SIMULATING CONTINUOUS FUZZY SYSTEMS: I

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ABSTRACT. In previous studies we first concentrated on utilizing crisp simulation to produce discrete event fuzzy systems simulations. Then we extended this research to the simulation of continuous fuzzy systems models. In this paper we continue our study of continuous fuzzy systems using crisp continuous simulation. Consider a crisp continuous system whose evolution depends on differential equations. Such a system contains a number of parameters that must be estimated. Usually point estimates are computed and used in the model. However these point estimates typically have uncertainty associated with them. We propose to incorporate uncertainty by using fuzzy numbers as estimates of these unknown parameters. Fuzzy parameters convert the crisp system into a fuzzy system. Trajectories describing the behavior of the system become fuzzy curves. We will employ crisp continuous simulation to estimate these fuzzy trajectories. Three examples are discussed.

1. Introduction

This paper is concerned with using crisp continuous simulation to estimate the evolution of continuous fuzzy systems. The continuous fuzzy systems we will look at are all governed by fuzzy differential equations. So, let us first look at our previous work in this area of using crisp simulation to study fuzzy systems([3],[4],[8],[9]).

We started with studying what are called discrete event (fuzzy) systems. These systems can usually be described as queuing networks. Items (transactions) arrive at various points in the system and go into a queue waiting for service. The service stations, preceded by a queue, are connected forming a network of queues and service, until the transaction finally exits the system. Examples considered included machine shops, emergency rooms, project networks, bus routes, etc. Analysis of all of these systems depends on parameters for arrival rates and service rates. These parameters are usually estimated from historical data. These estimators are generally point estimators. The point estimators are put into the model to compute system descriptors such as the mean time an item spends in the system, or the expected number of transactions leaving the system per unit time. We argued that these point estimators contain uncertainty not shown in the calculations. Our estimators of these parameters become fuzzy numbers constructed by placing a set of confidence intervals one on top of another. Using fuzzy number parameters in the model turns it into a fuzzy system. The system descriptors we want (time in system, number leaving per unit time) will be fuzzy numbers. In general computing

Invited Paper: Received in February 2005

Key words and phrases: Fuzzy systems, Fuzzy differential equations, Simulation, Uncertainty.

these fuzzy numbers can be difficult. We showed how crisp discrete event simulation can be used to estimate the fuzzy numbers used to describe system behavior.

Continuous systems are usually described by a system of ordinary differential equations (ODEs). Many parameters in the system of ODEs are not known precisely and must be estimated. To show the uncertainty in these parameter values we will use fuzzy number estimators. Fuzzy number parameter values produce a system of fuzzy ODEs to solve ([5]-[7]) and we have a continuous fuzzy system. Solution trajectories become fuzzy trajectories. We plan to use crisp continuous simulation to estimate these fuzzy trajectories.

We will start with a continuous crisp system whose description in time depends on crisp ordinary differential equations. Let us consider an example of a predator/prey model, also discussed in detail in Section 7. This is adopted from an example in ([14],[18]). The system of differential equations is

$$\dot{x} = -ax + bxy, \tag{1}$$

$$\dot{y} = dy - cxy, \tag{2}$$

for constants a, b, c, d all positive and initial conditions $x(0) = x_0$, $y(0) = y_0$. We write the time derivative of x (y) as $\dot{x}$ ($\dot{y}$). Non-trivial solutions to this system cannot be obtained in terms of elementary functions. So we would need to employ some software to obtain the graphs of $x(t)$ and $y(t)$. How do we get values for a, b, c, d? We will consider two cases in more detail in the next section: (1) their values are estimated by expert opinion; and (2) their values are estimated from data. In either case the estimators become fuzzy, or fuzzy numbers.

Our notation for a fuzzy set is to place a bar over a letter. So $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$ all represent fuzzy numbers. Usually we will be using triangular fuzzy numbers. A triangular fuzzy number is defined by three numbers $m < n < p$ where the base of the triangle is on the interval $[m, p]$ and the vertex (membership value one) is at $x = n$. We write $\overline{N} = (m/n/p)$ for triangular fuzzy number $\overline{N}$. A triangular shaped fuzzy number, written $\overline{N} \approx (m/n/p)$, has curves for its sides, instead of straight line segments. All fuzzy sets have membership functions written as $\overline{N}(x) =$ the membership value of x in the fuzzy set $\overline{N}$. An alpha-cut of $\overline{N}$, written $\overline{N}[\alpha]$, is defined as $\{x | \overline{N}(x) \geq \alpha\}$ for $0 < \alpha \leq 1$. $\overline{N}[0]$ is defined separately as the closure of the union of $\overline{N}[\alpha]$ for $\alpha \in (0, 1]$.

In the predator/prey model above we could have some, or all the parameters, fuzzy. Therefore, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{x}_0$ and $\overline{y}_0$ may all be fuzzy. Then we have a system of fuzzy nonlinear differential equations to solve. The trajectories for $\overline{x}(t)$ and $\overline{y}(t)$, $t \geq 0$, will be fuzzy which means that for each value of t, $\overline{x}(t)$ and $\overline{y}(t)$ will be fuzzy numbers. We have considered solving fuzzy differential equations before ([5]-[7]). However, in those papers/book we almost always allowed for only fuzzy initial conditions because the fuzzy solution became too difficult to obtain when more parameters became fuzzy. Now we may fuzzify more parameters because we are not finding a formula for the mathematical solution but instead we will use simulation.

The next section discusses fuzzy estimation. Then we briefly discuss the optimization problem associated with simulation used to estimate fuzzy trajectories and fuzzy numbers, followed by a discussion of the selection of the continuous simulation software. Then we present three examples: (1) an arms race model; (2) the M machine, R repair person queuing model; and (3) a predator/prey model. The last section contains a brief summary, conclusions and our plan for future research.

2. **Fuzzy Estimators**

In this paper we will consider only two methods of fuzzy estimation: (1) expert opinion and (2) from data using confidence intervals. First we discuss the expert opinion method. More information on fuzzy estimators is in [1].

Let us look at an example called the "arms race" model [11], also discussed in detail in Section 5. The system of differential equations is

$$\dot{x} = -ax + by + r, \tag{3}$$

$$\dot{y} = cx - dy + s, \tag{4}$$

where a, b, c, d are all positive constants, r, s are positive, or negative, constants, subject to initial conditions $x(0) = x_0$, $y(0) = y_0$. Here x (y) represents the yearly rates of armament expenditures of nation A (B) in dollars. Consider estimating the constant b. The basic assumption involving b is that the rate of change of x $(\dot{x})$ is directly proportional to the present expenditures of B, which is y. Or, $\dot{x} = by$. How shall we estimate b? Assume we do not have any recent data on these expenditures for these two countries. We turn to expert opinion.

We may obtain a value for b from some group of experts. This group could consist of only one expert. First assume we have only one expert and he/she is to estimate the value of b. We can solicit this estimate from the expert as is done in estimating job times in project scheduling ([15], Chapter 13). Let $b_1 =$ the "pessimistic" value of b, or the smallest possible value, let $b_3 =$ be the "optimistic" value of b, or the highest possible value, and let $b_2 =$ the most likely value of b. We then ask the expert to give values for b_1, b_2, b_3 and we construct the triangular fuzzy number $\overline{b} = (b_1/b_2/b_3)$ for b. If we have a group of N experts all to estimate the value of b we solicit the b_{1i}, b_{2i} and b_{3i}, $1 \leq i \leq N$, from them. Let b_1 be the average of the b_{1i}, b_2 is the mean of the b_{2i} and b_3 is the average of the b_{3i}. The simplest thing to do is to use $(b_1/b_2/b_3)$ for $\overline{b}$. We now assume, when necessary, this is how we employ expert opinion to obtain fuzzy estimators.

Let us next describe the construction of our fuzzy estimators out of a set of confidence intervals computed from data. More details can be found in ([1]-[4]). This type of fuzzy estimator will be needed in the queuing system model in Section 6. Let X be a random variable with probability density function $f(x; \theta)$ for single parameter θ. Assume that θ is unknown and it must be estimated from a random sample $X_1, ..., X_n$. Let $Y = u(X_1, ..., X_n)$ be a statistic used to estimate θ. Given the values of these random variables $X_i = x_i$, $1 \leq i \leq n$, we obtain a point estimate $\theta^* = y = u(x_1, ..., x_n)$ for θ. We would never expect this point estimate to exactly equal θ so we often also compute a $(1-\beta)100\%$ confidence interval for θ. In this confidence interval one usually sets β equal to 0.10, 0.05 or 0.01.

We propose to find the $(1-\beta)100\%$ confidence interval for all $0.01 \leq \beta < 1$. Starting at 0.01 is arbitrary and you could begin at 0.10 or 0.05 or 0.005, etc. Denote these confidence intervals as

$$[\theta_1(\beta), \theta_2(\beta)], \tag{5}$$

for $0.01 \leq \beta < 1$. Add to this the interval $[\theta^*, \theta^*]$ for the 0% confidence interval for θ. Then we have $(1-\beta)100\%$ confidence intervals for θ for $0.01 \leq \beta \leq 1$.

Now place these confidence intervals, one on top of the other, to produce a triangular shaped fuzzy number $\overline{\theta}$ whose α-cuts are the confidence intervals. We have

$$\overline{\theta}[\alpha] = [\theta_1(\alpha), \theta_2(\alpha)], \tag{6}$$

for $0.01 \leq \alpha \leq 1$. All that is needed is to finish the "bottom" of $\overline{\theta}$ to make it a complete fuzzy number. We will simply drop the graph of $\overline{\theta}$ straight down to complete its α-cuts so

$$\overline{\theta}[\alpha] = [\theta_1(0.01), \theta_2(0.01)], \tag{7}$$

for $0 \leq \alpha < 0.01$. In this way we are using more information in $\overline{\theta}$ than just a point estimate, or just a single interval estimate. Point estimators show no uncertainty in the estimator.

2.1. **Fuzzy Estimator of μ.** Consider X a random variable with probability density function $N(\mu, \sigma^2)$, with unknown mean μ and known variance σ^2. For unknown variance see [1]. To estimate μ we obtain a random sample $X_1, ..., X_n$ from $N(\mu, \sigma^2)$. Suppose the mean of this random sample turns out to be $\overline{x}$,which is a crisp number, not a fuzzy number. We know that $\overline{x}$ is $N(\mu, \sigma^2/n)$. So $(\overline{x}-\mu)/(\sigma/\sqrt{n})$ is $N(0,1)$. Therefore

$$P(-z_{\beta/2} \leq \frac{\overline{x}-\mu}{\sigma/\sqrt{n}} \leq z_{\beta/2}) = 1-\beta, \tag{8}$$

where $z_{\beta/2}$ is the z value so that the probability of a $N(0,1)$ random variable exceeding it is $\beta/2$. Now solve the inequality for μ producing

$$P(\overline{x} - z_{\beta/2}\sigma/\sqrt{n} \leq \mu \leq \overline{x} + z_{\beta/2}\sigma/\sqrt{n}) = 1-\beta. \tag{9}$$

This leads directly to the $(1-\beta)100\%$ confidence interval for μ

$$[\theta_1(\beta), \theta_2(\beta)] = [\overline{x} - z_{\beta/2}\sigma/\sqrt{n}, \overline{x} + z_{\beta/2}\sigma/\sqrt{n}], \tag{10}$$

where $z_{\beta/2}$ is defined as

$$\int_{-\infty}^{z_{\beta/2}} N(0,1)dx = 1-\beta/2, \tag{11}$$

and $N(0,1)$ denotes the normal density with mean zero and unit variance. Put these confidence intervals together as discussed above and we obtain $\overline{\mu}$ our fuzzy estimator of μ.

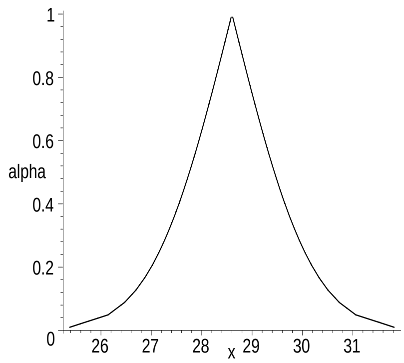

FIGURE 1. Fuzzy Estimator $\overline{\mu}$ in Example 2.1, $0.01 \leq \beta \leq 1$

The following example shows that the fuzzy estimator of the mean of the normal probability density will be a triangular shaped fuzzy number.

Example 2.1. Consider X a random variable with probability density function $N(\mu, 100)$, which is the normal probability density with unknown mean μ and known variance $\sigma^2 = 100$. To estimate μ we obtain a random sample $X_1, ..., X_n$ from $N(\mu, 100)$. Suppose the mean of this random sample turns out to be 28.6. Then a $(1-\beta)100\%$ confidence interval for μ is

$$[\theta_1(\beta), \theta_2(\beta)] = [28.6 - z_{\beta/2}10/\sqrt{n}, 28.6 + z_{\beta/2}10/\sqrt{n}]. \tag{12}$$

To obtain a graph of fuzzy μ, or $\overline{\mu}$, let $n = 64$ and assume that $0.01 \leq \beta \leq 1$. We evaluated equation (12) using Maple [10] and then the final graph of $\overline{\mu}$ is shown in Figure 1, without dropping the graph straight down to the x-axis at the end points.

For simplicity we will use triangular fuzzy numbers, instead of triangular shaped fuzzy numbers, for fuzzy estimators in the rest of the paper.

3. **Simulation Optimization**

Let $x(t)$ be the solution for a variable in the crisp continuous system that we wish to study. Fuzzy estimation of parameters in the system leads to a fuzzy system and a fuzzy trajectory $\overline{x}(t)$. For each $t \geq 0$, $\overline{x}(t)$ is a fuzzy number. Let $\overline{x}(t)[0] = [x_1(t), x_2(t)]$ and $\overline{x}(t)[1] = X(t)$. We want to look at the graph

$$(x_1(t), X(t), x_2(t)),\ 0 \leq t \leq t_m, \tag{13}$$

where t_m is some maximum time. This graph shows the uncertainty in the trajectory of x due to the uncertainty in the values of some of the parameters in the model. Now let us look at two examples of finding the $x_i(t)$. In the first example it is easy to find the $x_i(t)$ but in the second example finding these functions is more difficult.

Example 3.1. This is the simple fuzzy differential equation

$$\dot{x} = \overline{A}x, \tag{14}$$

where $\overline{A} = (a_1/a_2/a_3) > 0$ and $x(0) = x_0 > 0$. The solution is

$$\overline{x}(t) = x_0 exp(\overline{A}t). \tag{15}$$

Hence,

$$x_1(t) = x_0 exp(a_1 t), \tag{16}$$

$$X(t) = x_0 exp(a_2 t), \tag{17}$$

$$x_2(t) = x_0 exp(a_3 t). \tag{18}$$

Example 3.2. This is the "arms race" model in equations (3) and (4). Now the a,b,c,d,r,s of (3) and (4) are all triangular fuzzy numbers, $\overline{a}$,$\overline{b}$,$\overline{c}$,$\overline{d}$,$\overline{r}$,$\overline{s}$. We know how to get $X(t)$ and $Y(t)$: use the vertex values for all the triangular fuzzy numbers. But how shall we pick $a \in \overline{a}[0]$,...,$s \in \overline{s}[0]$ to obtain $x_i(t)$ $(y_i(t))$? We can make a guess, but we have no proof that these values will do the job. For $x_1(t)$ make the slope ($\dot{x}$) as small as possible so use a_2,b_1,r_1 where $\overline{a}[0] = [a_1, a_2]$, etc. We first "guess" and then verify that it gives the desired result by experimenting with lots of other values for the parameters.

4. Simulation Software

Now we come to the point were we need to select simulation software to do all the crisp continuous simulations starting in the next section. The authors are not experts in continuous simulation and do not know about all the products that are available. We will do a "search" for continuous simulation products.

In choosing a simulation package we will have these main constraints: (1) it must be inexpensive, hopefully at most 100 US dollars; (2) it must be easy to use; (3) it has to run on a desktop computer; and (4) the figures created by the software must be exportable to LaTeX. To start the search put "continuous simulation" into your web search engine and start looking through the pages. We did come up with "Buyer's Guide Simulation Software" [13] which contains most of the software companies marketing continuous simulation.

We also want the simulation software self contained and ready to use so we do not need to write code (C++, Java,...) to run a simulation. Obviously, it must have a good user's manual. We decided to use the "click-drag-drop" simulation software. The "click-drag-drop" method makes it very easy to construct a network. Each item in your system has an icon in a library which you click on, then drag it to where you want it in the network and then drop it. You connect the icons using the mouse. Separate windows open for you to input information about each icon. We found this the easiest method to build the systems we want to simulate. Our search was narrowed down to Simulink [17]. However, this product is expensive.

We then discovered a product similar to Simulink that can be downloaded free [12]. We used this continuous simulation package to build a few small continuous systems and then fuzzify them. In doing this we found that this product has very poor documentation and we will be unable to use it, in general, for larger systems.

Most simulation software comes in two types: (1) a "scaled down" version usually called the "student version"; and (2) the full version usually called the "standard"

Parameter	Fuzzy Value
a	$(1.5/2/2.5)$
b	$(0.5/1/1.5)$
c	$(5.5/6/6.5)$
d	$(1.5/2/2.5)$
r	$(-5.5/-5/-4.5)$
s	$(-12.5/-12/-11.5)$
x_0	15
y_0	15

Table 1. Fuzzy/Crisp Parameters in the Arms Race Model

or "professional" version. The professional software is too expensive, usually in the thousands of dollars, so we are interested in using the student software. We may be able to download the student (and professional) software and use it for free for a short time period. Since our university already has a licensing agreement for Matlab (needed for Simulink) all our continuous simulations will be performed with Simulink.

5. Example 1: An Arms Race Model

The arms race model was introduced in Section 2. We first need to look more closely at this model to understand the meaning of the constants a, b, c, d, r, s. $x(t)$ $(y(t))$ is the yearly rate of armament expenditures of nation A (B) in dollars. The first assumption is that $\dot{x}$ $(\dot{y})$ is directly proportional to y (x). Hence $\dot{x} = by$ $(\dot{y} = cx)$ for some positive constant b (c). The next assumption is that excessive expenditures places a "drag" on the economy, or $\dot{x}$ $(\dot{y})$ is directly and negatively proportional to x (y). That is $\dot{x} = -ax$ $(\dot{y} = -dy)$ for some positive constant a (d). So now we have $\dot{x} = -ax+by$ and $\dot{y} = cx-dy$. Finally, increasing (decreasing) arms expenditures is not the only mutual stimulation on expenditures, but underlying grievance (good will) of each nation against the other has important implications on expenditures and is in the constants r and s in equations (3) and (4). We also need initial values for x and y. If there is a value of t, say $t = t_0$, for which $x(t_0) = 0$ $(y(t_0) = 0)$, then $x(t) = 0$ $(y(t) = 0)$ for $t \geq t_0$.

We now need to estimate all the parameters in the model. The data we have is old, from before the last great war. So we turn to a group of experts in this field to help estimate all the parameters. We obtained the triangular fuzzy numbers shown in Table 1. Notice that $x_0 = y_0 = 15$ are known exactly and hence are not fuzzy. Since the parameters in the equations are now fuzzy, the solutions will also be fuzzy. Let $\overline{x}(t)[\alpha] = [x_1(t,\alpha), x_2(t,\alpha)]$ and $\overline{y}(t)[\alpha] = [y_1(t,\alpha), y_2(t,\alpha)]$, $t > 0$ and $0 \leq \alpha \leq 1$.

Next we need to choose the values of the parameters in their $\alpha = 0$ cut to get the outer boundary of the $\alpha = 0$ cut of the fuzzy trajectory for $x(t)$ and $y(t)$. Let us define, in general, $\overline{a}[0] = [a_1, a_2]$, $\overline{b}[0] = [b_1, b_2]$, $\overline{c}[0] = [c_1, c_2]$, $\overline{d}[0] = [d_1, d_2]$,

Boundary	Parameter Values
$x_1(t,\alpha)$	a_2, b_1, r_1
$x_2(t,\alpha)$	a_1, b_2, r_2
$y_1(t,\alpha)$	c_1, d_2, s_1
$y_2(t,\alpha)$	c_2, d_1, s_2

TABLE 2. Parameters for the Boundary of the Alpha Zero Cut of the Fuzzy Trajectories in Figures 3 and 4

$\overline{r}[0] = [r_1, r_2]$, and $\overline{s}[0] = [s_1, s_2]$. Then the solution to getting the outer boundary of the $\alpha = 0$ cut of the fuzzy trajectories is in Table 2.

These results are basically intuitive. To get the largest x we choose the smallest value for a, the largest value for b and the largest value of r. Similar reasoning for the other choices. Figure 2 shows the Simulink model for this example. The fuzzy trajectory for $\overline{x}(t)[0]$ ($\overline{y}(t)[0]$) is in Figure 3 (4).

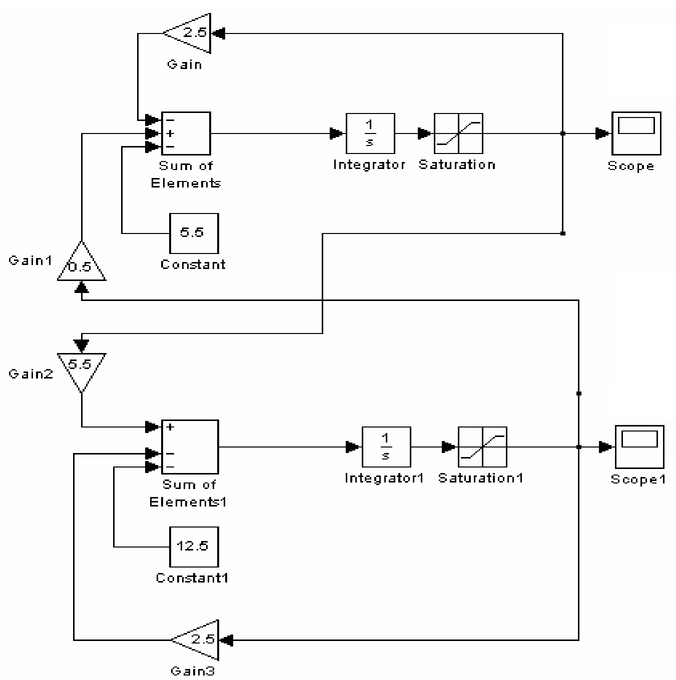

FIGURE 2. A Portion of the Simulink Diagram for the Arms Race Model (see text)

Let us first describe how Simulink solves this problem using Figure 2. The top of the figure is for $x(t)$ and the bottom is for $y(t)$. The input to the top "Integrator" is $-ax+by+r$ which is $\dot{x}$ and the output is $x(t)$. The "Saturation" just sets $x(t) = 0$ if

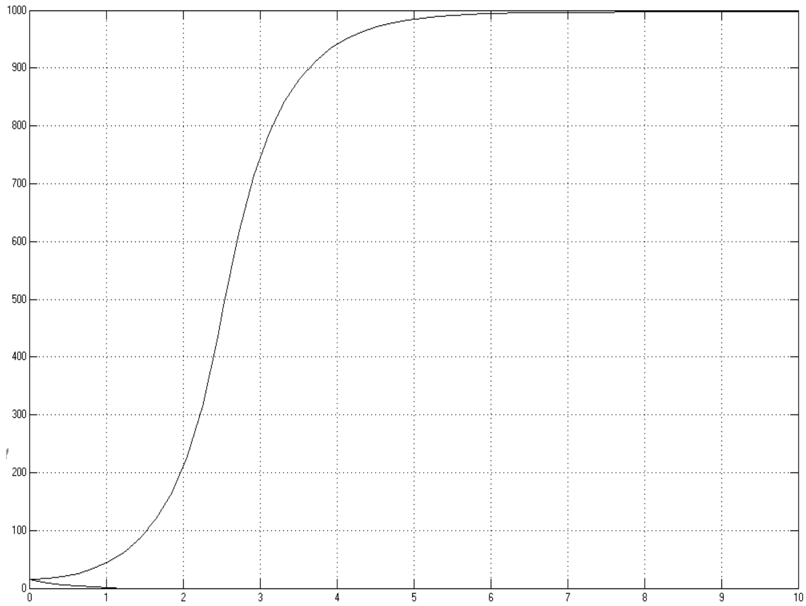

FIGURE 3. Fuzzy Trajectory for $\overline{x}(t)[0]$ in the Arms Race Model

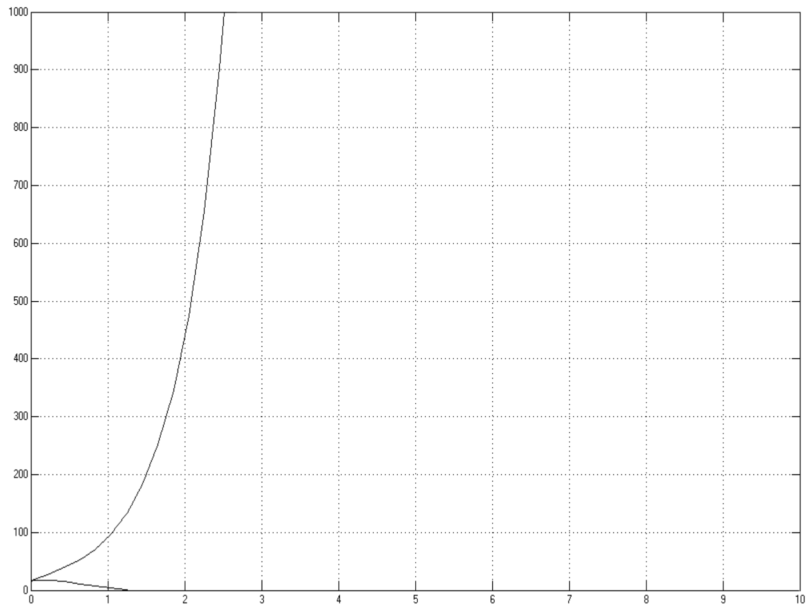

FIGURE 4. Fuzzy Trajectory for $\overline{y}(t)[0]$ in the Arms Race Model

$x(t)$ is negative. The top loop multiplies $x(t)$ by a for input to "Sum of Elements". The second input to "Sum of Elements" is from $y(t)$ after multiplying by b. The final input to "Sum of Elements" is the constant r. The numbers shown in the figure in the "Gains" are for $x_1(t,0)$ and $y_1(t,0)$. Similarly we have the bottom half of the figure for $y(t)$. The initial values for $x(t)$ and $y(t)$ are defined inside the "Integrators" and the "Scopes" produce the graphs.

Figure 3 is for $x_1(t,0)$ (lower graph) and $x_2(t,0)$ (upper graph) while Figure 4 is for $y_1(y,0)$ (lower graph) and $y_2(t,0)$ (upper graph). The horizontal axis is time and the vertical axis is \$. In both figures the spread between both graphs shows the uncertainty due to the uncertainty in the input parameters. In this example there is tremendous uncertainty in the results. The time period is in years. The lower graphs $\overline{x}_1(t)$ and $\overline{y}_1(t)$ both go to zero, which leads to disarmament, between one and two years. The upper graphs keep increasing and eventually go off of the top of the graph paper. It looks like $x_2(t,0)$ in Figure 3 approaches 1000 for $t \geq 7$, but this is not correct because the "Saturation" cuts the values off at 1000. That is, if $x_2(t,0) > 1000$, then "Saturation" resets the value to 1000. The uncertainty is too great going from disarmament to "run away" expenditures on armament. To reduce the uncertainty we need to see if we can reduce the uncertainty (less fuzzy) in the parameters.

6. **Example 2: Queuing Theory**

In this section we consider the classical M machine and R service person problem from queuing theory [15]. In this model we assume there are R repair persons available to service M machines. The calling source for the model, all the customers who can arrive at the queuing system, are the M machines which "arrive" for service when they break down. However, we do not bring a broken machine to the service station, the service station, consisting of one service person, travels to the broken machine to repair it. Assume that if a machine is operating correctly, then the time it breaks down is described by an exponential distribution that has rate λ. Assume also that any machine that has broken down requires the service of a single repair person and that the service time probability distribution is exponential with the same rate μ for each repair person and each machine.

Usually in queuing theory we investigate the behavior of the system during "steady state". Let $p_i(t)$ be the probability of i broken machines at time $t > 0$, for $0 \leq i \leq M$. If $p_i = \lim_{t\to\infty} p_i(t)$, $0 \leq i \leq M$, then the p_i are called the steady state probabilities and using the p_i in calculations produces steady state results. However, in this paper we do not go directly to the steady state probabilities but start with the differential equations defining the $p_i(t)$.

To simplify the discussion now assume that $M = 4$ and $R = 2$. We write $\dot{p}_i$ for the time derivative of $p_i(t)$. The following system of differential equations has been adopted from a discussion in [16].

$$\dot{p}_0 = -4\lambda p_0(t) + \mu p_1(t), \tag{19}$$

$$\dot{p}_1 = 4\lambda p_0(t) - [3\lambda + \mu]p_1(t) + 2\mu p_2(t), \tag{20}$$

$$\dot{p}_2 = 3\lambda p_1(t) - [2\lambda + 2\mu]p_2(t) + 2\mu p_3(t), \tag{21}$$

$$\dot{p}_3 = 2\lambda p_2(t) - [\lambda + 2\mu]p_3(t) + 2\mu p_4(t), \tag{22}$$

Boundary	Parameter Values
$p_{01}(t,\alpha)$	$\lambda_2(\alpha), \mu_1(\alpha)$
$p_{02}(t,\alpha)$	$\lambda_1(\alpha), \mu_2(\alpha)$
$n_1(t,\alpha)$	$\lambda_1(\alpha), \mu_2(\alpha)$
$n_2(t,\alpha)$	$\lambda_2(\alpha), \mu_1(\alpha)$

TABLE 3. Parameters for the Boundary of the Alpha-Cuts of $\overline{p}_0(t)$ and $\overline{N}(t)$

$$\dot{p}_4 = \lambda p_3(t) - 2\mu p_4. \tag{23}$$

Initial conditions are $p_0(0) = 1$ and $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 0$. The solution to this system produces the $p_i(t)$, $0 \le i \le 4$. We will be interested in finding $N(t) = p_1(t) + 2p_2(t) + 3p_3(t) + 4p_4(t)$ the expected number of broken machines in the system at time t.

However, the parameters λ and μ need to be estimated from data. So, from ([1]-[4], also Section 2) their estimators become fuzzy numbers which exhibit the uncertainty in their values. Let us now use $\overline{\lambda} = (0.3/0.5/0.7)$ as our fuzzy estimator of λ, or approximately one breakdown per machine every two hours. And let $\overline{\mu} = (1.5/2/2.5)$ our fuzzy estimator of μ, or approximately 30 minutes per repair. Time is in hours. Hence, the above differential equations become a system of fuzzy differential equations, the probabilities become fuzzy probabilities $\overline{p}_i(t)$, and the expected number of broken machines will also be fuzzy $\overline{N}(t)$.

Let $\overline{N}(t)[\alpha] = [n_1(t,\alpha), n_2(t,\alpha)]$, $t > 0$ and $0 \le \alpha \le 1$ and let $\overline{p}_i(t)[\alpha] = [p_{i1}(t,\alpha), p_{i2}(t,\alpha)]$ for $i = 0, 1, ..., 4$ and all α. Also let $\overline{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$ and $\overline{\mu}[\alpha] = [\mu_1(\alpha), \mu_2(\alpha)]$.

The solution to this problem is well suited for crisp continuous simulation. The choice of $\lambda \in \overline{\lambda}[\alpha]$ and $\mu \in \overline{\mu}[\alpha]$ to get the end points of the alpha-cuts of $\overline{p}_0(t)$ and $\overline{N}(t)$ are shown in Table 3. These results are all rather intuitive in that we use the smallest λ and the largest μ to get the smallest $N(t)$, etc.

The system diagram from Simulink in this application is rather complicated and therefore omitted. We selected one of the $\overline{p}_i(t)[0]$ to show the reader and $\overline{p}_0(t)[0]$ is in Figure 5. Notice how fast the system gets into steady state. The fuzzy trajectory of the $\alpha = 0$ cut of $\overline{N}(t)$ is in Figure 6.

In Figure 5 the lower curve is for $p_{01}(t,0)$ and the upper curve is $p_{02}(t,0)$. The horizontal axis is time and the vertical axis is probability. Steady state arrives when the curves go horizontal. The steady state probabilities are $\lim_{t\to\infty} p_{02}(t,0) = 0.6340$ and $\lim_{t\to\infty} p_{01}(t,0) = 0.2030$. The uncertainty band, the difference between both curves at steady state, is $0.6340 - 0.2030 = 0.4310$ which is quite large. This uncertainty on $p_0(t)$ results from the uncertainty in the values for λ and μ.

In Figure 6 the lower curve is for $n_1(t,0)$ and the upper curve is $n_2(t,0)$. The horizontal axis is time and the vertical axis is the number of broken machines being serviced or waiting for service. Steady state arrives when the curves go horizontal. The steady state values are $\lim_{t\to\infty} n_2(t,0) = 1.3965$ and $\lim_{t\to\infty} n_1(t,0) = 0.4353$. The uncertainty band, the difference between both curves at steady state, is $1.3965-$

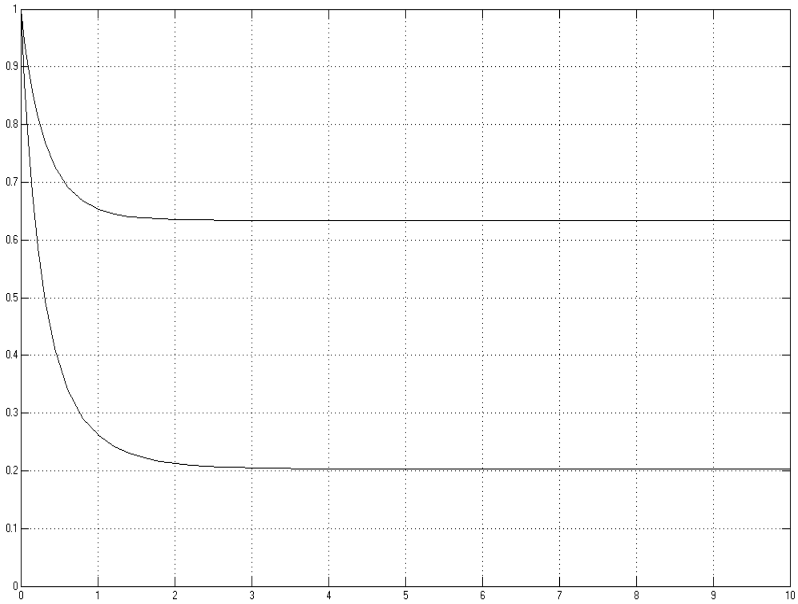

FIGURE 5. Fuzzy Trajectory of Fuzzy Probability $\overline{p}_0(t)[0]$ in the Queuing Model

0.4353 = 0.9612 which is quite large. This uncertainty on $N(t)$ results from the uncertainty in the values for λ and μ.

Since the uncertainty results are quite large in this example we would concentrate on obtaining more accurate estimates for λ and μ.

7. Example 3: Predator/Prey Model

The predator/prey model was introduced in Section 1. However, in this section we will modify it as discussed below. We first need to look more closely at this model to understand the meaning of the constants a, b, c, d. Let $x(t)$ ($y(t)$) denote the number of fox (rabbits) in a certain ecosystem at time t. Assume that in this region the fox only eat rabbits and therefore the $x(t)$ are the predators and the $y(t)$ are the prey. If there are no rabbits the fox population may decline according to

$$\dot{x} = -ax, \ a > 0. \tag{24}$$

If rabbits are present we assume that the fox population will grow at a rate proportional to the product xy, or at rate bxy. Hence

$$\dot{x} = -ax + bxy, \ a > 0, \ b > 0. \tag{25}$$

If there are no foxes, assuming unlimited food supply, then $\dot{y} = dy$, for some constant $d > 0$. But this allows the rabbit population to grow indefinitely. Suppose there is some theoretical maximum number (y_m) of rabbits that can live in this ecosystem; or the carrying capacity is y_m. Then we write $\dot{y} = dy(1 - (y/y_m))$ [14].

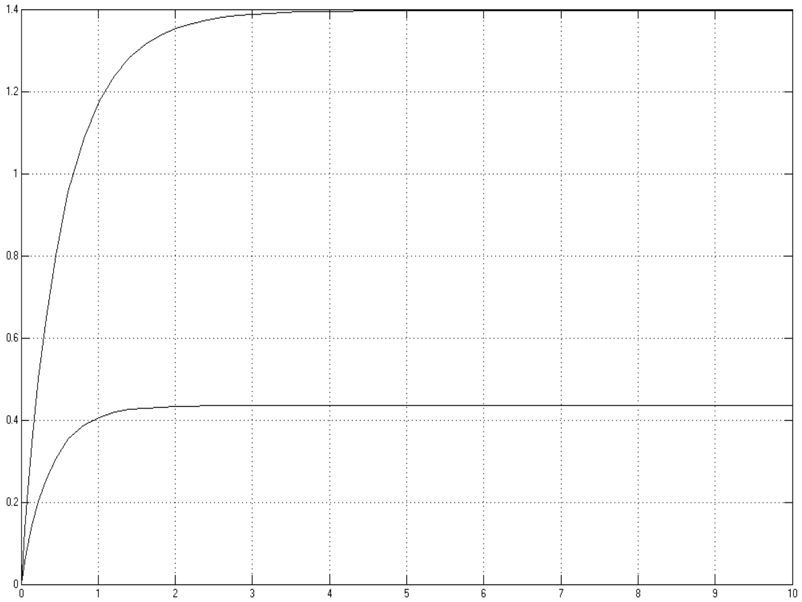

FIGURE 6. Fuzzy Trajectory of the Expected Number of Broken Machines $\overline{N}(t)[0]$ in the Queuing Model

When there are foxes present, the rabbit population will decline at rate cxy, $c > 0$. So

$$\dot{y} = -cxy + dy(1 - y/y_m), \; c > 0, \; d > 0. \tag{26}$$

The initial conditions are $x(0) = x_0$ and $y(0) = y_0$. If there is a value of t, say $t = t_0$, for which $x(t_0) < 1$ $(y(t_0) < 1)$, then $x(t) = 0$ $(y(t) = 0)$ for $t \geq t_0$.

Now we need to estimate all of these parameters. We have no data on this ecosystem so we turned to experts in this region to help estimate a, b, c, d, y_m and x_0, y_0 for a period starting January 1. We obtained the following triangular fuzzy numbers shown in Table 3. Notice that y_m, x_0 and y_0 are assumed to be known precisely and therefore are not fuzzy. Using fuzzy parameters we get fuzzy differential equations and the solutions $\overline{x}(t)$, $\overline{y}(t)$, are also fuzzy.

Next we need to choose the values of the parameters in their α cut to get the outer boundary of the α-cut of the fuzzy trajectory for the foxes and the rabbits. Let us define, in general, $\overline{a}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{b}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, $\overline{c}[\alpha] = [c_1(\alpha), c_2(\alpha)]$, $\overline{d}[\alpha] = [d_1(\alpha), d_2(\alpha)]$, $\overline{x}(t)[\alpha] = [x_1(t, \alpha), x_2(t, \alpha)]$ and $\overline{y}(t)[\alpha] = [y_1(t, \alpha), y_2(t, \alpha)]$. Then the solution to getting the outer boundary of the α cut of the fuzzy trajectories is in Table 5.

These results are basically intuitive. To get the largest fox population we choose: (1) the smallest value for a, the largest value for b, to maximize x; and (2) we also need the most rabbits for the fox to eat so use the smallest c and the largest d . However, we use the same settings for $x_1(t, 0)$ as for $y_1(t, 0)$ because these make $y_1(t, 0)$ go to zero and that forces $x_1(t, 0)$ also to zero. The results are shown

Parameter	Fuzzy Value
a	(0.2/0.3/0.4)
b	(0.04/0.05/0.06)
c	(0.3/0.4/0.5)
d	(5.5/6/6.5)
y_m	100
x_0	5
y_0	15

TABLE 4. Fuzzy/Crisp Parameters in the Predator/Prey Model

Outer Boundary	Parameter Values
$x_2(t,\alpha)$	a_1, b_2, c_1, d_2
$x_1(t,\alpha)$	a_1, b_2, c_2, d_1
$y_2(t,\alpha)$	a_2, b_1, c_1, d_2
$y_1(t,\alpha)$	a_1, b_2, c_2, d_1

TABLE 5. Parameters for the Boundary of the Alpha Cut of $\overline{x}(t)$ and $\overline{y}(t)$

in Figures 8 and 9. The systems diagram for Simulink is in Figure 7 without the needed part to implement the rule: if $x(t_0) < 1$ ($y(t_0) < 1$), then $x(t) = 0$ ($y(t) = 0$) for $t \geq t_0$.

The explanation of Simulink in Figure 7 is the same as in Figure 2 except now there are two new elements "Product of Elements" and "Fcn". In the "Product" we multiply $x(t)$ from the upper loop with $y(t)$ from the lower loop and then multiply by b (c) for the upper (lower) loop. The operation "Fcn" is to compute $dy(1-y/y_m)$.

In Figure 8 the lower graph is for $x_1(t,0)$ and the upper graph is for $x_2(t,0)$. The horizontal axis is time and the vertical axis is the number of foxes. We get steady state with $\lim_{t\to\infty} x_2(t,0) \approx 22$ and $\lim_{t\to\infty} x_1(t,0) = 0$. The fox population goes to zero because the rabbit population goes to zero (Figure 9). The uncertainty band, the difference between the two curves in steady state, is approximately 22 foxes.

The lower graph in Figure 9 is $y_1(t,0)$ and $y_2(t,0)$ is the upper graph. The vertical axis is the number of rabbits. The steady state limits are approximately 10 and zero making an uncertainty band of 10 rabbits.

The results show that there is a possibility that the population of foxes and rabbits will disappear. But there is a lot of uncertainty in the results due to the uncertainty in the values of the parameters in the model. We need to obtain more accurate estimates for these parameters which can then reduce the uncertainty in the results for $\overline{x}(t)$ and $\overline{y}(t)$.

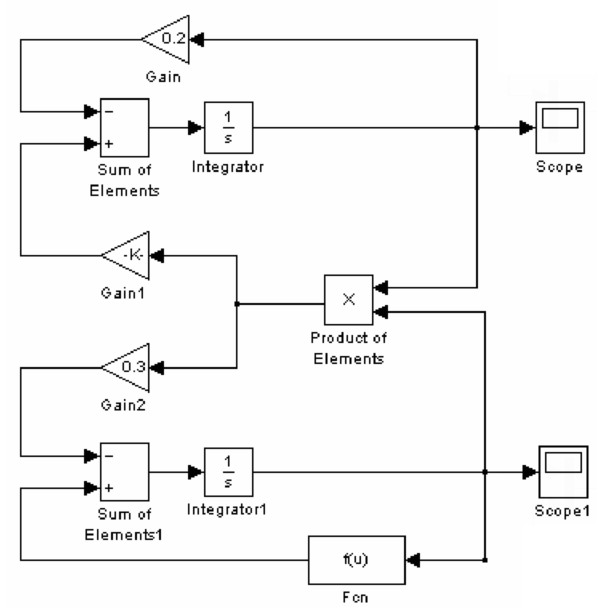

FIGURE 7. A Portion of the Simulink Diagram for the Predator/Prey Model (see text)

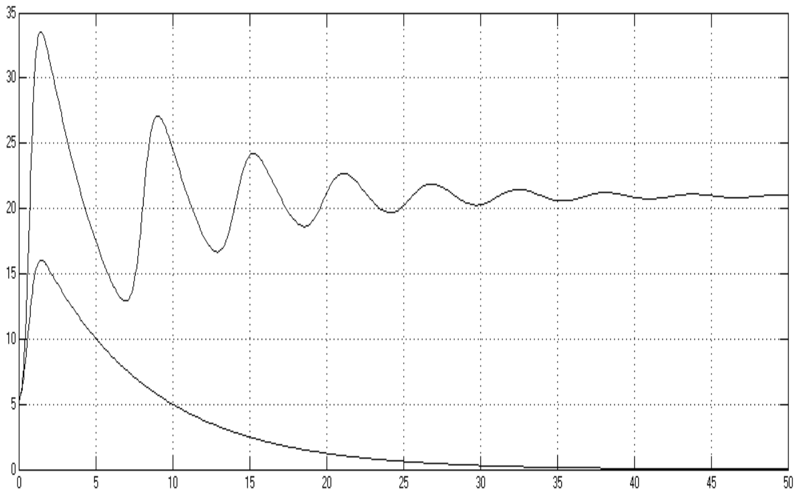

FIGURE 8. Fuzzy Trajectory for the Number of Foxes $\overline{x}(t)[0]$

8. **Summary, Conclusions and Future Research**

This paper continues our research into simulating fuzzy systems. We start with a crisp continuous system whose evolution depends on a system of ordinary differential equations. There will be a number of parameters in the system whose values

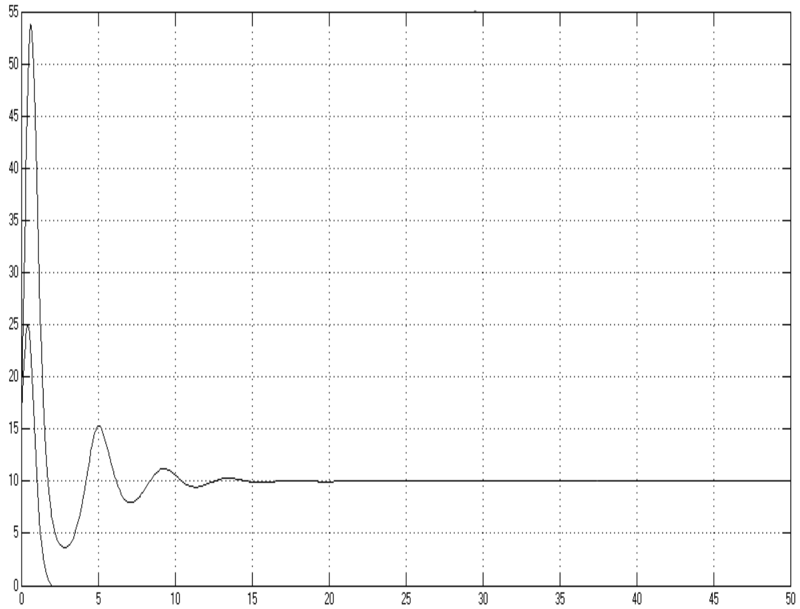

FIGURE 9. Fuzzy Trajectory for the Number of Rabbits $\overline{y}(t)[0]$

are not known precisely. These parameters need to be estimated and their estimators will contain uncertainties. We model these uncertainties using fuzzy numbers (see Section 2). Using fuzzy numbers for these parameters changes the system of differential equations into a system of fuzzy differential equations whose solution will be described by fuzzy trajectories. If we take a cut through the fuzzy trajectory for each fixed value of time we obtain a fuzzy number. The uncertainty in the results are now shown by the trajectory of the bases (alpha zero cut) of the fuzzy number solutions. Sometimes this uncertainty can be quite large. We approximate the fuzzy trajectories by employing the crisp continuous simulator Simulink. Three examples are described including an arms race model, a queuing theory model and predator/prey model. Future research will be concerned with extending these results to more complex continuous systems and obtaining more concrete results on the simulation optimization problem described in Section 3.

References

[1] J. J. Buckley, *Fuzzy statistics*, Springer-Verlag, Heidelberg, Germany, (2004).
[2] J. J. Buckley, *Fuzzy probabilities and fuzzy sets for web planning*, Springer-Verlag, Heidelberg, Germany, (2004).
[3] J. J. Buckley, *Simulating Fuzzy Systems*, Springer-Verlag, Heidelberg, Germany, To appear.
[4] J. J. Buckley, *Fuzzy systems*, Soft Computing, To appear.
[5] J. J. Buckley and T. Feuring, *Fuzzy initial value problem for nth order linear differential equations, Fuzzy Sets and Systems*, **121**(2001) 247-255.
[6] J. J. Buckley, E. Eslami and T. Feuring, *Fuzzy mathematics in economics and engineering*, Springer-Verlag, Heidelberg, Germany, (2002).

[7] J. J. Buckley, T. Feuring and Y. Hayashi, *Linear systems of first order ordinary differential equations: Fuzzy initial conditions*, Soft Computing, **6**(2002) 415-421.
[8] J. J. Buckley, K. Reilly and X. Zheng, *Simulating fuzzy systems I*, Applied Research in Uncertainty Modelling and Analysis, Eds. N.O.Attoh-Okine, B.Ayyub, Kluwer, (2004), To appear.
[9] J. J. Buckley, K. Reilly and X. Zheng, *Simulating fuzzy systems II*, Applied Research in Uncertainty Modelling and Analysis, Eds. N.O.Attoh-Okine, B.Ayyub, Kluwer, (2004), To appear.
[10] Maple 9, Waterloo Maple Inc., Waterloo, Canada.
[11] M. Olinick, *An introduction to mathematical models in the social and life sciences*, Addison-Wesley, Reading, MA, (1978).
[12] scilabsoft.inria.fr
[13] solutions.iienet.org
[14] M. R. Spiegel, *Applied differential equations*, Third Edition, Prentice-Hall, Englewood Cliffs, NJ, (1981).
[15] H. A. Taha, *Operations research*, Fifth Edition, Macmillan, N.Y., (1992).
[16] H. M. Wagner, *Principles of operations research*, Second Edition, Prentice Hall, Englewood Cliffs, N.J.. (1975).
[17] www.mathworks.com
[18] D. G. Zill, *A First course in differential equations*, Brooks/Cole, Pacific Grove, CA, (1997).tt

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