z * (0 * y))$,
 for all $x, y, z \in X$.

Example 2.2. [5] Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, *, 0)$ is a B -algebra.

Theorem 2.3. [7] If X is a B -algebra, then $x * y = x * (0 * (0 * y))$, for all $x, y \in X$;

A non-empty subset I of a B -algebra X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

A mapping $f : X \rightarrow Y$ of B -algebras is called a B -homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [10]).

Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B .

The inverse image of B , denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set in X with membership function μ_A . Then the image of A , denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy set A in the B -algebra X with the membership function μ_A is said to be have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t)$$

An interval-valued fuzzy set (briefly, i-v fuzzy set) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \forall x \in X.$$

Briefly, denoted by $A = [\mu_A^L, \mu_A^U]$ where μ_A^L and μ_A^U are any two fuzzy sets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and let $D[0, 1]$ denotes the family of all closed sub-intervals of $[0, 1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$ then $\bar{\mu}_A(x) = [c, c]$ is in $D[0, 1]$. Thus $\bar{\mu}_A(x) \in D[0, 1]$, for all $x \in X$. Therefore the i-v fuzzy set A is given by

$$A = \{(x, \bar{\mu}_A(x))\}, \forall x \in X$$

where

$$\bar{\mu}_A : X \longrightarrow D[0, 1]$$

Now we define refined minimum (briefly, rmin) and order " \leq " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0, 1]$ as:

$$\text{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$$

$$D_1 \leq D_2 \iff a_1 \leq a_2 \wedge b_1 \leq b_2$$

Similarly we can define \geq and $=$.

Definition 2.4. [5] Let μ be a fuzzy set in a B -algebra. Then μ is called a fuzzy B -subalgebra (B -algebra) of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

Example 2.5. [5] Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a B -subalgebra. Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by $\mu(0) = \mu(3) = 0.7 > 0.1 = \mu(x)$ for all $x \in X \setminus \{0, 3\}$. Then μ is a fuzzy B -subalgebra of X .

Example 2.6. [5] Let \mathcal{Z} be the group of integers under usual addition and let $\alpha \notin \mathcal{Z}$. We adjoin the special element α to \mathcal{Z} . Let $X := \mathcal{Z} \cup \{\alpha\}$. Define $\alpha + 0 = \alpha$, $\alpha + n = n - 1$ where $n \neq 0$ in \mathcal{Z} and $\alpha + \alpha$ is an arbitrary element in X . Define a mapping $\varphi : X \rightarrow X$ by $\varphi(\alpha) = 1$, $\varphi(n) = -n$ where $n \in \mathcal{Z}$. If we define a binary operation " $*$ " on X by $x * y := x + \varphi(y)$, then $(X, *, 0)$ is a B -algebra.

Now define $\mu : X \rightarrow [0, 1]$ as follows:

$$\mu(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0, \\ 1 & \text{if } x = \alpha, 0 \end{cases}$$

Then it is clear that μ is a fuzzy B -algebra that has sup property.

Proposition 2.7. [2] Let f be a B -homomorphism from X into Y and G be a fuzzy B -subalgebra of Y with the membership function μ_G . Then the inverse image $f^{-1}(G)$ of G is a fuzzy B -subalgebra of X .

Proposition 2.8. [2] Let f be a B -homomorphism from X onto Y and D be a fuzzy B -subalgebra of X with the sup property. Then the image $f(D)$ of D is a fuzzy B -subalgebra of Y .

3. Interval-valued Fuzzy B -algebra

From now on X is a B -algebra, unless otherwise is stated.

Definition 3.1. An i-v fuzzy set A in X is called an interval-valued fuzzy B -subalgebra (briefly i-v fuzzy B -subalgebra) of X if:

$$\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define $\bar{\mu}_A$ as:

$$\bar{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{Otherwise} \end{cases}$$

It is easy to check that A is an i-v fuzzy B -subalgebra of X .

Lemma 3.3. If A is an i-v fuzzy B -subalgebra of X , then for all $x \in X$

$$\bar{\mu}_A(0) \geq \bar{\mu}_A(x).$$

Proof. For all $x \in X$, we have

$$\begin{aligned} \bar{\mu}_A(0) &= \bar{\mu}_A(x * x) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(x)\} \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(x), \mu_A^U(x)]\} \\ &= [\mu_A^L(x), \mu_A^U(x)] = \bar{\mu}_A(x). \end{aligned}$$

Lemma 3.4. For any element x and y of X , let us write $\prod^n x * y$ for $x * (\dots * (x * (x * y)))$ where x occurs n times. We prove by induction

$$\prod^n x * x = \begin{cases} x & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k . By definition we get that $x * (x * x) = x$. Now suppose that $\prod^{2k-1} x * x = x$. Then by assumption

$$\begin{aligned} \prod^{2(k+1)-1} x * x &= \prod^{2k+1} x * x \\ &= \prod^{2k-1} x * (x * (x * x)) \\ &= \prod^{2k-1} x * x \\ &= x. \end{aligned}$$

Similarly we can prove theorem when n is even.

By above lemma we have

Proposition 3.5. Let A be an i-v fuzzy B -subalgebra of X , and let $n \in \mathcal{N}$. Then

- (i) $\bar{\mu}_A(\prod^n x * x) = \bar{\mu}_A(x)$, for any odd number n ,
- (ii) $\bar{\mu}_A(\prod^n x * x) = \bar{\mu}_A(0)$, for any even number n .

Theorem 3.6. Let A be an i-v fuzzy B -subalgebra of X . If there exists a sequence $\{x_n\}$ in X , such that

$$\lim_{n \rightarrow \infty} \bar{\mu}_A(x_n) = [1, 1]$$

Then $\bar{\mu}_A(0) = [1, 1]$.

Proof. By above lemma we have $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$, for all $x \in X$, thus $\bar{\mu}_A(0) \geq \bar{\mu}_A(x_n)$, for every positive integer n . Consider

$$[1, 1] \geq \bar{\mu}_A(0) \geq \lim_{n \rightarrow \infty} \bar{\mu}_A(x_n) = [1, 1].$$

Hence $\bar{\mu}_A(0) = [1, 1]$.

Theorem 3.7. An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in X is an i-v fuzzy B -subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy B -subalgebra of X .

Proof. Let μ_A^L and μ_A^U are fuzzy B -subalgebra of X and $x, y \in X$, consider

$$\begin{aligned} \bar{\mu}_A(x * y) &= [\bar{\mu}_A(x * y), \bar{\mu}_A(x * y)] \\ &\geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}] \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \end{aligned}$$

This completes the proof.

Conversely, suppose that A is an i-v fuzzy B -subalgebra of X . For any $x, y \in X$ we have

$$\begin{aligned} [\mu_A^L(x * y), \mu_A^U(x * y)] &= \bar{\mu}_A(x * y) \\ &\geq rmin[\bar{\mu}_A(x), \bar{\mu}_A(y)] \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]. \end{aligned}$$

Therefore $\mu_A^L(x * y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x * y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$, hence we get that μ_A^L and μ_A^U are fuzzy B -subalgebras of X .

Theorem 3.8. Let A_1 and A_2 are i-v fuzzy B -subalgebras of X . Then $A_1 \cap A_2$ is an i-v fuzzy B -subalgebra of X .

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2 , since A_1 and A_2 are i-v fuzzy B -subalgebras of X by above theorem we have:

$$\begin{aligned} \bar{\mu}_{A_1 \cap A_2}(x * y) &= [\mu_{A_1 \cap A_2}^L(x * y), \mu_{A_1 \cap A_2}^U(x * y)] \\ &= [\min(\mu_{A_1}^L(x * y), \mu_{A_2}^L(x * y)), \min(\mu_{A_1}^U(x * y), \mu_{A_2}^U(x * y))] \\ &\geq [\min((\mu_{A_1 \cap A_2}^L(x), \mu_{A_1 \cap A_2}^L(y)), \min((\mu_{A_1 \cap A_2}^U(x), \mu_{A_1 \cap A_2}^U(y)))] \\ &= rmin\{\bar{\mu}_{A_1 \cap A_2}(x), \bar{\mu}_{A_1 \cap A_2}(y)\} \end{aligned}$$

This completes the theorem.

Corollary 3.9. Let $\{A_i | i \in \Lambda\}$ be a family of i-v fuzzy B -subalgebras of X . Then $\bigcap_{i \in \Lambda} A_i$ is also an i-v fuzzy B -subalgebra of X .

Definition 3.10. Let A be an i-v fuzzy set in X and $[\delta_1, \delta_2] \in D[0, 1]$. Then the i-v level B -subalgebra $U(A; [\delta_1, \delta_2])$ of A and strong i-v B -subalgebra $U(A; >, [\delta_1, \delta_2])$ of X are defined as following:

$$U(A; [\delta_1, \delta_2]) := \{x \in X \mid \bar{\mu}_A(x) \geq [\delta_1, \delta_2]\},$$

$$U(A; >, [\delta_1, \delta_2]) := \{x \in X \mid \bar{\mu}_A(x) > [\delta_1, \delta_2]\}.$$

Theorem 3.11. Let A be an i-v fuzzy set of X and B be the closure of image of μ_A . Then the following condition are equivalent :

- (i) A is an i-v fuzzy B -subalgebra of X .
- (ii) For all $[\delta_1, \delta_2] \in Im(\mu_A)$, the nonempty level subset $U(A; [\delta_1, \delta_2])$ of A is a B -subalgebra of X .
- (iii) For all $[\delta_1, \delta_2] \in Im(\mu_A) \setminus B$, the nonempty strong level subset $U(A; >, [\delta_1, \delta_2])$ of A is a B -subalgebra of X .
- (iv) For all $[\delta_1, \delta_2] \in D[0, 1]$, the nonempty strong level subset $U(A; >, [\delta_1, \delta_2])$ of A is a B -subalgebra of X .
- (v) For all $[\delta_1, \delta_2] \in D[0, 1]$, the nonempty level subset $U(A; [\delta_1, \delta_2])$ of A is a B -subalgebra of X .

Proof. (i \longrightarrow iv) Let A be an i-v fuzzy B -subalgebra of X , $[\delta_1, \delta_2] \in D[0, 1]$ and $x, y \in U(A; <, [\delta_1, \delta_2])$, then we have

$$\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} > rmin\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2]$$

thus $x * y \in U(A; >, [\delta_1, \delta_2])$. Hence $U(A; >, [\delta_1, \delta_2])$ is a B -subalgebra of X .

(iv \longrightarrow iii) It is clear.

(iii \longrightarrow ii) Let $[\delta_1, \delta_2] \in Im(\mu_A)$. Then $U(A; [\delta_1, \delta_2])$ is a nonempty. Since $U(A; [\delta_1, \delta_2]) = \bigcap_{[\delta_1, \delta_2] > [\alpha_1, \alpha_2]} U(A; >, [\alpha_1, \alpha_2])$, where $[\alpha_1, \alpha_2] \in Im(\mu_A) \setminus B$. Then

by (iii) and Corollary 3.9, $U(A; [\delta_1, \delta_2])$ is a B -subalgebra of X .

(ii \longrightarrow v) Let $[\delta_1, \delta_2] \in D[0, 1]$ and $U(A; [\delta_1, \delta_2])$ be nonempty. Suppose $x, y \in U(A; [\delta_1, \delta_2])$. Let $[\beta_1, \beta_2] = min\{\mu_A(x), \mu_A(y)\}$, it is clear that $[\beta_1, \beta_2] = min\{\mu_A(x), \mu_A(y)\} \geq \{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2]$. Thus $x, y \in U(A; [\beta_1, \beta_2])$ and $[\beta_1, \beta_2] \in Im(\mu_A)$, by (ii) $U(A; [\beta_1, \beta_2])$ is a B -subalgebra of X , hence $x * y \in U(A; [\beta_1, \beta_2])$. Then we have

$$\bar{\mu}_A(x * y) \geq rmin\{\mu_A(x), \mu_A(y)\} \geq \{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2] \geq [\delta_1, \delta_2].$$

Therefore $x * y \in U(A; [\delta_1, \delta_2])$. Then $U(A; [\delta_1, \delta_2])$ is a B -subalgebra of X .

(v \longrightarrow i) Assume that the nonempty set $U(A; [\delta_1, \delta_2])$ is a B -subalgebra of X , for every $[\delta_1, \delta_2] \in D[0, 1]$. In contrary, let $x_0, y_0 \in X$ be such that

$$\bar{\mu}_A(x_0 * y_0) < rmin\{\bar{\mu}_A(x_0), \bar{\mu}_A(y_0)\}.$$

Let $\bar{\mu}_A(x_0) = [\gamma_1, \gamma_2]$, $\bar{\mu}_A(y_0) = [\gamma_3, \gamma_4]$ and $\bar{\mu}_A(x_0 * y_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] < rmin\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [min\{\gamma_1, \gamma_3\}, min\{\gamma_2, \gamma_4\}].$$

So $\delta_1 < min\{\gamma_1, \gamma_3\}$ and $\delta_2 < min\{\gamma_2, \gamma_4\}$.

Consider

$$[\lambda_1, \lambda_2] = \frac{1}{2}\bar{\mu}_A(x_0 * y_0) + rmin\{\bar{\mu}_A(x_0), \bar{\mu}_A(y_0)\}$$

We get that

$$\begin{aligned} [\lambda_1, \lambda_2] &= \frac{1}{2}([\delta_1, \delta_2] + [min\{\gamma_1, \gamma_3\}, min\{\gamma_2, \gamma_4\}]) \\ &= [\frac{1}{2}(\delta_1 + min\{\gamma_1, \gamma_3\}), \frac{1}{2}(\delta_2 + min\{\gamma_2, \gamma_4\})] \end{aligned}$$

Therefore

$$\begin{aligned} min\{\gamma_1, \gamma_3\} &> \lambda_1 = \frac{1}{2}(\delta_1 + min\{\gamma_1, \gamma_3\}) > \delta_1 \\ min\{\gamma_2, \gamma_4\} &> \lambda_2 = \frac{1}{2}(\delta_2 + min\{\gamma_2, \gamma_4\}) > \delta_2 \end{aligned}$$

Hence

$$[min\{\gamma_1, \gamma_3\}, min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \bar{\mu}_A(x_0 * y_0)$$

so that $x_0 * y_0 \notin U(A; [\delta_1, \delta_2])$

which is a contradiction, since

$$\bar{\mu}_A(x_0) = [\gamma_1, \gamma_2] \geq [min\{\gamma_1, \gamma_3\}, min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2]$$

$$\bar{\mu}_A(y_0) = [\gamma_3, \gamma_4] \geq [min\{\gamma_1, \gamma_3\}, min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2]$$

imply that $x_0, y_0 \in U(A; [\delta_1, \delta_2])$. Thus $\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$ for all $x, y \in X$. Which completes the proof.

Theorem 3.12. Each B -subalgebra of X is an i-v level B -subalgebra of an i-v fuzzy B -subalgebra of X .

Proof. Let Y be a B -subalgebra of X , and A be an i-v fuzzy set on X defined by

$$\bar{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in Y \\ [0, 0] & \text{Otherwise} \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 < \alpha_2$. It is clear that $U(A; [\alpha_1, \alpha_2]) = Y$. Let $x, y \in X$. We consider the following cases:

case 1) If $x, y \in Y$, then $x * y \in Y$. Therefore

$$\bar{\mu}_A(x * y) = [\alpha_1, \alpha_2] = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

case 2) If $x, y \notin Y$, then $\bar{\mu}_A(x) = [0, 0] = \bar{\mu}_A(y)$ and so

$$\bar{\mu}_A(x * y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

case 3) If $x \in Y$ and $y \notin Y$, then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\bar{\mu}_A(y) = [0, 0]$. Thus

$$\bar{\mu}_A(x * y) \geq [0, 0] = rmin\{[\alpha_1, \alpha_2], [0, 0]\} = rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

case 4) If $y \in Y$ and $x \notin Y$, then by the same argument as in case 3, we can conclude that $\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$.

Therefore A is an i-v fuzzy B -subalgebra of X .

Theorem 3.13. Let Y be a subset of X and A be an i-v fuzzy set on X which is given in the proof of Theorem 3.11. If A is an i-v fuzzy B -subalgebra of X , then Y is a B -subalgebra of X .

Proof. Let A be an i-v fuzzy B -subalgebra of X , and $x, y \in Y$. Then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2] = \bar{\mu}_A(y)$, thus

$$\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].$$

which implies that $x * y \in Y$.

Theorem 3.14. If A is an i-v fuzzy B -subalgebra of X , then the set

$$X_{\bar{\mu}_A} := \{x \in X \mid \bar{\mu}_A(x) = \bar{\mu}_A(0)\}$$

is a B -subalgebra of X .

Proof. Let $x, y \in X_{\bar{\mu}_A}$. Then $\bar{\mu}_A(x) = \bar{\mu}_A(0) = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x * y) \geq rmin\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = rmin\{\bar{\mu}_A(0), \bar{\mu}_A(0)\} = \bar{\mu}_A(0).$$

by Lemma 3.3, we get that $\bar{\mu}_A(x * y) = \bar{\mu}_A(0)$ which means that $x * y \in X_{\bar{\mu}_A}$.

Theorem 3.15. Let N be an i-v fuzzy subset of X . Let N be an i-v fuzzy set defined by $\bar{\mu}_A$ as:

$$\bar{\mu}_N(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in N \\ [\beta_1, \beta_2] & \text{Otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$. Then N is an i-v fuzzy B -subalgebra if and only if N is a B -subalgebra of X . Moreover, in this case $X_{\bar{\mu}_N} = N$.

Proof. Let N be an i-v fuzzy B -subalgebra. Let $x, y \in X$ be such that $x, y \in N$. Then

$$\bar{\mu}_N(x * y) \geq rmin\{\bar{\mu}_N(x), \bar{\mu}_N(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

and so $x * y \in N$.

Conversely, suppose that N is a B -subalgebra of X , let $x, y \in X$.

(i) If $x, y \in N$ then $x * y \in N$, thus

$$\bar{\mu}_N(x * y) = [\alpha_1, \alpha_2] = rmin\{\bar{\mu}_N(x), \bar{\mu}_N(y)\}$$

(ii) If $x \notin N$ or $y \notin N$, then

$$\bar{\mu}_N(x * y) \geq [\beta_1, \beta_2] = rmin\{\bar{\mu}_N(x), \bar{\mu}_N(y)\}$$

This show that N is an i-v fuzzy B -subalgebra.

Moreover, we have

$$X_{\bar{\mu}_N} := \{x \in X \mid \bar{\mu}_N(x) = \bar{\mu}_N(0)\} = \{x \in X \mid \bar{\mu}_N(x) = [\alpha_1, \alpha_2]\} = N.$$

Definition 3.16. [1] Let f be a mapping from the set X into a set Y . Let B be an i-v fuzzy set in Y . Then the inverse image of B , denoted by $f^{-1}[B]$, is the i-v fuzzy set in X with the membership function given by $\bar{\mu}_{f^{-1}[B]}(x) = \bar{\mu}_B(f(x))$, for all $x \in X$.

Lemma 3.17. [1] Let f be a mapping from the set X into a set Y . Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in X and Y respectively. Then

- (i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,
- (ii) $f(m) = [f(m^L), f(m^U)]$.

Proposition 3.18. Let f be a B -homomorphism from X into Y and G be an i-v fuzzy B -subalgebra of Y with the membership function μ_G . Then the inverse image $f^{-1}[G]$ of G is an i-v fuzzy B -subalgebra of X .

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy B -subalgebra of Y , by Theorem 3.7, we get that μ_B^L and μ_B^U are fuzzy B -subalgebra of Y . By Proposition 2.7, $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy B -subalgebra of X , by above lemma and Theorem 3.7, we can conclude that $f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]$ is an i-v fuzzy B -subalgebra of X .

Definition 3.19. [1] Let f be a mapping from the set X into a set Y , and A be an i-v fuzzy set in X with membership function μ_A . Then the image of A , denoted by $f[A]$, is the i-v fuzzy set in Y with membership function defined by:

$$\bar{\mu}_{f[A]}(y) = \begin{cases} rsup_{z \in f^{-1}(y)} \bar{\mu}_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\ [0, 0] & \text{otherwise} \end{cases}$$

Where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.20. Let f be a B -homomorphism from X onto Y . If A is an i-v fuzzy B -subalgebra of X with the sup property, then the image $f[A]$ of A is an i-v fuzzy B -subalgebra of Y .

Proof. Assume that A is an i-v fuzzy B -subalgebra of X , then $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy B -subalgebra of X if and only if μ_B^L and μ_B^U are fuzzy B -subalgebra of

X . By Proposition 2.8, $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy B -subalgebra of Y , by Lemma 3.17, and Theorem 3.7, we can conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy B -subalgebra of Y .

Acknowledgements. The author would like to express their sincere thanks to the referees for their valuable suggestions and comments.

REFERENCES

- [1] R. Biswas, *Rosenfeld's fuzzy subgroups with interval valued membership function*, Fuzzy Sets and Systems, **63**, **1**(1994), 87-90.
- [2] A. Borumand Saeid, *Fuzzy topological B-algebras*, (Submitted).
- [3] S. M. Hong, Y. B. Jun, S. J. Kim and G. I. Kim, *Fuzzy BCI-subalgebras with interval-valued membership functions*, IJMMS., **25**, **2** (2001), 135-143.
- [4] Y. Imai and K. Iseki, *On axiom systems of propositional calculi*, XIV Proc. Japan Academy, **42** (1966), 19-22.
- [5] Y. B. Jun, E. H. Roh, Chinju and H. S. Kim, *On Fuzzy B-algebras*, Czechoslovak Math. Journal, **52** (2002), 375-384.
- [6] J. Meng and Y.B. Jun, *BCK-algebras*, Kyung MoonSa, Seoul, Korea, (1994).
- [7] J. Neggers and H. S. Kim, *On B-algebras*, Math. Vensik, **54** (2002), 21-29.
- [8] ———, *On d-algebras*, Math. Slovaca, **49** (1999), 19-26.
- [9] A. Rosenfeld, *Fuzzy Groups*, J. Math. Anal. Appl., **35** (1971), 512-517.
- [10] L. A. Zadeh, *Fuzzy Sets*, Inform. Control, **8** (1965), 338-353.
- [11] ———, *The concept of a linguistic variable and its application to approximate reasoning. I*, Information Sci., **8** (1975), 199-249.

ARSHAM BORUMAND SAEID, DEPT. OF MATHEMATICS, ISLAMIC AZAD UNIVERSITY, KERMAN BRANCH, KERMAN, IRAN

E-mail address: arsham@iauk.ac.ir