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Solving LR fuzzy linear matrix equation†

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Abstract

In this paper, the fuzzy matrix equation $A\widetilde{X}B = \widetilde{C}$ in which A, B are $n \times n$ crisp matrices respectively and $\widetilde{C}$ is an $n \times n$ arbitrary LR fuzzy numbers matrix, is investigated. A new numerical procedure for calculating the fuzzy solution is designed and a sufficient condition for the existence of strong fuzzy solution is derived. Some examples are given to illustrate the proposed method.

Keywords: Fuzzy numbers, matrix analysis, fuzzy matrix equation, fuzzy approximate solutions.

1 Introduction

Imprecision is often involved in many real-world engineering systems. Fuzzy systems have been an essential role in the fuzzy modeling, which can formulate uncertainty in actual environment. In many matrix systems, some of the system parameters are vague or imprecise, and fuzzy number is a better tool than crisp one for modeling the problem, and hence solving a fuzzy matrix system is becoming more important. The concept of fuzzy numbers and their arithmetic operations were first introduced and investigated by Zadeh [40], Dubois et al.[19] and Nahmias [31]. A different approach to fuzzy numbers and the structure of fuzzy number spaces was given by Puri and Ralescu [37], Goetschell et al.[23] and Wu Congxin et al.[38, 39].

Since Friedman et al.[21] proposed a general model for solving an $n \times n$ fuzzy linear systems $A\widetilde{x} = \widetilde{b}$ by an embedding approach in 1998, lots of research works have been done about some more complicated fuzzy linear systems such as dual fuzzy linear systems (DFLS), general fuzzy linear systems (GFLS), fully fuzzy linear systems (FFLS)and general dual fuzzy linear systems (GDFLS) see $[1-6, 11, 16-17, 21, 30, 41]$. And some new theory and method for fuzzy linear systems still appeared recently$[7, 12, 20, 22, 32-34]$.

However, for a fuzzy linear matrix equation which always has a wide use in control theory and control engineering, few work has been done in the past decades. In 2009, Allahviranloo et al. [9] discussed the fuzzy linear matrix equations(FLME) with the form of $A\widetilde{X}B = \widetilde{C}$ by embedding approach. In 2011, Gong and Guo [24] investigated a class of inconsistent fuzzy matrix equations $A\widetilde{X} = \widetilde{B}$ and obtained its fuzzy least squares solutions by the same way. In 2017, Guo et al.[27] presented a computing method for fuzzy matrix equations $\widetilde{X}A = \widetilde{B}$. Recently, Amirfakhrian et al.[10] proposed a new method for solving fuzzy matrix equations $A\widetilde{X}B = \widetilde{C}$ based on triangular fuzzy numbers.

In traditional fuzzy linear systems, the uncertain elements were usually denoted by the parametric form of fuzzy numbers. Thus the extended linear equations always contains parameter $r, 0 \leq r \leq 1$, which makes their computation inconvenient in some sense. On the other hand, the weak fuzzy solution of fuzzy linear systems does not exist sometimes [8]. To make the multiplication of fuzzy numbers easy and handle the full fuzzy linear systems, Dubois and Prade [19] introduced the LR fuzzy number in 1978. We know that triangular fuzzy numbers is just specious cases of LR fuzzy numbers. In 2012, Otadi and Mosleh [35] set up a model for solving fully fuzzy matrix equations $\widetilde{A}\widetilde{X} = \widetilde{B}$. In 2013, Guo et al. [26] proposed a computing method for solving fuzzy Sylvester matrix equation $A\widetilde{X} + \widetilde{X}B = \widetilde{C}$ with LR fuzzy numbers. For fully fuzzy Sylvester matrix equation $\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{B} = \widetilde{C}$, Daud et al. [15] and Dookhitram et al. [18]

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carried on a thorough investigations in recent years. In 2014, Gong et al. [25] studied the general dual fuzzy matrix systems $A\widetilde{X}+\widetilde{B}=C\widetilde{X}+\widetilde{D}$ according to arithmetic operations of LR fuzzy numbers. In 2017, Kaur and Kumar [29] made a note on Gong et al.'s paper. Recently, Guo et al.[28] made a further investigation for dual fuzzy matrix systems $A\widetilde{X}+\widetilde{B}=C\widetilde{X}+\widetilde{D}$.

In this paper, we propose a general model for solving the fuzzy matrix equation $A\widetilde{X}B=\widetilde{C}$ in which A, B are $n\times n$ crisp matrices and $\widetilde{C}$ is an $n\times n$ arbitrary LR fuzzy numbers matrix respectively. We extend the fuzzy matrix equation into a crisp system of linear matrix equations. We obtain the fuzzy minimal solution of the original fuzzy matrix equation from solving the crisp function matrix system. Moreover, we discussed the existence condition of the strong fuzzy minimal solution. Finally, we given some examples to illustrate the proposed method. The structure of this paper is organized as follows:

In Section 2, we recall the fuzzy number and present the concept of the LR fuzzy linear matrix equation. In Section 3, the model to the fuzzy matrix equation is proposed in detail and the minimal fuzzy solution of the fuzzy linear matrix equation is obtained. Some examples are given to illustrate our method in Section 4 and the conclusion is drawn in Section 5.

2 Preliminaries

2.1 The LR fuzzy number

There are several definitions for the concept of LR fuzzy numbers (see [19, 31, 40]).

Definition 2.1. *A fuzzy number is a fuzzy set $u: R\to I=[0,1]$ that satisfies:*
(1) *u is upper semicontinuous,*
(2) *u is fuzzy convex, i.e., $u(\lambda x+(1-\lambda)y)\geq \min\{u(x),u(y)\}$ for all $x,y\in R, \lambda\in[0,1]$,*
(3) *u is normal, i.e., there exists $x_0\in R$ such that $u(x_0)=1$,*
(4) *$suppu=\{x\in R \mid u(x)>0\}$ is the support of the u, and its closure cl(suppu) is compact.*

Let E^1 be the set of all fuzzy numbers on R.

Definition 2.2. *A fuzzy number $\widetilde{M}$ is said to be a LR fuzzy number if*

$$\mu_{\widetilde{M}}(x)=\begin{cases} L(\frac{m-x}{\alpha}), & x\leq m, \quad \alpha>0,\\ R(\frac{x-m}{\beta}), & x\geq m, \quad \beta>0,\end{cases}$$

where m, α and β are called the mean value, left and right spreads of $\widetilde{M}$, respectively. The function $L(\cdot)$, which is called left shape function satisfies:
(1) *$L(x)=L(-x)$,*
(2) *$L(0)=1$ and $L(1)=0$,*
(3) *$L(x)$ is non increasing on $[0,\infty)$.*

The definition of a right shape function $R(\cdot)$ is similar to that of $L(\cdot)$.

Clearly, two LR fuzzy numbers $\widetilde{M}=(m,\alpha,\beta)_{LR}$ and $\widetilde{N}=(n,\gamma,\delta)_{LR}$ are said to be equal, if and only if $m=n, \alpha=\gamma$ and $\beta=\delta$. Also, $\widetilde{M}=(m,\alpha,\beta)_{LR}$ is positive(negative) if and only if $m-\alpha>0$($m+\beta<0$).

Definition 2.3. *For arbitrary LR fuzzy numbers$\widetilde{M}=(m,\alpha,\beta)_{LR}$ and $\widetilde{N}=(n,\gamma,\delta)_{LR}$, we have*
(1) *Addition*

$$\widetilde{M}+\widetilde{N}=(m,\alpha,\beta)_{LR}+(n,\gamma,\delta)_{LR}=(m+n,\alpha+\gamma,\beta+\delta)_{LR}. \tag{1}$$

(2) *Subtraction*

$$\widetilde{M}-\widetilde{N}=(m,\alpha,\beta)_{LR}-(n,\gamma,\delta)_{LR}=(m-n,\alpha-\delta,\beta-\gamma)_{LR}. \tag{2}$$

(3) *Scalar multiplication*

$$\lambda\widetilde{M}=\lambda(m,\alpha,\beta)_{LR}\cong\begin{cases} (\lambda m,\lambda\alpha,\lambda\beta), & \lambda\geq 0,\\ (\lambda m,-\lambda\beta,-\lambda\alpha)_{RL}, & \lambda<0.\end{cases} \tag{3}$$

2.2 LR fuzzy matrix equation

Definition 2.4. *A matrix $\widetilde{A}=(\widetilde{a}_{ij})$ is called a LR fuzzy matrix, if each element $\widetilde{a}_{ij}$ of $\widetilde{A}$ is a LR fuzzy number.*

Definition 2.5. *Let $\widetilde{A}$ and $\widetilde{B}$ be two LR fuzzy numbers matrix and denoted by $\widetilde{A}=(A,A^l,A^r)$ and $\widetilde{B}=(B,B^l,B^r)$ are said to be equal, if and only if $A=B, A^l=B^l$ and $A^r=B^r$. Also, $\widetilde{A}=(A,A^l,A^r)$ is positive(negative) if and only if $A-A^l>O(A+A^r<O)$.*

Definition 2.6. *The matrix system*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{1n} \\ \widetilde{x}_{21} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{x}_{n1} & \widetilde{x}_{n2} & \cdots & \widetilde{x}_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{12} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} \widetilde{c}_{11} & \widetilde{c}_{12} & \cdots & \widetilde{c}_{1m} \\ \widetilde{c}_{21} & \widetilde{c}_{12} & \cdots & \widetilde{c}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{c}_{n1} & \widetilde{c}_{n2} & \cdots & \widetilde{c}_{nn} \end{pmatrix}, \tag{4}$$

where $a_{ij}, b_{ij}, 1\le i,j\le n$ are crisp numbers and $\widetilde{c}_{ij}, 1\le i,j\le n$ are LR fuzzy numbers, is called a LR fuzzy linear matrix equations(LRFLMEs).

Using matrix notation, we have

$$A\widetilde{X}B=\widetilde{C}, \tag{5}$$

A fuzzy numbers matrix

$$\widetilde{X}=(\widetilde{x}_{ij})=(x_{ij},x_{ij}^l,x_{ij}^r), 1\le i,j\le n$$

is called a solution of the fuzzy matrix equation (5) *if and only if it satisfies $A\widetilde{X}B=\widetilde{C}$.*

3 Solving fuzzy matrix equation

In this section we investigate the fuzzy linear matrix equations (5). Firstly, we set up a computing model for solving LRCFLS. Then we define the LR fuzzy solution of LRCFLS and obtain its solution representation by means of generalized inverses of matrices. Finally, we give a sufficient condition for strong fuzzy approximate solution to the original fuzzy matrix equation.

Definition 3.1. *For two LR fuzzy numbers matrices$\widetilde{A}=(A,A^l,A^r)$ and $\widetilde{B}=(B,B^l,B^r)$, we have*

(1) *Addition*

$$\widetilde{A}+\widetilde{B}=(A,A^l,A^r)+(B,B^l,B^r)=(A+B,A^l+B^l,A^r+B^r). \tag{6}$$

(2) *Subtraction*

$$\widetilde{A}-\widetilde{B}=(A,A^l,A^r)-(B,B^l,B^r)=(A+B,A^l-B^r,A^r-B^l). \tag{7}$$

(3) *Scalar multiplication*

$$\lambda\widetilde{A}=\lambda(A,A^l,A^r)\cong\begin{cases} (\lambda A,\lambda A^l,\lambda A^r), & \lambda\ge 0, \\ (\lambda A,-\lambda A^r,-\lambda A^l), & \lambda<0. \end{cases} \tag{8}$$

Definition 3.2. *Let $A=(a_{ij})$ be a $m\times n$ crisp matrix and $\widetilde{B}=\widetilde{b}_{ij}$ be a $n\times p$ fuzzy matrix. The size of the product of two matrices is $m\times p$ and is written as follows:*

$$A\widetilde{B}=\widetilde{C}=(\widetilde{c}_{ij}). \tag{9}$$

where $\widetilde{c}_{ij}=\sum_{k=1,\ldots,n}^{+} a_{ik}\times\widetilde{b}_{kj}$. Likely, let $\widetilde{A}=(\widetilde{a}_{ij})$ be a $m\times n$ fuzzy matrix and $B=b_{ij}$ be a $n\times p$ crisp matrix. The size of the product of two matrices is $m\times p$ and is written as follows:

$$\widetilde{A}B=\widetilde{C}=(\widetilde{c}_{ij})$$

where $\widetilde{c}_{ij}=\sum_{k=1,\ldots,n}^{+}\widetilde{a}_{ik}\times b_{kj}$.

Theorem 3.3. *The fuzzy linear system $A\widetilde{X}B=\widetilde{C}$ can be extended into the following system of linear matrix equations*

$$\begin{cases} AXB=C, \\ A\begin{pmatrix} X^l, X^r \end{pmatrix}\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}=\begin{pmatrix} C^l, C^r \end{pmatrix}, \end{cases} \tag{10}$$

where $\widetilde{X}=(X,X^l,X^r), \widetilde{C}=(C,C^l,C^r)$. And the elements b_{ij}^+ of matrix B^+ and b_{ij}^- of matrix B^- are determined by this way: if $b_{ij}\ge 0, b_{ij}^+=b_{ij}$ else $b_{ij}^+=0, 1\le i,j\le n$; if $b_{ij}<0, b_{ij}^-=b_{ij}$ else $b_{ij}^-=0, 1\le i,j\le n$.

Proof. Let $\widetilde{C} = (C, C^l, C^r) = (c_{ij}, c^l_{ij}, c^r_{ij})_{n\times n}$, $\widetilde{X} = (X, X^l, X^r) = (x_{ij}, x^l_{ij}, x^r_{ij})_{n\times n}$. We also let $A = A^+ + A^-$ where the elements a^+_{ij} of matrix A^+ and a^-_{ij} of matrix A^- are determined by this way: if $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ else $a^+_{ij} = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a^-_{ij} = a_{ij}$ else $a^-_{ij} = 0, 1 \leq i, j \leq n$. For fuzzy matrix equation $A\widetilde{X}B = \widetilde{C}$, we can express it as

$$(A^+ + A^-)(X, X^l, X^r)B = (C, C^l, C^r). \tag{11}$$

Since

$$k\widetilde{x}_{ij} = \begin{cases} (kx_{ij}, kx^l_{ij}, kx^r_{ij}), & k \geq 0, \\ (kx_{ij}, -kx^r_{ij}, -kx^l_{ij}), & k < 0, \end{cases}$$

we have

$$A\widetilde{X} = \begin{cases} (AX, AX^l, AX^r), & A \geq 0, \\ (AX, -AX^r, -AX^l), & A < 0. \end{cases}$$

So the equation (11) can be rewritten as

$$A^+(X, X^l, X^r)B + A^-(X, X^l, X^r)B = (A^+X, A^+X^l, A^+X^r)B + (A^-X, -A^-X^r, -A^-X^l)B.$$

Since

$$k\widetilde{x}_{ij} = \begin{cases} (kx_{ij}, kx^l_{ij}, kx^r_{ij}), & k \geq 0, \\ (kx_{ij}, -kx^r_{ij}, -kx^l_{ij}), & k < 0, \end{cases}$$

we have

$$\widetilde{X}B = \begin{cases} (XB, X^lB, X^rB), & B \geq 0, \\ (XB, -X^rB, -X^lB), & B < 0. \end{cases}$$

Supposing $B = B^+ + B^-$ in which the elements b^+_{ij} of matrix B^+ and b^-_{ij} of matrix B^- are determined by this way: if $b_{ij} \geq 0, b^+_{ij} = b_{ij}$ else $b^+_{ij} = 0, 1 \leq i, j \leq n$; if $b_{ij} < 0, b^-_{ij} = b_{ij}$ else $b^-_{ij} = 0, 1 \leq i, j \leq n$, we have

$$\begin{gathered}(A^+XB^+, A^+X^lB^+, A^+X^rB^+) + (A^-XB^+, -A^-X^rB^+, -A^-X^lB^+) \\ +(A^+XB^-, -A^+X^rB^-, -A^+X^lB^-) + (A^-XB^-, A^-X^lB^-, A^-X^rB^-) = \\ (A^+XB^+ + A^-XB^+ + A^+XB^- + A^-XB^-, A^+X^lB^+ - A^-X^rB^+ - A^+X^rB^- \\ +A^-X^lB^-, A^+X^rB^+ - A^-X^lB^+ - A^+X^lB^- + A^-X^rB^-) = (C, C^l, C^r).\end{gathered}$$

Thus we get

$$\begin{cases} A^+XB^+ + A^-XB^+ + A^+XB^- + A^-XB^- = C, \\ A^+X^lB^+ - A^-X^rB^+ - A^+X^rB^- + A^-X^lB^- = C^l, \\ A^+X^rB^+ - A^-X^lB^+ - A^+X^lB^- + A^-X^rB^- = C^r. \end{cases} \tag{12}$$

Denoting (12) in matrix form, we have

$$\begin{cases} (A^+ + A^-)X(B^+ + B^-) = C, \\ \left(A^+X^l, A^+X^r\right)\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} + \left(-A^-X^r, -A^-X^l\right)\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \left(C^l, C^r\right), \end{cases}$$

or

$$\begin{cases} (A^+ + A^-)X(B^+ + B^-) = C, \\ A^+\left(X^l, X^r\right)\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} + A^-\left(X^l, X^r\right)\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \left(C^l, C^r\right), \end{cases}$$

i.e.,

$$\begin{cases} AXB = C, \\ A\left(X^l, X^r \right)\begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \left(C^l, C^r \right). \end{cases}$$

$\square$

In a similar way, we can obtain another model for solving the equation(5).

Theorem 3.4. *The fuzzy linear system* $A\widetilde{X}B = \widetilde{C}$ *can be extended into the following system of linear matrix equations*

$$\left\{ \begin{array}{c} AXB = C, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} X^l \\ X^r \end{pmatrix} B = \begin{pmatrix} C^l \\ C^r \end{pmatrix}, \end{array} \right. \tag{13}$$

where

$$\widetilde{X} = (X, X^l, X^r), \widetilde{C} = (C, C^l, C^r).$$

And the elements a^+_{ij} *of matrix* A^+ *and* a^-_{ij} *of matrix* A^- *are determined by the following way: if* $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ *else* $a^+_{ij} = 0, 1 \leq i, j \leq n$*; if* $a_{ij} < 0, a^-_{ij} = a_{ij}$ *else* $a^-_{ij} = 0, 1 \leq i, j \leq n$.

Proof. Let $\widetilde{C} = (C, C^l, C^r) = (c_{ij}, c^l_{ij}, c^r_{ij})_{n\times n}$, $\widetilde{X} = (X, X^l, X^r) = (x_{ij}, x^l_{ij}, x^r_{ij})_{n\times n}$. We also let $B = B^+ + B^-$ where the elements b^+_{ij} of matrix B^+ and b^-_{ij} of matrix B^- are determined by the way: if $b_{ij} \geq 0, b^+_{ij} = b_{ij}$ else $b^+_{ij} = 0, 1 \leq i, j \leq n$; if $b_{ij} < 0, b^-_{ij} = b_{ij}$ else $b^-_{ij} = 0, 1 \leq i, j \leq n$.

For fuzzy matrix equation $A\widetilde{X}B = \widetilde{C}$, we can express it as

$$A(X, X^l, X^r)(B^+ + B^-) = (C, C^l, C^r). \tag{14}$$

Since

$$\widetilde{x}_{ij}k = \left\{ \begin{array}{ll} (kx_{ij}, kx^l_{ij}, kx^r_{ij}), & k \geq 0, \\ (kx_{ij}, -kx^r_{ij}, -kx^l_{ij}), & k < 0 \end{array} \right. ,$$

we have

$$\widetilde{X}B = \left\{ \begin{array}{ll} (XB, X^lB, X^rB), & B \geq 0, \\ (XB, -X^rB, -X^lB), & B < 0 \end{array} \right.$$

So the equation(14) can be rewritten as

$$A(X, X^l, X^r)B^+ + A(X, X^l, X^r)B^- = A(XB^+, X^lB^+, X^rB^+) + A(XB^-, -X^rB^-, -X^lB^-).$$

Since

$$k\widetilde{x}_{ij} = \left\{ \begin{array}{ll} (kx_{ij}, kx^l_{ij}, kx^r_{ij}), & k \geq 0, \\ (kx_{ij}, -kx^r_{ij}, -kx^l_{ij}), & k < 0, \end{array} \right.$$

we have

$$A\widetilde{X} = \left\{ \begin{array}{ll} (AX, AX^l, AX^r), & A \geq 0, \\ (AX, -AX^r, -AX^l), & A < 0. \end{array} \right.$$

Suppose $A = A^+ + A^-$, in which the elements a^+_{ij} of matrix A^+ and a^-_{ij} of matrix A^- are determined by the way: if $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ else $a^+_{ij} = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a^-_{ij} = a_{ij}$ else $a^-_{ij} = 0, 1 \leq i, j \leq n$, we have

$$\begin{aligned} &(A^+XB^+, A^+X^lB^+, A^+X^rB^+) + (A^+XB^-, -A^+X^rB^-, -A^+X^lB^-) \\ &+ (A^-XB^+, -A^-X^rB^+, -A^-X^lB^+) + (A^-XB^-, A^-X^lB^-, A^-X^rB^-) \\ &= (A^+XB^+ + A^-XB^+ + A^+XB^- + A^-XB^-, A^+X^lB^+ - A^-X^rB^+ - A^+X^rB^- \\ &+ A^-X^lB^-, A^+X^rB^+ - A^-X^lB^+ - A^+X^lB^- + A^-X^rB^-) = (C, C^l, C^r). \end{aligned}$$

Thus we get

$$\left\{ \begin{array}{c} A^+XB^+ + A^-XB^+ + A^+XB^- + A^-XB^- = C, \\ A^+X^lB^+ - A^-X^rB^+ - A^+X^rB^- + A^-X^lB^- = C^l, \\ A^+X^rB^+ - A^-X^lB^+ - A^+X^lB^- + A^-X^rB^- = C^r, \end{array} \right. \tag{15}$$

which is the same as the equation(12).

Denoting (15) in matrix form, we have

$$\left\{ \begin{array}{c} (A^+ + A^-)X(B^+ + B^-) = C, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} X^lB^+ \\ X^rB^+ \end{pmatrix} + \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} -X^rB^- \\ -X^lB^- \end{pmatrix} = \begin{pmatrix} C^l \\ C^r \end{pmatrix}, \end{array} \right.$$

or

$$\begin{cases} (A^+ + A^-)X(B^+ + B^-) = C, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} X^l \\ X^r \end{pmatrix} B^+ + \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} X^l \\ X^r \end{pmatrix} B^- = \begin{pmatrix} C^l \\ C^r \end{pmatrix}, \end{cases}$$

i.e.,

$$\begin{cases} AXB = C, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} X^l \\ X^r \end{pmatrix} B = \begin{pmatrix} C^l \\ C^r \end{pmatrix}. \end{cases} .$$

□

By the matrix theory[13], we can prove the following result.

Theorem 3.5. *Let S belong to $R^{m\times n}$, T belong to $R^{p\times q}$ and C belong to $R^{m\times q}$. Then the minimal solution X^* of the matrix equation $SXT = C$ is expressed by*

$$X^* = S^\dagger C T^\dagger.$$

Proof. Let $Y = XT$, the matrix equation $SXT = C$ is expressed by $SY = C$. We can verify that the minimal solution Y^* of the matrix equation $SY = C$ is expressed by $Y^* = S^\dagger C$.

In fact, we write the matrix equation $SX = C$ in block form of matrix as follows:

$$SY_j = C_j, j = 1, 2, \dots, q$$

where Y_j, C_j are the jth of matrices Y, C respectively.

By the matrix theory, we know that the matrix equation $SY = C$ is consistent if and only if each linear equation $SY_j = C_j, j = 1, 2, \dots, q$ is consistent, and the matrix equation $SY = C$ is inconsistent if and only if at least one of linear equations $SY_j = C_j, j = 1, 2, \dots, q$ is inconsistent.

When $SY = C$ is consistent, the result that $Y^* = S^\dagger C$ is its minimal solution is straightforward.

When $SY = C$ is inconsistent, the expression

$$\|C - SY^*\|_F^2 = \min \|C - SY\|_F^2$$

holds is equivalent to the following conditions:

$$\|C - SY_j^*\|_F^2 = \min \|C_j - SY_j\|_F^2, j = 1, \dots, q$$

where $Y^* = [X_1^*, \dots, Y_q^*]$ is the least squares solution of the matrix equation $SY = C$ and Y_j^* is the least squares solution of linear equation $SY_j = C_j, j = 1, 2, \dots, q$ and

$$\|C - SY\|_F^2 = \sum_{j=1}^{q} \|C_j - SY_j\|_F^2 = \sum_{j=1}^{q}\sum_{i=1}^{m} \|c_{ij} - \sum_{k=1}^{p} s_{ik}y_{kj}\|_F^2.$$

From above analysis, we know that Y_j^* is the minimal solution of inconsistent linear equation $SY_j = C_j, j = 1, 2, \dots, q$ is equivalent to that $Y^* = [Y_1^*, \dots, Y_q^*]$ is the minimal solution of the inconsistent matrix equation $SY = C$. So we know that the minimal solution of the matrix equation $SY = C$ is expressed uniformly by $Y^* = S^\dagger C$.

On the other hand, we regard the matrix equation $XT = Y$ with $T^\top X^\top = Y^\top$ and denote it as $\overline{TX} = \overline{Y}$.

By the above analysis, we obtain that the minimal solution of the matrix equation $\overline{TX} = \overline{Y}$ is expressed uniformly by $\overline{X} = \overline{T}^\dagger \overline{Y}$, i.e., $X^* = YT^\dagger$. It means $X^* = YT^\dagger$ is the minimal solution of the matrix equation $XT = Y$.

Thus we prove that the minimal solution X^* of the matrix equation $SXT = C$ is expressed by $X^* = S^\dagger C T^\dagger$.

In order to solve the fuzzy matrix equation (5), we need to consider the systems of linear equations (10) or (13). It seems that we have obtained the minimal solutions of the linear system (10) and (13) as

$$\begin{cases} X = A^\dagger C B^\dagger, \\ \begin{pmatrix} X^l, X^r \end{pmatrix} = A^\dagger \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger \end{cases} \tag{16}$$

and

$$\begin{cases} X = A^\dagger C B^\dagger, \\ \begin{pmatrix} X^l \\ X^r \end{pmatrix} = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}^\dagger \begin{pmatrix} C^l \\ C^r \end{pmatrix} B^\dagger, \end{cases} \tag{17}$$

where $(.)^\dagger$ is the Moore-Penrose generalized inverse of matrix $(.)$. □

However, the solution matrix may still not be an appropriate LR fuzzy numbers matrix except for $X^l \geq O, X^r \geq O$. Now we give the definition of LR fuzzy solution to the equation(5) as follows:

Definition 3.6. *Let $\widetilde{X} = (X, X^l, X^r)$. If (X, X^l, X^r) is the minimal solution of equation(10) or (13), such that $X^l \geq O, X^r \geq O$, we call $\widetilde{X} = (X, X^l, X^r)$ is a strong LR fuzzy minimal solution of fuzzy matrix equation(5). Otherwise, $\widetilde{X} = (X, X^l, X^r)$ is said a weak LR fuzzy minimal solution of fuzzy matrix equation(5) given by*

$$\widetilde{X} = \widetilde{x_{ij}}$$

where

$$\widetilde{x}_{ij} = \begin{cases} (x_{ij}, x_{ij}^l, x_{ij}^r), & x_{ij}^l > 0, \quad x_{ij}^r > 0, \\ (x_{ij}, 0, \max\{-x_{ij}^l, x_{ij}^r\}), & x_{ij}^l < 0, \quad x_{ij}^r > 0, \\ (x_{ij}, \max\{x_{ij}^l, -x_{ij}^r\}, 0,), & x_{ij}^l > 0, \quad x_{ij}^r < 0, \\ (x_{ij}, -x_{ij}^l, -x_{ij}^r), & x_{ij}^l < 0, \quad x_{ij}^r < 0. \end{cases} \quad i, j = 1, \ldots, n. \tag{18}$$

To illustrate the expression (16) or (17) to be a LR fuzzy solution matrix, we now discuss the generalized inverse of non negative matrix

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}$$

in a special structure.

Lemma 3.7. [13] *Let*

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}.$$

Then the matrix

$$S^\dagger = \begin{pmatrix} (A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger & (A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger \\ (A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger & (A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger \end{pmatrix} \tag{19}$$

is the Moore-Penrose inverse of the matrix S, where $(A^+ + A^-)^\dagger, (A^+ - A^-)^\dagger$ are the Moore-Penrose inverse of matrices $A^+ + A^-$ and $A^+ - A^-$, respectively.

The key point to make the solution matrix a strong LR fuzzy solution is that $\widetilde{X} = (X, X^l, X^r)$ is LR fuzzy matrix, i.e., each element is a LR fuzzy number. By the following analysis, a sufficient condition is that $S^\dagger \geq O, B^\dagger \geq O$ or $A^\dagger \geq O, T^\dagger \geq O$..

Theorem 3.8. *If*

$$A^\dagger \geq O,$$
$$(B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger) \geq O, (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger) \geq O,$$

the fuzzy matrix equation (2.5) has a strong LR fuzzy minimal solution as follows:

$$\widetilde{X} = (X, X^l, X^r)$$

where

$$\begin{cases} X = A^\dagger C B^\dagger, \\ X^l = A^\dagger C^l E + A^\dagger C^r F, \\ X^r = A^\dagger C^l F + A^\dagger C^r E, \\ E = \frac{1}{2}((B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger), \\ F = \frac{1}{2}((B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger). \end{cases} \tag{20}$$

Proof. Since C^l and C^r are the left and right spreads fuzzy matrix $\widetilde{C}$, $C^l \geq O$ and $C^r \geq O$. It means $\begin{pmatrix} C^l, C^r \end{pmatrix}$ is a non negative matrix.

Let

$$S^\dagger = \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger & (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger \\ (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger & (B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger \end{pmatrix}.$$

We know the condition that $S^\dagger \geq 0$ is equivalent to $E \geq 0$ and $F \geq 0$.

Now that $E \geq O$ and $F \geq O$, the product of three non negative matrices

$$\begin{pmatrix} X^l, X^r \end{pmatrix} = A^\dagger \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger = A^\dagger \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \begin{pmatrix} A^\dagger C^l E + A^\dagger C^r F, A^\dagger C^l F + A^\dagger C^r E \end{pmatrix} \geq O$$

is non negative in nature. It means that $X^l \geq O$ and $X^r \geq O$. □

For the model equation(13), we have the following result by the same analysis.

Theorem 3.9. *If*

$$B^{\dagger} \geq O,$$
$$(A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}) \geq O, (A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger}) \geq O,$$

the fuzzy matrix equation (2.5) *has a strong LR fuzzy minimal solution as follows:*

$$\widetilde{X}=(X, X^{l}, X^{r})$$

where

$$\left\{\begin{array}{c} X=A^{\dagger}CB^{\dagger}, \\ X^{l}=EC^{l}B^{\dagger}+FC^{r}B^{\dagger}, \\ X^{r}=FC^{l}B^{\dagger}+EC^{r}B^{\dagger}, \\ E=\frac{1}{2}((A^{+}-A^{-})^{\dagger}+(A^{+}+A^{-})^{\dagger}), \\ F=\frac{1}{2}((A^{+}-A^{-})^{\dagger}-(A^{+}+A^{-})^{\dagger}). \end{array}\right. \tag{21}$$

Proof. The proof is straight forward. □

The following Theorems give some results for such S^{-1} and $S^{\dagger}$ to be nonnegative. As usual, $(.)^{\top}$ denotes the transpose of a matrix (.).

Theorem 3.10. [36] *The inverse S^{-1} of a nonnegative matrix S is nonnegative if and only if S is a generalized permutation matrix.*

Theorem 3.11. [14] *Let S be an $2n \times 2n$ nonnegative matrix with rank r. Then the following assertions are equivalent:*
(a). $S^{\dagger} \geq 0$.
(b). There exists a permutation matrix P, such that PS has the form

$$PS=\left(\begin{array}{c} T_{1} \\ T_{2} \\ \vdots \\ T_{r} \\ O \end{array}\right),$$

where each T_i has rank 1 and the rows of T_i are orthogonal to the rows of T_j, whenever $i \neq j$, the zero matrix may be absent.
(c). $S^{\dagger}=\left(\begin{array}{cc} GC^{\top} & GD^{\top} \\ GD^{\top} & GC^{\top} \end{array}\right)$ for some positive diagonal matrix G. In this case,

$$(C+D)^{\dagger}=G(C+D)^{\top}, (C-D)^{\dagger}=G(C-D)^{\top}.$$

4 Numerical examples

Example 4.1. *Consider the following fuzzy linear matrix system*

$$\left(\begin{array}{cc} 1 & -1 \\ 0 & 2 \end{array}\right)\left(\begin{array}{cc} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{array}\right)\left(\begin{array}{cc} -3 & 4 \\ 1 & 0 \end{array}\right)=\left(\begin{array}{cc} (3,2,1)_{LR} & (4,1,1)_{LR} \\ (5,2,2)_{LR} & (3,1,2)_{LR} \end{array}\right).$$

By the Theorem 3.1., *the original fuzzy matrix equation is extended into the following a system of linear matrix equations*(10)

$$\left\{\begin{array}{c} AXB=C, \\ A\left(\begin{array}{cc} X^{l}, X^{r} \end{array}\right)\left(\begin{array}{cc} B^{+} & -B^{-} \\ -B^{-} & B^{+} \end{array}\right)=\left(\begin{array}{cc} C^{l}, C^{r} \end{array}\right), \end{array}\right.$$

where

$$\widetilde{X}=(X, X^{l}, X^{r}), \widetilde{C}=(C, C^{l}, C^{r})$$

and

$$B^{+}=\left(\begin{array}{cc} 0 & 4 \\ 1 & 0 \end{array}\right), B^{-}=\left(\begin{array}{cc} -3 & 0 \\ 0 & 0 \end{array}\right), C=\left(\begin{array}{cc} 3 & 4 \\ 5 & 3 \end{array}\right), C^{l}=\left(\begin{array}{cc} 2 & 1 \\ 2 & 1 \end{array}\right), C^{r}=\left(\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\right).$$

From (16), *the solution of the computing model is*

$$X = A^{\dagger}CB^{\dagger} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{\dagger} \begin{pmatrix} 3 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} 1.3750 & 9.6250 \\ 0.3750 & 3.6250 \end{pmatrix}$$

and

$$\begin{pmatrix} X^l, X^r \end{pmatrix} = A^{\dagger} \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^{\dagger} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{\dagger} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\dagger}$$

$$= \begin{pmatrix} 0.3750 & 1.5000 & 0.5000 & 0.8750 \\ 0.1250 & 0.2500 & 0.2500 & 0.6250 \end{pmatrix}.$$

Since X^l, X^r *are nonnegative matrices and* $X - X^l > O$, *the solution we obtained is an appropriate LR fuzzy matrix*

$$\widetilde{X} = (X, X^l, X^r) = \left(\begin{pmatrix} 1.3750 & 9.6250 \\ 0.3750 & 3.6250 \end{pmatrix}, \begin{pmatrix} 0.3750 & 1.5000 \\ 0.1250 & 0.2500 \end{pmatrix}, \begin{pmatrix} 0.5000 & 0.8750 \\ 0.2500 & 0.6250 \end{pmatrix} \right)$$

i.e.,

$$\widetilde{X} = \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (1.3750, 0.3750, 0.5000)_{LR} & (9.6250, 1.5000, 0.8750)_{LR} \\ (0.3750, 0.1250, 0.2500)_{LR} & (3.6250, 0.2500, 0.6250)_{LR} \end{pmatrix},$$

which admits a nonnegative strong LR fuzzy solution of the original fuzzy matrix system.

Example 4.2. *Consider another fuzzy linear matrix system*

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} (2,2,1)_{LR} & (3,2,1)_{LR} \\ (3,1,1)_{LR} & (2,1,2)_{LR} \\ (1,1,1)_{LR} & (3,2,1)_{LR} \end{pmatrix}.$$

Suppose

$$\widetilde{X} = (X, X^l, X^r),$$

$$A = A^+ + A^- = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

$$B = B^+ + B^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and

$$\widetilde{C} = (C, C^l, C^r) = \left(\begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \right).$$

By the Theorem 3.1., *the original fuzzy matrix equation is extended into the following a system of linear matrix equations*(10)

$$\begin{cases} AXB = C, \\ A\begin{pmatrix} X^l, X^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} C^l, C^r \end{pmatrix}. \end{cases}$$

From (16), *the solution of the computing model is*

$$X = A^{\dagger}CB^{\dagger} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^{\dagger} = \begin{pmatrix} 0.0179 & -0.0179 \\ -0.3393 & 0.3393 \end{pmatrix}$$

and

$$\begin{pmatrix} X^l, X^r \end{pmatrix} = A^{\dagger} \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^{\dagger} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{\dagger}$$

$$= \begin{pmatrix} 0.4464 & 0.3571 & 0.3571 & 0.4464 \\ 0.0179 & 0.2143 & 0.2143 & 0.0179 \end{pmatrix}.$$

Since X^l, X^r are nonnegative matrices, the solution we obtained is an appropriate LR fuzzy matrix

$$\widetilde{X} = (X, X^l, X^r) = \left(\begin{pmatrix} 0.0179 & -0.0179 \\ -0.3393 & 0.3393 \end{pmatrix}, \begin{pmatrix} 0.4464 & 0.3571 \\ 0.0179 & 0.2143 \end{pmatrix}, \begin{pmatrix} 0.3571 & 0.4464 \\ 0.2143 & 0.0179 \end{pmatrix} \right),$$

i.e.,

$$\widetilde{X} = \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (0.0179, 0.4464, 0.3571)_{LR} & (-0.0179, 0.3571, 0.4464)_{LR} \\ (-0.3393, 0.0179, 0.2143)_{LR} & (0.3393, 0.4464, 0.0179)_{LR} \end{pmatrix},$$

which admits a strong LR fuzzy solution of the original fuzzy linear matrix equation.

5 Conclusions

In this work we presented a model for solving fuzzy linear matrix equations $A\widetilde{X}B = \widetilde{C}$ where A and B are crisp $m \times m$ and $n \times n$ matrices respectively and $\widetilde{C}$ is an $m \times n$ arbitrary LR fuzzy numbers matrix. The model was made of two crisp systems of linear equations which determined the mean value and the left and right spreads of the solution. The LR fuzzy minimal solution of the fuzzy linear matrix equation was derived from solving the crisp systems of linear equations. In addition, the existence condition of strong LR fuzzy solution was studied. Numerical examples showed that our method is feasible to solve this type of fuzzy matrix equations.

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