

## Horizontal representation of a hesitant fuzzy set and its application to multiple attribute decision making

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### Abstract

The main aim of this paper is to present a novel method for ranking hesitant fuzzy sets (HFSs) based on transforming HFSs into fuzzy sets (FSs). The idea behind the method is an interesting HFS decomposition which is referred here to as the horizontal representation in the current study. To show the validity of the proposed ranking method, we apply it to solve a multi-attribute decision-making problem under hesitant fuzzy environment. Interestingly, the results show that the proposed method gives the most accepted precedence of alternatives in comparison with the other existing methods.

*Keywords:* Hesitant fuzzy sets, ranking method for HFSs, hesitant multi-attribute decision-making.

## 1 Introduction

So far, a large number of ranking methods for fuzzy sets (FSs) have been suggested in the literature. To name a few, see the works of Bortolan and Degani [2] and Dubois and Prade [7], and the references cited therein. Moreover, Wang and Kerre [19, 20] performed a comprehensive study of the available ranking methods more than 35 indices in the fuzzy literature. Nowadays, one of the main areas of research in the decision making problems is to consider the extension of FSs, known as hesitant fuzzy set (HFS) [18]. The priority of the HFS compared to the concept of FS is to arising a possibility of values in a decision making situation. These days, the method of ranking alternatives has also received more attention, and such a discussion has been widely studied in the comparison of HFSs form different aspects. Farhadinia [8] in a comparison work has recently enriched the class of score values of HFSs by deriving a number of score functions for HFSs. Then, Farhadinia [9] classified hesitant fuzzy elements (HFEs) ranking methods into the algorithmic (see Chen, Xu and Xia [6] and that of Liao, Xu and Xia [13]) and the non-algorithmic techniques (see Xia and Xu [21], Xu and Xia's distance measures [22], and Farhadinia's techniques [11] and [8]) according to their performances. In the sequel, Farhadinia [9] proposed the lexicographical ranking technique for HFSs. In a recent work, Liu et al. [14] indicated that Farhadinia's [9] lexicographical ordering technique does not sometimes give valid results, and proposed another ranking technique. Different to the above methods, Alcantud and de Andres Calle [1] proposed a technique by introducing the segment method being mainly need to produce the distance and the similarity to the ideal HFS and to the anti-ideal HFS, respectively. The main aim of this paper is to present a new method for ranking HFSs, which is computationally simpler and easier to implement than previous ones. In this approach, the ranking of alternatives, which are provided in the form of HFSs, is specified by the use of the coming two phases. In the first phase, called *the decomposition phase*, we decompose any HFS into its  $\kappa$ -level sets being FSs. Then, in the second phase, named *the aggregation phase*, all the  $\kappa$ -level sets are unified by using a proper aggregation technique to a single FS. We will show that the ranking results of the aggregated FSs coincide to that of HFSs.

The significant difference between the proposed method of ranking and the existing ones is in how the ranking of HFSs is calculated. All the existing techniques are based on *the vertical representation of HFSs* which is known as

their HFEs for any fixed  $x_i \in X$ , meanwhile, we focus basically on *the horizontal representation of HFSs* which will be referred to as their  $\kappa$ -level sets for all  $x_i \in X$ . What needs to be mentioned here is that although Ngan in [16, 17] discussed on the horizontal (equivalently, u-map) representation of HFEs compared to the vertical representation, but these concepts are different conceptually from the horizontal and vertical representations introduced in this contribution, and the proposed horizontal representation can be further compared with that of Ngan's [16, 17] representation in the future work.

This paper is arranged as the following: Section 2 is devoted to the procedure of transforming a HFS to a FS. In Section 3, we introduce a ranking technique which is based on horizontal representation of HFSs. Section 4 is explained a multi-attribute decision-making under hesitant fuzzy environment, and illustrates the applicability of the proposed technique compared to the existing ones. In Section 5, conclusion is drawn.

## 2 Transformation of HFSs into FSs

Before going further with a more description of the proposed algorithm in this work, we need to state the following concepts.

Throughout this manuscript, we implement  $X = \{x_1, x_2, \dots, x_N\}$  to indicate the discourse set, and moreover, the notation  $FS(X)$  is used to denote a collection of all FSs defined on  $X$ .

In the following, we first recall some fuzzy aggregation operators defined on FSs. In the general form, an aggregation operator on a collection of  $n$  FSs  $A_1, A_2, \dots, A_n \in FS(X)$  is characterized by the mapping, for all  $x \in X$ ,  $\sigma : [0, 1]^n \rightarrow [0, 1]$ ,  $\sigma(A_1(x), A_2(x), \dots, A_n(x)) := A^\sigma(x)$ , where  $A^\sigma$  is called the aggregated fuzzy set and also  $A^\sigma \in FS(X)$ .

Here, we briefly restate some aggregation operators of FSs, and for the complete discussion, the interested reader is referred to [21],[23]-[28].

For a collection of FSs  $A_i (i = 1, 2, \dots, n)$  and the weight vector of  $A_i$ , denoted by  $W = (w_1, w_2, \dots, w_n)$  with  $w_i \in [0, 1]$ ,  $\sum_{i=1}^n w_i = 1$  and  $\lambda > 0$ , we then have at hand the following operations:  
The fuzzy weighted averaging (FWA) operator:

$$FWA(A_1, A_2, \dots, A_n) = \oplus_{i=1}^n (w_i A_i) = 1 - \prod_{i=1}^n (1 - A_i)^{w_i}; \quad (1)$$

The fuzzy weighted geometric (FWG) operator:

$$FWG(A_1, A_2, \dots, A_n) = \otimes_{i=1}^n (A_i^{w_i}) = \prod_{i=1}^n A_i^{w_i}; \quad (2)$$

The generalized fuzzy weighted averaging (GFWA) operator:

$$GFWA_\lambda(A_1, A_2, \dots, A_n) = (\oplus_{i=1}^n (w_i A_i^\lambda))^{\frac{1}{\lambda}} = (1 - \prod_{i=1}^n (1 - A_i^\lambda)^{w_i})^{\frac{1}{\lambda}}; \quad (3)$$

The generalized fuzzy weighted geometric (GFWG) operator:

$$GFWG_\lambda(A_1, A_2, \dots, A_n) = \frac{1}{\lambda} (\otimes_{i=1}^n (\lambda A_i)^{w_i}) = 1 - (1 - \prod_{i=1}^n (1 - (1 - A_i)^\lambda)^{w_i})^{\frac{1}{\lambda}}. \quad (4)$$

Let us now concern on the concept of HFS which is an important part of the present study.

**Definition 2.1.** [18] *A hesitant fuzzy set (HFS)  $H$  on  $X$  is defined as*

$$H = \{\langle x, h_H(x) \mid x \in X \rangle\}, \quad (5)$$

where  $h_H(x) \in [0, 1]$  is called a *hesitant fuzzy element (HFE)* [21], and represents all possible membership degrees of the element  $x$ .

**Example 2.2.** Assume that  $X = \{x_1, x_2, x_3\}$  is the universe of discourse,  $h_H(x_1) = \{0.2, 0.4, 0.5\}$ ,  $h_H(x_2) = \{0.3, 0.4\}$  and  $h_H(x_3) = \{0.3, 0.2, 0.5, 0.6\}$  are the HFEs of  $x_i$  ( $i = 1, 2, 3$ ) belonging to a set  $H$ , respectively. Then  $H$  can be considered as the following HFS  $H = \{\langle x_1, \{0.2, 0.4, 0.5\} \rangle, \langle x_2, \{0.3, 0.4\} \rangle, \langle x_3, \{0.3, 0.2, 0.5, 0.6\} \rangle\}$ .

Needless to say that any FS is a special type of a HFS, and this while there exists only one element in each HFEs  $h_H(x)$  of the HFS  $H$ .

**Assumption 2.3.** Let  $l(h_H(x))$  denote the number of values in  $h_H(x)$ . Hereafter, we take into account the following assumptions (see [10, 21, 22]): the arrangement of elements in each  $h_H(x)$  is considered to be in an increasing order, and the  $j$ th largest value in  $h_H(x)$  is then denoted by  $h_H^{\delta(j)}(x)$ ; all the HFEs  $h_H(x_i)$ , ( $i = 1, 2, \dots, N$ ) should have the same length  $l_H = \max_{x \in X} \{l(h_H(x))\}$ , that is, any HFE  $h_H(x_i)$  with less number of elements should be extended optimistically by repeating its maximum until it has the same length with others.

Now, we are in a position to introduce a new concept of a HFS which is called a  $\kappa$ -level set of the HFS.

**Definition 2.4.** For given  $H \in HFS(X)$ , we define the fuzzy set  $h_H^{[\kappa]}$  as the  $\kappa$ -level set of HFS  $H$  where

$$h_H^{[\kappa]}(x_i) = \begin{cases} h_H^{\delta(\kappa)}(x_i) & \text{if } \kappa \leq l_H, \\ h_H^{\delta(l_H)}(x_i) & \text{if } \kappa > l_H, \end{cases} \quad (6)$$

and moreover, as explained above, the notation  $h_H^{\delta(j)}(x_i)$  indicates the  $j$ th largest value in  $h_H(x_i)$ .

It is clear from Definition 2.4 that the number of  $\kappa$ -level sets of HFS  $H$  is the number  $l_H$ .

**Example 2.5.** Suppose that  $H \in HFS(X)$  is in the form of that given in Example 2.2. Then, by taking Definition 2.4 into consideration, we have (see FIGURE 1)

$$\begin{aligned} h_H^{[1]} &= \{\langle x_1, 0.2 \rangle, \langle x_2, 0.3 \rangle, \langle x_3, 0.2 \rangle\}; & h_H^{[2]} &= \{\langle x_1, 0.4 \rangle, \langle x_2, 0.4 \rangle, \langle x_3, 0.3 \rangle\}; \\ h_H^{[3]} &= \{\langle x_1, 0.5 \rangle, \langle x_2, 0.4 \rangle, \langle x_3, 0.5 \rangle\}; & h_H^{[4]} &= \{\langle x_1, 0.5 \rangle, \langle x_2, 0.4 \rangle, \langle x_3, 0.6 \rangle\}. \end{aligned}$$

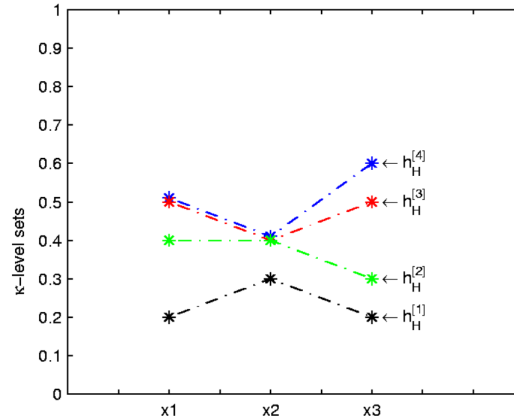


Figure 1:  $\kappa$ -level Sets of the HFS  $H$  in Example 2.5.

**Proposition 2.6.** The  $\kappa$ -level sets of the HFS  $H$  are increasing according to the parameter  $\kappa$ , that is, whenever  $\kappa_1 \leq \kappa_2$ , it holds that  $h_H^{[\kappa_1]} \preceq h_H^{[\kappa_2]}$ .

*Proof.* The proof comes directly from the second part of assumptions given immediately after Example 2.2.  $\square$

As a corollary of Proposition 2.6, we can conclude for any  $\kappa$ -level set of the HFS  $H$  that

$$h_H^{[1]} \preceq h_H^{[\kappa]} \preceq h_H^{[l_H]}, \quad 1 \leq \kappa \leq l_H, \quad (7)$$

which implies that any  $\kappa$ -level sets of a HFS  $H$  is bounded from below and bounded from above (of course this situation occurs when all HFEs of the HFS  $H$  are bounded).

The following argument shows that in order to representing a HFS there exist two ways. The first one is the most commonly used representation of a HFS, and referred to as *the vertical representation*. It is based on HFEs of HFS  $H$  and expressed by  $H = \bigcup_{x_i \in X} h_H(x_i)$ , where  $\bigcup_{x_i \in X}$  denotes the union operation over all  $x_i \in X$ .

The second one which seems to be a major missing discussion in the literature is referred here to as *the horizontal representation*.

Before giving any more attention to the second representation, it needs to consider the relation between HFEs and  $\kappa$ -level sets of HFS  $H$ , where for any fixed  $x_i \in X$

$$h_H(x_i) = \bigcup_{1 \leq \kappa \leq l_H} h_H^{[\kappa]}(x_i), \quad (8)$$

where  $\bigcup_{1 \leq \kappa \leq l_H}$  denotes the union operator over all  $1 \leq \kappa \leq l_H$ .

Now, we deal with introducing the second representation of a HFS as the following.

**Proposition 2.7.** (*Horizontal representation*) *A HFS  $H$  is represented using the union of all its  $\kappa$ -level sets in the form of*

$$H = \bigcup_{1 \leq \kappa \leq l_H} h_H^{[\kappa]}, \quad (9)$$

where  $\bigcup_{1 \leq \kappa \leq l_H}$  denotes the union operator over all  $1 \leq \kappa \leq l_H$ .

*Proof.* By the fact that  $H(x_i) = h_H(x_i)$  for any fixed  $x_i \in X$  and due to (8), the latter result can be deduced immediately.  $\square$

The horizontal representation is indeed an interesting HFS decomposition that have not been considered in the relevant literature. As can be seen later, such a decomposition is the basis of transforming HFSs into FSs.

Using the concept of  $\kappa$ -level sets given in Definition 2.4, we are now able to compare the sets in the form of HFSs.

**Definition 2.8.** *The two HFSs  $H_1$  and  $H_2$  are said to be comparable, and denoted by  $H_1 \preceq H_2$  (or  $H_1 \succeq H_2$ ) if and only if  $h_{H_1}^{[\kappa]} \preceq h_{H_2}^{[\kappa]}$  (or  $h_{H_1}^{[\kappa]} \succeq h_{H_2}^{[\kappa]}$ ) for any  $1 \leq \kappa \leq \max\{l_{H_1}, l_{H_2}\}$ .*

It should be noted that if  $l_{H_1} < l_{H_2}$ , then for operating correctly, we assume that  $h_{H_1}^{[\kappa]} = h_{H_1}^{[l_{H_1}]}$  for all  $\kappa > l_{H_1}$ .

In the case where  $h_{H_1}^{[\kappa]} \approx h_{H_2}^{[\kappa]}$  for any  $1 \leq \kappa \leq \max\{l_{H_1}, l_{H_2}\}$ , we result in  $H_1 \approx H_2$ , and vice versa.

Now, we develop an approach in order to transform a given HFS  $H$  into a FS which is in fact the aggregated fuzzy set of all  $\kappa$ -level sets of  $H$ .

**Definition 2.9.** *Let  $H \in \text{HFS}(X)$  and  $\sigma : [0, 1]^{l_H} \rightarrow [0, 1]$  be an aggregation operator given by (1)-(4). Then, corresponding to HFS  $H$  we define the following FS*

$$H^\sigma := \sigma(h_H^{[1]}, h_H^{[2]}, \dots, h_H^{[l_H]}), \quad (10)$$

where  $h_H^{[\kappa]}$ 's, ( $1 \leq \kappa \leq l_H$ ) are  $\kappa$ -level sets of HFS  $H$ .

**Remark 2.10.** *The interesting use of the approach introduced in Definition 2.9 is to deal with HFSs without concerning on the computational complexity of using their vertical representation in real-world applications. Indeed, we observe from Definition 2.9 that HFSs could be described by FSs and of course the whole machinery of FS analysis can be used to attack HFS situations.*

### 3 Ranking technique based on horizontal representation of HFSs

In group decision making under multiple attribute the goal is to find the best alternatives according to the preferences provided by the experts [3, 5]. This is achieved by applying ranking methods of alternatives in the selection processes of alternatives [4, 12, 15]. In this section, we focus on new ranking methods for HFSs. To do that, we will first review briefly the existing techniques of ranking hesitant fuzzy elements (HFEs), and we then propose a new technique of ranking hesitant fuzzy sets (HFSs) which is based on the concept of horizontal representation.

Recently, Farhadinia [8] dealt with a series of score functions for ranking hesitant fuzzy sets (HFSs). Subsequently, Farhadinia [9] defined a novel HFS ranking technique by using the idea of lexicographical ordering. In the sequel, Liu et al. [14] showed that the lexicographical ordering method of HFSs proposed by Farhadinia's does not work sometimes correctly, and then they presented a modified ranking method. In a recent paper, Alcantud and Calle [1] introduced a specific parametric expression of two indices of satisfaction, called the distance to an ideal or the similarity to an anti-ideal, to evaluate a HFS.

In this portion, we are eager to review the existing techniques of ranking HFEs or HFSs by emphasising on their shortcomings, and moreover, their shortages in exhibiting the consistency of human intuition in some situations. For further study in this regard, one may refer to [1],[8],[9],[14].

By referring to [21], we can observe that the generalized hesitant fuzzy weighted averaging (GHFWA) and also that of geometric (GHFWG) operators are denoted respectively by the aggregation operators  $GHFWA_\lambda$  and  $GHFWG_\lambda$  for the HFE in the form of  $h_H(x_i) = \{h_H^{\delta(1)}(x_i), \dots, h_H^{\delta(l)}(x_i)\}$  ( $i = 1, \dots, N$ ), and they are given by the following formulae

$$\begin{aligned} GHFWA_\lambda(h_H(x_1), \dots, h_H(x_N)) &= (\oplus_{i=1}^N (w_i h_H^\lambda(x_i)))^{\frac{1}{\lambda}} \\ &= \sum_{h_H^{\delta(j)}(x_1) \in h_H(x_1), \dots, h_H^{\delta(j)}(x_N) \in h_H(x_N)} \left( \{1 - \prod_{i=1}^N (1 - h_H^{\delta(j)}(x_i)^\lambda)^{w_i}\}^{\frac{1}{\lambda}} \right), \end{aligned} \quad (11)$$

$$\begin{aligned} GHFWG_\lambda(h_H(x_1), \dots, h_H(x_N)) &= \frac{1}{\lambda} (\otimes_{i=1}^N (\lambda h_H(x_i))^{w_i}) \\ &= \sum_{h_H^{\delta(j)}(x_1) \in h_H(x_1), \dots, h_H^{\delta(j)}(x_N) \in h_H(x_N)} \left( \{1 - (1 - \prod_{i=1}^N (1 - (1 - h_H^{\delta(j)}(x_i)^\lambda)^{w_i})^{\frac{1}{\lambda}}\} \right), \end{aligned} \quad (12)$$

where  $w_i \geq 0$  ( $i = 1, \dots, N$ ) are the weights with the property  $\sum_{i=1}^N w_i = 1$ . Furthermore, the HFS score functions  $\mathbf{S}_{WAM}^{AM}(H)$ ,  $\mathbf{S}_{WAM}^{GM}(H)$ ,  $\mathbf{S}_{WAM}^{Min}(H)$ , and  $\mathbf{S}_{WAM}^{Max}$  can be represented as (see [8])

$$\mathbf{S}_{WAM}^{AM}(H) = \sum_{i=1}^N w_i \left( \frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} h_H^{\delta(j)}(x_i) \right), \quad (13)$$

$$\mathbf{S}_{WAM}^{GM}(H) = \sum_{i=1}^N w_i \left( \left( \prod_{j=1}^{l_{x_i}} h_H^{\delta(j)}(x_i) \right)^{\frac{1}{l_{x_i}}} \right), \quad (14)$$

$$\mathbf{S}_{WAM}^{Min}(H) = \sum_{i=1}^N w_i (\min\{h_H^{\delta(1)}(x_i), \dots, h_H^{\delta(l_{x_i})}(x_i)\}), \quad (15)$$

$$\mathbf{S}_{WAM}^{Max}(H) = \sum_{i=1}^N w_i (\max\{h_H^{\delta(1)}(x_i), \dots, h_H^{\delta(l_{x_i})}(x_i)\}), \quad (16)$$

and furthermore the ranking functions  $\mathbf{S}_{xux}^{-d_{hnh}}$  and  $\mathbf{S}_{xux}^{-d_{hne}}$  which are indeed two distance-based techniques can be given as follows (see [22])

$$\mathbf{S}_{xux}^{-d_{hnh}}(H) = \sum_{i=1}^N w_i (S_{xux}^{-d_{hnh}}(h_H^{\delta(j)}(x_i))), \quad (17)$$

$$\mathbf{S}_{xux}^{-d_{hne}}(H) = \sum_{i=1}^N w_i (S_{xux}^{-d_{hne}}(h_H^{\delta(j)}(x_i))). \quad (18)$$

Besides the above-mentioned techniques, we can refer to Chen et al.'s [6] and Liao et al.'s [13] techniques which play role based on the two concepts of score and deviation functions, and they are represented respectively as

$$S_{c_{xx}}(h_H(x_i)) = \frac{1}{l_{x_i}} \sum_{k=1}^{l_{x_i}} h_H^{\delta(k)}(x_i), \quad (19)$$

$$\sigma_{c_{xx}}(h_H(x_i)) = \left( \frac{1}{l_{x_i}} \sum_{k=1}^{l_{x_i}} (h_H^{\delta(k)}(x_i) - S_{c_{xx}}(h_H(x_i)))^2 \right)^{\frac{1}{2}}; \quad (20)$$

and also

$$S_{l_{xx}}(h_H(x_i)) = \frac{1}{l_{x_i}} \sum_{k=1}^{l_{x_i}} h_H^{\delta(k)}(x_i), \quad (21)$$

$$\nu_{l_{xx}}(h_H(x_i)) = \left( \frac{2}{l_{x_i}(l_{x_i} - 1)} \sum_{p \neq q=1}^{l_{x_i}} (h_H^{\delta(p)}(x_i) - h_H^{\delta(q)}(x_i))^2 \right)^{\frac{1}{2}}. \quad (22)$$

Based on  $S_{c_{xx}}$  (or  $S_{l_{xx}}$ ) and  $\sigma_{c_{xx}}$  (or  $\nu_{l_{xx}}$ ), Chen et al.'s [6] (or Liao et al.'s [13]) introduced a comparison law between two HFEs  $h_{H_1}(x_i)$  and  $h_{H_2}(x_i)$  by the following steps:

- if  $S_{c_{xx}}(h_{H_1}(x_i)) > S_{c_{xx}}(h_{H_2}(x_i))$ , then  $h_{H_1}(x_i) > h_{H_2}(x_i)$ ;
- if  $S_{c_{xx}}(h_{H_1}(x_i)) = S_{c_{xx}}(h_{H_2}(x_i))$ , then
  1. if  $\sigma_{c_{xx}}(h_{H_1}(x_i)) = \sigma_{c_{xx}}(h_{H_2}(x_i))$ , then  $h_{H_1}(x_i) = h_{H_2}(x_i)$ ;
  2. if  $\sigma_{c_{xx}}(h_{H_1}(x_i)) > \sigma_{c_{xx}}(h_{H_2}(x_i))$ , then  $h_{H_1}(x_i) < h_{H_2}(x_i)$ ;
  3. if  $\sigma_{c_{xx}}(h_{H_1}(x_i)) < \sigma_{c_{xx}}(h_{H_2}(x_i))$ , then  $h_{H_1}(x_i) > h_{H_2}(x_i)$ .

Farhadinia [9] in a subsequent work verified that the techniques discussed in [6, 13, 21, 22] were not able to exhibit the consistency effect, and then he proposed a lexicographical-based ranking method as follows

$$\mathbf{R}(H) = \sum_{i=1}^n w_i (\mathbf{S}_{WAM}^{AM}(h_H(x_i), \nu_\phi(h_H(x_i)))) = \left( \sum_{i=1}^n w_i \mathbf{S}_{WAM}^{AM}(h_H(x_i)), \sum_{i=1}^n w_i (\nu_\phi(h_H(x_i))) \right), \quad (23)$$

where  $\mathbf{S}_{WAM}^{AM}$  is that given by (13), and

$$\nu_\phi(h_H(x_i)) = \sum_{k=1}^{N-1} \phi(h_H^{\delta(k+1)}(x_i) - h_H^{\delta(k)}(x_i)), \quad (24)$$

where  $\phi : [0, 1] \rightarrow [0, 1]$  is an increasing real function with  $\phi(0) = 0$ .

Taking the latter formulas for any two HFSs  $H_1 = \{\langle x_i, h_{H_1}(x_i) \rangle : x_i \in X\}$  and  $H_2 = \{\langle x_i, h_{H_2}(x_i) \rangle : x_i \in X\}$  into account, it holds that

1.  $H_1 < H_2$  if and only if  $\mathbf{R}(H_1) <_{lex} \mathbf{R}(H_2)$ ,
2.  $H_1 \leq H_2$  if and only if  $\mathbf{R}(H_1) \leq_{lex} \mathbf{R}(H_2)$ ,
3.  $H_1 = H_2$  if and only if  $\mathbf{R}(H_1) = \mathbf{R}(H_2)$ .

In the sequel, Liu et al. [14] using some examples verified that Farhadinia's lexicographical method [9] contains some shortcomings, and modified it as the following:

$$\mathbf{R}(H) = \sum_{i=1}^N w_i (\mathbf{S}_{WAM}^M(h(x_i)), \nu_\phi(h(x_i))) = \left( \sum_{i=1}^N w_i \mathbf{S}_{WAM}^M(h(x_i)), \sum_{i=1}^N w_i (\nu_\phi(h(x_i))) \right), \quad (25)$$

where the only difference between Liu et al. [14]'s technique and Farhadinia's [9] method is the first part of the vector  $\mathbf{R}(H)$  given by

$$\mathbf{S}_{WAM}^M(h(x_i)) = \sum_{i=1}^N w_i \left( \sum_{j=1}^{l_{x_i}} \frac{h_H^{\delta(j)}(x_i)}{\text{Count}(h_H^{\delta(j)}(x_i))} \right), \quad (26)$$

such that  $\text{Count}(h_H^{\delta(j)}(x_i))$  denotes the number of times of value  $h_H^{\delta(j)}(x_i)$  occurs in the HFE  $h_H(x_i)$ .

In a recent work, Alcantud and de Andres Calle [1] proposed the convex combination of distance to the ideal HFS together with the similarity to the anti-ideal HFS as follows

$$\Lambda_\alpha^{\lambda,w}(H) = \alpha \Delta^{\lambda,w}(H) + (1 - \alpha) \nabla^{\lambda,w}(H) \quad (27)$$

where

$$\Delta^{\lambda,w}(H) = \sum_{i=1}^N \frac{w_i}{l_{x_i}} \left( \sum_{j=1}^{l_{x_i}} (1 - h_H^{\delta(j)}(x_i))^\lambda \right)^{\frac{1}{\lambda}}, \quad (28)$$

$$\nabla^{\lambda,w}(H) = 1 - \sum_{i=1}^N \frac{w_i}{l_{x_i}} \left( \sum_{j=1}^{l_{x_i}} (h_H^{\delta(j)}(x_i))^\lambda \right)^{\frac{1}{\lambda}}. \quad (29)$$

Here, the higher value of  $\Lambda_\alpha^{\lambda,w}(H)$  stands for the worse suitability of that HFS  $H$ .

Now, we are in a position to indicate that the new approach for ranking HFSs is basically based on the HFS horizontal representation, and it satisfies some requirements.

At the first stage, let us take into account the relationship between the ranking order of HFSs and that of their aggregated FSs.

Hereafter and without losing any generality, we denote both the ordering relation on  $HFS(X)$  and the ordering relation on  $FS(X)$  by the same notation  $\prec$  in the following discussions.

**Proposition 3.1.** *Let  $H_1, H_2 \in HFS(X)$  with  $l_H := l_{H_1} = l_{H_2}$ , and  $FWA : [0, 1]^{l_H} \rightarrow [0, 1]$  be the aggregation operator given by (1). Corresponding to HFSs  $H_1, H_2$  we define*

$$H_1^{FWA} := FWA(h_{H_1}^{[1]}, h_{H_1}^{[2]}, \dots, h_{H_1}^{[l_H]}), \quad (30)$$

$$H_2^{FWA} := FWA(h_{H_2}^{[1]}, h_{H_2}^{[2]}, \dots, h_{H_2}^{[l_H]}), \quad (31)$$

where  $h_{H_i}^{[\kappa]}$ 's, ( $1 \leq \kappa \leq l_H$ ) are  $\kappa$ -level sets of HFSs  $H_i$ ,  $i = 1, 2$ . In this regard, there exists a correspondence relation between the ranking of HFSs  $H_1, H_2$  and that of their aggregated FSs  $H_1^{FWA}, H_2^{FWA}$  being stated as

$$H_1 \prec H_2 \quad \text{if and only if} \quad H_1^{FWA} \prec H_2^{FWA}, \quad (32)$$

$$H_1 \succ H_2 \quad \text{if and only if} \quad H_1^{FWA} \succ H_2^{FWA}, \quad (33)$$

$$H_1 \approx H_2 \quad \text{if and only if} \quad H_1^{FWA} \approx H_2^{FWA}. \quad (34)$$

*Proof.* The proof of the relations (32) and (33) is immediate from Definition 2.8.

The proof of the relation (34) comes from the monotonicity property of FWA operator which states that if  $h_{H_1}^{[k]} \preceq \widehat{h_{H_1}^{[k]}}$  for  $1 \leq k \leq l_H$ , then  $FWA(h_{H_1}^{[1]}, \dots, h_{H_1}^{[l_H]}) \preceq FWA(\widehat{h_{H_1}^{[1]}}, \dots, \widehat{h_{H_1}^{[l_H]}})$ .

Necessity of the relation (34): It is obvious by taking Definition 2.8 into account.

Sufficiency of the relation (34): Assume that  $H_1^{FWA} \approx H_2^{FWA}$ . This implies that

$$FWA(h_{H_1}^{[\delta(1)]}, h_{H_1}^{[\delta(2)]}, \dots, h_{H_1}^{[\delta(l_H)]}) = FWA(h_{H_2}^{[\delta(1)]}, h_{H_2}^{[\delta(2)]}, \dots, h_{H_2}^{[\delta(l_H)]}).$$

If there exists  $1 \leq \widehat{\kappa} \leq l_H$  such that  $h_{H_1}^{[\delta(\widehat{\kappa})]} < h_{H_2}^{[\delta(\widehat{\kappa})]}$  then, from the monotonicity property of FWA operator it follows that

$$FWA(h_{H_1}^{[\delta(1)]}, h_{H_1}^{[\delta(2)]}, \dots, h_{H_1}^{[\delta(l_H)]}) < FWA(h_{H_2}^{[\delta(1)]}, h_{H_2}^{[\delta(2)]}, \dots, h_{H_2}^{[\delta(l_H)]}),$$

which is a contradiction.

Once again, by a similar discussion, we find in the following that if there exists  $1 \leq \widehat{\kappa} \leq l_H$  such that  $h_{H_1}^{[\delta(\widehat{\kappa})]} > h_{H_2}^{[\delta(\widehat{\kappa})]}$ , then

$$FWA(h_{H_1}^{[\delta(1)]}, h_{H_1}^{[\delta(2)]}, \dots, h_{H_1}^{[\delta(l_H)]}) > FWA(h_{H_2}^{[\delta(1)]}, h_{H_2}^{[\delta(2)]}, \dots, h_{H_2}^{[\delta(l_H)]}),$$

which is also a contradiction. Hence, it must hold  $h_{H_1}^{[\delta(\kappa)]} = h_{H_2}^{[\delta(\kappa)]}$  for all  $1 \leq \kappa \leq l_H$ , and therefore by keeping Definition 2.8 into the mind, we imply that  $H_1 \approx H_2$ . This completes the proof.  $\square$

The latter proposition reveals that such a correspondence between the ranking of HFSs and that of FSs enables us to consider a variety of methods for ranking HFSs on the basis of existing ranking methods for FSs.

In the following, we derive a new ranking method of HFSs by taking the horizontal representation into account.

**Definition 3.2.** Let  $H \in HFS(X)$ , and  $FWA : [0, 1]^{l_H} \rightarrow [0, 1]$  be the aggregation operator given by (1). Then, corresponding to HFS  $H$ , we set

$$H^{FWA} := FWA(h_H^{[1]}, h_H^{[2]}, \dots, h_H^{[l_H]}), \quad (35)$$

where  $h_H^{[\kappa]}$ 's, ( $1 \leq \kappa \leq l_H$ ) are  $\kappa$ -level sets of HFS  $H$ . The ranking order of HFSs  $H_1$  and  $H_2$  is defined corresponding to the ranking order of their aggregated FSs  $H_1^{FWA}$  and  $H_2^{FWA}$  as follows:

$$\begin{aligned} H_1 \prec H_2 & \quad (\text{if and only if } H_1^{FWA} \prec H_2^{FWA}) & \quad \text{if and only if} \\ & & \quad \mathfrak{R}(H_1^{FWA}) < \mathfrak{R}(H_2^{FWA}); \\ H_1 \succ H_2 & \quad (\text{if and only if } H_1^{FWA} \succ H_2^{FWA}) & \quad \text{if and only if} \\ & & \quad \mathfrak{R}(H_1^{FWA}) > \mathfrak{R}(H_2^{FWA}); \\ H_1 \approx H_2 & \quad (\text{if and only if } H_1^{FWA} \approx H_2^{FWA}) & \quad \text{if and only if} \\ & & \quad \mathfrak{R}(H_1^{FWA}) = \mathfrak{R}(H_2^{FWA}), \end{aligned}$$

where the operator  $\mathfrak{R}$  is a ranking function that maps any FS into the real line.

Furthermore, we formulate the orders  $\preceq$  and  $\succeq$ , respectively as

$$\begin{aligned} H_1 \preceq H_2 & \quad \text{if and only if } H_1 \prec H_2 \quad \text{or} \quad H_1 \approx H_2; \\ H_1 \succeq H_2 & \quad \text{if and only if } H_1 \succ H_2 \quad \text{or} \quad H_1 \approx H_2. \end{aligned}$$

The above definition immediately gives rise to

**Lemma 3.3.** The relation  $\preceq$  is partial ordering on  $HFS(X)$  provided that its counterpart is a partial ordering relation on  $FS(X)$ .

*Proof.* Assume that the ordering relation on  $FS(X)$  given by Definition 3.2 is partial. Then, for any  $H, H_1, H_2, H_3 \in HFS(X)$ , one can easily verify that the relation  $\preceq$  satisfies the following properties:

(P1) Reflexivity:  $H \preceq H$ ;

(P2) Antisymmetry: If  $H_1 \preceq H_2$  and  $H_2 \preceq H_1$ , then  $H_1 \approx H_2$ ;

(P3) Transitivity: If  $H_1 \preceq H_2$  and  $H_2 \preceq H_3$ , then  $H_1 \preceq H_3$ .

Hence, the relation  $\preceq$  is partial ordering on HFSs.  $\square$



There exist other properties reported in [19, 20] for the ordering approaches which are not included here since they sound weird.

In what follows, we present an algorithmic form of the proposed ranking method for HFSs.

**Algorithm 3.4.** (A new ranking method for HFSs on the basis of the horizontal representation)

Let  $\{H_1, H_2, \dots, H_m\}$  be a collection of  $m$  HFSs on  $X = \{x_1, x_2, \dots, x_N\}$ . Then, we implement the following steps to find the ranking order of the given HFSs.

Step 1. For any HFS  $H_i$  ( $i = 1, 2, \dots, m$ ), we construct the corresponding  $\kappa$ -level sets  $h_{H_i}^{[\kappa]}$ 's ( $\kappa = 1, 2, \dots, l_{H_i}$ ) by the use of Definition 2.4.

Step 2. Taking Definition 2.9 into account, we construct the aggregated FS  $H_i^{FWA} := FWA(h_{H_i}^{[1]}, h_{H_i}^{[2]}, \dots, h_{H_i}^{[l_{H_i}]})$ , corresponding to each HFS  $H_i$  ( $i = 1, 2, \dots, m$ ).

Step 3. Following from Definition 3.2, we can specify the ordering of HFSs

$\{H_1, H_2, \dots, H_m\}$  according to the ordering of their aggregated FSs

$\{H_1^{FWA}, H_2^{FWA}, \dots, H_m^{FWA}\}$ .

## 4 Multiple attribute decision-making with hesitant fuzzy information

Here, we employ a multiple attribute decision-making problem with hesitant fuzzy information that was discussed in [9] to compare the result of the proposed approach with that of the existing techniques.

**Example 4.1.** Suppose that an enterprise's board of directors including five members is interested in developing a five-year strategy initiatives. This team evaluates four projects  $Y_i$  ( $i = 1, 2, 3, 4$ ) with respect to four attributes that are specified by the Balanced Scorecard methodology as:  $G_1$ : financial perspective,  $G_2$ : the customer satisfaction,  $G_3$ : internal business process perspective, and  $G_4$ : learning and growth perspective. Moreover, such attributes are assumed to have the weight vector  $w = (0.2, 0.3, 0.15, 0.35)$ .

Considering the hesitant fuzzy decision matrix which evaluates the alternative  $Y_i$  under the attribute  $G_j$  as below,

	$G_1$	$G_2$	$G_3$	$G_4$
$Y_1$	{0.2, 0.4, 0.7}	{0.2, 0.6, 0.8}	{0.2, 0.3, 0.6, 0.7, 0.9}	{0.3, 0.4, 0.5, 0.7, 0.8}
$Y_2$	{0.2, 0.4, 0.7, 0.9}	{0.1, 0.2, 0.4, 0.5}	{0.3, 0.4, 0.6, 0.9}	{0.5, 0.6, 0.8, 0.9}
$Y_3$	{0.3, 0.5, 0.6, 0.7}	{0.2, 0.4, 0.5, 0.6}	{0.3, 0.5, 0.7, 0.8}	{0.2, 0.5, 0.6, 0.7}
$Y_4$	{0.3, 0.5, 0.6}	{0.2, 0.4}	{0.5, 0.6, 0.7}	{0.8, 0.9}

Table 1: Hesitant Fuzzy Decision Matrix

We implement Algorithm 3.4 to get the ranking order of alternatives  $Y_i$  ( $i = 1, 2, 3, 4, 5$ ) being obtained through the following steps:

**Steps 1-2.** By the help of Definition 2.9, we construct the aggregated FS  $H_i^{FWA} := FWA(h_{H_i}^{[1]}, h_{H_i}^{[2]}, \dots, h_{H_i}^{[l_{H_i}]})$ , corresponding to each HFS  $H_i$  ( $i = 1, 2, 3, 4$ ).

For instance, take the HFS  $H_1$  together with the aggregation weight vector of  $h_{H_1}^{[\kappa]}$  ( $\kappa = 1, 2, 3, 4, 5$ ) that is in the form of  $W^{H_1} = (w_1^{H_1}, w_2^{H_1}, w_3^{H_1}, w_4^{H_1}, w_5^{H_1})$ .

Before doing any further processing, we first obtain the weighted form of hesitant fuzzy decision matrix by the help of the weight vector  $w = (0.2, 0.3, 0.15, 0.35)$  where  $H_1^w = \{0.2 \times H_1(G_1), 0.3 \times H_1(G_2), 0.15 \times H_1(G_3), 0.35 \times H_1(G_4)\}$ , denotes the first row of the weighted hesitant fuzzy decision matrix. Hereafter, in order to avoid confusion, we denote the resulted HFS  $H_1^w$  again by  $H_1$ .

Now, by taking the aggregation weight vector of  $h_{H_1}^{[\kappa]}$  ( $\kappa = 1, 2, 3, 4, 5$ ) as  $W^{H_1} = (w_1^{H_1}, w_2^{H_1}, w_3^{H_1}, w_4^{H_1}, w_5^{H_1})$ , we have

$$\begin{aligned}
H_1^{FWA} &:= FWA(h_{H_1}^{[1]}, h_{H_1}^{[2]}, h_{H_1}^{[3]}, h_{H_1}^{[4]}, h_{H_1}^{[5]}) \\
&= FWA(\{0.2, 0.2, 0.2, 0.3\}, \{0.4, 0.6, 0.3, 0.4\}, \{0.7, 0.8, 0.6, 0.5\}, \\
&\quad \{0.7, 0.8, 0.7, 0.7\}, \{0.7, 0.8, 0.9, 0.8\}) \\
&= \oplus_{\kappa=1}^5 (w_{\kappa}^{H_1} h_{H_1}^{[\kappa]}) = 1 - \prod_{\kappa=1}^5 (1 - h_{H_1}^{[\kappa]})^{w_{\kappa}^{H_1}}
\end{aligned}$$

$$\begin{aligned}
&= \{1 - [(1 - 0.2)^{w_1^{H_1}} (1 - 0.4)^{w_2^{H_1}} (1 - 0.7)^{w_3^{H_1}} (1 - 0.7)^{w_4^{H_1}} (1 - 0.7)^{w_5^{H_1}}], \\
&\quad 1 - [(1 - 0.2)^{w_1^{H_1}} (1 - 0.6)^{w_2^{H_1}} (1 - 0.8)^{w_3^{H_1}} (1 - 0.8)^{w_4^{H_1}} (1 - 0.8)^{w_5^{H_1}}], \\
&\quad 1 - [(1 - 0.2)^{w_1^{H_1}} (1 - 0.3)^{w_2^{H_1}} (1 - 0.6)^{w_3^{H_1}} (1 - 0.7)^{w_4^{H_1}} (1 - 0.9)^{w_5^{H_1}}], \\
&\quad 1 - [(1 - 0.3)^{w_1^{H_1}} (1 - 0.4)^{w_2^{H_1}} (1 - 0.5)^{w_3^{H_1}} (1 - 0.7)^{w_4^{H_1}} (1 - 0.8)^{w_5^{H_1}}]\}.
\end{aligned}$$

If we set  $w_1^{H_1} = w_2^{H_1} = w_3^{H_1} = w_4^{H_1} = w_5^{H_1} = \frac{1}{5}$ , where  $\sum_{\kappa=1}^5 w_{\kappa}^{H_1} = 1$  then one comes to

$$H_1^{FWA} = \{\langle G_1, 0.1089 \rangle, \langle G_2, 0.1949 \rangle, \langle G_3, 0.0818 \rangle, \langle G_4, 0.1916 \rangle\}.$$

Just in the same way, we are able to construct the aggregated FFSs  $H_2^{FWA}$ ,  $H_3^{FWA}$  and  $H_4^{FWA}$  as follows:

$$\begin{aligned}
H_2^{FWA} &= \{\langle G_1, 0.1116 \rangle, \langle G_2, 0.0912 \rangle, \langle G_3, 0.0832 \rangle, \langle G_4, 0.2470 \rangle\}; \\
H_3^{FWA} &= \{\langle G_1, 0.1055 \rangle, \langle G_2, 0.1286 \rangle, \langle G_3, 0.0867 \rangle, \langle G_4, 0.1775 \rangle\}; \\
H_4^{FWA} &= \{\langle G_1, 0.0937 \rangle, \langle G_2, 0.1004 \rangle, \langle G_3, 0.0901 \rangle, \langle G_4, 0.3035 \rangle\}.
\end{aligned}$$

**Step 3.** By employing the average value of components of each  $H_i^{FWA}$  ( $i = 1, 2, 3, 4$ ), that is,

$$\begin{aligned}
R(H_1^{FWA}) &= 0.1049, & R(H_2^{FWA}) &= 0.1288, & R(H_3^{FWA}) &= 0.0854, \\
R(H_4^{FWA}) &= 0.2299,
\end{aligned}$$

we conclude that the ranking order of alternatives  $Y_i$ 's is determined as the following

$$Y_4 \succ Y_2 \succ Y_1 \succ Y_3.$$

Recently, Farhadinia [8, 9] has collected the results of the existing HFS ranking techniques whose ranking values are represented in TABLE 2. In order to have a better understanding of what results are to be produced by the proposed horizontal representation-based ranking method compared to the exiting ones, we have dedicated the last row of TABLE 2 to the results of the proposed technique.

From TABLE 2, one can observe that the most repeated project as the best one is  $Y_4$ , and the other priority order is as  $Y_2$ ,  $Y_1$  and  $Y_3$ , respectively. This result coincides exactly with that given by Chen, Xu and Xia's technique [6], Liao, Xu and Xia's technique [13], Farhadinia's lexicographical ordering [9], Liu, Wang and Zhang's technique [14], Alcantud and Calle's technique [1] and of course the proposed technique in this contribution.

What should be considered here is that all the above-mentioned techniques are based on the HFE representation (equivalently, vertical representation), while, the proposed technique needs not to consider the HFE representation, and it only works with the HFS representation (equivalently, horizontal representation), called here as the  $\kappa$ -level sets. This clearly reduces the process of converting the HFE representations into the HFS representation as that performed in the existing techniques.

## 5 Conclusions

In this manuscript, we mainly proposed a transformation of HFSs into FFSs. This transformation is based on a HFS decomposition that have not been discussed by the researchers in the relevant literature. We referred to such a HFS decomposition as the horizontal representation. Such a transformation allows the user to consider all the concerns that may be arisen in HFS theory, and of course they had been originally planed in the FS counterparts. Therefore, considering the proposed horizontal representation of HFSs will be very much attractive, and it will be helpful for the researchers to create more reasonable and less computational complexity applications.

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Technique	$Y_1$	$Y_2$	$Y_3$
$GHFWA_1$ (via $S_{xix}$ )	0.5634	0.6009	0.5178
$GHFWA_2$ (via $S_{xix}$ )	0.5847	0.6278	0.5337
$GHFWA_5$ (via $S_{xix}$ )	0.6324	0.6807	0.5723
$GHFWA_{10}$ (via $S_{xix}$ )	0.6730	0.7235	0.6087
$GHFWA_{20}$ (via $S_{xix}$ )	0.7058	0.7576	0.6410
$GHFWG_1$ (via $S_{xix}$ )	0.4783	0.4625	0.4661
$GHFWG_2$ (via $S_{xix}$ )	0.4546	0.4295	0.4526
$GHFWG_5$ (via $S_{xix}$ )	0.4011	0.3706	0.4170
$GHFWG_{10}$ (via $S_{xix}$ )	0.3564	0.3264	0.3809
$GHFWG_{20}$ (via $S_{xix}$ )	0.3221	0.2919	0.3507
$GHFWA_1$ (via $S_{Nia}$ )	0.9018	0.9101	0.8921
$S_{WAM}^{AM}$	0.5167	0.5275	0.4937
$S_{GM}^{AM}$	0.4618	0.4845	0.4575
$S_{Min}^{WAM}$	0.2350	0.2900	0.2350
$S_{Max}^{WAM}$	0.7950	0.7800	0.6850
Xu and Xia's technique (via $S_{xux}^{-d_{hnh}}$ )	0.5099	0.4725	0.5062
Xu and Xia's technique (via $S_{xux}^{-d_{hne}}$ )	0.5024	0.5366	0.5456
Chen, Xu and Xia's technique (via $S_{cxx}$ )	0.5167	0.5275	0.4937
Liao, Xu and Xia's technique (via $S_{lxx}$ )	0.5167	0.5275	0.4937
The HFS lexicographical ranking	(0.5167,0.133)	(0.5275,0.072)	(0.4937,0.155)
Liu, Wang and Zhang's technique	(0.5197, 0.5600)	(0.5275, 0.4900)	(0.4937, 0.4500)
Alcantud and Calle's technique ( $\lambda = 1$ )	0.4803	0.4725	0.5063
<b>The proposed technique</b>	0.1049	0.1288	0.0854

Table 2: The Values of HFS Ranking Functions together with the Alternatives Precedence (The First Portion).

$Y_4$	Ranking
0.6524	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.6781	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.7314	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.7745	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.8077	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.5130	$Y_4 > Y_1 > Y_3 > Y_2$
0.4755	$Y_4 > Y_1 > Y_3 > Y_2$
0.4082	$Y_3 > Y_4 > Y_1 > Y_2$
0.3609	$Y_3 > Y_4 > Y_1 > Y_2$
0.3266	$Y_3 > Y_4 > Y_1 > Y_2$
0.9528	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.5708	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.5606	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.4750	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.6600	$Y_1 > Y_2 > Y_3 > Y_4$
0.4291	$Y_4 > Y_2 > Y_3 > Y_1$
0.5101	$Y_1 > Y_4 > Y_2 > Y_3$
0.5708	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.5708	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
(0.5708, 0.0285)	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
(0.5708, 0.1850)	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.4292	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$
0.2299	$\mathbf{Y}_4 > \mathbf{Y}_2 > \mathbf{Y}_1 > \mathbf{Y}_3$

Table 3: (The Second Portion of TABLE 2.)