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Multi-granulation fuzzy probabilistic rough sets and their corresponding three-way decisions over two universes

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Abstract

This article introduces a general framework of multi-granulation fuzzy probabilistic rough sets (MG-FPRSs) models in multi-granulation fuzzy probabilistic approximation space over two universes. Four types of MG-FPRSs are established, by the four different conditional probabilities of fuzzy event. For different constraints on parameters, we obtain four kinds of each type MG-FPRSs over two universes. To find a suitable way of explaining and determining these parameters in each kind of each type MG-FPRS, three-way decisions (3WDs) are studied based on Bayesian minimum-risk procedure, i.e., the decision-theoretic rough set (DTRS) approach. The main contribution of this paper is twofold. One is to extend the fuzzy probabilistic rough set (FPRS) to MG-FPRS model over two universes. Another is to present an approach to select parameters in MG-FPRS modeling by using the process of decision-making under conditions of risk.

Keywords: Rough set, fuzzy event, multi-granulation fuzzy probabilistic rough set, three-way decisions.

1 Introduction

This article introduces the general framework of MG-FPRS models based on multi-granulation fuzzy probabilistic approximation space. The proposed models and general frameworks are established inspired by FPRS models as given Zhao et al. in [44] and a general framework of rough sets in a multi-granulation space as given Yao et al. in [36]. The FPRSs, as a combination of fuzzy set theory [37], rough set theory [17] and probability theory [5]. In fact, FPRSs are extension of probabilistic rough sets (PRSs) studied at length in literature's [13, 14, 15, 18, 23, 26, 33, 46, 47, 50] to fuzzy environment. The pair of fuzzy probabilistic approximation operators are built in terms of fuzzy conditional probabilities and parameters (representing to what degree we can bear the uncertainty or misclassification). When applying FPRSs to some certain situation, parameters, playing a key role in establishing fuzzy probabilistic approximation operators, are usually provided by experts who are familiar with that situation. Without the help of experts, it is not confirmed which way is most reasonable to choose the values of parameters. The solution of this problem, Yao et al. [32, 34] proposed the DTRS approach and provided a mathematic as well as systematic way to explain and calculate the parameters on the basis of losses or cost of various decisions. Thus the introduction of MG-FPRS models are less important without any mathematic and systematic way to explain and calculate the parameters. For this region we also introduce, multi-granulation fuzzy decision-theoretic rough set (MG-FDTRS) approach, i.e., the corresponding 3WDs of our models.

The method of granular computing is proposed by Zadeh [39], which is based on a single-granulation structure. Recently, rough set theory becomes a popular mathematical framework for granular computing. In this theory, concepts are expressed by upper and lower approximations induced by a single-granulation structure [19, 31], i.e., the concepts are depicted by known knowledge from a single binary relation on the universe. This kind of rough set models is called single-granulation rough set. Based on different requirements for users , Qian et al. [20] first proposed multi-granulation rough set (MGRS). It has a more widely application scope, for example, decision making, feature selection and so on [9, 28].

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Since for different requirements, a concept can be described by different multiple binary relations, many extensions of MGRSs have been proposed. For example, Qian et al. generalized classical MGRSs to neighborhood-based ones [7] and covering ones [8]. The neighborhood MGRSs are useful for hybrid data sets. Dou et al. [3] investigated variable precision MGRSs. Yao et al. [36] discussed rough set models in multi-granulation space. Qian et al. [21] discussed multi-granulation decision-theoretic rough sets (MG-DTRSs). The topological structures of MGRs were discussed by She et al. [22]. Li et al. [11] made a detailed comparison between MGRSs and concept lattices via rule acquisition.

However, the decision and the most of knowledge in the real life applications are often fuzzy and one often encounters a kind of special information system in which data come from different sources, such an information system is called multi-source information system. Thus, it is necessary to introduce the fuzzy rough methodology into the classical DTRS for wider applications. In literature's [1, 4, 12, 40, 41, 42] dealt with the real-valued data sets by applying a fuzzy rough technique to solve the problem. For example, Chen et al. [1] and Zhao et al. [42] used fuzzy rough sets to propose a novel method for attribute reduction and rule induction. Liang et al. [12] has proposed the triangular fuzzy decision-theoretic rough set by considering the losses being expressed by triangular fuzzy numbers. But they still can not be used to analyze data in the context of fuzzy multi-granulation, which limits its further applications of many problems under the framework of the fuzzy environment. To solve this problem, Lin et al. [10] proposed fuzzy multi-granulation decision-theoretic approach to multi-source fuzzy information system.

Furthermore, in the real world, we often encounter situations both probabilistic uncertainty and fuzziness coexist. The conventional fuzzy modeling or probabilistic techniques are individually not sufficient for representing such situations. The researchers [2, 24, 25, 27, 44] to handling this situations by applying fuzzy probabilistic rough technique. For example Deng et al. [2] proposed decision-theoretic three-way approximation of fuzzy set. They deal with fuzzy sets actually based on cut sets of fuzzy sets. Sun et al. [24] investigated rough approximation of a fuzzy concepts on a probabilistic approximation space. They established the model of probabilistic rough fuzzy sets based on crisp equivalence relation on the universe. Yang et al. [27] proposed FPRS model based on fuzzy relations. Even through the fuzzy relation was adopted in their model, it is the cut set of fuzzy relation that really works. That means it is based on classical relations. Using fuzzy set directly, Zhao et al. [44] first presented FPRS models in details. They also discussed 3WDs their models. Sun et al. [24] discussed fuzzy rough set on probabilistic approximation space over two universes and its application in the area of emergency decision-making. However, they used single-granulation structure for elaborating their models, which limits further applications in many problems under the multiple information sources. Thus it is necessary for us to study the MG-FPRS models. For instance some special information systems, such as multi-source information system, distributive information systems and group of intelligent agents, the single granulation FPRSs can not be used to data mining from these information systems, but MG-FPRSs can.

The remainder of this paper is organized as follows. Section 2 reviews the basic notions of fuzzy set, fuzzy event, etc. Section 3, propose the general framework of MG-FPRS models. The MG-FPRSs and their three-way decisions are studied in Section 4. Brief examples are attached in Section 4 to illustrate our models. The differences between our study and the existing ones concerned in Section 5. Section 6 concludes the paper.

2 Fuzzy sets and probability theory

Let U be a universe of discourse. A fuzzy set A is a mapping from U into $[0,1]$, i.e., $A : U \to [0,1], x \mapsto A(x) \in [0,1]$, $\forall x \in U$ [6,37]. The family of all fuzzy sets on U is denoted by $F(U)$. Given a fuzzy set $A \in F(U)$ and $\alpha \in [0,1]$, the α-cut set of A is defined as $A_\alpha = \{x \in U : A(x) \geq \alpha\}$, which is a classical subset of U. The operations on fuzzy sets are defined as follows: for all $A, B \in F(U)$ and $x \in U$,
$(AB)(x) = A(x)B(x)$, $(A \cap B)(x) = A(x) \wedge B(x)$, $(A \cup B)(x) = A(x) \vee B(x)$, $A^c(x) = 1 - A(x)$.
The order of fuzzy sets are defined by the natural order of real numbers, i.e., for fuzzy sets $A, B \in F(U), A \subseteq B$ iff $A(x) \leq B(x), \forall x \in U$. Given two universes of discourse U and W, let R be a mapping, i.e., $R : U \times W \to [0,1]$. Then R is a fuzzy relation from U to W. If $U = W$, then R is a fuzzy relation on U, i.e., $R \in F(U \times U)$.

Definition 2.1. [38] *Let $(U, \mathcal{A}, P)$ be a probability space in which U is a sample space of events, $\mathcal{A}$ is a σ-algebra of events (i.e., the measurable subsets of U) and P is a probability measure defined on $\mathcal{A}$. For a fuzzy set $A \in F(U)$, if $A \in \zeta(\mathcal{A}) = \{A \in F(U) : A_\alpha \in \mathcal{A}, \alpha \in [0,1]\}$ then A is a fuzzy event on U. The probability of fuzzy event A (called the fuzzy probability of A) is $P(A) = \int_U A(x)dP$.*

If $U = \{x_1, x_2, \cdots, x_n\}$, and $p_i = P(\{x_i\})(i = 1, 2, \cdots, n)$, then

$$P(A) = \sum_{i=1}^{n} A(x_i)p_i.$$

(1)

In general, the membership function of the fuzzy set A can be obtained by using the existing definition for the given concept (or fuzzy event) or established by the experts.

In [38], Zadeh pointed out that, the independence of two fuzzy events, A and B, are defined as $P(AB) = P(A)P(B)$, which means that joint probability of two fuzzy events is defined based on the algebraic product of these two fuzzy events instead of the standard intersection of them. Therefore the conditional probability is defined as follows.

Definition 2.2. [38] *Let $(U, \mathcal{A}, P)$ be a probability space and A, B be two fuzzy events on U. If $P(B) \neq 0$ then the conditional probability of A given B (called fuzzy conditional probability) is defined as* $P(A \mid B) = \dfrac{P(AB)}{P(B)}$.

If $U = \{x_1, x_2, \cdots, x_n\}$, and $p_i = P(\{x_i\})(i = 1, 2, \cdots, n)$, then

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\sum_{i=1}^{n} A(x_i)B(x_i)p_i}{\sum_{i=1}^{n} B(x_i)p_i} \tag{2}$$

for all fuzzy events $A, B \in \zeta(\mathcal{A})$.

Proposition 2.3. [43] *Let $(U, \mathcal{A}, P)$ be a probability space and A, B, C be three fuzzy events on U. If $P(A) \neq 0$ then the following conclusions hold.*
(1) $P(\emptyset \mid A) = 0, P(U \mid A) = 1$;
(2) If $B \subseteq C$, then $P(B) \leq P(C)$ and $P(B \mid A) \leq P(C \mid A)$;
(3) $P(B \mid A) + P(B^c \mid A) = 1$.

3 Models of MG-FPRSs over two universes

In this section, first we give multi-granulation fuzzy probabilistic approximation space over two universes. Then classify four type of MG-FPRSs models based on this approximation space.

Definition 3.1. [16] *Let U and W be two non-empty finite universes. $R_k (1 \leq k \leq m)$ are arbitrary m fuzzy relations on $U \times W$. P is a probability measure defined on the σ-algebra formed by the image (that is, the subset classes of the universe W) of element $x (x \in U)$. Then, we call $(U, W, R_k (1 \leq k \leq m), P)$ a multi-granulation fuzzy probabilistic approximation space over two universes.*

Remark 3.2. *In Definition 3.1, the multi-granulation fuzzy probabilistic approximation space over two universes become the fuzzy probabilistic approximation space over two universes as given in [25] and the generalized fuzzy probabilistic approximation space as given in [44] if multi-granulation fuzzy probabilistic approximation space over two universes degenerates a single granulation, i.e., $m = 1$.*

Definition 3.3. [16] *Let $R_k (1 \leq k \leq m)$ be arbitrary m fuzzy relations on $U \times W$. For each $x \in U$, two fuzzy sets $[x]_{\cap_{k=1}^{m} R_k}$ and $[x]_{\cup_{k=1}^{m} R_k}$, for all $y \in W$, are defined as follows:*

$$[x]_{\cap_{k=1}^{m} R_k}(y) = \wedge_{k=1}^{m} R_k(x, y) \text{ and } [x]_{\cup_{k=1}^{m} R_k}(y) = \vee_{k=1}^{m} R_k(x, y)$$

Now we are ready to propose a common framework of four type MG-FPRSs models over two universes. This models are present by Figure 1. The first two models, called Type-I multi-granulation fuzzy probabilistic rough set (Type-I MG-FPRS) and Type-II multi-granulation fuzzy probabilistic rough set (Type-II MG-FPRS), are a class based on combination of relation first and then construction of approximations, as demonstrated in the upper half Figure 1. The next two models, called Type-III multi-granulation fuzzy probabilistic rough set (Type-III MG-FPRS) and Type-IV multi-granulation fuzzy probabilistic rough set (Type-IV MG-FPRS), are another class that used the reverse order, as demonstrated in the lower half Figure 1.

In the next section, we only consider Type-I MG-FPRSs model and their corresponding 3WDs, based on the multi-granulation fuzzy probabilistic approximation space over two universes.

4 Type-I MG-FPRS models and their corresponding 3WDs over two universes

The four kinds of Type-I MG-FPRSs in terms of four groups of parameters are here mainly discussed in this section. We study DTRSs approach for fuzzy events within the framework of multi-granulation fuzzy probabilistic approximation spaces. To provided a sensible way to determine the values of parameters or to determine which kind of Type-I MG-FPRSs is to be adopted.

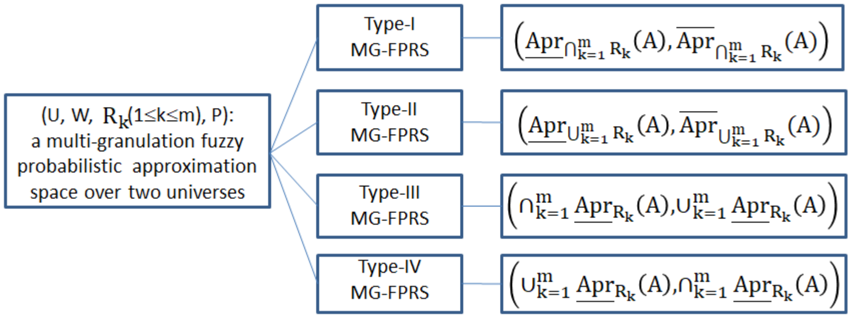

Figure 1: A general framework of MG-FPRSs models in a multi-granulation fuzzy probabilistic approximation space over two universes.

4.1 Four kinds of Type-I MG-FPRSs over two universes

The aim of proposing Type-I MG-FPRSs is to characterize fuzzy events in terms of the available knowledge represented by arbitrary m fuzzy relations.

Definition 4.1. *Let $(U, W, R_k(1 \leq k \leq m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A \in F(W)$ be a fuzzy event. For a pair of parameters α and β with $0 \leq \beta < P(A) < \alpha \leq 1$, the Type-I α-multi-granulation fuzzy probabilistic lower approximation and Type-I β-multi-granulation fuzzy probabilistic upper approximation of A are defined, respectively, as follows:*

$$\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\}, \quad \overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > \beta\}.$$

The pair $(\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A), \overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A))$, is called Type-I (α, β)-multi-granulation fuzzy probabilistic rough set (Type-I (α, β)-MG-FPRS) over two universes of A.

In the aforementioned definition of the lower and upper approximations of A with respect to $(U, W, R_k(1 \leq k \leq m), P)$, the parameters α and β can be viewed as the least thresholds given in advance by the decision-makers. Moreover, the value of parameters α and β are selected by the decision-makers according to the requirements of the problem in reality.
Furthermore, we can also present the positive, negative and boundary regions of A with respect to $(U, W, R_k(1 \leq k \leq m), P)$ and the parameters α and β as follows, respectively.

$$POS^{\alpha}(A) = \underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\},$$
$$NEG^{\beta}(A) = (\overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \leq \beta\},$$
$$BND^{(\alpha,\beta)}(A) = \overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A) - \underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : \beta < P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < \alpha\}.$$

Remark 4.2. *(1) If $W = \{y_1, y_2, \cdots, y_n\}$, and $P(y_i) = p_i (i = 1, 2, \cdots, n)$, then it follows from (2) that*

$$P(A \mid [x]_{\cap_{k=1}^{m} R_k}) = \frac{\sum_{i=1}^{n} (\wedge_{k=1}^{m} R_k(x, y_i)) A(y_i) p_i}{\sum_{i=1}^{n} (\wedge_{k=1}^{m} R_k(x, y_i)) p_i}, \forall x \in U. \tag{3}$$

(2) In special case of $m = 1$, the Type-I (α, β)-MG-FPRS is called Type-I (α, β)-single-granulation fuzzy probabilistic rough set (Type-I (α, β)-SG-FPRS) over two universes.
(3) When $m = 1$, $U = W$ and R_k is a fuzzy $\top$-equivalence relation ($\top$ is a t-norm) then the Type-I (α, β)-SG-FPRS degenerates into the (α, β)-FPRS as given in [44].
(4) When $m = 1$, $U = W$, R_k is a classical equivalence relation and A is a classical event then the Type-I (α, β)-SG-FPRS degenerates into the (α, β)-PRS as given in [49].

Proposition 4.3. *Let $(U, W, R_k(1 \le k \le m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A, B \in F(W)$ with $P(A) \neq 0$. The following properties hold.*
(1) If $0 \le \beta < P(A) < \alpha \le 1$, then $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(A)$.
(2) $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(\emptyset) = \emptyset$, $\overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(W) = U$ for all $\alpha \in (0,1]$ and $\beta \in [0,1)$.
(3) If $A \subseteq B$ and $0 \le \beta < min\{P(A), P(B)\}$, $max\{P(A), P(B)\} < \alpha \le 1$, then $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(B)$ and $\overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(B)$.
(4) If $P(A) < \alpha_1 \le \alpha_2 \le 1$ and $0 \le \beta_1 \le \beta_2 < P(A)$, then $\underline{Apr}^{\alpha_2}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha_1}_{\cap_{k=1}^m R_k}(A)$ and $\overline{Apr}^{\beta_2}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta_1}_{\cap_{k=1}^m R_k}(A)$.

Proof. If $x \in \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A)$, $\forall x \in U$, then $P(A \mid [x]_{\cap_{k=1}^m R_k}) \ge \alpha > \beta$. It shows that $x \in \overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(A)$. Therefore, we can prove $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(A)$.
(2) From Proposition 2.3(1), we have $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(\emptyset) = \{x \in U : P(\emptyset \mid [x]_{\cap_{k=1}^m R_k}) \ge \alpha\} = \{x \in U : 0 \ge \alpha\} = \emptyset$, $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(W) = \{x \in U : P(W \mid [x]_{\cap_{k=1}^m R_k}) > \beta\} = \{x \in U : 1 > \beta\} = U$.
(3) If $x \in \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A)$, $\forall x \in U$, then $P(A \mid [x]_{\cap_{k=1}^m R_k}) \ge \alpha$. Since $A \subseteq B$, then from Proposition 2.3(2), we have $P(A \mid [x]_{\cap_{k=1}^m R_k}) \le P(B \mid [x]_{\cap_{k=1}^m R_k})$. Therefore, we have $P(B \mid [x]_{\cap_{k=1}^m R_k}) \ge \alpha$. It shows that $x \in \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(B)$. Therefore, we can prove $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(B)$. Similarly, we can prove $\overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta}_{\cap_{k=1}^m R_k}(B)$.
(4) If $x \in \underline{Apr}^{\alpha_2}_{\cap_{k=1}^m R_k}(A)$, $\forall x \in U$, then $P(A \mid [x]_{\cap_{k=1}^m R_k}) \ge \alpha_2 \ge \alpha_1$. It shows that $x \in \underline{Apr}^{\alpha_1}_{\cap_{k=1}^m R_k}(A)$. Therefore, we can prove $\underline{Apr}^{\alpha_2}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha_1}_{\cap_{k=1}^m R_k}(A)$. Similarly, we can prove $\overline{Apr}^{\beta_2}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\beta_1}_{\cap_{k=1}^m R_k}(A)$. □

Now, for the sake of illustration, we consider the following emergency decision-making problem as given in [25].

Example 4.4. *Consider an emergency decision-making problem during an unconventional emergency event such as an earthquake or flood. Suppose that there occurs an earthquake in some area. The area affected by the earthquake can be divided into different disaster areas according to some criteria given in advance. The important factors such as the affected population, economic loss and number of destroyed facilities can be presented according to the general characteristics of an earthquake. Then, one can obtain the degree of the relationship between these factors and specific disaster areas by looking at historical records about past earthquakes. Thus, one can present a quantitative description using the numerical values of general factors based on available information to reflect the degree of the disaster if an earthquake should affect these areas. The quantitative description of these general factors will be the sole available information for emergency decision-making when a new earthquake occurs. Since the incomplete available information and the possible existence of random available information are the most basic feature of unconventional emergency events. The incomplete available information will be fuzzy and inaccurate and often is qualitative rather than quantitative. The possible existence of random available information will be probability distribution rather than weight. Meanwhile, decision-makers have to make quick decisions in response to unconventional emergency events based on the fuzzy, inaccurate, qualitative and existence of random information as available. Let $U = \{x_1, x_2, \cdots, x_{10}\}$ be the set of disaster areas for an actual unconventional emergency event. The specific disaster areas $x_i(i = 1, 2, \cdots, 10)$ could be divided according to the administrative district or the space-time distribution of geography. That is, $x_1, x_2, \cdots, x_{10}$ are the places which need rescue quickly. $W = \{y_1, y_2, \cdots, y_{10}\}$ be a set of general characteristic factors describing the unconventional emergency event, such as the affected population, economic loss and number of destroyed facilities. The fuzzy relations, $R_k(x_i, y_j) \in F(U \times W)(1 \le k \le 5)$ shows the degree of specific disaster areas $x_i(i = 1, 2, \cdots, 10)$ and the general characteristics of an earthquake, factors $y_j(j = 1, 2, \cdots, 10)$ to last consecutive five years historical records about similar emergencies in this area, are shown in Tables 1 to 5. Broadly speaking, each factor that characterizes an unconventional emergency event can be seen as a profitable index, which means that the more important that factor is, the higher is the value assigned to it. So, the higher the value of $R_k(x_i, y_j)$, the more important the characteristic y_j for the specific disaster areas x_i. Let fuzzy set A be the quantitative description of all the characteristic factors $y_j(j = 1, 2, \cdots, 10)$ of unconventional emergency events according to the available inaccurate and insufficient information. It is easy to know that A is the fuzzy set of universe W, and the membership function is given as follows:*

$$A = \frac{0.8}{y_1} + \frac{0.6}{y_2} + \frac{0.25}{y_3} + \frac{0.84}{y_4} + \frac{0.6}{y_5} + \frac{0.6}{y_6} + \frac{0.82}{y_7} + \frac{0.75}{y_8} + \frac{0.25}{y_9} + \frac{0.6}{y_{10}}.$$

The membership function represents the fuzzy event that the area is seriously effected. The quantitative description of all the characteristic factors $y_j(j = 1, 2, \cdots, 10)$ according to the possible existence of random available information *is given by* $\{0.15, 0.08, 0.035, 0.16, 0.08, 0.08, 0.1, 0.15, 0.035, 0.13\}$*, which is the actually probability distribution on* W*. This type of information we can not avoid for better judgement of optimal areas. The computation of* $\wedge_{k=1}^{5} R_k(x_i, y_j)$

$R_1(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.70	0.35	0.75	0.80	0.40	0.60	0.65	0.45	0.40	0.35
x_2	0.50	0.40	0.90	0.75	0.35	0.55	0.75	0.65	0.70	0.45
x_3	0.90	0.70	0.75	0.60	0.75	0.75	0.60	0.70	0.65	0.35
x_4	0.75	0.60	0.65	0.40	0.55	0.80	0.35	0.85	0.45	0.40
x_5	0.70	0.95	0.45	0.60	0.40	0.40	0.85	0.45	0.60	0.80
x_6	0.90	1.00	0.50	0.55	0.85	0.70	0.90	0.35	0.50	0.65
x_7	0.45	0.70	0.85	0.80	0.80	0.80	0.75	0.65	0.55	0.60
x_8	0.80	0.75	0.65	0.60	0.75	0.70	0.40	0.40	0.25	0.45
x_9	0.85	0.60	0.40	0.45	0.70	0.80	0.75	0.35	0.85	0.35
x_{10}	0.65	0.90	0.55	0.80	0.60	0.60	0.35	0.80	0.75	0.60

Table 1: The fuzzy relation R_1

$R_2(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.60	0.45	0.75	0.45	0.65	0.60	0.50	0.60	0.65	0.85
x_2	0.65	0.35	0.60	0.50	0.75	0.70	0.35	0.50	0.75	0.70
x_3	0.70	0.65	0.35	0.40	0.45	0.35	0.50	0.45	0.70	0.65
x_4	0.80	0.55	0.40	0.30	0.35	0.45	0.45	0.35	0.40	0.70
x_5	0.45	0.60	0.75	0.45	0.85	0.80	0.75	0.75	0.35	0.45
x_6	0.75	0.75	0.30	0.25	0.45	0.90	0.65	0.60	0.65	0.35
x_7	0.65	0.90	0.75	0.60	0.40	0.75	0.40	0.85	0.70	0.25
x_8	0.60	0.50	0.65	0.75	0.35	0.35	0.35	0.45	0.65	0.50
x_9	0.45	0.35	0.60	0.65	0.50	0.40	0.75	0.30	0.30	0.75
x_{10}	0.85	0.40	0.90	0.70	0.55	0.75	0.85	0.70	0.45	0.60

Table 2: The fuzzy relation R_2

$R_3(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.75	0.45	0.35	0.65	0.50	0.35	0.75	0.25	0.65	0.45
x_2	0.60	0.50	0.75	0.35	0.75	0.45	0.60	0.50	0.60	0.70
x_3	0.70	0.35	0.85	0.55	0.60	0.65	0.40	0.45	0.70	0.85
x_4	0.65	0.60	0.50	0.85	0.55	0.25	0.25	0.65	0.85	0.65
x_5	0.70	0.45	0.95	0.70	0.25	0.85	0.65	0.50	0.65	0.45
x_6	0.45	0.80	0.70	0.60	0.30	0.40	0.25	0.45	0.35	0.95
x_7	0.65	0.45	0.60	0.45	0.60	0.60	0.50	0.25	0.40	0.90
x_8	0.35	0.30	0.25	0.30	0.50	0.30	0.65	0.85	0.30	0.35
x_9	0.55	0.70	0.75	0.25	0.45	0.25	0.75	0.60	0.20	0.65
x_{10}	0.80	0.45	0.85	0.80	0.35	0.45	0.70	0.50	0.70	0.35

Table 3: The fuzzy relation R_3

$R_4(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.45	0.60	0.55	0.45	0.60	0.45	0.55	0.35	0.25	0.90
x_2	0.70	0.45	0.65	0.60	0.55	0.40	0.75	0.25	0.60	0.80
x_3	0.60	0.65	0.60	0.75	0.75	0.30	0.85	0.75	0.90	0.25
x_4	0.35	0.75	0.70	0.85	0.65	0.25	0.65	0.60	0.70	0.85
x_5	0.60	0.70	0.85	0.40	0.60	0.70	0.60	0.75	0.45	0.35
x_6	0.50	0.30	0.70	0.35	0.40	0.75	0.45	0.50	0.80	0.20
x_7	0.55	0.25	0.25	0.45	0.30	0.85	0.60	0.40	0.25	0.45
x_8	0.60	0.35	0.40	0.65	0.25	0.25	0.70	0.85	0.70	0.65
x_9	0.35	0.40	0.85	0.75	0.65	0.75	0.85	0.75	0.45	0.70
x_{10}	0.90	0.55	0.60	0.85	0.75	0.60	0.90	0.65	0.80	0.65

Table 4: The fuzzy relation R_4

$R_5(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.75	0.25	0.65	0.95	0.65	0.55	0.75	0.60	0.55	0.35
x_2	0.60	0.70	0.45	0.40	0.60	0.45	0.65	0.45	0.60	0.85
x_3	0.75	0.65	0.55	0.75	0.75	0.45	0.45	0.35	0.45	0.95
x_4	0.25	0.35	0.50	0.60	0.70	0.40	0.60	0.40	0.40	0.90
x_5	0.60	0.25	0.45	0.25	0.70	0.60	0.45	0.55	0.60	0.80
x_6	0.45	0.65	0.40	0.65	0.85	0.65	0.50	0.65	0.65	0.70
x_7	0.35	0.70	0.25	0.75	0.80	0.65	0.35	0.45	0.75	0.75
x_8	0.40	0.80	0.35	0.55	0.95	0.40	0.45	0.60	0.85	0.50
x_9	0.80	0.65	0.70	0.50	0.65	0.35	0.60	0.60	0.70	0.60
x_{10}	0.65	0.25	0.85	0.50	0.60	0.55	0.25	0.75	0.65	0.70

Table 5: The fuzzy relation R_5

is shown in Table 6. Applying equation (1) follows that $P(A) = 0.6884$*. The fuzzy conditional probabilities of* A *can be*

$\wedge_{k=1}^{5} R_k(x_i, y_j)$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0.45	0.25	0.35	0.45	0.40	0.35	0.50	0.25	0.25	0.35
x_2	0.50	0.35	0.45	0.35	0.35	0.40	0.35	0.25	0.60	0.45
x_3	0.60	0.35	0.35	0.40	0.45	0.30	0.40	0.35	0.45	0.25
x_4	0.25	0.35	0.40	0.30	0.35	0.25	0.25	0.35	0.40	0.40
x_5	0.45	0.25	0.45	0.25	0.25	0.40	0.45	0.45	0.35	0.35
x_6	0.45	0.30	0.30	0.25	0.30	0.40	0.25	0.35	0.35	0.20
x_7	0.35	0.25	0.25	0.45	0.30	0.60	0.35	0.25	0.25	0.25
x_8	0.35	0.30	0.25	0.30	0.25	0.25	0.35	0.40	0.25	0.35
x_9	0.35	0.35	0.40	0.25	0.45	0.25	0.60	0.30	0.20	0.35
x_{10}	0.60	0.25	0.55	0.50	0.35	0.45	0.25	0.50	0.45	0.35

Table 6: Computation of $\wedge_{k=1}^{5} R_k(x_i, y_j)$

calculated for each $x_i \in U$ by (3) as follows:

$$\begin{aligned}
P(A \mid [x_1]_{\cap_{k=1}^{5} R_k}) = 0.7072 \quad &, \quad P(A \mid [x_2]_{\cap_{k=1}^{5} R_k}) = 0.6747,\\
P(A \mid [x_3]_{\cap_{k=1}^{5} R_k}) = 0.7019 \quad &, \quad P(A \mid [x_4]_{\cap_{k=1}^{5} R_k}) = 0.6708,\\
P(A \mid [x_5]_{\cap_{k=1}^{5} R_k}) = 0.6912 \quad &, \quad P(A \mid [x_6]_{\cap_{k=1}^{5} R_k}) = 0.6909,\\
P(A \mid [x_7]_{\cap_{k=1}^{5} R_k}) = 0.7032 \quad &, \quad P(A \mid [x_8]_{\cap_{k=1}^{5} R_k}) = 0.7011,\\
P(A \mid [x_9]_{\cap_{k=1}^{5} R_k}) = 0.6940 \quad &, \quad P(A \mid [x_{10}]_{\cap_{k=1}^{5} R_k}) = 0.6977.
\end{aligned}$$

Since a high level of prediction accuracy is expected, we set $\alpha = 0.7$ and $\beta = 0.68$. Then, the following results are obtained according to Definition 4.1:

$$\underline{Apr}^{0.7}_{\cap_{k=1}^{m} R_k}(A) = \{x_1, x_3, x_7, x_8\}, \overline{Apr}^{0.68}_{\cap_{k=1}^{m} R_k}(A) = \{x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\} \text{ and}$$
$$POS^{0.7}(A) = \{x_1, x_3, x_7, x_8\}, NEG^{0.68}(A) = \{x_2, x_4\}, BND^{(0.7,0.68)}(A) = \{x_5, x_6, x_9, x_{10}\},$$

which means the areas x_1, x_3, x_7 and x_8 are seriously effected and need to be rescued; the areas x_2 and x_4 are not the seriously effected and does not need rescued; and we are not sure about the areas x_5, x_6, x_9 and x_{10} with current evaluation.

In the following, we present the Type-I MG-FPRS model with symmetric bounds i.e., the special case of Type-I MG-FPRS model with asymmetric bounds.

Definition 4.5. *Let $(U, W, R_k(1 \le k \le m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A \in F(W)$ be a fuzzy event. For $0.5 < \alpha \le 1$, the Type-I α-multi-granulation fuzzy probabilistic lower and upper approximations of A are defined, respectively, as follows:*

$$\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \ge \alpha\}, \quad \overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > 1 - \alpha\}.$$

The pair $(\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A), \overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A))$, is called Type-I α-multi-granulation fuzzy probabilistic rough set (Type I α-MG-FPRS) over two universes of A.

The positive, negative and boundary regions of A are defined, respectively, as follows:

$$\begin{aligned}
POS^{\alpha}(A) &= \underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \ge \alpha\},\\
NEG^{\alpha}(A) &= (\overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \le 1 - \alpha\},\\
BND^{\alpha}(A) &= \overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) - \underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = \{x \in U : 1 - \alpha < P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < \alpha\}.
\end{aligned}$$

The Type-I α-MG-FPRS is adopted when the decision-maker only one parameter needs to be decided.

Remark 4.6. *(1) In special case of $m = 1$, then the Type-I α-MG-FPRS is called Type-I α-single-granulation fuzzy probabilistic rough set (Type-I α-SG-FPRS) over two universes.*
(2) When $m = 1$, $U = W$ and R_k is a fuzzy $\top$-equivalence relation then the Type-I α-SG-FPRS degenerates into the α-FPRS as given in [44].
(3) When $m = 1$, $U = W$, R_k is a classical equivalence relation and A is a classical event then the Type-I α-SG-FPRS degenerates into the α-PRS as given in [50].
(4) When $m = 1$, $U = W$, R_k is a classical equivalence relation, P is a uniform distribution on U and A is a classical event then the Type-I α-SG-FPRS degenerates into the VPRS as given in [48].

From Definition 4.5, the following assertions are clear.

Proposition 4.7. *Let $(U, W, R_k(1 \le k \le m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A, B \in F(W)$ with $0.5 < \alpha \le 1$. The following properties hold.*
(1) $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A)$.
(2) $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(\emptyset) = \overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(\emptyset) = \emptyset$, $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(W) = \overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(W) = U$.
(3) If $A \subseteq B$, *then* $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(B)$ *and* $\overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(B)$.
(4) If $0.5 < \alpha_1 \le \alpha_2 \le 1$, *then* $\underline{Apr}^{\alpha_2}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{\alpha_1}_{\cap_{k=1}^m R_k}(A)$ *and* $\overline{Apr}^{\alpha_1}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{\alpha_2}_{\cap_{k=1}^m R_k}(A)$.
(5) $\underline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A) = (\overline{Apr}^{\alpha}_{\cap_{k=1}^m R_k}(A^c))^c$.

Example 4.8. *(Continued from Example 4.4) Let us consider another fuzzy event on W is denoted by* $B = \frac{0.63}{y_1} + \frac{0.35}{y_2} + \frac{0.10}{y_3} + \frac{0.70}{y_4} + \frac{0.42}{y_5} + \frac{0.42}{y_6} + \frac{0.80}{y_7} + \frac{0.65}{y_8} + \frac{0.08}{y_9} + \frac{0.43}{y_{10}}$ *represent the area is most seriously effected. The fuzzy conditional probabilities of B can be calculated for each $x_i \in U$ by (3) as follows:*

$$
\begin{aligned}
P(B \mid [x_1]_{\cap_{k=1}^5 R_k}) &= 0.5649 \quad , \quad P(B \mid [x_2]_{\cap_{k=1}^5 R_k}) = 0.5229,\\
P(B \mid [x_3]_{\cap_{k=1}^5 R_k}) &= 0.5546 \quad , \quad P(B \mid [x_4]_{\cap_{k=1}^5 R_k}) = 0.5210,\\
P(B \mid [x_5]_{\cap_{k=1}^5 R_k}) &= 0.5509 \quad , \quad P(B \mid [x_6]_{\cap_{k=1}^5 R_k}) = 0.5411,\\
P(B \mid [x_7]_{\cap_{k=1}^5 R_k}) &= 0.5565 \quad , \quad P(B \mid [x_8]_{\cap_{k=1}^5 R_k}) = 0.5582,\\
P(B \mid [x_9]_{\cap_{k=1}^5 R_k}) &= 0.5551 \quad , \quad P(B \mid [x_{10}]_{\cap_{k=1}^5 R_k}) = 0.5493.
\end{aligned}
$$

If $\alpha = 0.55$, then from Definition 4.5 we obtain $POS^{0.5}(B) = \{x_1, x_3, x_5, x_7, x_8, x_9\}$, $NEG^{0.5}(B) = \{x_2, x_4, x_6, x_{10}\}$ *and* $BND^{0.5}(B) = \emptyset$. *That is the areas x_1, x_3, x_5, x_7, x_8 and x_9 are most seriously effected and need to be rescued immediately with the probability at least* 0.55.

The above two kinds of Type-I MG-FPRSs, Type I (α, β)-MG-FPRS and Type-I α-MG-FPRS are parameter related i.e., all needs to evaluate values of parameters when applying them. In the following we introduce another two kinds of Type-I MG-FPRSs, their parameter free, i.e., which do not have undetermined parameter.

Definition 4.9. *Let $(U, W, R_k(1 \le k \le m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A \in F(W)$ be a fuzzy event. The Type-I 0.5-multi-granulation fuzzy probabilistic lower and upper approximations of A are defined, respectively, as follows:*

$$\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > 0.5\} \quad , \quad \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) \ge 0.5\}.$$

The pair $(\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A), \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A))$, *is called Type-I 0.5-multi-granulation fuzzy probabilistic rough set (Type I 0.5-MG-FPRS) over two universes of A.*

The positive, negative and boundary regions of A are defined, respectively, as follows:

$$
\begin{aligned}
POS^{0.5}(A) &= \underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > 0.5\},\\
NEG^{0.5}(A) &= (\overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) < 0.5\},\\
BND^{\alpha}(A) &= \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) - \underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) = 0.5\}.
\end{aligned}
$$

Remark 4.10. *(1) In special case of $m = 1$, the Type-I 0.5-MG-FPRS is called Type-I 0.5-single-granulation fuzzy probabilistic rough set (Type-I 0.5-SG-FPRS) over two universes.*
(2) When $m = 1$, $U = W$ and R_k is a fuzzy $\top$-equivalence relation then the Type-I 0.5-SG-FPRS degenerates into the 0.5-FPRS as given in [44].
(3) When $m = 1$, $U = W$, R_k is a classical equivalence relation and A is a classical event then the Type-I 0.5-SG-FPRS degenerates into the 0.5-PRS as given in [18].

From Definition 4.9, the following assertions are clear.

Proposition 4.11. *Let $(U, W, R_k(1 \leq k \leq m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A, B \in F(W)$. The following properties hold.*
(1) $\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A)$.
(2) $\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(\emptyset) = \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(\emptyset) = \emptyset$, $\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(W) = \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(W) = U$.
(3) If $A \subseteq B$, *then* $\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) \subseteq \underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(B)$ *and* $\overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(B)$.
(4) $\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A) = (\overline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(A^c))^c$.

Definition 4.12. *Let $(U, W, R_k(1 \leq k \leq m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A \in F(W)$ be a fuzzy event. The Type-I Bayesian multi-granulation fuzzy probabilistic lower and upper approximations of A are defined, respectively, as follows:*

$$\underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > P(A)\} \quad , \quad \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq P(A)\}.$$

The pair $(\underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A), \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A))$, is called Type-I Bayesian multi-granulation fuzzy probabilistic rough set (Type-I B-MG-FPRS) over two universes of A.

The positive, negative and boundary regions of A are defined, respectively, as follows:

$$POS^{P(A)}(A) = \underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > P(A)\},$$
$$NEG^{P(A)}(A) = (\overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) < P(A)\},$$
$$BND^{P(A)}(A) = \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) - \underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) = P(A)\}.$$

Remark 4.13. *(1) In special case of $m = 1$, the Type-I B-MG-FPRS is called Type-I Bayesian single-granulation fuzzy probabilistic rough set (Type-I B-SG-FPRS) over two universes.*
(2) When $m = 1$, $U = W$ and R_k is a fuzzy $\top$-equivalence relation then the Type-I B-SG-FPRS degenerates into the B-FPRS as given in [44].
(3) When $m = 1$, $U = W$, R_k is a classical equivalence relation and A is a classical event then the Type-I B-SG-FPRS degenerates into the BRS as given in [23].
(4) If $P(A) = 0.5$, then the Type-I B-MG-FPRS is just the 0.5-MG-FPRS in Definition 4.9.

From Definition 4.12, the following assertions are clear.

Proposition 4.14. *Let $(U, W, R_k(1 \leq k \leq m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes and $A, B \in F(W)$. The following properties hold.*
(1) $\underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) \subseteq \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A)$.
(2) $\underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(\emptyset) = \underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(W) = \emptyset$, $\overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(\emptyset) = \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(W) = U$.

Example 4.15. *(Continued from Example 4.4) Let us consider another fuzzy event on W denoted by $C = \frac{0.75}{y_1} + \frac{0.15}{y_2} + \frac{0.65}{y_3} + \frac{0.20}{y_4} + \frac{0.80}{y_5} + \frac{0.85}{y_6} + \frac{0.22}{y_7} + \frac{0.25}{y_8} + \frac{0.45}{y_9} + \frac{0.72}{y_{10}}$ represent the areas is most uncertain to effected. Applying equation (1) follows that $P(C) = 0.4801$. The fuzzy conditional probabilities of C can be calculated for each $x_i \in U$ by (3) as follows:*

$$P(C \mid [x_1]_{\cap_{k=1}^5 R_k}) = 0.4885 \quad , \quad P(C \mid [x_2]_{\cap_{k=1}^5 R_k}) = 0.5175,$$
$$P(C \mid [x_3]_{\cap_{k=1}^5 R_k}) = 0.4913 \quad , \quad P(C \mid [x_4]_{\cap_{k=1}^5 R_k}) = 0.4790,$$
$$P(C \mid [x_5]_{\cap_{k=1}^5 R_k}) = 0.4928 \quad , \quad P(C \mid [x_6]_{\cap_{k=1}^5 R_k}) = 0.5046,$$
$$P(C \mid [x_7]_{\cap_{k=1}^5 R_k}) = 0.4923 \quad , \quad P(C \mid [x_8]_{\cap_{k=1}^5 R_k}) = 0.4672,$$
$$P(C \mid [x_9]_{\cap_{k=1}^5 R_k}) = 0.4794 \quad , \quad P(C \mid [x_{10}]_{\cap_{k=1}^5 R_k}) = 0.5017.$$

From Definition 4.9 we obtain $POS^{0.5}(C) = \{x_2, x_6, x_{10}\}$, $NEG^{0.5}(C) = \{x_1, x_3, x_4, x_5, x_7, x_8, x_9\}$ and $BND^{0.5}(C) = \emptyset$. Which means the areas x_2, x_6 and x_{10} are effected with a probability more than 0.5. Furthermore, from Definition 4.12 we obtain

$$POS^{0.4801}(C) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_{10}\}, NEG^{0.4801}(C) = \{x_4, x_8, x_9\} \text{ and } BND^{0.4801}(C) = \emptyset.$$

Which means the areas x_1, x_2, x_3, x_5, x_6, x_7 and x_{10} are effected with a probability more than $P(C)$.

4.2 Bayesian decision procedure based on multi-granulation fuzzy probabilistic approximation space over two universes

Let $(U, W, R_k(1 \leq k \leq m), P)$ be a multi-granulation fuzzy probabilistic approximation space over two universes. The Bayesian decision procedure adopts two states and three actions to describe the decision process. The set of states is given by $\Omega = \{A, A^c\}$, where A is a fuzzy set on W. The set of actions is $\mathcal{A} = \{a_P, a_N, a_B\}$, where a_P, a_N and a_B represent the three actions in classifying an object, namely, deciding $POS(A)$, deciding $NEG(A)$, and deciding $BND(A)$, are respectively. The loss function λ^x_A, is given by a 3×2 matrix shown in Table 7. The denotation λ^x_A implies that the loss function may depend on the fuzzy state $A \in F(W)$ and the object $x \in U$. But for the following discussion convenience, the subscript $_A$ and the superscript ^{x} are omitted. For each element $x \in U$, the fuzzy set

	A: positive	A^c: negative
a_P: accept	$\lambda_{PP} = \lambda(a_P \mid A)$	$\lambda_{PN} = \lambda(a_P \mid A^c)$
a_N: reject	$\lambda_{NP} = \lambda(a_N \mid A)$	$\lambda_{NN} = \lambda(a_N \mid A^c)$
a_B: defer	$\lambda_{BP} = \lambda(a_B \mid A)$	$\lambda_{BN} = \lambda(a_B \mid A^c)$

Table 7: Loss function λ^x_A, for A

$[x]_{\cap_{k=1}^m R_k}$ is adopted as the description of x and $[x]_{\cap_{k=1}^m R_k}(y) = \wedge_{k=1}^m R(x, y)$ for all $y \in W$. The expected losses of taking the individual actions for element x are computed (expressed) as follows:

$$\mathcal{L}_P = \mathcal{L}(a_P \mid [x]_{\cap_{k=1}^m R_k}) = \lambda_{PP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{PN} P(A^c \mid [x]_{\cap_{k=1}^m R_k}), \tag{4}$$

$$\mathcal{L}_N = \mathcal{L}(a_N \mid [x]_{\cap_{k=1}^m R_k}) = \lambda_{NP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{NN} P(A^c \mid [x]_{\cap_{k=1}^m R_k}), \tag{5}$$

$$\mathcal{L}_B = \mathcal{L}(a_B \mid [x]_{\cap_{k=1}^m R_k}) = \lambda_{BP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{BN} P(A^c \mid [x]_{\cap_{k=1}^m R_k}). \tag{6}$$

The Bayesian decision procedure leads to the following three minimum-risk decision rules:
(P1) If $\mathcal{L}_P \leq \mathcal{L}_B$ and $\mathcal{L}_P \leq \mathcal{L}_N$, then decide $x \in POS(A)$.
(N1) If $\mathcal{L}_N \leq \mathcal{L}_P$ and $\mathcal{L}_N \leq \mathcal{L}_B$, then decide $x \in NEG(A)$.
(B1) If $\mathcal{L}_B \leq \mathcal{L}_P$ and $\mathcal{L}_B \leq \mathcal{L}_N$, then decide $x \in BND(A)$.
When any two or all actions have the same risk, different tie-breaking criteria is adopted depending on concrete situations. From Proposition 2.3(3), we have $P(A \mid [x]_{\cap_{k=1}^m R_k}) + P(A^c \mid [x]_{\cap_{k=1}^m R_k}) = 1$. Thus, the equations (4)-(6) can be simplified as follows:
$\mathcal{L}_P = \lambda_{PP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{PN}(1 - P(A \mid [x]_{\cap_{k=1}^m R_k}))$,
$\mathcal{L}_N = \lambda_{NP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{NN}(1 - P(A \mid [x]_{\cap_{k=1}^m R_k}))$,
$\mathcal{L}_B = \lambda_{BP} P(A \mid [x]_{\cap_{k=1}^m R_k}) + \lambda_{BN}(1 - P(A \mid [x]_{\cap_{k=1}^m R_k}))$.
Consider a special kind of loss function as given in [18, 29, **?**] with

$$\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP} \quad \text{and} \quad \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}. \tag{7}$$

That is, the loss of classifying an object x in state A into the positive region $POS(A)$ is less than or equal to the loss of classifying x into the boundary region $BND(A)$, and both of these losses are less than the loss of classifying x into the negative region $NEG(A)$. The reverse order of losses is used for classifying an object that does not in state A. For this type of loss functions, the minimum-risk decision rules (P1)- (B1) can be written as:
(P2) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq \alpha$ and $P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq \gamma$, then decide $x \in POS(A)$;
(N2) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) \leq \gamma$ and $P(A \mid [x]_{\cap_{k=1}^m R_k}) \leq \beta$, then decide $x \in NEG(A)$;
(B2) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) \leq \alpha$ and $P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq \beta$, then decide $x \in BND(A)$;
where

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \tag{8}$$

$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}, \tag{9}$$

$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}. \tag{10}$$

Remark 4.16. *The study of above approach decision-theoretic rough set within multi-granulation fuzzy probabilistic approximation space over two universes is called Type-I multi-granulation fuzzy decision-theoretic rough set (Type-I MG-FDTRS) approach.*
(1) In special case of $m = 1$, the Type-I MG-FDTRS approach is called Typr-I single-granulation fuzzy decision-theoretic rough set (Type-I SG-FDTRS) approach over two universes.
(2) When $m = 1$, $U = W$ and R_k is a fuzzy $\top$-equivalence relation then the Type-I SG-FDTRS approach degenerates into the FDTRS approach as given in [44].
(3) When $m = 1$, $U = W$, R_k is a classical equivalence relation and A is a classical event then the Type-I SG-FDTRS approach degenerates into the DTRS approach as given in [29].

4.3 The interrelationship between Type-I MG-FDTRS approach and Type-I MG-FPRSs over two universes

Here we study derivations of four kinds Type-I MG-FPRSs through Type-I MG-FDTRS approach. For this study we are considered four additional conditions on the loss functions as given in [29, 32, 33].

$$(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{BN} - \lambda_{NN})(\lambda_{BP} - \lambda_{PP}), \tag{11}$$

$$(\lambda_{BP} - \lambda_{PP})(\lambda_{NP} - \lambda_{BP}) = (\lambda_{BN} - \lambda_{NN})(\lambda_{PN} - \lambda_{BN}), \tag{12}$$

$$\lambda_{PN} - \lambda_{BN} > \lambda_{BP} - \lambda_{PP}, \tag{13}$$

$$\lambda_{BP} - \lambda_{PP} = \lambda_{PN} - \lambda_{BN}, \quad \lambda_{BN} - \lambda_{NN} = \lambda_{NP} - \lambda_{BP}. \tag{14}$$

4.3.1 Type-I (α, β)-MG-FPRS

Suppose $P(A) > 0$ and the loss function satisfies the conditions (7) and (11), then $1 \geq \alpha > \gamma > \beta \geq 0$. From the conditions (7) and (11), we can-not infer the desired relationship α, β and $P(A)$. In order to make sure that $\alpha > P(A) > \beta$, we have further imposed the conditions:

$$P(A)(\lambda_{BP} - \lambda_{PP}) < (1 - P(A))(\lambda_{PN} - \lambda_{BN}), \tag{15}$$

$$P(A)(\lambda_{NP} - \lambda_{BP}) > (1 - P(A))(\lambda_{BN} - \lambda_{NN}). \tag{16}$$

If the risk of classifying an element $x \in U$ into the positive (or, negative) region of A equals to that of classifying x into the boundary region of A, we decide $x \in POS(A)$ (or, $x \in NEG(A)$). For this tie-breaking criteria, using the decision rules (P2) - (B2), we are obtained the following equivalent forms:
(P3) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq \alpha$, then decide $x \in POS(A)$,
(N3) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) \leq \beta$, then decide $x \in NEG(A)$,
(B3) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) < \alpha$, then decide $x \in BND(A)$.

By rules (P3) - (B3), we therefore obtain the decision regions i.e., positive, negative and boundary regions for $A \in F(W)$, as follows:

$$POS^{\alpha}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\},$$
$$NEG^{\beta}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \leq \beta\},$$
$$BND^{(\alpha,\beta)}(A) = \{x \in U : \beta < P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < \alpha\}.$$

Now we are derived multi-granulation fuzzy probabilistic lower and upper approximations of A, as follows, through decisions regions:

$$\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = POS^{\alpha}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\}\ ,\ \overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A) = (NEG^{\beta}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > \beta\}.$$

The pair $(\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A), \overline{Apr}^{\beta}_{\cap_{k=1}^{m} R_k}(A))$ is the Type-I (α, β)-MG-FPRS in Definition 4.1.

4.3.2 Type-I α-MG-FPRS

If the loss function satisfies the conditions (7), (12) and (13), then $\beta = 1 - \alpha$ and $\alpha > 0.5$. Using the same tie-breaking criteria adopted in Section 4.3.1 and the decision rules (P2) - (B2), we are obtained following equivalent forms:
(P4) If $P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha$, then decide $x \in POS(A)$,
(N4) If $P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \leq 1 - \alpha$, then decide $x \in NEG(A)$,
(B4) If $1 - \alpha < P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < \alpha$, then decide $x \in BND(A)$.
By rules (P4) - (B4), we therefore obtain the decision regions i.e., positive, negative and boundary regions for $A \in F(W)$, as follows:

$$POS^{\alpha}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\},$$
$$NEG^{\alpha}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \leq 1 - \alpha\},$$
$$BND^{\alpha}(A) = \{x \in U : 1 - \alpha < P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < \alpha\}.$$

Furthermore, the multi-granulation fuzzy probabilistic lower and upper approximations of A, as follows, through decisions regions:

$$\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = POS^{\alpha}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq \alpha\}, \overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A) = (NEG^{\alpha}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > 1 - \alpha\}.$$

The pair $(\underline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A), \overline{Apr}^{\alpha}_{\cap_{k=1}^{m} R_k}(A))$ is the Type-I α-MG-FPRS in Definition 4.5.

4.3.3 Type-I 0.5-MG-FPRS

If the loss function satisfies the conditions (7) and (14), then $\alpha = \gamma = \beta = 0.5$. When the risk of classifying an element $x \in U$ into the positive (or, negative) region of A is equal to that of classifying it into the boundary region of A, we decide $x \in BND(A)$. For this tie-breaking criteria, using the decision rules (P2) - (B2) we are obtained the following equivalent forms:
(P5) If $P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > 0.5$, then decide $x \in POS(A)$,
(N5) If $P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < 0.5$, then decide $x \in NEG(A)$,
(B5) If $P(A \mid [x]_{\cap_{k=1}^{m} R_k}) = 0.5$, then decide $x \in BND(A)$.
The decision regions i.e., positive, negative and boundary regions of $A \in F(W)$ are obtained by the rules (P5) - (B5), respectively as follows:

$$POS^{0.5}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > 0.5\},$$
$$NEG^{0.5}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) < 0.5\},$$
$$BND^{0.5}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) = 0.5\}.$$

The corresponding multi-granulation fuzzy probabilistic lower and upper approximations are derived as follows:

$$\underline{Apr}^{0.5}_{\cap_{k=1}^{m} R_k}(A) = POS^{0.5}(A) = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) > 0.5\}, \overline{Apr}^{0.5}_{\cap_{k=1}^{m} R_k}(A) = (NEG^{0.5}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^{m} R_k}) \geq 0.5\}.$$

The pair $(\underline{Apr}^{0.5}_{\cap_{k=1}^{m} R_k}(A), \overline{Apr}^{0.5}_{\cap_{k=1}^{m} R_k}(A))$ is the Type-I 0.5-MG-FPRS in Definition 4.9.

4.3.4 Type-I B-MG-FPRS

Assume $P(A) > 0$ and the loss function satisfies conditions (7) and the following two conditions:

$$P(A)(\lambda_{BP} - \lambda_{PP}) = (1 - P(A))(\lambda_{PN} - \lambda_{BN}) \text{ and } P(A)(\lambda_{NP} - \lambda_{BP}) = (1 - P(A))(\lambda_{BN} - \lambda_{NN}).$$

Then $\alpha = \beta = \gamma = P(A)$. Using the same tie-breaking in Section 4.3.3 yields the following simplified and equivalent rules of (P2) - (B2):
(P6) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) > P(A)$, then decide $x \in POS(A)$,
(N6) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) < P(A)$, then decide $x \in NEG(A)$,
(B6) If $P(A \mid [x]_{\cap_{k=1}^m R_k}) = P(A)$, then decide $x \in BND(A)$.
The positive, negative and boundary regions of $A \in F(W)$ are obtained by the rules (P6) - (B6), respectively as follows:

$$\begin{aligned} POS^{P(A)}(A) &= \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > P(A)\}, \\ NEG^{P(A)}(A) &= \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) < P(A)\}, \\ BND^{P(A)}(A) &= \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) = P(A)\}. \end{aligned}$$

The corresponding multi-granulation fuzzy probabilistic lower and upper approximations are derived as follows:

$$\begin{aligned} \underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) &= POS^{P(A)(A)} = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) > P(A)\}, \\ \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A) &= (NEG^{P(A)}(A))^c = \{x \in U : P(A \mid [x]_{\cap_{k=1}^m R_k}) \geq P(A)\}. \end{aligned}$$

The pair $(\underline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A), \overline{Apr}^{P(A)}_{\cap_{k=1}^m R_k}(A))$ is the Type-I $P(A)$-MG-FPRS in Definition 4.12.

Example 4.17. *(Continued from Example 4.15) Suppose the loss function for fuzzy event C is estimated by an decision-maker as follows:*
$\lambda_{PP} = 0.2$, $\lambda_{BP} = 0.3$, $\lambda_{NP} = 0.6$, $\lambda_{NN} = 0.23$, $\lambda_{BN} = 0.5$, $\lambda_{PN} = 0.6$.
Which satisfies the conditions (7), (11), (15) and (16). Using equations (8) and (10), we can easily get $\alpha = 0.5$ and $\beta = 0.47$. The conditional probabilities, $P(C \mid [x_i]_{\cap_{k=1}^5 R_k})$ $(i = 1, 2, \cdots, 10)$, are given in Example 4.15. Therefore we obtain the decision regions of the fuzzy event C,

$$\begin{aligned} &POS^{0.5}(C) = \{x_2, x_6, x_{10}\}, \\ &NEG^{0.47}(C) = \{x_8\} \\ &BND^{(0.5,0.47)}(C) = \{x_1, x_3, x_4, x_5, x_7, x_9\}. \end{aligned}$$

and the multi-granulation fuzzy probabilistic approximations of the fuzzy event C

$$\underline{Apr}^{0.5}_{\cap_{k=1}^m R_k}(C) = \{x_2, x_6, x_{10}\} \quad \textit{and} \quad \overline{Apr}^{0.47}_{\cap_{k=1}^m R_k}(C) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}\}.$$

Remark 4.18. *The other three type multi-granulation fuzzy probabilistic rough sets models and their corresponding three-way decisions over two universes are not considered in this paper.*
Our four type models of multi-granulation fuzzy probabilistic rough sets and their corresponding three-way decisions are exactly depends on four different conditional probabilities $P(A \mid [x]_{\cap_{k=1}^m R_k})$, $P(A \mid [x]_{\cup_{k=1}^m R_k})$, $\cap_{k=1}^m P(A \mid [x]_{R_k})$ and $\cup_{k=1}^m P(A \mid [x]_{R_k})$ are respectively.

If $W = \{y_1, y_2, \cdots, y_n\}$, and $P(y_i) = p_i (i = 1, 2, \cdots, n)$, then the computation of conditional probabilities $P(A \mid [x]_{\cup_{k=1}^m R_k})$, $\cap_{k=1}^m P(A \mid [x]_{R_k})$ and $\cup_{k=1}^m P(A \mid [x]_{R_k})$ are obtain from (2), respectively as follows:

$$P(A \mid [x]_{\cup_{k=1}^m R_k}) = \frac{\sum_{i=1}^n (\vee_{k=1}^m R_k(x, y_i)) A(y_i) p_i}{\sum_{i=1}^n (\vee_{k=1}^m R_k(x, y_i)) p_i}, \ \forall x \in U. \tag{17}$$

$$\cap_{k=1}^m P(A \mid [x]_{R_k}) = \wedge_{k=1}^m \frac{\sum_{i=1}^n R_k(x, y_i) A(y_i) p_i}{\sum_{i=1}^n R_k(x, y_i) p_i}, \ \forall x \in U. \tag{18}$$

$$\cup_{k=1}^m P(A \mid [x]_{R_k}) = \vee_{k=1}^m \frac{\sum_{i=1}^n R_k(x, y_i) A(y_i) p_i}{\sum_{i=1}^n R_k(x, y_i) p_i}, \ \forall x \in U. \tag{19}$$

The following example shows their differences.

Example 4.19. *(Continued from Example 4.4) The fuzzy conditional probabilities of A can be calculated for $x_1 \in U$ by (3), (17), (18) and (19) as follows:*

$P(A \mid [x_1]_{\cap_{k=1}^5 R_k}) = 0.7072$, $P(A \mid [x_1]_{\cup_{k=1}^5 R_k}) = 0.6966$, $\cap_{k=1}^5 P(A \mid [x_1]_{R_k}) = 0.6713$, $\cup_{k=1}^5 P(A \mid [x_1]_{R_k}) = 0.7173$.

5 The differences between our study and the existing ones

In what follows, we mainly distinguish the differences between our study and the existing ones with respect to computation of conditional probabilities, granulation environment and research objective studies.
(1) Our work is different from the ones in [16, 30, 32, 34, 35] with regard to the computation of the conditional probability. Infact the models are constructed in terms of fuzzy probability instead of the cardinality-based estimation in [16, 30, 32, 34, 35].
(2) Our research is different from the one in [44] as far as the granulation environment is concerned. To be more concentrate, our study was made under multi-granulation fuzzy relations and proposed four types FPRSs, while in [44] were done under single-granulation fuzzy relations and established one type FPRS.
(3) Our study is different from the one in [25] in terms of research objective. More specifically, in the current study we count the incomplete available information and the possible existence of random available information, while the research in [25] avoid the possible existence of random available information. In decision making problem, we consider three situations for optimal results such as seriously affected areas (Example 4.4), most seriously affected areas (Example 4.8) and affected areas (Example 4.15), while reference [25] only considers one situation such as most seriously affected areas (Section 4.4 [24]).

6 Conclusions

In this paper, we systematically discuss the basic theory of the models over two universes. Our main conclusive decisions are listed below:
(1) From the view point of Remarks 4.2, 4.6, 4.10, 4.13 and 4.16, we can observe that the many existing results as given in [18, 23, 29, 44, **?**] are special case of our proposed models, which is shown as Figure 2 and 3.

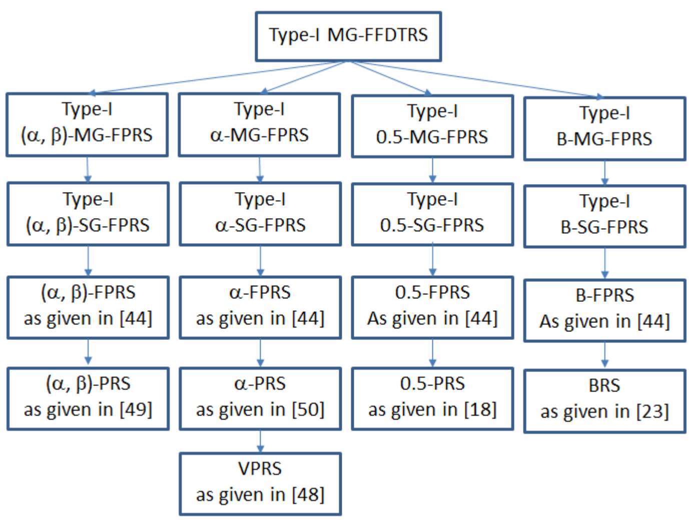

Figure 2: Relationship among proposed, Type-I MG-FDTRS approach, other Type-I MG-FPRS models and exiting models.

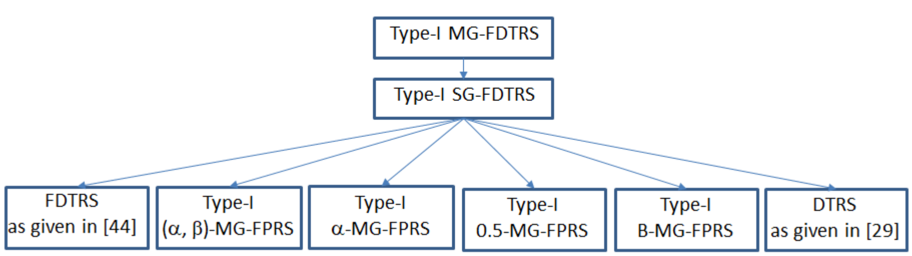

Figure 3: Relationship among proposed Type-I MG-FDTRS, Type-I SG-FDTRS approaches and existing DTRS approaches.

(2) From the view point of MG-FPRS models we can observe that, the concepts are expressed by lower approximation and upper approximation induced by a multi-granulation structure, i.e., the concepts are depicted by known knowledge induced from a

multiple fuzzy binary relations on the universes. We can also observe that, from Section 4, 3WDs are directly induced by lower and upper approximations of MG-FPRSs and the lower and upper approximation induced by the 3WDs.

In the future work we will study other three type MG-FPRSs models and their corresponding 3WDs over two universes, and their interrelationships including applications. We also apply our obtained results in [45].

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