Iranian Journal of Fuzzy Systems

Volume 16, Number 5, (2019), pp. 139-153

University of Sistan and Baluchestan

Monoidal closedness of L-generalized convergence spaces

L. Zhang[1], J. Fang[2] and W. Wang[3]

[1,2,3] *Department of Mathematics, Ocean University of China, 238 Songling Road, 266100, Qingdao, P.R. China*

1308626321@qq.com, jmfang@ouc.edu.cn, 962927488@qq.com

Abstract

In this paper, it is shown that the category of stratified L-generalized convergence spaces is monoidal closed if the underlying truth-value table L is a complete residuated lattice. In particular, if the underlying truth-value table L is a complete Heyting Algebra, the Cartesian closedness of the category is recaptured by our result.

Keywords: Complete residuated lattice, stratified L-generalized convergence space, monoidal closedness, stratified L-filter, category theory.

1 Introduction

Since classical topology mainly deals with the category **Top** of topological spaces, it is inconvenient with respect to the formation of function spaces because **Top** isn't Cartesian-closed. So, convergence theory was proposed in favour of the existence of function spaces (Cf. [39]). Stratified $[0,1]$-topological spaces (Cf. [16]) had been studied over several decades motivated by Zadeh [43] who introduced the concept of a fuzzy set in order to handle incomplete or inexact information, here the unit interval [0,1] is considered as the truth-value table. As had been pointed out in [31], for stratified $[0,1]$-topological spaces, the function space doesn't exist in general. [30, 31] consider fuzzy convergence spaces as a generalization of Choquet convergence spaces [4], and then the resulting category was shown to be Cartesian-closed. As Höhle [15] pointed out, the theory may turned out to be void in case of more general truth-value table, e.g. a complete residuated lattice [3], instead of the unit interval [0,1]. In 2001, Jäger [17], by using the notion of L-filter defined in case L is a complete Heyting Algebra, developed the theory of stratified L-generalized convergence spaces and the resulting category SL-**GCS** of stratified L-generalized convergence spaces has the desired structural property of Cartesian-closedness and in 2010, with the same truth-value context, Fang [6] defined a subcategory SL-**OCS** of the SL-**GCS**, namely the category of stratified L-ordered convergence spaces, which is also Cartesian-closed. As a continuation of Jäger's works, the theory of these spaces was developed to a significant extent in recent years contributed by Craig and Jäger [5], Boustique and Richardson [2], Fang [7, 8, 9, 10, 11], Flores and others [12], Jäger [18, 19, 20, 21, 22], Jäger and Burton [23], Li [26, 27, 28, 29], Pang [34, 35, 36, 37, 38], Yao [40, 41], D.Orpen and G. Jäger [33] and others.

It is valued to note in case of the underlying lattice L being a complete Heyting Algebra, we see that Jäger and Fang all got nice conclusion. Regretfully, the idempotency about the meet operation $\wedge$ of L is indispensable for Cartesian-closedness of the resulting categories. The requirement of idempotency is not very convenient because the semigroup operation in most of truth-value tables (e.g., complete MV-algebras, complete residuated lattices) is not idempotent.

Besides Cartesian-closedness, monoidal closedness in general is another property of fundamental importance in category theory [24, 25, 32] and monoidal categories are also frequently investigated as mathematical structures both in category theory and theoretical computer science. In addition, Cartesian-closedness is a special case of monoidal closedness. So it is more general to study monoidal closedness. Furthmore, we can also get the expected functional space if the category is monoidal closed. What's more, we remind the reader of the fact that a complete residuated lattice is itself a monoidal closed category. So, it is natural to research monoidal closedness of the category SL-**GCS** of stratified L-generalized convergence spaces when the underlying truth-value table is complete residuated lattice.

Corresponding Author: J. Fang

Received: October 2017; Revised: June 2018; Accepted: April 2019.

In this paper, let $L = (L, *)$ be a complete residuated lattice, then we propose the notion of tensor product, denoted by $\otimes$, between two stratified L-generalized convergence spaces. Thus we can conclude that (SL-**GCS**,$\otimes$) is monoidal category. Further, we concern with closedness of the category SL-**GCS** of stratified L-generalized convergence spaces. As shall be shown, this category is monoidal closed indeed, and if the tensor operation on the underlying complete residuated lattice L is idempotent, equivalently L is a complete Heyting Algebra, the monoidal closedness degenerates into the Cartesian-closedness [17] of the category SL-**GCS** of stratified L-generalized convergence spaces.

The content is organized as follows. Section 2 recalls some basic definitions and results of complete residuated lattices and stratified L-filters. Section 3 focuses on the tensor product of stratified L-generalized convergence spaces, and we conclude that the class of all stratified L-generalized convergence spaces forms a symmetric monoidal category. The monoidal closedness of the category SL-**GCS** of stratified L-generalized convergence spaces is presented in Section 4.

2 Preliminaries

In this paper, the triple $(L, \leq, *)$ denoted by L sometimes, is a complete residuated lattice, which means that $(L, \leq)$ is a complete lattice with the top element $\top$ and bottom element $\bot$ $(\neq \top)$, and a binary operation $* : L \times L \to L$ satisfying the following axioms[3],

(CR1) $\alpha * \top = \alpha$ for all $\alpha \in L$;

(CR2) $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in L$;

(CR3) $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ for all $\alpha, \beta, \gamma \in L$;

(CR4) $\alpha * \bigvee_{j\in J} \beta = \bigvee_{j\in J} \alpha * \beta$ for $\alpha \in L, \{\beta_j\}_{j\in J} \subseteq L$.

From (CR4), there exists the binary operation $\longrightarrow: L \times L \to L$ corresponding to $*$, computed by

$$\alpha \longrightarrow \beta = \bigvee\{\gamma \in L \mid \alpha * \gamma \leq \beta\} \text{ for all } \alpha, \beta \in L.$$

The binary operation $\longrightarrow$ is called the implication about $*$. Further, for each $\alpha \in L$, the monotone mapping $\alpha * (-) : L \to L$ is the left adjoint $\alpha \longrightarrow (-) : L \to L$ in the sense that

$$\alpha * \beta \leq \gamma \Leftrightarrow \beta \leq \alpha \longrightarrow \gamma \text{ for all } \beta, \gamma \in L.$$

For $B \subseteq L$, write $\bigvee B$ for the least upper bound of B and $\bigwedge B$ for the greatest lower bound of B. In particular, $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$.
The implication plays a particular role in the sequel. We list some of its properties.

Lemma 2.1 ([3, 16])**.** *Suppose that $(L, \leq, *)$ is a complete residuated lattice and $\longrightarrow$ is the implication corresponding to $*$. Then for all $\alpha, \beta, \gamma, \delta \in L$, $\{\alpha_j\}_{j\in J} \subseteq L$, the following results hold:*

(1) $\top \longrightarrow \alpha = \alpha$, *and* $0 * \alpha = 0$;

(2) $\alpha \leq \beta$ *if and only if* $\alpha \longrightarrow \beta = \top$;

(3) $(\alpha \longrightarrow \beta) * (\beta \longrightarrow \gamma) \leq (\alpha \longrightarrow \gamma)$;

(4) $\alpha \longrightarrow \bigwedge_{j\in J} \alpha_j = \bigwedge_{j\in J}(\alpha \longrightarrow \alpha_j)$, *hence* $(\alpha \longrightarrow \beta) \leq (\alpha \longrightarrow \gamma)$ *whenever* $\beta \leq \gamma$;

(5) $\left(\bigvee_{j\in J} \alpha_j\right) \longrightarrow \beta = \bigwedge_{j\in J}(\alpha_j \longrightarrow \beta)$, *hence* $(\alpha \longrightarrow \gamma) \geq (\beta \longrightarrow \gamma)$ *whenever* $\alpha \leq \beta$;

(6) $\alpha \longrightarrow (\beta \longrightarrow \gamma) = \beta \longrightarrow (\alpha \longrightarrow \gamma) = (\alpha * \beta) \longrightarrow \gamma$;

(7) $\alpha * (\alpha \longrightarrow \beta) \leq \beta$;

(8) $(\alpha \longrightarrow \beta) * (\gamma \longrightarrow \delta) \leq \alpha * \gamma \longrightarrow \beta * \delta$ *and*
$(\alpha \wedge \gamma) * (\beta \wedge \delta) \leq (\alpha * \beta) \wedge (\gamma * \delta)$.

An L-subset on a set X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. By $\bot_X$ and $\top_X$, we denote the constant L-subsets on X taking the value $\bot$ and $\top$, respectively. We don't distinguish an element $\alpha \in L$ and the constant function $\alpha : X \to L$ such that $\alpha(x) = \alpha$ for all $x \in X$. Further, a subset A of X is identified with its *characteristic function* $\chi_A : X \to \{\bot, \top\}$ defined by $\chi_A(x) = \top$ if $x \in A$ and $= \bot$ otherwise. All algebraic operations on L can be extended to the L-power set L^X pointwise. That is, for all $a, b \in L^X$, $\alpha \in L$ and $x \in X$,

1. $(a \vee b)(x) = a(x) \vee b(x)$,
2. $(a \wedge b)(x) = a(x) \wedge b(x)$,
3. $(a * b)(x) = a(x) * b(x)$,
4. $(a \longrightarrow b)(x) = a(x) \longrightarrow b(x)$.

Obviously, $(L^X, \leq)$ (here, $a \leq b$ if and only if $a(x) \leq b(x)$ for all $x \in X$) is still a complete lattice and for a family $\mathcal{A} \subseteq L^X$ of L-subsets, the infimum $\bigwedge \mathcal{A}$ is given by $(\bigwedge \mathcal{A})(x)= \bigwedge\{f(x) \in L \mid f \in \mathcal{A}\}$ and the supremum $\bigvee \mathcal{A}$ by $(\bigvee \mathcal{A})(x) = \bigvee\{f(x) \in L \mid f \in \mathcal{A}\}$ for all $x \in X$.

For a given set X, define a binary mapping $S_X(-,-) : L^X \times L^X \to L$ by $S_X(A, B) = \bigwedge_{x \in X}(A(x) \longrightarrow B(x))$ for each pair $(A, B) \in L^X \times L^X$, where " $\longrightarrow$ " is the implication corresponding to $*$. For all $A,\ B \in L^X$, $S_X(A, B)$ can be interpreted as the degree to which A is a subset of B. It is called subsethood degree [14] or fuzzy inclusion order [42] of L-subsets.

Let $\varphi : X \to Y$ be a map. Define $\varphi^{\to} : L^X \to L^Y$ and $\varphi^{\leftarrow} : L^Y \to L^X$ by $\varphi^{\to}(f)(y) = \bigvee_{\varphi(x)=y} f(x)$ for $f \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(g) = g \circ \varphi$ for $g \in L^Y$, respectively.

Definition 2.2 ([16])**.** *Let X be a nonempty set. A mapping $\mathcal{F} : L^X \to L$ is called a stratified L-filter on X if it satisfies the following conditions:*

(F1) $\mathcal{F}(\top_X) = \top$, $\mathcal{F}(\bot_X) = \bot$*;*

(F2) $A \leq B$ *means* $\mathcal{F}(A) \leq \mathcal{F}(B)$ *for all* $A, B \in L^X$*;*

(F3) $\mathcal{F}(A) * \mathcal{F}(B) \leq \mathcal{F}(A * B)$ *for all* $A, B \in L^X$*;*

(Fs) $\alpha * \mathcal{F}(A) \leq \mathcal{F}(\alpha * A)$ *for* $\alpha \in L, A \in L^X$*.*

The set of all stratified L-filters on X is denoted by $F^s_L(X)$.

Example 2.3. *Let X be a nonempty set.*

(1) Given a point $x \in X$, the mapping $[x] : L^X \to L$ in [16], defined by for all $A \in L^X, [x](A) = A(x)$, is a stratified L-filter on X, which is called the point L-filter of x.

(2) The mapping $\mathcal{F}_0 : L^X \to L$ in [16], defined by $\mathcal{F}_0(A) = \bigwedge_{x \in X} A(x)$ for $A \in L^X$, is the smallest stratified L-filter with respect to the pointwise order " $\leq$ " on $F^s_L(X)$, i.e.,

$$\mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall A \in L^X, \quad \mathcal{F}(A) \leq \mathcal{G}(A).$$

The infimum of a collection of stratified L-filters $\{\mathcal{F}_j\}_{j \in J}$ with respect to " $\leq$ " always exists and it is determined by $(\bigwedge_{j \in J} \mathcal{F}_j)(A) = \bigwedge_{j \in J} \mathcal{F}_j(A)$ for each $A \in L^X$.

The supremum of two stratified L-filters does not always exist. For its existence, we introduce the following proposition.

Proposition 2.4 ([13])**.** *For two stratified L-filters $\mathcal{F}$ and $\mathcal{G}$ on a set X, the supremum $\mathcal{F} \vee \mathcal{G}$ exists if and only if $\mathcal{F}(A) * \mathcal{G}(B) = \bot$ for all $A, B \in L^X$ whenever $A * B = \bot_X$. In particular, if exists, then the supremum is computed by*

$$(\mathcal{F} \vee \mathcal{G})(C) = \bigvee\{\mathcal{F}(A) * \mathcal{G}(B) \mid A, B \in L^X \text{with } A * B \leq C\}$$

for all $C \in L^X$.

Remark 2.5 (Images and inverse images of stratified L-filters,[16])**.** *Let $\varphi : X \to Y$ be a mapping, $\mathcal{F} \in F^s_L(X)$, and $\mathcal{G} \in F^s_L(Y)$.*

(a) *The mapping* $\varphi^{\Rightarrow}(\mathcal{F}) : L^Y \to L$ *defined for each* $B \in L^Y$ *by* $\varphi^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(\varphi^{\leftarrow}(B)) = \mathcal{F}(B \circ \varphi)$*, is a stratified L-filter, called the image of* $\mathcal{F}$ *under* φ*.*

(b) *The mapping* $\varphi^{\Leftarrow}(\mathcal{G}) : L^X \to L$ *defined for each* $A \in L^X$ *by*

$$\varphi^{\Leftarrow}(\mathcal{G})(A) = \bigvee\{\mathcal{G}(B) \mid \varphi^{\leftarrow}(B) \leq A\},$$

is a stratified L-filter on X*, if and only if* $\mathcal{G}(B) = \bot$ *whenever* $\varphi^{\leftarrow}(B) = B \circ \varphi = \bot_X$*. When* $\varphi^{\Leftarrow}(\mathcal{G}) \in F_L^s(X)$*, it is said to be the inverse image of* $\mathcal{G}$ *under* φ*.*

(c) *In particular,* $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(\mathcal{F}))$ *always exists and* $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(\mathcal{F})) \leq \mathcal{F}$*. If* φ *is injective, then we have* $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(\mathcal{F})) = \mathcal{F}$*.*

(d) *If the inverse image of* $\mathcal{G}$ *is a stratified L-filter on* X*, then* $\varphi^{\Rightarrow}(\varphi^{\Leftarrow}(\mathcal{G})) \geq \mathcal{G}$ *is true. Further, if* φ *is surjective, then* $\varphi^{\Leftarrow}(\mathcal{G})$ *is always a stratified L-filter on* X *and we have* $\varphi^{\Rightarrow}(\varphi^{\Leftarrow}(\mathcal{G})) = \mathcal{G}$*.*

(e) *Let* $A \subseteq X$ *be a nonempty subset and* $i_A : A \to X$ *the canonical inclusion. Then* $i_A^{\Leftarrow}(\mathcal{F})$ *is a stratified L-filter on* A *whenever* $\mathcal{F}(\top_{X-A}) = \bot$*, in the case, call it the restriction of* $\mathcal{F}$ *to* A *and denote it by* $\mathcal{F}_A$*.*

Let X and Y be two nonempty sets, $\mathcal{F} \in F_L^s(X)$ and $\mathcal{G} \in F_L^s(Y)$. From the proof of Proposition 3.6 in [5], we know $p_X^{\Leftarrow}(\mathcal{F})(A) * p_Y^{\Leftarrow}(\mathcal{G})(B) = \bot$ for all $A, B \in L^{X\times Y}$ such that $A * B = \bot_{X\times Y}$, here $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projection mappings to X and to Y, respectively. Thus, $p_X^{\Leftarrow}(\mathcal{F}) \vee p_Y^{\Leftarrow}(\mathcal{G})$ exists with respect to the pointwise order on $F_L^s(X \times Y)$.

Proposition 2.6. *Let* X *and* Y *be two nonempty sets,* $\mathcal{F} \in F_L^s(X)$ *and* $\mathcal{G} \in F_L^s(Y)$*. If we denote their supremum* $p_X^{\Leftarrow}(\mathcal{F}) \vee p_Y^{\Leftarrow}(\mathcal{G})$ *by* $\mathcal{F} \otimes \mathcal{G}$*, then for each* $A \in L^{X\times Y}$

$$\mathcal{F} \otimes \mathcal{G}(A) = \bigvee\{\mathcal{F}(C) * \mathcal{G}(D) \mid C \in L^X, D \in L^Y, C \otimes D \leq A\},$$

here, $C \otimes D \in L^{X\times Y}$ *defined by* $(x, y) \in X \times Y$, $C \otimes D(x, y) = p_X^{\leftarrow}(C) * p_Y^{\leftarrow}(D)(x, y) = C(x) * D(y)$ *and called the tensor product of* C *and* D *in the following.*

Proof. From the Proposition 2.4, we conclude that for each $A \in L^{X\times Y}$,

$$\begin{aligned}
& p_X^{\Leftarrow}(\mathcal{F}) \vee p_Y^{\Leftarrow}(\mathcal{G})(A) \\
= & \bigvee \left\{p_X^{\Leftarrow}(\mathcal{F})(A_1) * p_Y^{\Leftarrow}(\mathcal{G})(A_2) \mid A_1, A_2 \in L^{X\times Y}, A_1 * A_2 \leq A\right\} \\
= & \bigvee \Big\{ \bigvee_{p_X^{\leftarrow}(C)\leq A_1} \mathcal{F}(C) * \bigvee_{p_Y^{\leftarrow}(D)\leq A_2} \mathcal{G}(D) \mid C \in L^X, D \in L^Y, A_1, A_2 \in L^{X\times Y}, \\
& \qquad A_1 * A_2 \leq A\Big\} \\
= & \bigvee \Big\{ \bigvee_{p_X^{\leftarrow}(C)\leq A_1, p_Y^{\leftarrow}(D)\leq A_2} \mathcal{F}(C) * \mathcal{G}(D) \mid C \in L^X, D \in L^Y, A_1, A_2 \in L^{X\times Y}, \\
& \qquad A_1 * A_2 \leq A\Big\} \\
= & \bigvee \left\{\mathcal{F}(C) * \mathcal{G}(D) \mid C \in L^X, D \in L^Y, p_X^{\leftarrow}(C) * p_Y^{\leftarrow}(D) \leq A\right\} \\
= & \bigvee \left\{\mathcal{F}(C) * \mathcal{G}(D) \mid C \in L^X, D \in L^Y, C \otimes D \leq A\right\},
\end{aligned}$$

then the proposition is proved. □

Following the proposition above, $\mathcal{F} \otimes \mathcal{G}$ will be called the tensor product of stratified L-filter $\mathcal{F} \in F_L^s(X)$ and $\mathcal{G} \in F_L^s(Y)$ in the paper.

Following conclusion is a $\otimes$-version of Proposition 3.7 in Jäger[15] and by the idea in the proof of Proposition 3.7 in [15], we can prove the following proposition.

Proposition 2.7. *Let* $\varphi : X \to U$ *and* $\psi : Y \to V$ *be two mappings, and* $\mathcal{F} \in F_L^s(X)$*,* $\mathcal{G} \in F_L^s(Y)$*. Then*

$$(\varphi \times \psi)^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = \varphi^{\Rightarrow}(\mathcal{F}) \otimes \psi^{\Rightarrow}(\mathcal{G}).$$

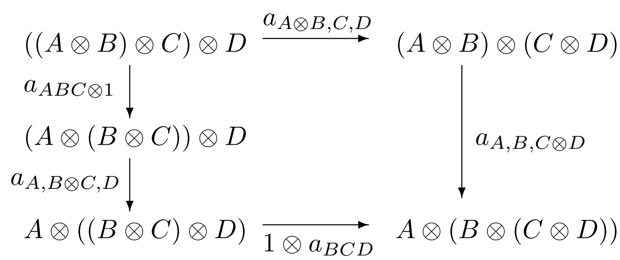

Figure 1: Diagram of Definition 2.8

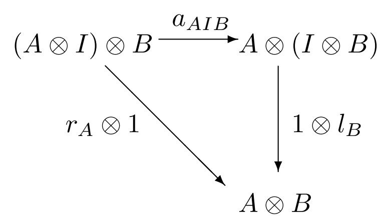

Figure 2: Diagram of Definition 2.8

The reader is referred to the classical text [32] for ideas and results in category theory. We only list the definition of monoidal category and an adjunction between categories, and its characteristic theorem as follows.

Definition 2.8 ([1])**.** *A monoidal category* $\mathbb{C}$ *consists in giving:*

(1) a category $\mathbb{C}$*;*

(2) a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, called the tensor product - we write* $A \otimes B$ *for the image under* $\otimes$ *of the pair* (A, B)*;*

(3) an object $I \in \mathbb{C}$*, called the unit;*

(4) for every triple A, B, C *of objects, an "associativity" isomorphism*

$$a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C);$$

(5) for every object A*, a "left unit" isomorphism*

$$l_A : I \otimes A \to A;$$

(6) for every object A*, a "right unit" isomorphism*

$$r_A : A \otimes I \to A.$$

These date must satisfy the following requirements:

(1) the isomorphisms a_{ABC} *are natural in* A, B, C;

(2) the isomorphisms l_A *are natural in* A;

(3) the isomorphisms r_A *are natural in* A;

(4) Figure 1 is commutative for every quadruple of objects A, B, C, D *(associativity coherence);*

(5) Figure 2 is commutative for every pair A, B *of objects (unit coherence).*

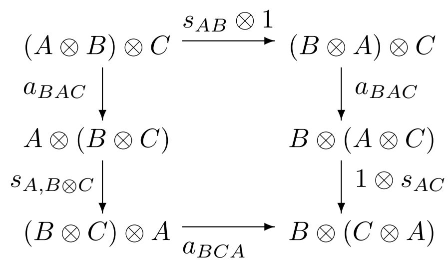

Figure 3: Diagram of Definition 2.9

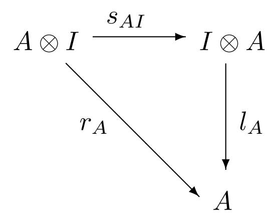

Figure 4: Diagram of Definition 2.9

Definition 2.9 ([1])**.** *A monoidal category $\mathbb{C}$ is symmetric when an isomorphism*

$$s_{AB} : A \otimes B \to B \otimes A$$

is given for every pair of A, B of objects. These isomorphisms must be such that:

(1) the morphisms s_{AB} are natural in A, B;

(2) Figure 3 is commutative for every triple A, B, C of objects (associativity coherence);

(3) Figure 4 is commutative for every object A (unit coherence);

(4) Figure 5 is commutative (symmetry axiom) for every pair A, B of objects.

Definition 2.10. *Let $\mathbf{F} : \mathbb{C} \to \mathbb{D}$ and $\mathbf{G} : \mathbb{D} \to \mathbb{C}$ be functors. If for each object C in $\mathbb{C}$ and object D in $\mathbb{D}$, there exists a bijection*

$$\varphi := \varphi_{C,D} : [\mathbf{F}C, D]_{\mathbb{D}} \to [C, \mathbf{G}D]_{\mathbb{C}}$$

such that φ is natural in C and D, then the triple $\langle \mathbf{F}, \mathbf{G}, \varphi \rangle$ is said to be an adjunction from $\mathbb{C}$ and $\mathbb{D}$. In this case, the functor $\mathbf{F}$ is said to be a left adjoint for $\mathbf{G}$ and $\mathbf{G}$ is called a right adjoint for $\mathbf{F}$.

Theorem 2.11. *Let $\mathbb{C}$ and $\mathbb{D}$ be categories, $\mathbf{F} : \mathbb{C} \to \mathbb{D}$ and $\mathbf{G} : \mathbb{D} \to \mathbb{C}$ be functors. $\mathbf{F}$ is a left adjoint for $\mathbf{G}$ if and only if for each object D in $\mathbb{D}$, there exists a morphism $\varepsilon_D : \mathbf{F} \circ \mathbf{G}D \to D$ in $\mathbb{D}$ such that for each object A in $\mathbb{C}$ and a morphism $f : \mathbf{F}A \to D$ in $\mathbb{D}$, there exists a unique morphism $\overline{f} : A \to \mathbf{G}D$ in $\mathbb{C}$ such that $f = \varepsilon_D \circ \mathbf{F}\overline{f}$, in the other words, the pair $(D,\ \varepsilon_D)$ is a universal morphism from $\mathbf{F}$ to D.*

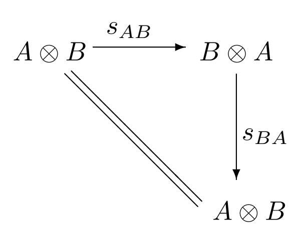

Figure 5: Diagram of Definition 2.9

3 The tensor product of stratified L-generalized convergences

In this section, we will focus on the monoidal property of SL-**GCS**.

Definition 3.1 ([17])**.** *Let X be a nonempty set. A mapping* lim *from* $F_L^s(X)$ *to* L^X *subject to the conditions*

(GL1) $\lim[x](x) = \top$ *for all* $x \in X$,

(GL2) $\mathcal{F} \leq \mathcal{G}$ *means* $\lim\mathcal{F} \leq \lim\mathcal{G}$ *for all* $\mathcal{F}, \mathcal{G} \in F_L^s(X)$,

is called a stratified L-generalized convergence on X, and then the pair (X, lim*) a stratified L-generalized convergence space (in [15], those spaces were called stratified L-fuzzy convergence space). A mapping* $\varphi : (X, \lim^X) \to (Y, \lim^Y)$ *is called continuous if and only if for all* $\mathcal{F} \in F_L^s(X)$, *we have*

$$\lim^X \mathcal{F} \leq \varphi^{\leftarrow}(\lim^Y \varphi^{\Rightarrow}(\mathcal{F})), \ \ \textit{or equivalently, } \varphi^{\rightarrow}(\lim^X \mathcal{F}) \leq \lim^Y \varphi^{\Rightarrow}(\mathcal{F}).$$

The class of all stratified L-generalized convergence spaces and continuous mappings forms a category, which is denoted by SL-**GCS**.

For a nonempty set X, Jäger in [2] defined an order $\leq$ on the fibre w.r.t. X, i.e.,

$$\{\lim \mid \lim \text{is a stratified } L\text{-generalized convergence on } X\},$$

$\lim_{lg}(X)$ for short, in the sense that $\lim_1 \leq \lim_2$ for $\lim_1, \lim_2 \in \lim_{lg}(X)$ if and only if the identity mapping

$$\mathrm{id}_X : (X, \lim_2) \to (X, \lim_1)$$

is continuous, that is, $\lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x)$ for all $\mathcal{F} \in F_L^s(X)$ and $x \in X$. We also write (X, $\lim_1$)$\leq$ (X, $\lim_2$) in this case. If $\lim_1 \leq \lim_2$ for $\lim_1, \lim_2 \in \lim_{lg}(X)$, we say $(X, \lim_2)$ is finer than (X, $\lim_1$), or (X, $\lim_1$) is coarser than (X, $\lim_2$), as usual.

Example 3.2 ([18])**.** *Let X be a nonempty set.*

(1) If we define $\lim_{ind}\mathcal{F}(x) = \top$ *for all* $\mathcal{F} \in F_L^s(X)$, $x \in X$, *then* $\lim_{ind}$ *is the coarsest stratified L-generalized convergence on X, called the indiscrete convergence on X.*

(2) We define the discrete stratified L-generalized convergence on X by $\lim_l = \top$ *whenever* $\mathcal{F} = [x]$ *for some* $x \in X$, *and* $= \bot$, *otherwise. Then it is the finest stratified L-generalized convergence on X.*

In particular, if X is a single point set, say $X = \{\infty\}$, then $[\infty] \leq \mathcal{F}$ holds for any stratified L-filter $\mathcal{F}$ on the single set $\{\infty\}$. So, as a consequence of (GL1) and (GL2), there exists a unique stratified L-generalized convergence on $\{\infty\}$, which will be denoted by $\lim_\infty$, and the corresponding space is denoted by $\lim_\infty$ also.

Let $\{\lim_j\}_{j\in J}$ be a nonempty family of stratified L-generalized convergences on a set X. It is easy to see that the supremum $\sup_{j\in J} \lim_j$ in $(\lim_{lg}(X), \leq)$ is determined by

$$\forall \mathcal{F} \in F_L^s(X), \ x \in X, \ \ (\sup_{j\in J} \lim_j)\mathcal{F}(x) = \textstyle\bigwedge_{j\in J} \lim_j \mathcal{F}(x).$$

Moreover, by Example 3.2, $\lim_{ind}$ is the coarsest L-generalized convergence on X. Hence, we have

Theorem 3.3. *For a nonempty set X,* $(\lim_{lg}(X), \leq)$ *is a complete lattice.*

The following theorem is shown in [17] by Jäger, which could also confirm Proposition 3.3 is true.

Theorem 3.4 ([17])**.** *SL-**GCS** is a topological category over* **SET**.

Now we are in the position to define the tensor of two stratified L-generalized convergence spaces. At first, we need to explore a technical proposition, the proof of which is omitted.

Proposition 3.5. *Let* $(X, \lim^X)$ *and* $(Y, \lim^Y)$ *be two objects in SL-**GCS**. If we define a mapping from* $F_L^s(X \times Y)$ *to* $L^{X\times Y}$, *denoted by* $\lim^X \otimes \lim^Y$, *as follows:*

$$(\lim^X \otimes \lim^Y)(\mathcal{H}) = \lim^X p_X^{\Rightarrow}(\mathcal{H}) \otimes \lim^Y p_Y^{\Rightarrow}(\mathcal{H})$$

for $\mathcal{H} \in F_L^s(X \times Y)$, here $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. Then $\lim^X \otimes \lim^Y$ is a stratified L-generalized convergence on $X \times Y$, called $\otimes$ the tensor operation of stratified L-generalized convergence spaces in the following.

Following Proposition 3.5, we will call $(X \times Y, \lim^X \otimes \lim^Y)$ the tensor space of $(X, \lim^X)$ and $(Y, \lim^Y)$, denoted by $(X, \lim^X) \otimes (Y, \lim^Y)$. In order to confirm that the tensor operation of stratified L-generalized convergence spaces has the desired properties, such as associativity, symmetry, and has the unit in particular, we present some of results needed. The propositions listed below follow from the definition of tensor operation $\otimes$, directly.

Proposition 3.6. *Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two objects in SL-**GCS**. Then $p_X : (X, \lim^X) \otimes (Y, \lim^Y) \to (X, \lim^X)$ and $p_Y : (X, \lim^X) \otimes (Y, \lim^Y) \to (Y, \lim^Y)$ are continuous.*

Then by using Proposition 3.6, the associativity and symmetry of $\otimes$ are obtained as described in the following proposition, and their routine proofs are omitted.

Proposition 3.7. *Let $(X, \lim^X)$, $(Y, \lim^Y)$ and $(Z, \lim^Z)$ be objects in SL-**GCS**. Then the canonical mapping*

$$j_{(XY)Z} : \big((X, \lim^X) \otimes (Y, \lim^Y)\big) \otimes (Z, \lim^Z) \to (X, \lim^X) \otimes \big((Y, \lim^Y) \otimes (Z, \lim^Z)\big),$$

*given by $j_{(XY)Z}((x, y), z) = (x, (y, z))$ for $x \in X, y \in Y, z \in Z$, is an isomorphism in SL-**GCS**. In particular,*

$$\big((X, \lim^X) \otimes (Y, \lim^Y)\big) \otimes (Z, \lim^Z) \cong (X, \lim^X) \otimes \big((Y, \lim^Y) \otimes (Z, \lim^Z)\big).$$

Proposition 3.8. *Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be two objects in SL-**GCS**. Then the canonical mapping*

$$S_{XY} : (X, \lim^X) \otimes (Y, \lim^Y) \to (Y, \lim^Y) \otimes (X, \lim^X),$$

*given by $S_{XY}(x, y) = (y, x)$ for $x \in X$ and $y \in Y$, is an isomorphism in SL-**GCS**. In particular,*

$$(X \times Y, \lim^X \otimes \lim^Y) \cong (Y \times X, \lim^Y \otimes \lim^X).$$

The unitary property of $\otimes$ could be shown as follows.

Proposition 3.9. *If $(X, \lim^X)$ is an object in SL-**GCS**, then $(X, \lim^X) \otimes \lim_\infty \cong (X, \lim^X)$.*

Proof. Clearly, the projection $p_X : X \times \{\infty\} \to X$ is bijection and $p_X : (X, \lim) \otimes \lim_\infty \to (X, \lim)$ is continuous. To confirm that p_X is an isomorphism from $(X, \lim) \otimes \lim_\infty$ to $(X, \lim)$, it suffices to check its inverse mapping $I_X : (X, \lim) \to (X, \lim) \otimes \lim_\infty$, such that $I_X(x) = (x, \infty)$ is continuous, which could be checked by

$$(\lim \otimes \lim_\infty) I_X^{\Rightarrow}(\mathcal{F})(I_X(x)) = \lim p_X^{\Rightarrow}(I_X^{\Rightarrow}(\mathcal{F}))(x) * \top = \lim \mathcal{F}(x)$$

for $\mathcal{F} \in F_L^s(X)$ and $x \in X$. □

Further, the tensor operation of stratified L-generalized convergence spaces is a bifunctor. In fact, we have

Proposition 3.10. *The tensor operation $\otimes$ of stratified L-generalized convergence spaces from the product category SL-**GCS**$\times SL$-**GCS** to SL-**GCS** is a bifunctor.*

Proof. It suffices to show the product map

$$f \times g : X \times U \to Y \times V,$$

defined by $f \times g(x, u) = (f(x), g(u))$ for all $(x, u) \in X \times U$, is continuous from $(X \times U, \lim^X \otimes \lim^U)$ to $(Y \times V, \lim^Y \otimes \lim^V)$ for two continuous mappings $f : (X, \lim^X) \to (Y, \lim^Y)$ and $g : (U, \lim^U) \to (V, \lim^V)$. For this, we take any stratified L-filter $\mathcal{H}$ on $X \times U$. Then, by using the definition of $\lim^Y \otimes \lim^V$, we have

$$\begin{aligned}
& \lim^Y \otimes \lim^V (f \times g)^{\Rightarrow}(\mathcal{H}) \\
= \ & \lim^Y p_Y^{\Rightarrow}((f \times g)^{\Rightarrow}(\mathcal{H})) \otimes \lim^V p_V^{\Rightarrow}((f \times g)^{\Rightarrow}(\mathcal{H})) \\
= \ & \lim^Y f^{\Rightarrow}(p_X^{\Rightarrow}(\mathcal{H})) \otimes \lim^V g^{\Rightarrow}(p_U^{\Rightarrow}(\mathcal{H})) \\
\geq \ & f^{\rightarrow}(\lim^X p_X^{\Rightarrow}(\mathcal{H})) \otimes g^{\rightarrow}(\lim^U p_U^{\Rightarrow}(\mathcal{H})) \\
= \ & (f \times g)^{\rightarrow}(\lim^X p_X^{\Rightarrow}(\mathcal{H}) \otimes \lim^U p_U^{\Rightarrow}(\mathcal{H})) \\
= \ & (f \times g)^{\rightarrow}((\lim^X \otimes \lim^U)\mathcal{H}),
\end{aligned}$$

here, $p_U : X \times U \to U$, $p_X : X \times U \to X$, $p_V : Y \times V \to V$ and $p_Y : Y \times V \to Y$ are the projections, which means that $f \times g : (X \times U, \lim^X \otimes \lim^U) \to (Y \times V, \lim^Y \otimes \lim^V)$ is continuous. □

Corollary 3.11. *Let $(X, \lim^X)$ be an object in SL-**GCS**. Define $\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ by for each object $(Y, \lim^Y)$ in SL-**GCS**,*

$$\mathbf{F}_\otimes(Y, \lim^Y) = (Y, \lim^Y) \otimes (X, \lim^X)$$

and for each continuous mapping $\varphi : (Y, \lim^Y) \to (Z, \lim^Z)$,

$$\mathbf{F}_\otimes(\varphi) = \varphi \times id_X : (Y, \lim^Y) \otimes (X, \lim^X) \to (Z, \lim^Z) \otimes (X, \lim^X).$$

Then $\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ is a functor.

By using Propositions 3.7-3.10, standard argument can be employed to show:

(a) The coherent isomorphism $j_{(XY)Z}$:

$$((X, \lim^X) \otimes (Y, \lim^Y)) \otimes (Z, \lim^Z) \to (X, \lim^X) \otimes ((Y, \lim^Y) \otimes (Z, \lim^Z))$$

is natural for three objects $(X, \lim^X)$, $(Y, \lim^Y)$, $(Z, \lim^Z)$ in SL-**GCS**.

(b) $s_{XY} : (X, \lim^X) \otimes (Y, \lim^Y) \to (Y, \lim^Y) \otimes (X, \lim^X)$ as a coherent isomorphism is natural for two objects $(X, \lim^X), (Y, \lim^Y)$ in SL-**GCS**.

(c) $p_X : (X, \lim) \otimes \lim_\infty \to (X, \lim)$ as a coherent isomorphism is natural for each object $(X, \lim^X)$ in SL-**GCS**.

Following (a), (b) and (c) above, we obtain the main result in the section.

Theorem 3.12. *The category SL-**GCS** of all stratified L-generalized convergence spaces is a symmetric monoidal category with respect to the tensor operation $\otimes$.*

At the end of the section, we point out that Proposition 3.10 tells us that for each object $(X, \lim^X)$ in SL-**GCS**, we obtain a functor

$$\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$$

determined by $(X, \lim^X)$ in the sense of

$$\mathbf{F}_\otimes(Y, \lim^Y) = (Y, \lim^Y) \otimes (X, \lim^X)$$

for each object $(Y, \lim^Y)$ in SL-**GCS**, and

$$\mathbf{F}_\otimes(\varphi) = \varphi \times id_X : (Y, \lim^Y) \otimes (X, \lim^X) \to (Z, \lim^Z) \otimes (X, \lim^X)$$

for each continuous mapping $\varphi : (Y, \lim^Y) \to (Z, \lim^Z)$.

In next section, we will show that there exists a functor from SL-**GCS** to SL-**GCS**, which is the right adjoint of the functor $\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ determined by $(X, \lim^X)$. In this way, we will confirm the symmetric monoidal category SL-**GCS** relative to the our tensor operation $\otimes$ is $\otimes$-closed in the sense of the definition below.

Definition 3.13 ([1])**.** *A symmetric monoidal category $\mathbb{C}$ is said to be $\otimes$-closed with respect to the tensor operation $\otimes$ if for each object B in $\mathbb{C}$, the functor $(-) \otimes B : \mathbb{C} \to \mathbb{C}$ has a right adjoint $(-)^B : \mathbb{C} \to \mathbb{C}$.*

4 $\otimes$-closedness of SL-GCS

In the last section, we denote the set of morphisms from an object $(X, \lim^X)$ to another object $(Y, \lim^Y)$ in SL-**GCS** by $[X, Y]_\otimes$, i.e.,

$$[X, Y]_\otimes := \{\varphi : (X, \lim^X) \to (Y, \lim^Y) \mid \varphi \text{ is continuous}\}.$$

Then we construct a natural function space structure $\otimes$-lim on the set $[X, Y]_\otimes$, which assures that the function space is obtained. Following this, our goal is to look for a functor from SL-**GCS** to itself, which is a right adjoint of the functor $\mathbf{F}_\otimes$ determined by a space $(X, \lim^X)$ introduced at the end of Section 3. Finally, the monoidal closedness of the category SL-**GCS** of stratified L-generalized convergence spaces is explored.

Lemma 4.1 ([17] for $* = \wedge$)**.** *If $\varphi : X \to Y$ is a mapping, and $A \in L^X$, then it holds that $\varphi^\to(A) = ev_{XY}(\top_{\{\varphi\}} \otimes A)$.*

Lemma 4.2. *Let $\varphi : X \to Y$ be a mapping and $\mathcal{G} \in F_L^s(X)$. Then*

$$\varphi^{\Rightarrow}(\mathcal{G}) \leq ev_{XY}^{\Rightarrow}([\varphi] \otimes \mathcal{G}).$$

Proof. This conclusion is a $\otimes$-version of Lemma 8.2 [17]. We include here a simple proof for the convenience of the reader.

For an L-subset $B \in L^Y$, we observe that

$$\begin{aligned}
& ev_{XY}^{\Rightarrow}([\varphi] \otimes \mathcal{G})(B) = ([\varphi] \otimes \mathcal{G})(B \circ ev_{XY}) \\
= & \bigvee \left\{ [\varphi](C) * \mathcal{G}(D) \mid C \in L^{[X,Y]_\otimes}, D \in L^X \text{with } C \otimes D \leq ev_{XY}^{\leftarrow}(B) \right\} \\
\geq & \bigvee \left\{ [\varphi](\top_{\{\varphi\}}) * \mathcal{G}(D) \mid D \in L^X \text{with } \top_{\{\varphi\}} \otimes D \leq ev_{XY}^{\leftarrow}(B) \right\} \\
= & \bigvee \left\{ \top * \mathcal{G}(D) \mid D \in L^X \text{with } ev_{XY}^{\rightarrow}(\top_{\{\varphi\}} \otimes D) \leq B \right\} \\
= & \bigvee \left\{ \mathcal{G}(D) \mid D \in L^X \text{with } \varphi^{\rightarrow}(D) \leq B \right\} \quad \text{(By Lemma 4.1)} \\
= & \bigvee \left\{ \mathcal{G}(D) \mid D \in L^X \text{with } D \leq \varphi^{\leftarrow}(B) \right\} = \mathcal{G}(\varphi^{\leftarrow}(B)) \\
= & \varphi^{\Rightarrow}(\mathcal{G})(B),
\end{aligned}$$

that is to say $ev_{XY}^{\Rightarrow}([\varphi] \otimes \mathcal{G}) \geq \varphi^{\Rightarrow}(\mathcal{G})$ and the proof is completed. □

When L is a frame, i.e., $* = \wedge$ in this paper, Jäger in [15] showed there is a natural function space structure on the set $[X, Y]_\wedge$, and from this he concluded that the category of all stratified L-generalized convergence spaces is Cartesian closed if the underlying lattice L is a frame. Following Jäger's work, for two objects $(X, \lim^X), (Y, \lim^Y)$ in SL-**GCS**, we define a mapping

$$\otimes\text{-}\lim : F_L^s([X, Y]_\otimes) \to L^{[X,Y]_\otimes}$$

by for each $\mathcal{H} \in F_L^s([X, Y]_\otimes), \varphi \in [X, Y]_\otimes$,

$$\otimes\text{-}\lim \mathcal{H}(\varphi) = \textstyle\bigwedge_{\mathcal{G} \in F_L^s(X)} S_X(\lim^X \mathcal{G}, \varphi^{\leftarrow}(\lim^Y ev_{XY}(\mathcal{H} \otimes \mathcal{G})))$$

Then we conclude

Lemma 4.3. *Let $(X, \lim^X), (Y, \lim^Y)$ be two stratified L-generalized convergence spaces. Then $([X, Y]_\otimes, \otimes\text{-}\lim)$ is a stratified L-generalized convergence space.*

Proof. For the conclusion of the lemma, we have to check that the structure $\otimes$-lim satisfies the axioms (GL1)-(GL2). From the definition of $\otimes$-lim, it is routine to check it satifies (GL2). In order to check (GL1), we take any element $\varphi \in [X, Y]_\otimes$. And then it holds that

$$\begin{aligned}
\otimes\text{-}\lim[\varphi](\varphi) &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\big(\lim^X \mathcal{G}, \varphi^{\leftarrow}(\lim^Y ev_{XY}([\varphi] \otimes \mathcal{G}))\big) \\
&\geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\big(\varphi^{\leftarrow}(\lim^Y \varphi^{\Rightarrow}(\mathcal{G})), \varphi^{\leftarrow}(\lim^Y ev_{XY}^{\Rightarrow}([\varphi] \otimes \mathcal{G}))\big) \\
&\qquad\qquad \text{(by using the continuity of } \varphi) \\
&\geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_Y\big(\lim^Y \varphi^{\Rightarrow}(\mathcal{G}), \lim^Y ev_{XY}^{\Rightarrow}([\varphi] \otimes \mathcal{G})\big) \\
&\geq \top,
\end{aligned}$$

where, the last inequality is true by Lemma 4.2 and the property of $\lim^Y$ satisfying (GL2). Thus from this, we see (GL1) is satisfied by $\otimes$-lim and the proof is completed. □

Now, by Lemma 4.3 above, we realize that for each fixed object $(X, \lim^X)$, an object-mapping $\mathbf{G}_\otimes$ from SL-**GCS** to itself is given by

$$\mathbf{G}_\otimes(Y, \lim^Y) = ([X, Y]_\otimes, \otimes\text{-}\lim), \ \forall (Y, \lim^Y) \in |SL\text{-}\mathbf{GCS}|.$$

In order to show the object-mapping $\mathbf{G}_\otimes$ from SL-**GCS** to itself is functor, let us define the morphism-action of $\mathbf{G}_\otimes$ in the sense of

$$\forall \varphi \in [X, Y]_\otimes, \quad \mathbf{G}_\otimes(\Phi)(\varphi) = \Phi \circ \varphi$$

for each continuous mapping Φ from $(Y, \lim^Y)$ to $(Z, \lim^Z)$ in SL-**GCS**. Since the continuity of $\Phi : (Y, \lim^Y) \to (Z, \lim^Z)$ and $\varphi : (X, \lim^X) \to (Y, \lim^Y)$ assures the continuity of the composition $\Phi \circ \varphi$, we can conclude that $\mathbf{G}_\otimes(\Phi)$ is well-defined. The following Lemma 4.4 confirms that it is a functor indeed, and hence it will be called a power-functor determined by the space $(X, \lim^X)$.

Lemma 4.4. *Let $(X, \lim^X)$ be an object in SL-**GCS**. Then*

$$\mathbf{G}_\otimes(\Phi) : ([X, Y]_\otimes, \otimes\text{-}\lim_{XY}) \to ([X, Z]_\otimes, \otimes\text{-}\lim_{XZ})$$

is continuous for each continuous mapping $\Phi : (Y, \lim^Y) \to (Z, \lim^Z)$.

Proof. Let $\Phi : (Y, \lim^Y) \to (Z, \lim^Z)$ be a continuous mapping. In order to show that $\mathbf{G}_\otimes(\Phi)$ is continuous, let us take any stratified L-filter $\mathcal{H} \in F_L^s([X, Y]_\otimes)$. By using the definition of $\otimes\text{-}\lim_{XY}$ and $\otimes\text{-}\lim_{XZ}$, we observe that for any $\varphi \in [X, Y]_\otimes$,

$$\begin{aligned}
& \mathbf{G}_\otimes(\Phi)^\leftarrow\Big(\otimes\text{-}\lim_{XZ}\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H})\Big)(\varphi) \\
= \; & \otimes\text{-}\lim_{XZ}\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H})\Big(\Phi \circ \varphi\Big) \\
= \; & \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\Big(\lim^X \mathcal{G},\ (\Phi \circ \varphi)^\leftarrow(\lim^Z ev_{XZ}^\Rightarrow(\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \otimes \mathcal{G}))\Big) \\
= \; & \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\Big(\lim^X \mathcal{G},\ \varphi^\leftarrow \circ \Phi^\leftarrow(\lim^Z ev_{XZ}^\Rightarrow(\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \otimes \mathcal{G}))\Big) := P.
\end{aligned}$$

$\Phi \circ ev_{XY} = ev_{XZ} \circ (\mathbf{G}_\otimes(\Phi) \times id_X)$ to produce the following equality:

$$\begin{aligned}
P \; = \; & \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\Big(\lim^X \mathcal{G},\ \varphi^\leftarrow \circ \Phi^\leftarrow(\lim^Z ev_{XZ}^\Rightarrow \circ (\mathbf{G}_\otimes(\Phi) \times id_X)^\Rightarrow(\mathcal{H} \otimes \mathcal{G}))\Big) \\
= \; & \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\Big(\lim^X \mathcal{G},\ \varphi^\leftarrow \circ \Phi^\leftarrow(\lim^Z \Phi^\Rightarrow \circ ev_{XY}^\Rightarrow(\mathcal{H} \otimes \mathcal{G}))\Big) := Q.
\end{aligned}$$

by using the continuity of Φ:

$$Q \;\geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X\Big(\lim^X \mathcal{G},\ \varphi^\leftarrow(\lim^Y ev_{XY}^\Rightarrow(\mathcal{H} \otimes \mathcal{G}))\Big) = \otimes\text{-}\lim_{XY}\mathcal{H}(\varphi),$$

which implies that $\mathbf{G}_\otimes(\Phi) : ([X, Y]_\otimes, \otimes\text{-}\lim_{XY}) \to ([X, Z]_\otimes, \otimes\text{-}\lim_{XZ})$ is continuous, and the proof is completed. □

Let $(X, \lim^X)$ be an object in SL-**GCS**. Recall that there exists a functor $\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ determined by the space $(X, \lim^X)$. In order to confirm the power-functor $\mathbf{G}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ determined by the space $(X, \lim^X)$ is a right adjoint of the functor $\mathbf{F}_\otimes$, By Theorem 2.11 we have to complete the following tasks:

(i) to find a continuous map

$$\varepsilon_{(Y, \lim^Y)} : \mathbf{F}_\otimes \circ \mathbf{G}_\otimes(Y, \lim^Y) \to (Y, \lim^Y)$$

for each space $(Y, \lim^Y)$.

(ii) for a continuous map $\varphi : \mathbf{F}_\otimes(Z, \lim^Z) \to (Y, \lim^Y)$, there is a unique continuous map $\overline{\varphi} : (Z, \lim^Z) \to \mathbf{G}_\otimes(Y, \lim^Y)$ such that $\varphi = \varepsilon_{(Y, \lim^Y)} \circ \mathbf{F}_\otimes(\overline{\varphi})$.

For tasks (i) and (ii), we firstly, confirm that the evaluation mapping $ev_{XY} : [X, Y]_\otimes \times X \to Y$ such that

$$\forall (\varphi, x) \in [X, Y]_\otimes \times\ X, \quad ev_{XY}(\varphi, x) = \varphi(x),$$

is a candidate of the $\varepsilon_{(Y, \lim^Y)}$ by Proposition 4.5 below, and then complete the task (ii) after it.

Proposition 4.5. *Let $(X, \lim^X)$ be an object in SL-**GCS**. Then for each object $(Y, \lim^Y)$, the evaluation mapping*

$$ev_{XY} : \mathbf{F}_\otimes\mathbf{G}_\otimes(Y, \lim^Y) = ([X, Y]_\otimes, \otimes\text{-}\lim) \otimes (X, \lim^X) \to (Y, \lim^Y)$$

is continuous.

Proof. In order to show the continuity of ev_{XY}, let us take any stratified L-filter $\mathcal{H} \in F_L^s([X,Y]_\otimes \times X)$. Now, by means of the definition w.r.t. $(\otimes\text{-}\lim) \otimes \lim^X$, we observe that for each $(\varphi, x) \in [X,Y]_\otimes \times X$,

$$\begin{aligned} & ((\otimes\text{-}\lim) \otimes \lim^X)\mathcal{H}(\varphi, x) \\ = \ & \otimes\text{-}\lim p_{[X,Y]_\otimes}^{\Rightarrow}(\mathcal{H})(\varphi) * \lim^X p_X^{\Rightarrow}(\mathcal{H})(x) \\ \leq \ & S_X\Big(\lim^X p_X^{\Rightarrow}(\mathcal{H}),\ \varphi^{\leftarrow}(\lim^Y ev_{XY}^{\Rightarrow}(p_{[X,Y]_\otimes}^{\Rightarrow}(\mathcal{H}) \otimes p_X^{\Rightarrow}(\mathcal{H})))\Big) * \\ & \qquad \lim^X p_X^{\Rightarrow}(\mathcal{H})(x) \\ \leq \ & \lim^Y ev_{XY}^{\Rightarrow}(p_{[X,Y]_\otimes}^{\Rightarrow}(\mathcal{H}) \otimes p_X^{\Rightarrow}(\mathcal{H}))(\varphi(x)) \\ \leq \ & \lim^Y ev_{XY}^{\Rightarrow}(\mathcal{H})(\varphi(x)) \quad (\text{since } p_{[X,Y]_\otimes}^{\Rightarrow}(\mathcal{H}) \otimes p_X^{\Rightarrow}(\mathcal{H}) \leq \mathcal{H} \text{ in general.}) \\ = \ & ev_{XY}^{\leftarrow}(\lim^Y ev_{XY}^{\Rightarrow}(\mathcal{H}))(\varphi, x), \end{aligned}$$

that is to say $((\otimes\text{-}\lim) \otimes \lim^X)\mathcal{H} \leq ev_{XY}^{\leftarrow}(\lim^Y ev_{XY}^{\Rightarrow}(\mathcal{H}))$ holds for each $\mathcal{H} \in F_L^s([X,Y]_\otimes \times X)$. From this observation, we conclude that ev_{XY} is continuous, so the proof is completed. □

Let $\varphi : Z \times X \to Y$ be a mapping. For each $z \in Z$, there exists a mapping $\varphi(z,-) : X \to Y$ such that $\varphi(z,-)(x) = \varphi(z,x)$ for any $x \in X$. Using the similar method of Lemma 8.5 [17], we obtain the inequality

$$\varphi(z,-)^{\Rightarrow}(\mathcal{F}) \geq \varphi^{\Rightarrow}([z] \otimes \mathcal{F})$$

for a stratified L-filter $\mathcal{F} \in F_L^s(X)$. Then if the mapping

$$\varphi : (Z, \lim{}^Z) \otimes (X, \lim{}^X) \to (Y, \lim{}^Y)$$

is continuous, we could conclude that for each $z \in Z, \varphi(z,-) : (X, \lim^X) \to (Y, \lim^Y)$ is continuous, which in fact, can be proved as follows:

$$\begin{aligned} \lim^X \mathcal{F}(x) &= \lim^Z [z](z) * \lim^X \mathcal{F}(x) \\ &\leq (\lim^Z \otimes \lim^X)([z] \otimes \mathcal{F})(z,x) \\ &\leq \lim^Y \varphi^{\Rightarrow}([z] \otimes \mathcal{F})(\varphi(z,x)) \\ &\leq \lim^Y \varphi(z,-)^{\Rightarrow}(\mathcal{F})(\varphi(z,-)(x)), \end{aligned}$$

for every stratified L-filter $\mathcal{F} \in F_L^s(X)$. In this way, for every continuous mapping $\varphi : (Z, \lim^Z) \otimes (X, \lim^X) \to (Y, \lim^Y)$, there is a well-defined $\overline{\varphi} : (Z, \lim^Z) \to ([X,Y]_\otimes, \otimes\text{-}\lim)$ such that $\overline{\varphi}(z) = \varphi(z,-)$ for all $z \in Z$. Moreover, the following proposition confirms that it is continuous.

Proposition 4.6 ([17] for $* = \wedge$). *Let* $(X, \lim^X), (Y, \lim^Y)$ *and* $(Z, \lim^Z)$ *be objects in* SL-**GCS**. *If* $\varphi : (Z, \lim^Z) \otimes (X, \lim^X) \to (Y, \lim^Y)$ *is continuous, then* $\overline{\varphi} : (Z, \lim^Z) \to ([X,Y]_\otimes, \otimes\text{-}\lim)$ *is continuous.*

Proof. First of all, note that $ev_{XY} \circ (\overline{\varphi} \times id_X) = \varphi$. From this and Proposition 2.7, we have for stratified L-filters $\mathcal{F} \in F_L^s(Z)$ and $\mathcal{G} \in F_L^s(X)$, that

$$\varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) \;=\; (ev_{XY} \circ (\overline{\varphi} \times id_X))^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = ev_{XY}^{\Rightarrow}(\overline{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G}).$$

Then by using the equality $\varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = ev_{XY}^{\Rightarrow}(\overline{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G})$ obtained above, we observe that for each $z \in Z$

$$\begin{aligned} \lim^Z \mathcal{F}(z) &\leq S_X\Big(\lim^X \mathcal{G},\ \lim^Z \mathcal{F}(z) * \lim^X \mathcal{G}\Big) \\ &= \bigwedge_{x \in X} \Big(\lim^X \mathcal{G}(x) \longrightarrow \lim^Z \mathcal{F}(z) * \lim^X \mathcal{G}(x)\Big) \\ &\leq \bigwedge_{x \in X} \Big(\lim^X \mathcal{G}(x) \longrightarrow (\lim^Z \otimes \lim^X)(\mathcal{F} \otimes \mathcal{G})(z,x)\Big) \\ &\leq \bigwedge_{x \in X} \Big(\lim^X \mathcal{G}(x) \longrightarrow \lim^Y \varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G})(\varphi(z,x))\Big) \\ &\qquad \text{(By means of the continuity of } \varphi) \\ &= \bigwedge_{x \in X} \Big(\lim^X \mathcal{G}(x) \longrightarrow \lim^Y ev_{XY}^{\Rightarrow}(\overline{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G})(\overline{\varphi}(z)(x))\Big) \\ &= \otimes\text{-}\lim \overline{\varphi}^{\Rightarrow}(\mathcal{F})(\overline{\varphi}(z)), \end{aligned}$$

which implies $\lim^Z \mathcal{F} \leq \overline{\varphi}^{\leftarrow}(\otimes\text{-}\lim \overline{\varphi}^{\Rightarrow}(\mathcal{F}))$. So, we confirm that

$$\overline{\varphi} : (Z, \lim^Z) \to ([X,Y]_\otimes, \otimes\text{-}\lim)$$

is continuous, finally. □

From Propositions 4.5, 4.6 as well as Theorem 2.11, we could conclude that for each object $(X, \lim^X)$, the power-functor $\mathbf{G}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$ determined by it has a left adjoint $\mathbf{F}_\otimes : SL\text{-}\mathbf{GCS} \to SL\text{-}\mathbf{GCS}$. Then according to Theorem 3.12 and Definition 3.13, we have the following main result of the paper.

Theorem 4.7. *The category SL-**GCS** of stratified L-generalized convergence spaces is $\otimes$-closed with respect to the tensor operation $\otimes$.*

If the underlying lattice L is a complete Heyting Algebra, we recapture the Cartesian-closedness of the category SL-**GCS** by Theorem 4.7. In details, we have

Corollary 4.8 (Jäger [17])**.** *If L is a complete Heyting Algebra, then the category SL-**GCS** of stratified L-generalized convergence spaces is Cartesian closed.*

5 Conclusions

In this paper, the lattice valued environment L is complete residuated lattice with nonidempotent semigroup operation. By defining the tensor operation of stratified L-generalized convergence spaces, we have proved that the category of stratified L-generalized convergence spaces is monoidal category, further certified that the category of stratified L-generalized convergence spaces is monoidal closed, thus we can get the desired function space. In particular, if the lattice valued environment L is a complete Heyting Algebra, the Cartesian closedness of the category is recaptured by our result.

These results fill the blank in the research of convergence space about monoidal closedness, and the tensor operation of stratified L-generalized convergence spaces is a bifunctor, which has elicitation for defining the tensor operation of general space objects.

Acknowledgement

This work is supported by National Natural Science Foundations (11471297) of China.

References

[1] F. Borceux, *Handbook of categorical algebra, categories and structures*, Cambridge University Press, Cambridge, **2** (1994), 291–348.

[2] H. Boustique, G. Richardson, *Regularity: Lattice-valued Cauchy spaces*, Fuzzy Sets and Systems, **190** (2012), 94–104.

[3] R. Bělohlávek, *Fuzzy relation systems, foundation and principles,* Klumer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.

[4] G. Choquet, *Convergences*, Anna. University Grenoble, **23** (1948), 57–112.

[5] A. Craig, G. Jäger, *A common framework for latticed-valued uniform spaces and probabilistic uniform limit spaces*, Fuzzy Sets and Systems, **160** (2009), 1177–1203.

[6] J. M. Fang, *Stratified L-ordered convergence structures*, Fuzzy Sets and Systems, **161** (2010), 2130–2149.

[7] J. M. Fang, *Relationships between L-ordered convergence structures and strong L-topologies*, Fuzzy Sets and Systems, **161** (2010), 2923–2944.

[8] J. M. Fang, *Lattice-valued semiuniform convergence spaces*, Fuzzy Sets and Systems, **195** (2012), 33–57.

[9] J. M. Fang, *Stratified L-ordered quasiuniform limit spaces*, Fuzzy Sets and Systems, **251** (2013), 51–73.

[10] J. M. Fang, *Lattice-valued preuniform convergence spaces*, Fuzzy Sets and Systems, **251** (2014), 52–70.

[11] J. M. Fang, Y. Yue, $\top$-*Diagonal conditions and continuous extension theorem*, Fuzzy Sets and Systems, **321** (2017), 73–89.

[12] P. V. Flores, R. N. Mohapatra, G. Richardson, *Lattice-valued spaces: Fuzzy convergence*, Fuzzy Sets and Systems, **157** (2006), 2706–2714.

[13] J. Gutiérrez García, M. A. De Prada Vicente, A. P. Šostak, *A unified approach to the concept of fuzzy L-uniform space*, Topological and Algebraic Structures in Fuzzy Sets (S.E. Rodabaugh, E.P. Klement Eds.), Kluwer Academic Publishers, **20** (2003), 81–114.

[14] J. A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18** (1967), 145–174.

[15] U. Höhle, *Probabilistic topologies induced by L-fuzzy uniformities*, Manuscripta Mathematica, **38** (1982), 289–323.

[16] U. Höhle, A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, in: U. Höhle, S. E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory, Kluwer Academic Publishers, Boston, **3** (1999), 123–272.

[17] G. Jäger, *A category of L-fuzzy convergence spaces*, Quaestiones Mathematicae, **24** (2001), 501–507.

[18] G. Jäger, *Subcategories of lattice-valued convergence spaces*, Fuzzy Sets and Systems, **156** (2005), 1–24.

[19] G. Jäger, *Pretopological and topological lattice-valued convergence spaces*, Fuzzy Sets and Systems, **158** (2007), 424–435.

[20] G. Jäger, *Fischer's diagonal condition for lattice-valued convergence spaces*, Quaestiones Mathematicae, **31** (2008), 11–25.

[21] G. Jäger, *Lattice-valued convergence spaces and regularity*, Fuzzy Sets and Systems, **159** (2008), 2488–2502.

[22] G. Jäger, *Diagonal conditions for lattice-valued uniform convergence spaces*, Fuzzy Sets and Systems, **210** (2013), 39–53.

[23] G. Jäger, M. H. Burton, *Stratified L-uniform convergence spaces*, Quaestiones Mathematicae, **28** (2005), 11–36.

[24] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. del Seminario Matematico e Fisico di Milano, **43** (1973), 135–166.

[25] H. Lai, D. Zhang, *Closedness of the category of liminf complete fuzzy orders*, Fuzzy Sets and Systems, **282** (2016), 86–98.

[26] L. Li, D. Zhang, *On the relationship between limit spaces, many valued topological spaces, and many valued preorders*, Fuzzy Sets and Systems, **160** (2009), 1204–1217.

[27] L. Li, Q. Jin, *On adjunctions between Lim, SL-Top, and SL-Lim*, Fuzzy Sets and Systems, **182** (2011), 66–78.

[28] L. Li, Q. Jin, *On stratified L-convergence spaces: Pretopological axioms and diagonal axioms*, Fuzzy Sets and Systems, **204** (2012), 40–52.

[29] L. Li, Q. Jin, *p-Topologicalness and p-regularity for lattice-valued convergence spaces*, Fuzzy Sets and Systems, **238** (2014), 26–45.

[30] E. Lowen, R. Lowen, P. Wuyts, *The categorical topology approach to fuzzy topology and fuzzy convergence*, Fuzzy Sets and Systems, **40** (1991), 347–373.

[31] E. Lowen, R. Lowen, *A topological universe extension of FTS*, In:Applications of category theory to fuzzy sets, (S.E. Rodabaugh, E.P. Klement, U. Höhle, eds.), Kluwer, Dordrecht, **14** (1992), 153–176.

[32] S. MacLane, *Categories for the working mathematician*, Second Edition, Graduate Texts in Mathematics, Springer, New York, **5** (1998), 251–266.

[33] D. Orpen, G. Jäger, *Lattice-valued convergence spaces: Extending the lattice context*, Fuzzy Sets and Systems, **190** (2012), 1–20.

[34] B. Pang, J. M. Fang, *L-fuzzy Q-convergence structures*, Fuzzy Sets and Systems, **182** (2011), 53–65.

[35] B. Pang, *On (L,M)-fuzzy convergence spaces*, Fuzzy Sets and Systems, **238** (2014), 46–70.

[36] B. Pang, *The category of stratified L-filter spaces*, Fuzzy Sets and Systems, **247** (2014), 108–126.

[37] B. Pang, *Degrees of separation properties in stratified L-generalized convergence spaces using residual implication*, Filomat, **31**(20) (2017), 6293-6305.

[38] B. Pang, *Stratified L-ordered filter spaces*, Quaestiones Mathematicae, **40**(5) (2017), 661-678.

[39] G. Preuss, *Foundations of topology*, Kluwer Academic Publishers, Dordrecht, Boston, London 2002.

[40] W. Yao, *On many-valued stratified L-fuzzy convergence spaces*, Fuzzy Sets and Systems, **159** (2008), 2503–2519.

[41] W. Yao, *On L-fuzzifying convergence spaces*, Iranian Journal of Fuzzy Systems, **6**(1) (2009), 63–80.

[42] D. Zhang, *An enriched category approach to many valued topology*, Fuzzy Sets and Systems, **158** (2007), 349–366.

[43] L. A. Zadeh, *Fuzzy sets*, Information Control, **8**(3) (1965), 338–353.