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The KKT optimality conditions for constrained programming problem with generalized convex fuzzy mappings

F. Song[1] and Z. Z Wu[2]

[1,2] *College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, P.R. China*

sonnfe@163.com, zezhongwu@cuit.edu.cn

Abstract

The aim of present paper is to study a constrained programming with generalized $\alpha-$univex fuzzy mappings. In this paper we introduce the concepts of $\alpha-$univex, $\alpha-$preunivex, pseudo $\alpha-$univex and $\alpha-$unicave fuzzy mappings, and we discover that $\alpha-$univex fuzzy mappings are more general than univex fuzzy mappings. Then, we discuss the relationships of generalized $\alpha-$univex fuzzy mappings and get some properties. In the last, we derive necessary and sufficient Karush-Kuhn-Tucker conditions and its dual problems with generalized differentiable $\alpha-$univex fuzzy mappings for fuzzy constrained programming problem.

Keywords: Fuzzy mappings, triangular fuzzy number, $\alpha-$univex, $g-$differentiability, fuzzy optimization.

1 Introduction

In classical mathematics, it is well-known that the convexity plays a very important role in the optimization problem. In recent years, more and more scholars are constantly exploring various generalized convex functions, and studying their optimization and duality problems. Since invex functions has been proposed as a generalized convex functions by Hanson [17] in 1981, the optimal sufficient conditions and dual problems with inequalities constraints under the new generalized convex functions were studied. Subsequently, some other generalized convexity functions were continually found, for example, Ben Isral and Mond [6] introduced the preinvex function f with respect to vector value function $\eta(x, y)$. Further properties, applications of preinvexity and its some generalizations were studied by Mohn and Neogy [25], Yang and Li [46], Antczak and Tadeusz [2]. In 1991, Bector and Singh [7] proposed a class of functions, called B-vex functions which is defined by relaxing the definition of convexity of a function. And then, generalized B-vex functions and generalized B-vex programming were studied by Tapia and Bector [9] and other authors. Later, Bector et. al [4] presented the notion of univex and preunivex functions as a generalized convexity of the B-vex function in 1992. Since then, the univex functions and its applications in optimality conditions were studied by many scholars. For instance, the class of generalized type I functions for a differentiable multiobjective programming problem and its duality results were introduced by Aghezzaf and Hachimi [1]. In [24], Mishra et. al extended the class of generalized type I univexity. In recent years, Anurag Jayswal and Rajnish Kumar [18] put forward a new class of functions called d-V-type-I univex. The generalized (d, ρ, η, θ)-type I univex function was proposed by Tripathy and Devi [40]. In [28], Noor and Muhammad introduced some classes of $\alpha-$invex functions by relaxing the definition of invex functions, and then, some other generalizations of $\alpha-$invex we can see [33, 22]. Later, Rautela and Pant [35] extended the classes of generalized α-invex functions to generalized α-univex functions and utilized these notions they derived a Karush-Kuhn-Tucker type sufficient optimality condition and established Mond-Weir type duality results for the mathematical programming problem. And in [23], Mishra, Pant and Rautela presented the concepts of $\alpha-$univex, pseudo $\alpha-$univex, strict pseudo $\alpha-$univex, quasi $\alpha-$univex functions and derived Karush-Kuhn-Tucker-Type sufficient optimality conditions and established weak, strong and converse duality theorems.

Corresponding Author: Z. Z Wu

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In fuzzy mathematics, some generalized convex fuzzy mappings have been studied by many scholars since Chang and Zadeh in [14] come up with fuzzy mappings. Nanda and Kar [26] introduced convex fuzzy mapping and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. In [27], some different types of fuzzy preinvexity functions are defined and their properties are studied. Syau [39] gave the notion of pseudo-convexity, invexity and pseudo-invexity for fuzzy mappings of one variable based on the notion of differentiability proposed by Goetschel and Voxman [16], investigated the relationship between convex fuzzy mappings and preinvex fuzzy mappings at the sametime. In 2001, Syau [38] introduced a class of pseudo-B-vex, B-invex, and pseudo-B-invex fuzzy mappings are deemed as a generalization of pseudo-convex, invex, and pseudo-invex fuzzy mappings. It is important for the research and application of various fuzzy differences and fuzzy differentials due to the appearance of fuzzy numbers. In 1983, Puri and Ralescu [34] defined the derivative and H-derivative of fuzzy mappings and Hukuhara difference. In 1987, Kaleva [19] discussed the H-derivative and obtained a sufficient and necessary condition of the H-differentiability of fuzzy mappings. Later, Wang and Wu [41] put forward the concepts of directional derivative, differential and subdifferential of fuzzy mapping from R^n into E^1. Based on the concept of differentiability of fuzzy mappings, Wu and Xu [45] gave the concepts of fuzzy pseudoconvex, fuzzy invex, fuzzy pseudoinvex and fuzzy preinvex mapping from R^n to the set of fuzzy numbers, and put forward the concepts of fuzzy extended directional differential, subgradient and subdifferential, and in [44] a new criterion be obtained for the existence of a fuzzy preinvex mapping under the condition of upper or lower semicontinuity and discussed the relationship between the fuzzy variational-like inequality and fuzzy optimization problems. The new concept is called strongly generalized differentiable fuzzy mappings is presented in [5, 30]. Since $H-$difference and $gH-$difference may not exist for any pair of fuzzy numbers, such as triangular fuzzy numbers $\mu =< 0, 2, 4 >$ and trapezoidal fuzzy numbers $\nu =< 0, 1, 2, 3 >$. Therefore, in 2010, L. Stefanini [37] putted forward $g-$difference to overcome such a problem. Subsequently, a more general differentiable $g-$differentiable was proposed by B.Bede and L.Stefanini in [8].

With the intensive study of fuzzy functions, the fuzzy optimization problem has become an important research for scholars. Particularly, in [43, 42], the concepts of level-wise differentiable and Hukuhara differentiable fuzzy functions were used for obtaining KKT type optimality conditions for fuzzy optimization problems. Panigrahi [32] extended and generalized some concepts to fuzzy mappings of several variables using BuckleyCFeuring approach for fuzzy differentiation and derived KKT condition for the constrained fuzzy minimization problem. In [11][10], Chalco-Cano et. al obtained KKT type optimality conditions by using strongly generalized differentiable (GH-differentiable, for short) fuzzy functions and generalized Hukuhara differentiable (gH-differentiable, for short) fuzzy functions, respectively. In recent years, some articles begin to pay attention to the necessary conditions of fuzzy optimization with $gH-$differentiable fuzzy functions, we can refer to the literatures [29, 31]. In [36], L.Stefanini et.al proposed directional gH-differentiability and formulated KKT-like necessary and sufficient conditions for non-dominated solutions in constrained optimization problems.

Motivated by the recent works going on these filed, and there are no articles have studied $\alpha-$univex fuzzy mappings so far. Therefore, in this paper, we mainly report the following three points: (i) we state a class of new concepts, $\alpha-$preunivex, $\alpha-$univex and pseudo $\alpha-$univex fuzzy mappings with $g-$difference. It is different from the concept of weakly univex fuzzy mappings in [20]. From the definition of $\alpha-$univex fuzzy mappings, it is easy to understand that $\alpha-$univex fuzzy mappings is univex fuzzy mappings when let $\alpha(x, y) = 1$. In other words, univex fuzzy mappings can be seen as $\alpha-$univex fuzzy mappings, but not all $\alpha-$univex fuzzy mappings is univex fuzzy mappings. (ii) We obtain some relationships of the generalized $\alpha-$univex fuzzy mappings, such as $\alpha-$preunivex fuzzy mappings is $\alpha-$univex fuzzy mappings, pseudo $\alpha-$univex fuzzy mappings under some conditions. (iii) We get the necessary and sufficient conditions for constrained programming under the assumption that $\alpha-$univex is generalized differentiable ($g-$differentiable, for short) fuzzy mapping, and also get its duality problems.

This paper is organized as follows. Section 2 mainly recalls the definitions and properties of fuzzy numbers. Section 3 introduces the generalized Hukuhara differentiability and generalized α-univex fuzzy functions. Simultaneously, we discusses some properties of the generalized $\alpha-$univex fuzzy mappings. Section 4 considers optimal conditions. Section 5 gets several dual theorems. Section 6 is conclusion.

2 Preliminaries

A fuzzy set μ on R^n is a mapping $\mu : R^n \to [0, 1]$. We call $[\mu]^r = \{x \in R^n : \mu(x) \geq r\}$, r-cut for any $r \in (0, 1]$, and $supp\mu = \{x \in R^n : \mu(x) > 0\}$ is called the support of μ. We defined $[\mu]^0$ is the closure of $supp\mu$.

Definition 2.1. *A compact and convex fuzzy number μ on R^n is a fuzzy set with the following properties:*

(1) μ is normal, i.e. there exists $x_0 \in R^n$ such that $\mu(x_0) = 1$;

(2) μ *is an upper semi-continuous function;*

(3) $\mu(\lambda x+(1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}, x, y \in R^n, \lambda \in [0,1]$*;*

(4) $[\mu]^0$ *is compact.*

We denote the set of all fuzzy numbers by E. Obviously, $\mu \in E$ is a fuzzy number if and only if $[\mu]^r$ is a nonempty compact convex subset of R^1 (denoted by $\left[\mu^L(r), \mu^U(r)\right]$) for each $r \in [0,1]$. Therefore, a fuzzy number μ is determined by the endpoints of the interval $\left[\mu^L(r), \mu^U(r)\right]$.

A real number a is a special case of fuzzy number encoded as

$$\tilde{a}(t)=\begin{cases}1, & t=a\\ 0, & t\neq a\end{cases}$$

In particular, the fuzzy number $\tilde{0}$ is defined as $\tilde{0}(t)=1$ if $t=0$, and $\tilde{0}(t)=0$ if $t\neq 0$.

Because each fuzzy number μ can be parameterized by $\left\{(\mu^L(r), \mu^U(r), r) : r \in [0,1]\right\}$, there are some conditions of the two endpoint functions $\mu^L(r)$ and $\mu^U(r)$ we have to know.

Definition 2.2. *A fuzzy number* μ *is said to be a triangular fuzzy number if* $\mu^L(1)=\mu^U(1)$*. Moreover, we say a linear triangular fuzzy number if* $\mu^L(r)$*, and* $\mu^U(r)$ *are linear. And we denote a linear triangular fuzzy number by* $<\mu^L(0), \mu^L(1), \mu^U(0)>$*.*

Lemma 2.3. [16] *Assume that* $I=[0,1]$*,* μ *is a fuzzy number, the endpoint functions* $\mu^L : I \to R$ *and* $\mu^U : I \to R$ *satisfy the following conditions:*

(1) $\mu^L : I \to R$ *is a bounded increasing function;*

(2) $\mu^U : I \to R$ *is a bounded decreasing function;*

(3) $\mu^L(1) \leq \mu^U(1)$*;*

(4) for $0<k\leq 1$*,* $\lim_{r\to k^-}\mu^L(r)=\mu^L(k)$ *and* $\lim_{r\to k^-}\mu^U(r)=\mu^U(k)$*;*

(5) $\lim_{r\to 0^+}\mu^L(r)=\mu^L(0)$ *and* $\lim_{r\to 0^+}\mu^U(r)=\mu^U(0)$*.*

In the following, we give the fuzzy addition and scalar multiplication for the fuzzy numbers $\mu, \nu \in E$, $\lambda \in R$, respectively, for $x \in R$,

$$(\mu\tilde{+}\nu)(x)=\sup_{y+z=x}\min[\mu(y), \nu(z)], (\lambda\mu)(x)=\begin{cases}\mu(\lambda^{-1}x), & \lambda\neq 0\\ 0, & \lambda=0\end{cases}$$

Also, it is well known that for any two fuzzy numbers $\mu, \nu \in E$ represented by $[\mu^L(r), \mu^U(r)]$ and $[\nu^L(r), \nu^U(r)]$ for every $r \in [0,1]$, and for any real number λ, $\lambda\mu \in E$ have $[\mu\tilde{+}\nu]^r=[\mu]^r\tilde{+}[\nu]^r$, and $[\lambda\mu]^r=\lambda[\mu]^r$,

$$(\mu\tilde{+}\nu)^L(r)=\mu^L(r)+\nu^L(r), \quad (\mu\tilde{+}\nu)^U(r)=\mu^U(r)+\nu^U(r),$$

$$(\lambda\mu)^L(r)=\begin{cases}\lambda\mu^L(r), \lambda\geq 0\\ \lambda\mu^U(r), \lambda<0\end{cases}, (\lambda\mu)^U(r)=\begin{cases}\lambda\mu^U(r), \lambda\geq 0\\ \lambda\mu^L(r), \lambda<0\end{cases}$$

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Definition 2.4. [34]*(H-difference) For* $\mu, \nu \in E$*, there exists* $\omega \in E$ *such that* $\mu=\nu+\omega$*, then it is said that the Hukuhara difference between* μ *and* ν *exists. So,* ω *is called the H-difference between* μ *and* ν*, and is denoted by* $\mu -_H \nu$*. If the H-difference* $\mu -_H \nu=\omega$ *exists, then* $(\mu -_H \nu)^L(r)=\mu^L(r)-\nu^L(r)$*,* $(\mu -_H \nu)^U(r)=\mu^U(r)-\nu^U(r)$*.*

Definition 2.5. [37]*(gH-difference) Let* $\mu, \nu \in E$*. The generalized Hukuhara difference(gH-difference) in is defined as*

$$\mu\ominus_{gH}\nu=\omega \Leftrightarrow \begin{cases}(i) & \mu=\nu+\omega, \quad or\\ (ii) & \nu=\mu+(-1)\omega.\end{cases}$$

For any two intervals $[\mu]^r=[\mu^L(r), \mu^U(r)]$ *and* $[\nu]^r=[\nu^L(r), \nu^U(r)]$*, if* $\mu\ominus_{gH}\nu$ *exists, then*

$$[\mu\ominus_{gH}\nu]^r=\left[\min\left\{\mu_r^L-\nu_r^L, \mu_r^U-\nu_r^U\right\}, \max\left\{\mu_r^L-\nu_r^L, \mu_r^U-\nu_r^U\right\}\right].$$

Since $H-$difference and $gH-$difference may not exist for any pair of fuzzy numbers, such as triangular fuzzy numbers $\mu =< 0, 2, 4 >$ and trapezoidal fuzzy numbers $\nu =< 0, 1, 2, 3 >$. Therefore, in 2010, L. Stefanini [37] putted forward $g-$difference to overcome such a problem.

Definition 2.6. [37]*(g-difference) The generalized difference ($g-$difference for short) $\mu \ominus_g \nu$ is defined as*

$$[\mu \ominus_g \nu]^r = cl \bigcup_{\beta \geq r} ([\mu]_\beta \ominus_{gH} [\nu]_\beta).$$

for all $r \in [0,1]$. Note that, for any closed intervals μ, ν, $\mu \ominus_g \nu = \mu \ominus_{gH} \nu$.

Definition 2.7. *For $\mu, \nu \in E$, we say $\mu \preceq \nu$, if for every $r \in [0,1]$, $\mu^L(r) \leq \nu^L(r)$ and $\mu^U(r) \leq \nu^U(r)$. If $\mu \preceq \nu, \nu \preceq \mu$, then $\mu = \nu$. We say $\mu \prec \nu$, if $\mu \preceq \nu$ and $\exists r \in [0,1]$, such that $\mu^L(r) < \nu^L(r)$ or $\mu^U(r) < \nu^U(r)$. For $\mu, \nu \in E$, if either $\mu \preceq \nu$ or $\nu \preceq \mu$, then we say that μ and ν are comparable; otherwise they are incomparable. In this paper, we assume that the fuzzy numbers used are all comparable.*

Given two fuzzy numbers $u, v \in E$, we use the Pompeiu-Hausdorff metric to define the distance between u and v by

$$D(u,v) = \sup_{r \in [0,1]} H([u]^r, [v]^r) = \sup_{r \in [0,1]} \max \left\{ \left| u^L(r) - v^L(r) \right|, \left| u^U(r) - v^U(r) \right| \right\}.$$

It is well-known that (E, D) is a complete metric space [15].

3 Generalized α-univex fuzzy mappings and properties

It is well known that differentiability and gradient are important contents for the study of generalized convexity, and they have been studied by many scholars in recent years. Different differentiability will bring different results to the research. In the following, we mainly apply the $g-$differentiable to this paper.

Definition 3.1. *A mapping $\tilde{f} : X \to E$ is said to be a fuzzy mapping. For any $r \in [0,1]$, denote $\tilde{f}(x,r) = \left[f^L(x,r), f^U(x,r)\right]$.*

Definition 3.2. [14] *Let X be an open subset of R^n, and $\tilde{f} : X \to E$ be a fuzzy mapping. The r-cut of $\tilde{f}$ at $x \in X$ can be denoted by $\tilde{f}(x,r) = [f^L(x,r), f^U(x,r)]$, where $f^L(x,r) = \min \tilde{f}(x,r)$ and $f^U(x,r) = \max \tilde{f}(x,r)$. The two functions $f^L(x,r)$ and $f^U(x,r)$ are functions:$X \times [0,1] \to R$, $f^L(x,r)$ is a bounded increasing function of r and $f^U(x,r)$ is a bounded decreasing function of r, moreover, $f^L(x,r) \leq f^U(x,r)$ for each $r \in [0,1]$. Furthermore, if for each $r \in [0,1]$ both $f^L(x,r)$ and $f^U(x,r)$ are continuous functions at $x \in X$, then we say $\tilde{f}$ is continuous at x.*

Definition 3.3. [8] *Let $\tilde{f} : X \to E$ be a fuzzy mapping, it is said that $\tilde{f}$ is generalized Hukuhara differentiable ($g-$differentiable, for short) at $x_0 \in X$ if for any h small enough, there exists an element $\tilde{f}'(x_0) \in E$ such that*

$$\lim_{h \to 0} \frac{\tilde{f}(x_0 + h) \ominus_g \tilde{f}(x_0)}{h} = \tilde{f}'(x_0).$$

From the definition, we can see $g-$differentiable is based on the existence of $g-$difference between two fuzzy numbers, therefore, $g-$differentiable is more general than $H-$differentiable, $gH-$differentiable and $S-$differentiable. Following, we will give the gradient of a fuzzy mapping $\tilde{f}(x)$.

Definition 3.4. *Let $\tilde{f} : X(\subseteq R^n) \to E$ be a fuzzy mapping, if $\tilde{f}$ is $g-$differentiable about variables $x_i, i = 1, 2, \cdots, n$, and denoted by $D_{x_i}\tilde{f}(x), (i = 1, 2, \cdots, n)$, then*

$$\bigtriangledown \tilde{f}(x) = (D_{x_1}\tilde{f}(x), D_{x_2}\tilde{f}(x), \cdots, D_{x_n}\tilde{f}(x)).$$

We call $\bigtriangledown \tilde{f}(x)$ the gradient of the fuzzy mapping $\tilde{f}$ at x.

Example 3.5. *$\tilde{f}$ is a fuzzy mapping defined as follow*

$$\tilde{f}(x) =< 0, 1, 2 > x, x \in R.$$

For all $r \in [0,1]$, we have

$$\tilde{f}(x,r) = \begin{cases} [rx, (2-r)x], x \geq 0, \\ [(2-r)x, rx], x < 0. \end{cases}$$

We can easily check that f^L, f^U are not differentiable at $x = 0$, if let $r = 0$, because $\lim_{x \to 0^+} f^L(x,r) = r \neq \lim_{x \to 0^-} f^L(x,r) = 2 - r$, and $\lim_{x \to 0^+} f^U(x,r) = 2 - r \neq \lim_{x \to 0^-} f^U(x,r) = r$. But it is g-differentiable for all $x \in X$, because $\tilde{f}(x) = C \cdot x$, $[C]^r = [r, 2-r]$, according to the Definition 3.3, $\tilde{f}'(x) = C, C \in E$.

From the example, we imply that g-differentiability is more general than weakly differentiability [38]. Therefore, in this paper, we mainly apply the g-differentiability to the generalized α-univex fuzzy mappings. In the following, we let: X be a nonempty open set in R^n, $\eta : X \times X \to R^n$, $\alpha : X \times X \to R \setminus \{0\}$, $\Phi : R \to R$, $b : X \times X \to R^+$.

Definition 3.6. [28] *A subset $X(\subseteq R^n)$ is said to be an α-invex set with respect to $\eta : X \times X \to R^n$, $\alpha(x,y) : X \times X \to R \setminus \{0\}$, if for each $x, y \in X$,*

$$y + \lambda\alpha(x,y)\eta(x,y) \in X,\ \ \lambda \in [0,1].$$

Definition 3.7. *Let X be a nonempty open set in R^n, which is α-invex for all $y \in X$ with respect to η. Then, a fuzzy mapping $\tilde{f}$ is said to be $\alpha-$preunivex with respect to η, α, Φ, b, i.e., for all $x \in X$,*

$$\tilde{f}(y + \lambda\alpha(x,y)\eta(x,y)) \preceq \tilde{f}(y) + \lambda b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right].$$

Example 3.8. *Consider a fuzzy mapping*

$$\tilde{f}(x) =< 0, 1, 2 > x, x \in X = (-\infty, 0).$$

It is equivalent to $\tilde{f}(x,r) = [(2-r)x, rx], r \in [0,1]$. For any $x, y \in X$, let $\eta(x,y) = \begin{cases} y - x, & x \geq y \\ x - y, & x < y \end{cases}$, $\alpha(x,y) = \begin{cases} -\frac{3}{2}, & x \geq y \\ \frac{1}{2}, & x < y \end{cases}$, obviously, $y + \lambda\alpha(x,y)\eta(x,y) \in X$. Furthermore, let $b(x,y) = \begin{cases} 1, & x \geq y \\ \frac{1}{3}, & x < y \end{cases}$, $\Phi(\mu) = \mu$, $\tilde{f}$ is $\alpha-$preunivex with the same η, α. Such as, for $x = -1, y = -2$, then $y + \lambda\alpha(x,y)\eta(x,y) = -2 - \frac{3}{2}\lambda \in X$, for all $\lambda \in [0,1]$, and $[(2-r)(-2-\frac{3}{2}\lambda), r(-2-\frac{3}{2}\lambda)] \preceq [(2-r)(\lambda-2), r(\lambda-2)]$, satisfies the Definition 3.7 for each $r \in [0,1]$. Therefore, X is $\alpha-$invex and $\tilde{f}$ is $\alpha-$preunivex.

Definition 3.9. *Let $\tilde{f}$ be a differentiable fuzzy mapping. Then $\tilde{f}$ is said to be univex with respect to η, Φ, b, i.e., for all $x, y \in X$,*

$$b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right] \succeq \eta(x,y)^T \bigtriangledown \tilde{f}(y).$$

Definition 3.10. *Let $\tilde{f}$ be a differentiable fuzzy mapping. Then $\tilde{f}$ is said to be $\alpha-$univex with respect to η, α, Φ, b, i.e., for all $x, y \in X$,*

$$b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right] \succeq \eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y).$$

Remark 3.11. *If a fuzzy mapping $\tilde{f}$ is univex, then it is $\alpha-$univex with respect to $\alpha(x,y) = 1$; However, if a fuzzy mapping $\tilde{f}$ is $\alpha-$univex, it may not be univex. Therefore, $\alpha-$univex fuzzy mapping is a more general convexity comparing to univex fuzzy mapping. The following example explains it.*

Example 3.12. *Consider the following fuzzy mapping*

$$\tilde{f}(x,r) = [1, 2-r]x^2, (r \in [0,1], x > 0).$$

Firstly, we can check that $\tilde{f}$ is not a univex with respect to

$$b(x,y) = \begin{cases} \frac{1}{x+y}, & x \geq y \\ 1, & x < y \end{cases}, \eta(x,y) = \begin{cases} 1, & x \geq y > 0 \\ x - y, & y > x > 0 \end{cases}, \Phi(\mu) = \mu.$$

For example, let $x = y = 1$, we have $0 \not\succeq [2, 2(2-r)]$ for each $r \in [0,1]$. However, $\tilde{f}$ is fuzzy $\alpha-$univex mapping with $\alpha = \begin{cases} \frac{1}{3}, & x \geq y \\ \frac{x}{y} + 1, & x < y \end{cases}$ and the same η, b and Φ. Such as, let $x = 1, y = 2$, obviously, $[-3(2-r), -3r] \succeq [-6(2-r), -6r]$ for each $r \in [0,1]$.

Definition 3.13. *Let $\tilde{f}$ be a differentiable fuzzy mapping. Then $\tilde{f}$ is said to be $\alpha-$unincave with respect to η, α, Φ, b, i.e., for all $x, y \in X$,*

$$b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right] \preceq \eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y).$$

Definition 3.14. *Let $\tilde{f}$ be a differentiable fuzzy mapping. Then $\tilde{f}$ is said to be pseudo $\alpha-$univex with respect to η, α, Φ, b, i.e., for all $x, y \in X$,*

$$\eta(x,y)^T \alpha(x,y) \bigtriangledown \tilde{f}(y) \succeq 0 \Rightarrow b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right] \succeq 0,$$

or

$$b(x,y)\Phi\left[\tilde{f}(x) \ominus_g \tilde{f}(y)\right] \prec 0 \Rightarrow \eta(x,y)^T \alpha(x,y) \bigtriangledown \tilde{f}(y) \prec 0.$$

Example 3.15. *Consider the follow fuzzy mapping*

$$\tilde{f}(x_1,x_2) = [1, 2-r]x_1^2 + [1, 2-r]x_2^2.$$

Let $X = \left\{(x_1,x_2)|x_2 > x_1 > 0, (x_1, x_2 \in R^2)\right\}$, $\eta(x,y) = x - \frac{1}{2}y$, $\alpha(x,y) = p \in (0, +\infty)$, $b(x,y) = 1$ and $\Phi(\mu) = \mu$. Take $x = (\frac{3}{2}, 2)$, $y = (1,2)$, then, $\eta(x,y) = (1,1)^T$, we get $\eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y) = 6p[1, 2-r] \succeq 0$ for all $r \in [0,1]$, at the same time, $b(x,y)\Phi\left[\tilde{f}(x) \ominus_{gH} \tilde{f}(y)\right] = \frac{5}{4}[1, 2-r] \succeq 0$ for all $r \in [0,1]$. Therefore, $\tilde{f}(x)$ is pseudo $\alpha-$univex.

In the following, we discuss some relations of $\alpha-$preunivex, $\alpha-$univex and pseudo $\alpha-$univex fuzzy mappings. There are some interesting results about generalized $\alpha-$univex fuzzy mappings.

Condition C [28]. Let X be an $\alpha-$invex set. For any $x, y \in X$ and $\lambda \in [0,1]$, we have

$$\eta\left(y, y + \lambda\alpha(x,y)\eta(x,y)\right) = -\lambda\eta(x,y), \quad \eta\left(x, y + \lambda\alpha(x,y)\eta(x,y)\right) = (1-\lambda)\eta(x,y).$$

Theorem 3.16. *Let $X \subseteq R^n$ be an $\alpha-$invex set with respect to α, η. If Condition C hold and the fuzzy mapping $\tilde{f}$ is $\alpha-$preunivex with respect to b, Φ and the same α, η on X, then the convex fuzzy mapping $\tilde{\varphi}(\lambda) = \tilde{f}(y + \lambda\alpha(x,y)\eta(x,y))$ is preunivex on $[0,1]$.*

Proof. To proof the fuzzy mapping $\tilde{\varphi}(\lambda)$ is preunivex on $[0,1]$, we just need to proof for any $\lambda_1, \lambda_2, k \in [0,1]$, the following formula hold true,

$$\tilde{\varphi}\left(\lambda_2 + k\eta(\lambda_1,\lambda_2)\right) \preceq \tilde{\varphi}(\lambda_2) + kb\Phi[\tilde{\varphi}(\lambda_1) \ominus_g \tilde{\varphi}(\lambda_2)].$$

By the Lemma ([21]) and the $\alpha-$preunivexity of $\tilde{f}$, we get

$$\begin{aligned}
&\tilde{\varphi}\left(\lambda_2 + k\eta(\lambda_1,\lambda_2)\right)\\
=&\tilde{\varphi}\left(\lambda_2 + k(\lambda_1 - \lambda_2)\right)\\
=&\tilde{f}\left(y + (\lambda_2 + k(\lambda_1-\lambda_2))\alpha(x,y)\eta(x,y)\right)\\
=&\tilde{f}\left(y + \lambda_2\alpha(x,y)\eta(x,y) + k(\lambda_1-\lambda_2)\alpha(x,y)\eta(x,y)\right)\\
=&\tilde{f}\left(y + \lambda_2\alpha(x,y)\eta(x,y) + k\alpha(x,y)\eta(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y))\right)\\
=&\tilde{f}(y + \lambda_2\alpha(x,y)\eta(x,y) + k\alpha(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y))\eta(y+\lambda_1\\
&\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y)))\\
\preceq&\tilde{f}(y+\lambda_2\alpha(x,y)\eta(x,y)) + kb\Phi\left[\tilde{f}(y+\lambda_1\alpha(x,y)\eta(x,y)) \ominus_g \tilde{f}(y+\lambda_2\alpha(x,y)\eta(x,y))\right]\\
=&\tilde{\varphi}(\lambda_2) + kb\Phi\left[\tilde{\varphi}(\lambda_1) \ominus_g \tilde{\varphi}(\lambda_2)\right].
\end{aligned}$$

Hence, $\tilde{\varphi}(\lambda)$ is preunivex on $[0,1]$ with respect to α, η, and Φ. □

Theorem 3.17. *Let $\tilde{f}$ be a g-differentiable fuzzy mapping on an open $\alpha-$invex set $X \subseteq R^n$ with respect to α, η, and f^L, f^U are also differentiable. If $\tilde{f}$ is $\alpha-$preunivex, then $\tilde{f}$ is $\alpha-$univex.*

Proof. Since $\tilde{f}$ is $\alpha-$preunivex, there exist α, η, b and Φ, $\lambda \in [0,1]$ such that

$$\tilde{f}(y + \lambda(x,y)\alpha(x,y)\eta(x,y)) \preceq \tilde{f}(y) + \lambda b(x,y)\Phi[\tilde{f}(x) \ominus_g \tilde{f}(y)].$$

For all $r \in [0,1]$, we have

$$f^L(y + \lambda(x,y)\alpha(x,y)\eta(x,y), r) \leq f^L(y,r) + \lambda b(x,y)\Phi^L[\tilde{f}(x,r) \ominus_g \tilde{f}(y,r)],$$

$$f^U(y + \lambda(x,y)\alpha(x,y)\eta(x,y), r) \leq f^U(y,r) + \lambda b(x,y)\Phi^U[\tilde{f}(x,r) \ominus_g \tilde{f}(y,r)].$$

Since $\tilde{f}$ is g-differentiable and f^L, f^U are also differentiable, we have

$$\alpha(x,y)\eta(x,y)^T \bigtriangledown f^L(y,r) = \lim_{\lambda \to 0^+} \frac{f^L(y+\lambda(x,y)\alpha(x,y)\eta(x,y),r) - f^L(y,r)}{\lambda} \leq b(x,y)\Phi^L[\tilde{f}(x,r) \ominus_g \tilde{f}(y,r)],$$

$$\alpha(x,y)\eta(x,y)^T \bigtriangledown f^U(y,r) = \lim_{\lambda \to 0^+} \frac{f^U(y+\lambda(x,y)\alpha(x,y)\eta(x,y),r) - f^U(y,r)}{\lambda} \leq b(x,y)\Phi^U[\tilde{f}(x,r) \ominus_g \tilde{f}(y,r)].$$

Thus, we get

$$b(x,y)\Phi[\tilde{f}(x) \ominus_g \tilde{f}(y)] \succeq \eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y).$$

Therefore, $\tilde{f}$ is $\alpha-$univex. □

Theorem 3.18. *Let $\tilde{f}$ be a g-differentiable fuzzy mapping on an open $\alpha-$invex set $X \subseteq R^n$ with respect to α, η. If $\tilde{f}$ is $\alpha-$univex, then $\tilde{f}$ is pseudo $\alpha-$univex.*

Proof. By contradiction, suppose $\tilde{f}$ is not a pseudo $\alpha-$univex fuzzy mapping, then there exist α, η, b and Φ such that

$$\eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y) \succeq 0 \Rightarrow b(x,y)\Phi[\tilde{f}(x) \ominus_g \tilde{f}(y)] \not\succeq 0.$$

Since $\tilde{f}$ is $\alpha-$univex, that is

$$b(x,y)\Phi[\tilde{f}(x) \ominus_g \tilde{f}(y)] \succeq \eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y).$$

Because $\eta(x,y)^T\alpha(x,y) \bigtriangledown \tilde{f}(y) \succeq 0$, we get

$$b(x,y)\Phi[\tilde{f}(x) \ominus_g \tilde{f}(y)] \succeq 0.$$

Obviously, this contradicts the assumption. Therefore, $\tilde{f}$ is pseudo $\alpha-$univex. □

Theorem 3.19. *Let $\tilde{f}$ be a g-differentiable fuzzy mapping on an open $\alpha-$invex set $X \subseteq R^n$ with respect to α, η. If $\tilde{f}$ is $\alpha-$preunivex, then $\tilde{f}$ is pseudo $\alpha-$univex.*

Proof. Since $\tilde{f}$ is $\alpha-$preunivex, from Theorem 3.17, we can get that $\tilde{f}$ is $\alpha-$univex. Then from Theorem 3.18, $\tilde{f}$ is pseudo $\alpha-$univex. The proof is completed. □

4 Sufficient and necessary optimality conditions of constrained optimization problem

In this section, we establish some sufficient and necessary Karush-Kuhn-Tucker conditions for a $\bar{x} \in X$ to be a feasible solution of constrained optimization problem (FP). Let $\tilde{f}(x)$, $\tilde{g}_i(x)(i = 1, \cdots, m)$, $\tilde{h}_k(x)(k = 1, \cdots, l)$ be g-differentiable fuzzy mappings defined on a nonempty open set $X \subseteq R^n$. In the following, we consider the primal problem (FP):

$$\begin{aligned} \min \quad & \tilde{f}(x) \\ s.t. \quad & \tilde{g}_i(x) \preceq 0, i = 1, \cdots, m; \\ & \tilde{h}_k(x) = 0, k = 1, \cdots, l; \\ & x \in X. \end{aligned}$$

We denote the feasible set of the primal problem (FP) by

$$FP := \left\{ x \in X : \tilde{g}(x) \preceq 0, \tilde{h}(x) = 0 \right\}.$$

Since $\preceq$ is a partial order, there may not exist optimal solution for some situations. Therefore, we mainly consider the non-dominated solution as following:

Definition 4.1. *Let $\bar{x}$ be a feasible solution of (FP), then* *(i) $\bar{x}$ is said to be a local non-dominated solution of problem (FP) if there exists $\delta > 0$, and no $x \in N(x,\delta)$ such that $\tilde{f}(x) \prec \tilde{f}(\bar{x})$;*

(ii) $\bar{x}$ is said to be a non-dominated solution of problem (FP) if and only if there does not exist $x \in X$ such that $\tilde{f}(x) \prec \tilde{f}(\bar{x})$.

Theorem 4.2. *(Necessary Condition) Let X be a nonempty open set of R^n. $\bar{x}$ be a non-dominated solution of (FP). And let $I = \{i : \tilde{g}_i(\bar{x}) = 0\}$. Assume that $\tilde{f}, \tilde{g}_i, \tilde{h}_k$ are g−differentiable fuzzy mappings at $\bar{x}$. Furthermore, we assume that the constraints qualification $T^U = G^U \cap H^U$ holds true, where T^U is the cone of tangents of the feasible region at $\bar{x}$, and*

$$G^U = \left\{\eta : \{\triangledown\tilde{g}_i(\bar{x})^T\eta(x,y)\}^U \leq 0,\ i \in I\right\},$$
$$H^U = \left\{\eta : \left\{\triangledown\tilde{h}_k(\bar{x})^T\eta(x,y)\right\}^U = 0,\ k = 1,\cdots l\right\}.$$

If $F^U \cap T^U = \varnothing$, there, $F^U = \left\{\eta : \left\{\triangledown\tilde{f}(\bar{x})^T\eta(x,y)\right\}^U < 0\right\}$, then, there exist scalars $u_i \geq 0$ for $i \in I$ and v_k for $k = 1,\cdots,l$ such that

$$\triangledown\tilde{f}(\bar{x}) + \sum_{i\in I} u_i \triangledown \tilde{g}_i(\bar{x}) + \sum_{k=1}^{l} v_k \triangledown \tilde{h}_k(\bar{x}) = 0.$$

Proof. Since $F^U = \left\{\eta : \left\{\triangledown\tilde{f}(\bar{x})^T\eta(x,y)\right\}^U < 0\right\}$, then there must have $F^L = \left\{\eta : \left\{\triangledown\tilde{f}(\bar{x})^T\eta(x,y)\right\}^L < 0\right\}$. Also, we have

$$G^L = \left\{\eta : \{\triangledown\tilde{g}_i(\bar{x})^T\eta(x,y)\}^L \leq 0,\ i \in I\right\},$$
$$H^L = \left\{\eta : \left\{\triangledown\tilde{h}_k(\bar{x})^T\eta(x,y)\right\}^L = 0,\ k = 1,\cdots l\right\}.$$

By the constraint qualification, we get $F^L \cap G^L \cap H^L = \varnothing$, that is the following system has no solution:

$$\left\{\eta(x,y)^T \triangledown \tilde{f}(\bar{x},r)\right\}^L < 0,\ \ \{\eta(x,y)^T \triangledown \tilde{g}_i(\bar{x},r)\}^L \leq 0,\ \ \left\{\eta(x,y)^T \triangledown \tilde{h}_k(\bar{x},r)\right\}^L = 0.$$

(i) If $\left\{\eta(x,y)^T \triangledown \tilde{f}(\bar{x},r)\right\}^L = \eta(x,y)^T \left\{\triangledown\tilde{f}(\bar{x},r)\right\}^L$, i.e.

$$\{\eta(x,y)^T \triangledown \tilde{g}_i(\bar{x},r)\}^L = \eta(x,y)^T \{\triangledown\tilde{g}_i(\bar{x},r)\}^L,$$
$$\left\{\eta(x,y)^T \triangledown \tilde{h}_k(\bar{x},r)\right\}^L = \eta(x,y)^T \left\{\triangledown\tilde{h}_k(\bar{x},r)\right\}^L.$$

Let the rows of A be given by $\{\triangledown\tilde{g}_i(\bar{x},r)\}^L$, $i \in I$, $\{\triangledown h_k(\bar{x},r)\}^L$ and $-\{\triangledown h_k(\bar{x},r)\}^L$ for $k = 1,\cdots,l$, and $c = -\{\triangledown f(\bar{x},r)\}^L$, then the above system can be simplified to the following system:

$$A\eta \leq 0, \quad c^T\eta > 0.$$

By the Farkas's Theorem ([3], page 55), the system $A^T\xi = c, \xi \geq 0$ has a solution, that is there exist $u_i \geq 0$ for $i \in I$ and v_k, such that

$$\left\{\triangledown\tilde{f}(\bar{x},r)\right\}^L + \sum_{i\in I} u_i \{\triangledown\tilde{g}_i(\bar{x},r)\}^L + \sum_{k=1}^{l} v_k \left\{\triangledown\tilde{h}_k(\bar{x},r)\right\}^L = 0.$$

(ii) If $\left\{\eta(x,y)^T \triangledown \tilde{f}(\bar{x},r)\right\}^L = \eta(x,y)^T \left\{\triangledown\tilde{f}(\bar{x},r)\right\}^U$, i.e.

$$\{\eta(x,y)^T \triangledown \tilde{g}_i(\bar{x},r)\}^L = \eta(x,y)^T \{\triangledown\tilde{g}_i(\bar{x},r)\}^U,$$
$$\left\{\eta(x,y)^T \triangledown \tilde{h}_k(\bar{x},r)\right\}^L = \eta(x,y)^T \left\{\triangledown\tilde{h}_k(\bar{x},r)\right\}^U.$$

We also can get the same conclusion.

In the same way, we can get

$$\left\{\bigtriangledown\tilde{f}(\bar{x},r)\right\}^U + \sum_{i\in I} u_i \left\{\bigtriangledown\tilde{g}_i(\bar{x},r)\right\}^U + \sum_{k=1}^{l} v_k \left\{\bigtriangledown\tilde{h}_k(\bar{x},r)\right\}^U = 0.$$

Therefore, we have

$$\bigtriangledown\tilde{f}(\bar{x}) + \sum_{i\in I} u_i \bigtriangledown \tilde{g}_i(\bar{x}) + \sum_{k=1}^{l} v_k \bigtriangledown \tilde{h}_k(\bar{x}) = 0.$$

□

In Theorem 4.2, we can see the necessary condition is suitable for any generalized convex fuzzy mappings. But in the following, we will limit $\tilde{f}, \tilde{g}_i, \tilde{h}_k$ with generalized $\alpha-$univex fuzzy mappings, we also get necessary Karush-Kuhn-Tucker condition.

Theorem 4.3. *(Necessary Condition) Let X be a nonempty open set of R^n, $\bar{x}$ be a non-dominated solution of (FP). And let $I = \{i : \tilde{g}_i(\bar{x}) = 0\}$. Assume that $\tilde{f}, \tilde{g}_i, \tilde{h}_k$ are differentiable at $\bar{x}$.*

(i) we assume $\tilde{f}, \tilde{g}$ satisfy the following conditions at $\bar{x}$

$$b_0(x,\bar{x})\Phi_0\left[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})\right] \succeq \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{f}(\bar{x})\ ,\ -b_i(x,\bar{x})\Phi_i\left[\tilde{g}_i(\bar{x})\right] \succeq \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x}).$$

(ii) $\tilde{h}_k(x)$ is $\alpha-$univex in $J = \{k : v_k > 0\}$, and $\tilde{h}_k(x)$ is $\alpha-$unicave in

$K = \{k : v_k < 0\}$.

Furthermore, suppose $\mu \not\preceq 0 \Rightarrow \Phi_0(\mu) \not\preceq 0$, $\quad \mu \preceq 0 \Rightarrow \Phi_i(\mu) \succeq 0$. Then, there exist scalars $u_i \geq 0$ for $i \in I$ and v_k for $k = 1, \cdots, l$ such that

$$\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L + \sum_{i\in I} u_i \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L + \sum_{k=1}^{l} \left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L = 0.$$

$b_0(x,\bar{x}) > 0, b_i(x,\bar{x}) \geq 0, b_k(x,\bar{x}) \geq 0.$

Proof. Since $\bar{x}$ is a non-dominated solution of (FP), then we have $\tilde{f}(x) \not\preceq \tilde{f}(\bar{x})$, and $\tilde{f}(x) \ominus_g \tilde{f}(\bar{x}) \not\preceq 0$. In addition, $\bar{x}$ is also a feasible solution, so, we have $\tilde{g}_i(\bar{x}) \preceq 0$, then $\Phi_i[\tilde{g}_i(\bar{x})] \succeq 0$. Furthermore, we have $\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x}) \preceq 0$. By the assumption of $\tilde{h}(x)$, there exist $\alpha, \eta, \Phi_k, b_k$ such that

$$0 = b_k(x,\bar{x})\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(\bar{x})] \succeq \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{h}_k(\bar{x}), \quad k \in J,$$

$$0 = b_k(x,\bar{x})\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(\bar{x})] \preceq \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{h}_k(\bar{x}), \quad k \in K.$$

Multiplying the above two inequalities by $v_k > 0$ and $v_k < 0$, respectively, we have

$$v_k\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{h}_k(\bar{x}) \preceq 0, \quad k \in J \cup K.$$

Noting that, $v_k = 0$ when $k \notin J \cup K$. Since $\tilde{f}(x)$ is $\alpha-$univex fuzzy mapping, we have

$$\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{f}(\bar{x}) \preceq b_0(x,\bar{x})\Phi_0\left[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})\right] \not\preceq 0.$$

Therefore, there must exist $\alpha(x,y)$ and $\eta(x,y)$ such that $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{f}(\bar{x})\right\}^U \geq 0$.

(i) If $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{f}(\bar{x})\right\}^U = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$, i.e.

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x})\right\}^U = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^U,$$

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^U = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^U.$$

Then, there have $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$, i.e.

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L,$$

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L.$$

Thus, there exist scalars $u_i\geq 0$ for $i\in I$ and v_k for $k=1,\cdots,l$ such that

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^L+\sum_{i\in I}u_i\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L+$$
$$\sum_{k=1}^{l}\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L+$$
$$\sum_{i\in I}u_i\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L+\sum_{k=1}^{l}\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L=0.$$

Therefore, multiply by $\eta^{-T}(x,y)\alpha^{-1}(x,y)$ on both sides of the equation, we get

$$\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L+\sum_{i\in I}u_i\left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L+\sum_{k=1}^{l}\left\{v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L=0.$$

(ii) If $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$, i.e.

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^U,$$

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^U.$$

Then, we have $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^U=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$, i.e.,

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{g}_i(\bar{x})\right\}^U=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L,$$

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^U=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{v_k\bigtriangledown\tilde{h}_k(\bar{x})\right\}^L.$$

We also can get the conclusion of the case (i). Similarly, we can discuss the case

(iii) $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^U=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$,

(iv) $\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x})\right\}^L=\eta(x,\bar{x})^T\alpha(x,\bar{x})\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$.

□

In Theorem 4.2 and Theorem 4.3, we get necessary Karush-Kuhn-Tucker conditions, in the following Theorem 4.4 and Theorem 4.5 we will get sufficient Karush-Kuhn-Tucker conditions.

Theorem 4.4. *(Sufficient Condition) Let $\bar{x}$ be a feasible solution of the primal problem (FP). Suppose that:*

(i) There exist $\eta,\alpha,\Phi_0,b_0,\Phi_i,b_i$ such that $\tilde{f}(x),\tilde{g}_i(x)$ satisfy the following

conditions at $\bar{x}$

$$b_0(x,\bar{x})\Phi_0\left[\tilde{f}(x)\ominus_g\tilde{f}(\bar{x})\right]\succeq\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{f}(\bar{x}),\tag{1}$$

$$-b_i(x,\bar{x})\Phi_i\left[\tilde{g}_i(\bar{x})\right]\succeq\eta(x,\bar{x})^T\alpha(x,\bar{x})\bigtriangledown\tilde{g}_i(\bar{x}).\tag{2}$$

(ii) $\tilde{h}_k(x)$ *is* $\alpha-$*univex in* $J = \{k : v_k > 0\}$, *and* $\tilde{h}_k(x)$ *is* $\alpha-$*unincave in* $K = \{k : v_k < 0\}$.

(iii) There exist $u_i, v_k \in R^m$ *such that*

$$\left\{\triangledown \tilde{f}(\bar{x})\right\}^L + \sum_{i\in I} u_i \left\{\triangledown \tilde{g}_i(\bar{x})\right\}^L + \sum_{k\in J\cup K} \left\{v_k \triangledown \tilde{h}_k(\bar{x})\right\}^L = 0,\ u_i \geq 0. \tag{3}$$

Further suppose that

$$\Phi_0(\mu) \not\prec 0 \Rightarrow \mu \not\prec 0, \tag{4}$$

$$\mu \preceq 0 \Rightarrow \Phi_i(\mu) \succeq 0, \tag{5}$$

and

$$b_0(x, \bar{x}) > 0, b_i(x, \bar{x}) \geq 0, b_k(x, \bar{x}) \geq 0,$$

for all feasible x*. Then,* $\bar{x}$ *is a non-dominated solution of (FP).*

Proof. Because $\bar{x}$ is a feasible point of (FP), there have $\tilde{g}_i(\bar{x}) \preceq 0$. From (5), we have $\Phi_i[\tilde{g}_i(\bar{x})] \succeq 0$, and from (2), we get

$$\eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{g}_i(\bar{x}) \preceq 0.$$

So, we have

$$-\eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{g}_i(\bar{x}) \succeq 0.$$

From suppose (ii), there exist $\alpha, \eta, \Phi_k, b_k$ such that

$$0 = b_k(x, \bar{x})\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(\bar{x})] \succeq \eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{h}_k(\bar{x}), \quad k \in J,$$

$$0 = b_k(x, \bar{x})\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(\bar{x})] \preceq \eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{h}_k(\bar{x}), \quad k \in K.$$

Multiplying the above two inequalities by $v_k > 0$ and $v_k < 0$, respectively, we have

$$v_k \eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{h}_k(\bar{x}) \preceq 0, \quad k \in J \cup K.$$

So, $-v_k \eta(x, \bar{x})^T \alpha(x, \bar{x}) \triangledown \tilde{h}_k(\bar{x}) \succeq 0$. Noting that $v_k = 0$, when $k \notin J \cup K$.

(i) If $\left\{\eta^T(x, y)\alpha(x, y) \triangledown \tilde{f}(\bar{x})\right\}^L = \eta(x, y)^T \alpha(x, y) \left\{\triangledown \tilde{f}(\bar{x})\right\}^L$, from (1), (3), we have

$$b(x, y)\Phi_0^L[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \geq \left\{\eta^T(x, y)\alpha(x, y) \triangledown \tilde{f}(\bar{x})\right\}^L = \eta(x, y)^T \alpha(x, y) \left\{\triangledown \tilde{f}(\bar{x})\right\}^L$$
$$= -\eta(x, y)^T \alpha(x, y) \sum_{i\in I} u_i \left\{\triangledown \tilde{g}_i(\bar{x})\right\}^L - \eta(x, y)^T \alpha(x, y) \sum_{k\in J\cup K} \left\{v_k \triangledown \tilde{h}_k(\bar{x})\right\}^L \geq 0.$$

Because $b(x, y)\Phi_0^L[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \geq 0$, thus, there must have $b(x, y)\Phi_0^U[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \geq 0$. Therefore, we get $b(x, y)\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \succeq 0$. Then from (4), it implies, $\tilde{f}(x) \ominus_g \tilde{f}(\bar{x}) \not\prec 0$. Thus, $\tilde{f}(x) \not\prec \tilde{f}(\bar{x})$. Therefore, $\bar{x}$ is a non-dominated solution of (FP).

(ii) If $\left\{\eta^T(x, y)\alpha(x, y) \triangledown \tilde{f}(\bar{x})\right\}^U = \eta(x, y)^T \alpha(x, y) \left\{\triangledown \tilde{f}(\bar{x})\right\}^L$, from (1), (3), we have

$$b(x, y)\Phi_0^U[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \geq \left\{\eta^T(x, y)\alpha(x, y) \triangledown \tilde{f}(\bar{x})\right\}^U = \eta(x, y)^T \alpha(x, y) \left\{\triangledown \tilde{f}(\bar{x})\right\}^L$$
$$= -\eta(x, y)^T \alpha(x, y) \sum_{i\in I} u_i \left\{\triangledown \tilde{g}_i(\bar{x})\right\}^L - \eta(x, y)^T \alpha(x, y) \sum_{k\in J\cup K} \left\{v_k \triangledown \tilde{h}_k(\bar{x})\right\}^L \geq 0.$$

It implies,

$$b(x, y)\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \not\prec 0,$$

From (4), we get

$$\tilde{f}(x) \ominus_g \tilde{f}(\bar{x}) \not\prec 0.$$

Thus, $\tilde{f}(x) \not\prec \tilde{f}(\bar{x})$. Therefore, $\bar{x}$ is a non-dominated solution of (FP).

Similarly, we can verify cases

(iii) if $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L = \eta(x,y)^T\alpha(x,y)\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$, i.e.,

(iv) if $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^U = \eta(x,y)^T\alpha(x,y)\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$.

□

In Theorem 4.4 we get sufficient condition under $\tilde{f}, \tilde{g}$ are α-univex fuzzy mapping and $\tilde{h}$ is $\alpha-$univex in $J = \{k : v_k > 0\}$, and $\tilde{h}_k(x)$ is $\alpha-$unincave in $K = \{k : v_k < 0\}$. The following sufficient condition will get when fuzzy mapping $\tilde{f}, \tilde{g}, \tilde{f}, \tilde{h}$ are pseudo $\alpha-$univex.

Theorem 4.5. *(Sufficient Condition) Let $\bar{x}$ be a feasible solution of the primal problem (FP). Suppose that:*

(i) $\tilde{f}(x)$ is a pseudo $\alpha-$univex fuzzy mapping,that is there exist $\eta, \alpha, \Phi_0, b_0$ such that

$$\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{f}(\bar{x}) \succeq 0 \Rightarrow b_0(x,\bar{x})\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \succeq 0. \tag{6}$$

(ii) There exist $\eta, \alpha, b_i, b_k, \Phi_i, \Phi_k, i \in I, k = 1, \cdots, l$ such that

$$-b_i(x,\bar{x})\Phi_i[\tilde{g}_i(\bar{x})] \preceq 0 \Rightarrow \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x}) \preceq 0, \tag{7}$$

$$-b_k(x,\bar{x})\Phi_k[\tilde{h}_k(\bar{x})] = 0 \Rightarrow \begin{cases} \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{h}_k(\bar{x}) \preceq 0, or \\ \eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{h}_k(\bar{x}) \succeq 0. \end{cases} \tag{8}$$

(iii) There exist $u_i, v_k \in R^m$ and $u_i \geq 0$ such that

$$\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L + \sum_{i\in I} u_i \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L + \sum_{k\in J\cup K}\left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L = 0, \tag{9}$$

where, $J = \{k : v_k > 0\}, K = \{k : v_k < 0\}$.
Further suppose that

$$\Phi_0(\mu) \not\prec 0 \Rightarrow \mu \not\prec 0, \tag{10}$$

$$\mu \preceq 0 \Rightarrow \Phi_i(\mu) \succeq 0, \tag{11}$$

and $b_0(x,\bar{x}) > 0, b_i(x,\bar{x}) \geq 0, b_k(x,\bar{x}) \geq 0$, for all feasible x. Then, $\bar{x}$ is a non-dominated solution of (FP).

Proof. Let $\bar{x}$ be a feasible solution of (FP). Then, for $i \in I$, $\tilde{g}_i(\bar{x}) \preceq 0$, $\tilde{h}_k(\bar{x}) = 0$, from(11), we have

$$-b_i(x,\bar{x})\Phi_i[\tilde{g}_i(\bar{x})] \preceq 0 \quad - b_k(x,\bar{x})\Phi_k[\tilde{h}_k(\bar{x})] = 0.$$

So, multiplying the (7) by u_i, we get

$$\eta(x,\bar{x})^T\alpha(x,\bar{x})u_i \bigtriangledown \tilde{g}_i(\bar{x}) \preceq 0.$$

Multiplying the first inequality of (8) by $v_k > 0$ and the second inequality of (8) by $v_k < 0$, we have

$$\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k \bigtriangledown \tilde{h}_k(\bar{x}) \preceq 0, \quad k \in J\cup K,$$

and noting that, when $k \notin J \cup K$, $v_k = 0$.
Multiplying (9) by $\alpha\eta$ and the above conclusions, we can get

(i) If $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L = \eta(x,y)^T\alpha(x,y)\left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$, i.e.,

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x})\right\}^L = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L,$$
$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L.$$

Then, we have

$$\begin{aligned}&\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L = \eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L \\ =& -\eta(x,y)^T\alpha(x,y)\sum_{i\in I} u_i \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L - \eta(x,y)^T\alpha(x,y) \sum_{k\in J\cup K}\left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L \geq 0.\end{aligned}$$

Since $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L \geq 0$, we have $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^U \geq 0$. The two formulas imply $\eta^T(x,y)\alpha(x,y)\bigtriangledown \tilde{f}(\bar{x}) \succeq 0$. So, from (6), we have $b_0(x,\bar{x})\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \succeq 0$. From (10), we have $\Phi_0[\tilde{f}(x) \ominus_{gH} \tilde{f}(\bar{x})] \nprec 0$, and $\tilde{f}(x) \ominus_g \tilde{f}(\bar{x}) \nprec 0$. Therefore, we get $\tilde{f}(x) \nprec \tilde{f}(\bar{x})$. So, $\bar{x}$ is a non-dominated solution of (FP).

(ii) If $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L = \eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$, i.e.,

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x})\right\}^L = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^U,$$
$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^U.$$

Then, $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^U = \eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L$, i.e.,

$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x}) \bigtriangledown \tilde{g}_i(\bar{x})\right\}^U = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L,$$
$$\left\{\eta(x,\bar{x})^T\alpha(x,\bar{x})v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^U = \eta(x,\bar{x})^T\alpha(x,\bar{x}) \left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L.$$

We have

$$\begin{aligned}\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^U =&\eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^L \\ =& -\eta(x,y)^T\alpha(x,y)\sum_{i\in I} u_i \left\{\bigtriangledown\tilde{g}_i(\bar{x})\right\}^L - \eta(x,y)^T\alpha(x,y) \sum_{k\in J\cup K}\left\{v_k \bigtriangledown \tilde{h}_k(\bar{x})\right\}^L \geq 0.\end{aligned}$$

Therefore, we get $\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x}) \nprec 0$. From (6), we have $\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(\bar{x})] \nprec 0$. Thus, $\tilde{f}(x) \ominus_g \tilde{f}(\bar{x}) \nprec 0$ and so, $\tilde{f}(x) \nprec \tilde{f}(\bar{x})$.

In the same way, we can discuss the cases

(iii) if $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^L = \eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$,

(iv) if $\left\{\eta^T(x,y)\alpha(x,y) \bigtriangledown \tilde{f}(\bar{x})\right\}^U = \eta(x,y)^T\alpha(x,y) \left\{\bigtriangledown\tilde{f}(\bar{x})\right\}^U$.

□

Example 4.6. *Consider the following fuzzy programming problem:*

$$\begin{aligned}\min \quad & \tilde{f}(x,r) = [r-2, 1-2r]x^2 (x>0) \\ s.t. \quad & \tilde{g}(x,r) = [r, 2-r](x-2) \preceq 0 \\ & \tilde{h}(x,r) = [r, 2-r](x^2-1) = 0 \\ & r \in [0,1]\end{aligned}$$

Obviously, fuzzy mapping $\tilde{f}(x)$ is $g-$differentiable and $gH-$differentiable, due to $f^L(x,r)$, $f^U(x,r)$, and $f^L(x,r) + f^U(x,r) = (-r-1)x^2$ are not convex functions, so, the methods in [32] and [12] cannot be used.

Let $\Phi_0(\mu)=\mu$, $b_0(x,y)=\begin{cases}\frac{y}{x+y}, & x\geq y\\ 1, & x<y\end{cases}$, $\eta(x,y)=\begin{cases}x-y, & x\geq y\\ x+y, & x<y\end{cases}$, *then* $\tilde{f}(x)$ *is not univex fuzzy mapping, but is* $\alpha-$*univex fuzzy mapping with* $\alpha=\begin{cases}\frac{1}{4}, & x\geq y\\ 1, & x<y\end{cases}$. $\tilde{g}(x),\tilde{h}(x)$ *satisfy Theorem 4.4 with respect to* $\Phi_1(\mu)=|\mu|, b_1(x,y)=1, \Phi_2(\mu)=\Phi_0(\mu), b_2=b_0$ *and the same* α,η. *Thus, let* $r=0$, *we get the feasible point* $x=1$ *is the non-dominated solution, and* $(u,v)=(0,1)$.

Example 4.7. *Consider the following fuzzy programming problem:*

$$\begin{aligned}\min\quad & \tilde{f}(x,r)=[2r-1,2-r]x_1\\ s.t.\quad & \tilde{g}(x,r)=[2r-1,2-r](x_2-1)\preceq 0\\ & \tilde{h}(x,r)=[2r-1,2-r](x_1+x_2)=0\\ & r\in[0,1]\end{aligned}$$

Given $X=\left\{(x_1,x_2),(y_1,y_2)|(x_1,x_2)\geq(y_1,y_2),(x_1,x_2),(y_1,y_2)\in R^2\right\}$. *Obviously,* $f^L(x,r)$, $f^U(x,r)$ *are not differentiable at* $x=0$, $\tilde{f}(x)$ *is not weakly differentiable. Therefore, the method in [20] cannot be used.*

However, $\tilde{f}(x),\tilde{g}(x),\tilde{h}(x)$ *are* $g-$*differentiable. Note that the function* $\tilde{f}(x)$ *is* $\alpha-$*univex fuzzy mapping with* $\alpha(x,y)=\begin{cases}-1, & xy\geq 0\\ 1, & xy<0\end{cases}$, $\eta(x,y)=\begin{cases}x-y, & xy\geq 0\\ x+y, & xy<0\end{cases}$, *and* $b_0=1,\Phi_0(\mu)=\mu$. *For* $\tilde{g}(x),\tilde{h}(x)$ *are also* $\alpha-$*univex with the same* α,η *and* $b_1=1,\Phi_1(\mu)=|\mu|, b_2=1,\Phi_2(\mu)=\mu$, *respectively.*

Let $r=0$, $\tilde{f}(x),\tilde{g}(x),\tilde{h}(x)$ *satisfy the Theorem 4.4. So, the point* $\bar{x}=(-1,1)$ *is a non-dominated solution, and* $(u,v)=(1,-1)$.

5 Duality problem for generalized $\alpha-$univex fuzzy mappings

Consider the following dual problem (DFP) of the problem (FP)

$$\begin{aligned}\max\quad & \tilde{f}(a)\\ s.t.\quad & \bigtriangledown\tilde{f}(a)+\sum_{i\in I}\bar{u}_i\bigtriangledown\tilde{g}_i(a)+\sum_{k=1}^{l}\bar{v}_k\bigtriangledown h_k(a)=0\\ & \bar{u}_i\tilde{g}_i(a)=0\\ & \bar{u}_i\geq 0.\end{aligned}$$

We denote the feasible set of the dual problem (DFP) by

$$D=\{(a,\bar{u}_i,\bar{v}_k)\in R^n\times R^m\times R^m\}$$

It satisfies $\bigtriangledown\tilde{f}(a)+\sum_{i\in I}\bar{u}_i\bigtriangledown\tilde{g}_i(a)+\sum_{k=1}^{l}\bar{v}_k\bigtriangledown h_k(a)=0$, $\bar{u}_i\tilde{g}_i(a)=0$, and $\bar{u}_i\geq 0$. From Theorem 4.4. we have the following weakly duality.

Theorem 5.1. *(Weakly Duality) Let* x *be FP-feasible,* $(a,\bar{u}_i,\bar{v}_k)$ *be DFP-feasible.*

(i) Assume that there exist $\eta,\alpha,\Phi_0,b_0,\Phi_i,b_i,i=1,\cdots,m$, *such that for any*

x, $\tilde{f}(x),\tilde{g}_i(x)$ *at* a *satisfy the following conditions*

$$b_0(x,a)\Phi_0\left[\tilde{f}(x)\ominus_g\tilde{f}(a)\right]\succeq\eta(x,a)^T\alpha(x,a)\bigtriangledown\tilde{f}(a), \tag{12}$$

$$-b_i(x,a)\Phi_i\left[\tilde{g}_i(a)\right]\succeq\eta(x,a)^T\alpha(x,a)\bigtriangledown\tilde{g}_i(a). \tag{13}$$

(ii) $\tilde{h}_k(x)$ *is* $\alpha-$*univex in* $J=\{k:\bar{v}_k>0\}$, *and* $\tilde{h}_k(x)$ *is* $\alpha-$*unicave in* $K=$

$\{k:\bar{v}_k<0\}$ *at* a *for all feasible* x.
Further suppose that

$$\Phi_0(\mu)\not\prec 0\Rightarrow\mu\ \not\prec 0,$$

$$\tag{14}$$

$$\mu = 0 \Rightarrow \Phi_i(\mu) \succeq 0, \tag{15}$$

and

$$b_0(x,a) > 0, b_i(x,\bar{x}) \geq 0, b_k(x,a) \geq 0.$$

Then, $\tilde{f}(x) \not\prec \tilde{f}(a)$.

Proof. Since $(a, \bar{u}_i, \bar{v}_k)$ is DFP-feasible and $\bar{u}_i \geq 0$. Then we have

$$\left\{\triangledown \tilde{f}(a)\right\}^L + \sum_{i\in I} \bar{u}_i \left\{\triangledown \tilde{g}_i(a)\right\}^L + \sum_{k=1}^{l} \left\{\bar{v}_k \triangledown h_k(a)\right\}^L = 0,\ \bar{u}_i\tilde{g}_i(a) = 0,$$

and from (15), we get $\Phi_i[\bar{u}_i\tilde{g}_i(a)] \preceq 0$. So, we have $-b_i(x,a)\Phi_i[\bar{u}_i\tilde{g}_i(a)] \preceq 0$. Thus, from(13), we have $\eta(x,a)^T\alpha(x,a) \triangledown \tilde{g}_i(a) \preceq 0$. From (ii), there exist $\alpha, \eta, \Phi_k, b_k$ such that

$$0 = b_k(x,a)\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(a)] \succeq \eta(x,a)^T\alpha(x,a) \triangledown \tilde{h}_k(a),$$

$$0 = b_k(x,a)\Phi_k[\tilde{h}_k(x) \ominus_g \tilde{h}_k(a)] \preceq \eta(x,a)^T\alpha(x,a) \triangledown \tilde{h}_k(a).$$

Multiplying the above two inequalities by $\bar{v}_k > 0$ and $\bar{v}_k < 0$, respectively, we have

$$\bar{v}_k\eta(x,a)^T\alpha(x,a) \triangledown \tilde{h}_k(a) \preceq 0, \quad k \in \bar{J} \cup \bar{K}.$$

Where, $\bar{J} = \{k : \bar{v}_k > 0\}, \bar{K} = \{k : \bar{v}_k < 0\}$. Noting that, $\bar{v}_k = 0$ when $k \in \bar{J} \cup \bar{K}$.
From (12), we have

(i) If $\left\{\eta(x,a)^T\alpha(x,a) \triangledown \tilde{f}(a)\right\}^L = \eta(x,a)^T\alpha(x,a)\left\{\triangledown \tilde{f}(a)\right\}^L$, we have

$$b_0(x,a)\Phi_0^L\left[\tilde{f}(x) \ominus_g \tilde{f}(a)\right] \geq \left\{\eta(x,a)^T\alpha(x,a) \triangledown \tilde{f}(a)\right\}^L$$
$$= -\eta(x,a)^T\alpha(x,a)\sum_{i\in I} \bar{u}_i \left\{\triangledown \tilde{g}_i(a)\right\}^L - \eta(x,a)^T\alpha(x,a) \sum_{k=J\cup K} \left\{\bar{v}_k \triangledown \tilde{h}_k(a)\right\}^L \geq 0.$$

Since $b_0(x,a)\Phi_0^L\left[\tilde{f}(x) \ominus_g \tilde{f}(a)\right] \geq 0$, there must have

$$b_0(x,a)\Phi_0^U\left[\tilde{f}(x) \ominus_g \tilde{f}(a)\right] \geq 0.$$

The two formulas imply $\Phi_0\left[\tilde{f}(x) \ominus_g \tilde{f}(a)\right] \succeq 0$. Then from (14), we have $\Phi_0\left[\tilde{f}(x) \ominus_g \tilde{f}(a)\right] \not\prec 0$. Thus, we get $\tilde{f}(x) \ominus_g \tilde{f}(a) \not\prec 0$. Therefore, $\tilde{f}(x) \not\prec \tilde{f}(a)$.
Similar to Theorem 4.4, we can discuss these cases (ii), (iii) and (iv).

□

From Theorem 4.5. we have the following weakly duality.

Theorem 5.2. *(Weakly Duality) Let x be FP-feasible, $(a, \bar{u}_i, \bar{v}_k)$ be DFP-feasible.*

(i) Assume that there exist $\eta, \alpha, \Phi_0, b_0, \Phi_i, b_i, \Phi_k, b_k, i = 1, \cdots, m, k = 1, \cdots, l,$

such that for any x, $\tilde{f}(x)$, $\tilde{g}(x)$, $\tilde{h}(x)$ at a satisfy the following conditions

$$\eta(x,a)^T\alpha(x,a) \triangledown \tilde{f}(a) \succeq 0 \Rightarrow b_0(x,a)\Phi_0\left[\tilde{f}(x) -_g \tilde{f}(a)\right] \succeq 0, \tag{16}$$

$$-b_i(x,a)\Phi_i\left[\tilde{g}_i(a)\right] \preceq 0 \Rightarrow \eta(x,a)^T\alpha(x,a) \triangledown \tilde{g}_i(a) \preceq 0. \tag{17}$$

$$-b_k(x,a)\Phi_k[\tilde{h}_k(a)] = 0 \Rightarrow \begin{cases} \eta(x,a)^T\alpha(x,a) \triangledown \tilde{h}_k(a) \preceq 0, or \\ \eta(x,a)^T\alpha(x,a) \triangledown \tilde{h}_k(a) \succeq 0. \end{cases}$$

$$ \tag{18} $$

(ii) Further suppose that

$$\Phi_0(\mu) \not\preceq 0 \Rightarrow \mu \not\preceq 0, \tag{19}$$

$$\mu = 0 \Rightarrow \Phi_i(\mu) \succeq 0, \tag{20}$$

and

$$b_0(x,a) > 0, b_i(x,a) \geq 0, b_k(x,a) \geq 0,$$

for all feasible x. Then, $\tilde{f}(x) \not\preceq \tilde{f}(a)$.

Proof. Since $(a, \bar{u}_i, \bar{v}_k)$ is a feasible solution of (DFP) and $\bar{u}_i \geq 0$. Then,

$$\left\{\bigtriangledown \tilde{f}(a)\right\}^L + \sum_{i \in I} \bar{u}_i \left\{\bigtriangledown \tilde{g}_i(a)\right\}^L + \sum_{k=1}^{l} \left\{\bar{v}_k \bigtriangledown h_k(a)\right\}^L = 0,\ \bar{u}_i \tilde{g}_i(a) = 0,\ \tilde{h}_k(a) = 0,$$

from(20), we have

$$-b_i(x,a)\Phi_i[\bar{u}_i \tilde{g}_i(a)] \preceq 0\ ,\ -b_k(x,a)\Phi_k[\bar{u}_i \tilde{h}_k(a)] \preceq 0.$$

So, multiplying (17) by $\bar{u}_i$, we get

$$\eta(x,a)^T \alpha(x,a) \bar{u}_i \bigtriangledown \tilde{g}_i(a) \preceq 0.$$

Multiplying the first inequality of (18) by $\bar{v}_k > 0$ and the second inequality of (18) by $\bar{v}_k < 0$, we get

$$\eta(x,a)^T \alpha(x,a) \bar{v}_k \bigtriangledown \tilde{h}_k(a) \preceq 0, \quad k \in \bar{J} \cup \bar{K},$$

where $\bar{J} = \{k : \bar{v}_k > 0\}, \bar{K} = \{k : \bar{v}_k < 0\}$ and noting that, when $k \notin \bar{J} \cup \bar{K}$, $\bar{v}_k = 0$.
Combining the above conclusions, we get

(i) If $\left\{\eta(x,a)^T \alpha(x,a) \bigtriangledown \tilde{f}(a)\right\}^L = \eta(x,a)^T \alpha(x,a) \left\{\bigtriangledown \tilde{f}(a)\right\}^L$, then we have

$$\begin{aligned}\left\{\eta(x,a)^T \alpha(x,a) \bigtriangledown \tilde{f}(a)\right\}^L &= \eta(x,a)^T \alpha(x,a) \left\{\bigtriangledown \tilde{f}(a)\right\}^L \\ &= -\eta(x,a)^T \alpha(x,a) \sum_{i \in I} \bar{u}_i \left\{\bigtriangledown \tilde{g}_i(a)\right\}^L - \eta(x,a)^T \alpha(x,a) \sum_{k = J \cup K} \left\{\bar{v}_k \bigtriangledown \tilde{h}_k(a)\right\}^L \geq 0.\end{aligned}$$

It is equivalent to $\eta(x,a)^T \alpha(x,a) \bigtriangledown \tilde{f}(a) \succeq 0$. Then, from (16), we get $b_0(x,a)\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(a)] \succeq 0$. From (19), we have $\Phi_0[\tilde{f}(x) \ominus_g \tilde{f}(a)] \not\preceq 0$,and $\tilde{f}(x) \ominus_g \tilde{f}(a) \not\preceq 0$. Therefore, we get $\tilde{f}(x) \not\preceq \tilde{f}(a)$. Similar to Theorem 4.5, we can discuss cases (ii), (iii) and (iv). □

Example 5.3. *Consider the dual problem of Example 5.1*

$$\begin{aligned} \max \quad & \tilde{f}(w,r) = [r-2, 1-2r]w^2, (w > 0) \\ s.t. \quad & [r-2, 1-2r] \cdot 2w + \bar{u}[r, 2-r] + \bar{v}[r, 2-r] \cdot 2w = 0 \\ & \bar{u}[r, 2-r](w-2) = 0 \\ & \bar{u} \geq 0,\ r \in [0,1] \end{aligned}$$

By calculating, we get that the point $(w, \bar{u}, \bar{v}) = (1, 0, 1)$ is a dual feasible solution. Furthermore, $(\tilde{f}, \tilde{g}, \tilde{h})$ satisfies by hypotheses of Theorem 6.1. So, the weak duality holds, and for any $x \in X$, $\tilde{f}(x) \not\preceq \tilde{f}(1)$.

6 Conclusions

Generalized convex fuzzy mapping has a very important research position in nonlinear fuzzy programming, and it is also a hot topic of research. In this paper, we put forward a new class of generalized univex fuzzy mapping which is defined as $\alpha-$univex fuzzy mapping. $\alpha-$univex fuzzy mapping is more general than univex fuzzy mapping, specially, we say $\alpha-$univex fuzzy mapping is univex fuzzy mapping when $\alpha(x, y) = 1$. In addition, we discuss some relationships of generalized $\alpha-$nivex fuzzy mappings. In the last, we get the necessary and sufficient Karush-Kuhn-Tucker conditions of fuzzy optimization problem which includes equalities and inequalities constraints. We also get its duality problems in the Section 5. In future research, we will continue to study other optimization conditions for $\alpha-$univex fuzzy mappings and fuzzy optimization problems. In addition, extending the one-dimensional fuzzy mappings of $\alpha-$univex to n-dimensional fuzzy mappings is also a very meaningful research work.

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