POINTWISE PSEUDO-METRIC ON THE L-REAL LINE

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ABSTRACT. In this paper, a pointwise pseudo-metric function on the L-real line is constructed. It is proved that the topology induced by this pointwise pseudo-metric is the usual topology.

1. **Introduction**

The L-fuzzy unit interval and the L-fuzzy real line are two important L-topological spaces. The L-fuzzy unit interval was defined by Hutton [2]. The L-fuzzy real line was respectively defined by Höhle [3] and Gantner et al. [4]. They are important not only in L-topology, but also in other fields.

To reflect the characteristics of pointwise L-topology, i.e., the relation between a fuzzy point and its Q-neighborhoods (or R-neighborhoods) [5], a theory of pointwise uniformities and a theory of pointwise metrics were introduced on completely distributive lattices and in L-fuzzy set theory(see [6, 7, 8, 9]). Many ideal results in general topology were generalized to L-topology. In [9], it was proved that the L-fuzzy real line is pointwise pseudo-metrizable, but no pointwise pseudo-metric function on the L-fuzzy real line was given. In this paper, our aim is to construct a pointwise pseudo-metric function in the L-real line and prove that the topology induced by this pointwise pseudo-metric function is the usual topology.

2. **Preliminaries**

Throughout this paper, L always denotes a completely distributive lattice with an order-reversing involution. $M(L^X)$ denotes the set of all non-zero $\vee$-irreducible elements in L^X. For $A \in L^X$, $\beta(A)$ denotes the maximal minimal family of A (see [5]) and $\beta^*(A) = \beta(A) \bigcap M(L^X)$. It is easy to verify that for $e \in M(L)$, $e \in \beta^*(A)$ if and only if $a \ll A$, where $\ll$ is the way below relation ([1]).

Definition 2.1 ([9]). A pointwise pseudo-quasi-metric on L^X is a mapping $d : M(L^X) \times M(L^X) \to [0, +\infty)$ satisfying the following (M1)–(M3):

(M1) $\forall a \in M(L^X),\ d(a, a) = 0.$

(M2) $\forall a, b, c \in M(L^X),\ d(a, c) \leq d(a, b) + d(b, c).$

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(M3) $\forall a, b \in M(L^X),\ d(a,b) = \bigwedge_{c \ll b} d(a,c)$.

A pointwise pseudo-quasi-metric d is called a pointwise pseudo-metric if it satisfies the following conditions.

(M4) $\forall a, b, c \in M(L^X),\ a \leq b$ implies $d(a,c) \leq d(b,c)$.

(M5) $\forall \lambda, \mu \in M(L^X),\ \bigwedge_{a \not\leq \lambda'} d(a,\mu) < r$ if and only if $\bigwedge_{b \not\leq \mu'} d(b,\lambda) < r$.

Theorem 2.2 ([9]). *Let d be a pointwise pseudo-metric on L^X. $\forall r \in (0,+\infty)$, define a mapping $P_r : M(L^X) \to L^X$ by*

$$P_r(a) = \bigvee\{b \in M(L^X) \mid d(a,b) \geq r\}.$$

Then the family $\{P_r \mid r \in (0,+\infty)\}$ of R-nbd mappings of d satisfies the following conditions.

$(R1)$ $\forall a \in M(L^X), \bigwedge_{r>0} P_r(a) = 0$;

$(R2)$ $\forall a \in M(L^X), \forall r \in (0,+\infty), a \not\leq P_r(a)$;

$(R3)$ $\forall r, s \in (0,+\infty), P_s \odot P_r \geq P_{r+s}$;

$(R4)$ $\forall a \in M, P_r(a) = \bigwedge_{s<r} P_s(a)$;

$(R5)$ $\forall r \in (0,+\infty)$, *$P_r$ is symmetric.*

Theorem 2.3 ([9]). *If $\{P_r \mid P_r : M(L^X) \to L^X, r \in (0,+\infty)\}$ is a family of mappings satisfying* (R1)–(R5), *and we define $d : M(L^X) \times M(L^X) \to [0,+\infty)$ by*

$$d(a,b) = \bigwedge\{\ r \mid b \not\leq P_r(a)\},$$

then d is a pointwise pseudo-metric on L^X and the family of R-nbd mappings of d is exactly $\{\ P_r \mid r \in (0,+\infty)\}$.

Theorem 2.4 ([9]). *If d is a pointwise pseudo-quasi-metric on L^X, then*

(1) *$\{P_r(a) \mid a \in M(L^X), r \in (0,+\infty)\}$ is a base for a co-topology on L^X. This co-topology is denoted by η_d;*

(2) *$\{P_r(a) \mid r > 0\}$ is a locally R-neighborhood base at a in the co-topology η_d.*

Definition 2.5 ([3, 4]). The L-(fuzzy) real line $\mathbb{R}(L)$ is defined as the set of all equivalence classes of antitone maps $\lambda : \mathbb{R} \to L$ satisfying

$$\bigvee_{t \in R} \lambda(t) = 1 \text{ and } \bigwedge_{t \in R} \lambda(t) = 0,$$

where the equivalence identifies two maps $\lambda and \mu$ if and only if $\forall t \in I$, $\lambda(t+) = \mu(t+)$. The canonical L-topology on $\mathbb{R}(L)$ is generated from the subbase $\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}$, where

$$\mathcal{L}_t : I(L) \to L \ \text{ by } \ \mathcal{L}_t(\lambda) = \lambda(t-)'$$

$$\mathcal{R}_t : I(L) \to L \ \text{ by } \ \mathcal{R}_t(\lambda) = \lambda(t+).$$

3. Pointwise Pseudo-metric on the L-real Line

Lemma 3.1. *Let $\mathbb{R}(L)$ be the L-real line. Define a mapping $\varepsilon : M(L^{\mathbb{R}(L)}) \to \mathbb{R}$ and a mapping $\sigma : M(L^{\mathbb{R}(L)}) \to \mathbb{R}$ such that for all $e \in M(L^{\mathbb{R}(L)})$,*

$$\varepsilon(e) = \sup\{t \mid e \leq \mathcal{L}'_t\}, \quad \sigma(e) = \inf\{t \mid e \leq \mathcal{R}'_t\},$$

Then we have the following results:

(1) $\varepsilon(e) = \max\{t \mid e \leq \mathcal{L}'_t\}$, $\sigma(e) = \min\{t \mid e \leq \mathcal{R}'_t\}$.

(2) *If $a, b \in M(L^{\mathbb{R}(L)})$ and $a \leq b$, then $\varepsilon(a) \geq \varepsilon(b)$ and $\sigma(a) \leq \sigma(b)$.*

(3) *If $b \in M(L^{\mathbb{R}(L)})$, then $\varepsilon(b) = \bigwedge_{c \ll b} \varepsilon(c)$ and $\sigma(b) = \bigvee_{c \ll b} \sigma(c)$.*

(4) *$\forall \lambda, \mu \in M(L^{\mathbb{R}(L)})$, there exists $a \not\leq \lambda'$ such that $\varepsilon(\mu) < \varepsilon(a) + r$ if and only if there exists $b \not\leq \mu'$ such that $\sigma(\lambda) > \sigma(b) - r$.*

Proof. (1) and (2) are obvious. By (2) we can obtain that $\varepsilon(b) \leq \bigwedge_{c \ll b} \varepsilon(c)$ and $\sigma(b) \geq \bigvee_{c \ll b} \sigma(c)$. Thus in order to prove (3) we need only to prove that

$$\varepsilon(b) \geq \bigwedge_{c \ll b} \varepsilon(c) \ \text{ and } \ \sigma(b) \leq \bigvee_{c \ll b} \sigma(c).$$

Suppose that $\varepsilon(b) < \bigwedge_{c \ll b} \varepsilon(c)$. Then there exists $s \in \mathbb{R}$ such that

$$\varepsilon(b) = \max\{t \mid b \leq \mathcal{L}'_t\} < s < \bigwedge_{c \ll b} \varepsilon(c).$$

This implies that $b \not\leq \mathcal{L}'_s$. Further there exists $c \ll b$ such that $c \not\leq \mathcal{L}'_s$. Thus we have that $\varepsilon(c) < s$. By $s < \bigwedge_{c \ll b} \varepsilon(c)$ we obtain a contradiction. Therefore $\varepsilon(b) \geq \bigwedge_{c \ll b} \varepsilon(c)$. Similarly we can prove that $\sigma(b) \leq \bigvee_{c \ll b} \sigma(c)$. Hence (3) follows.

To prove (4) suppose that $\varepsilon(\mu) < \varepsilon(a) + r$. Then there is $t > 0$ such that $\varepsilon(\mu) < \varepsilon(a) + r - t$. This implies that

$$\mu \not\leq \mathcal{L}'_{\varepsilon(a)+r-t} \ \text{ or } \ \mathcal{L}_{\varepsilon(a)+r-t} \not\leq \mu'.$$

So there exists a point $b \leq \mathcal{L}_{\varepsilon(a)+r-t}$ such that $b \not\leq \mu'$. We obtain

$$\sigma(b) \leq \varepsilon(a) + r - t \ \text{ or } \ \sigma(b) - r < \varepsilon(a)$$

since $\mathcal{L}_{\varepsilon(a)+r-t} \leq \mathcal{R}'_{\varepsilon(a)+r-t}$. By $a \leq \mathcal{L}'_{\varepsilon(a)}$ we have that

$$\lambda \not\leq a' \geq \mathcal{L}_{\varepsilon(a)} \geq \mathcal{R}'_{\sigma(b)-r}.$$

Therefore $\sigma(\lambda) > \sigma(b) - r$.

□

Theorem 3.2. *Let $\mathbb{R}(L)$ be the L-real line. For all $a, b \in M(L^{\mathbb{R}(L)})$, define*

$$d_1(a,b) = \max\{\varepsilon(b) - \varepsilon(a), 0\}, \ d_2(a,b) = \max\{\sigma(a) - \sigma(b), 0\},$$

Then d_1, d_2 are pointwise pseudo-quasi-metrics, $\{\mathcal{L}_t \mid t \in \mathbb{R}\}$ is the topology induced by d_1 and $\{\mathcal{R}_t \mid t \in \mathbb{R}\}$ is the topology induced by d_2.

Proof. We only prove that d_1 is a pointwise pseudo-quasi-metric. The proof for d_2 is similar. Obviously, by (2) in Lemma 3.1 we know that $a \leq b \Rightarrow d_1(a,b) = 0$. Thus (M1) is true. (M2) can be obtained as follows.

$$\begin{aligned} d_1(a,c) &= \max\{\varepsilon(c) - \varepsilon(a), 0\} \\ &= \max\{\varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), 0\} \\ &\leq \max\{\varepsilon(b) - \varepsilon(a), 0\} + \max\{\varepsilon(c) - \varepsilon(b), 0\} \\ &= d_1(a,b) + d_1(b,c) \end{aligned}$$

(M3) can be obtained as follows:

$$\begin{aligned} d_1(a,b) &= \max\{\varepsilon(b) - \varepsilon(a), 0\} \\ &= \max\{\bigwedge_{c \ll b} \varepsilon(c) - \varepsilon(a), 0\} \\ &= \max\{\bigwedge_{c \ll b} (\varepsilon(c) - \varepsilon(a)), 0\} \\ &= \bigwedge_{c \ll b} \max\{\varepsilon(c) - \varepsilon(a), 0\} = \bigwedge_{c \ll b} d_1(a,c). \end{aligned}$$

In order to prove that $\{\mathcal{L}_t \mid t \in \mathbb{R}\}$ is the topology induced by d_1 and $\{\mathcal{R}_t \mid t \in \mathbb{R}\}$ is the topology induced by d_2, we only need to prove that the family $\{P_r^{d_1} \mid r > 0\}$ of R-nbd mappings of d_1 and the family $\{P_r^{d_2} \mid r > 0\}$ of R-nbd mappings of d_2 satisfy the following condition:

$$P_r^{d_1}(a) = \mathcal{L}'_{\varepsilon(a)+r} \quad \text{and} \quad P_r^{d_2}(a) = \mathcal{R}'_{\sigma(a)-r}.$$

In fact, $\forall a, b \in M(L^{\mathbb{R}(L)})$, we have:

$$\begin{aligned} b \leq P_r^{d_1}(a) &\Leftrightarrow \quad d_1(a,b) \geq r \\ &\Leftrightarrow \varepsilon(b) - \varepsilon(a) \geq r \\ &\Leftrightarrow \varepsilon(b) \geq \varepsilon(a) + r \quad \Leftrightarrow \quad b \leq \mathcal{L}'_{\varepsilon(a)+r} \end{aligned}$$

and

$$\begin{aligned} b \leq P_r^{d_2}(a) &\Leftrightarrow \quad d_2(a,b) \geq r \\ &\Leftrightarrow \sigma(a) - \sigma(b) \geq r \\ &\Leftrightarrow \sigma(b) \leq \sigma(a) - r \quad \Leftrightarrow \quad b \leq \mathcal{R}'_{\sigma(a)-r} \end{aligned}$$

The result follows. □

Remark 3.3. When $L = 2$, d_1 and d_2 are conjugate pseudo-quasi-metrics in the usual sense.

Theorem 3.4. *Let $\mathbb{R}(L)$ be the L-real line. For all $a, b \in M(L^{\mathbb{R}(L)})$, define*

$$d(a,b) = \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} = \max\{d_1(a,b), d_2(a,b)\}.$$

Then d is a pointwise pseudo-metric and d exactly induces the topology on $\mathbb{R}(L)$.

Proof. By (2) in Lemma 3.1 it is obvious that we know that $a \leq b \Rightarrow d(a,b) = 0$. Thus (M1) is true. (M2) can be obtained as follows:

$$\begin{aligned} d(a,c) &= \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} \\ &= \max\{\varepsilon(c) - \varepsilon(b) + \varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b) + \sigma(b) - \sigma(c), 0\} \\ &\leq \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} + \max\{\varepsilon(c) - \varepsilon(b), \sigma(b) - \sigma(c), 0\} \\ &= d(a,b) + d(b,c) \end{aligned}$$

(M3) can be obtained as follows:

$$\begin{aligned} d(a,b) &= \max\{\varepsilon(b) - \varepsilon(a), \sigma(a) - \sigma(b), 0\} \\ &= \max\{\bigwedge_{c \ll b} \varepsilon(c) - \varepsilon(a), \sigma(a) - \bigvee_{c \ll b} \sigma(c), 0\} \qquad \text{by Lemma 3.1} \\ &= \max\{\bigwedge_{c \ll b} (\varepsilon(c) - \varepsilon(a)), \bigwedge_{c \ll b} (\sigma(a) - \sigma(c)), 0\} \\ &= \bigwedge_{c \ll b} \max\{\varepsilon(c) - \varepsilon(a), \sigma(a) - \sigma(c), 0\} = \bigwedge_{c \ll b} d(a,c) \end{aligned}$$

(M4) can be obtained from (2) in Lemma 3.1.

To prove (M5), we note that $\forall \lambda, \mu \in M(L^{\mathbb{R}(L)})$, if

$$\bigwedge_{a \not\leq \lambda'} d(a,\mu) = \bigwedge_{a \not\leq \lambda'} \max\{\varepsilon(\mu) - \varepsilon(a), \sigma(a) - \sigma(\mu), 0\} < r,$$

then there exists $a \not\leq \lambda'$ such that

$$\max\{\varepsilon(\mu) - \varepsilon(a), \sigma(a) - \sigma(\mu), 0\} < r,$$

i.e.,

$$\varepsilon(\mu) - \varepsilon(a) < r, \ \sigma(a) - \sigma(\mu) < r.$$

Hence we have that

$$\varepsilon(\mu) < \varepsilon(a) + r, \ \sigma(\mu) > \sigma(a) - r.$$

By (4) in Lemma 3.1 we know that there exist $b \not\leq \mu'$ and $c \not\leq \mu'$ such that

$$\sigma(\lambda) > \sigma(b) - r, \ \varepsilon(\lambda) < \varepsilon(c) + r.$$

Thus, since μ' is a prime element, $b \wedge c \not\leq \mu'$. Take a point $d \leq b \wedge c$ such that $d \not\leq \mu'$. Then

$$\sigma(\lambda) > \sigma(b) - r \geq \sigma(d) - r, \ \varepsilon(\lambda) < \varepsilon(c) + r \leq \varepsilon(d) + r.$$

This implies that

$$\bigwedge_{d \not\leq \mu'} d(d,\lambda) = \bigwedge_{d \not\leq \mu'} \max\{\varepsilon(\lambda) - \varepsilon(d), \sigma(d) - \sigma(\lambda), 0\} < r.$$

In order to prove that $\{\mathcal{L}_t, \mathcal{R}_t \mid t \in \mathbb{R}\}$ is a subbase of the topology induced by d, we only need to prove that the family $\{P_r^d \mid r > 0\}$ of R-nbd mappings of d satisfies the following condition:

$$P_r^d(a) = \mathcal{L}'_{\varepsilon(a)+r} \vee \mathcal{R}'_{\sigma(a)-r}.$$

In fact, $\forall a, b \in M(L^{\mathbb{R}(L)})$ we have:

$$\begin{array}{rcl} b \leq P_r^d(a) & \Leftrightarrow & d(a,b) \geq r \\ & \Leftrightarrow & \varepsilon(b) - \varepsilon(a) \geq r \quad \text{or} \quad \sigma(a) - \sigma(b) \geq r \\ & \Leftrightarrow & \varepsilon(b) \geq \varepsilon(a) + r \quad \text{or} \quad \sigma(b) \leq \sigma(a) - r \\ & \Leftrightarrow & b \leq \mathcal{L}'_{\varepsilon(a)+r} \vee \mathcal{R}'_{\sigma(a)-r} \end{array}$$

The result follows. □

Remark 3.5. When $L = 2$, the pointwise pseudo-metric d in Theorem 3.4 can be regarded as the usual pseudo-metric defined by $d(a,b) = |a - b|$.

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