Iranian Journal of Fuzzy Systems

Volume 16, Number 5, (2019), pp. 155-167

University of Sistan and Baluchestan

Multidimensional fuzzy finite tree automata

S. Moghari[1] and M. M. Zahedi[2]

[1] *Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran.*
[2] *Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.*

s.moghari@shahroodut.ac.ir, zahedi_mm@modares.ac.ir

Abstract

This paper introduces the notion of multidimensional fuzzy finite tree automata (MFFTA) and investigates its closure properties from the area of automata and language theory. MFFTA are a superclass of fuzzy tree automata whose behavior is generalized to adapt to multidimensional fuzzy sets. An MFFTA recognizes a multidimensional fuzzy tree language which is a regular tree language so that for each dimension, a fuzzy membership grade is assigned to each tree. We study MFFTA by extending some classical problems and properties of automata and regular languages such as determinization, reduction, duality and operations on languages. Furthermore, we provided the method of converting every complete fuzzy tree automata to an MFFTA as well as an example to show the efficiency of MFFTA in comparison to FFTA.

Keywords: Multidimensional fuzzy tree languages, fuzzy dual languages, closure properties, determinization, reduction.

1 Introduction

Automata are abstract machines for solving computational problems [16, 21, 22]. They also provide mathematical tools for analyzing the input, output and behavior of systems. They are often classified by the class of their languages which is the set of recognized inputs by them. One of the well-known classifications on automata is Chomsky hierarchy [5] which describes the relations between various languages and kinds of formalized logics. After presenting this hierarchy, a variety of new automata are introduced that many of them extend previous models. *Fuzzy finite automata* (FFA) [21] and *finite tree automata* (FTA) [7, 8, 24, 25] are two of the newer models, each of them have extended finite automaton [14] in a different way.

FFA are used to model uncertainty which is inherent in many applications [9]. The input of an FFA is a word (a sequence of input alphabets called actions) and the output is a real number in interval $[0, 1]$ as the grade of acceptance of the input. The set of words recognized by an FFA is called a fuzzy regular language [21]. The operations on fuzzy regular languages and their closure properties are well-studied [1, 21]. On the other hand, it can be said that the finite automata are spacial case of FFA in which their output (the grade of acceptance of the input) is 0 (reject) or 1 (accept).

FTA are mathematical models of computation that process tree structures. An FTA recognizes a set of trees called regular tree language [6]. In other words, the input of an FTA is a tree structure constructed from input (ranked) alphabets and the output is 0 or 1. Since words are spacial case of trees (each node has at most one child), finite automata are spacial case of FTA. Studies on closure properties, operations on regular languages and related algorithms have also done well in the field of regular tree languages [6].

One of the generalizations of both FFA and FTA is *fuzzy FTA* (FFTA) [15, 21, 10] in which provides tools for dealing with vaguely defined trees [11, 22]. The behavior of an FFTA is called fuzzy regular tree language which is a mapping from a set of regular trees on ranked alphabets to the membership values that (generally) lie in the range $[0, 1]$. Naturally, FTA are a spacial class of FFTA that the membership value generated for each tree is 0 or 1. Also, FFA are a subclass of FFTA that each node of their input tree has at most one child.

Corresponding Author: S. Moghari

Received: November 2017; Revised: September 2018; Accepted: April 2019.

Numerous researchers and authors have contributed to the notion of FFTA. Esik and Liu [10] studied the algebraic aspects of FFTA. They defined a fuzzy recognizable tree language with membership in a completely distributive lattice and derived its Kleene theorem. In the following, Bozapalidis and Bozapalidoy [2, 3, 4] considered FFTA whose behavior computed with respect to a pair of a *t-conorm* and a *t-norm* operations. Also, some studies focused on closure properties of fuzzy regular tree languages and developed some algorithms for efficient reduction [18] and minimization [12, 13, 17, 19] of FFTA. Furthermore, their works on determinization of FFTA lead to introducing the notion of *complex FFTA* (CFFTA) [20]. The remarkable point about the definition of CFFTA is using multidimensional states instead of symbols. However, they proved that the classes of CFFTA and FFTA are equivalent in terms of the languages they accept. Figure 1 illustrates the relation between the types of automata we discussed.

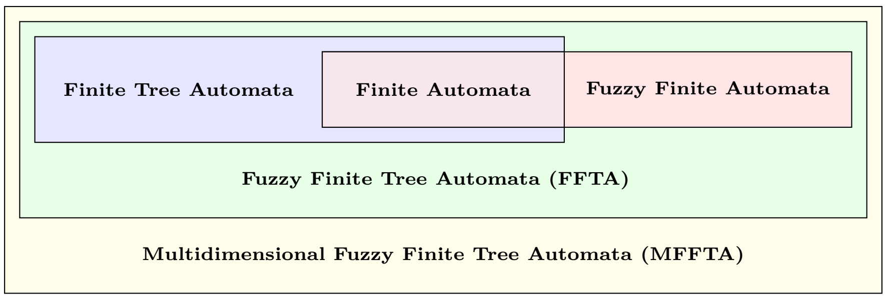

Figure 1: The relations between languages of finite automata, FFA, FTA, FFTA and MFFTA.

In this paper, we have improved the definition of CFFTA [20] and introduced the notion of *multidimensional FFTA*. Figure (1) shows that MFFTA is the superclass of several automata types that recognize (fuzzy) regular languages. Therefore, any result that is proved for the class of MFFTA will be for the classes of finite automata, FFA, FTA and FFTA.

The aim of this paper is to introduce fuzzy tree automata theory based on the multidimensional fuzzy sets. The main difference between the class of FFTA and the class of MFFTA is their behavior, where an FFTA recognizes a 1-dimensional fuzzy tree language rather than an MFFTA that could recognizes a k-dimensional fuzzy tree language with $k \geq 1$. We also develop closure properties and related classical problems such as set operations, determinization, reduction and duality for MFFTA and multidimensional fuzzy tree languages. Furthermore, some new properties for MFFTA and proof of related theorems are achieved and the relationship between MFFTA and several kinds of automata is discussed.

2 Preliminaries

2.1 Fuzzy sets

Basic concepts of Zadeh's fuzzy logic and fuzzy set theory are borrowed from [26, 27, 23]. Also, we use ℓ as the unit interval $[0, 1] \subseteq \mathbb{R}$.

Definition 2.1. *Let X be a collection of elements or objects. A fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A(x)$ is called the membership function or grade of membership of x in A and (generally) lies in ℓ. The empty fuzzy set $\emptyset$ on X is defined as $\mu_\emptyset(x) = 0$ for all $x \in X$. The largest fuzzy set in X, called universal fuzzy set in X and is denoted by 1_X, is defined as $\mu_{1_X}(x) = 1$ for all $x \in X$. As well, we denote by $\widetilde{X}$ the set of all fuzzy sets on X.*

Fuzzy set operations *complement*, *union* and *intersection* on sets A and B can be defined by the membership functions $\overline{\mu}_A$, $\mu_{A\cup B}$ and $\mu_{A\cap B}$, as follows:

$$\overline{\mu}_A(x) = 1 - \mu_A(x), \forall x \in X,$$

$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x), \forall x \in X,$$

$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x), \forall x \in X,$$

where the symbols $\vee$ and $\wedge$ are used as *max* and *min* functions, respectively.

Definition 2.2. *Let e be a fuzzy expression over $(\vee, \wedge, \ell)$. We denote by e^* the dual of e which is obtained by exchanging $(\vee$ and $\wedge)$ and (every constant $\alpha \in \ell$ and $\overline{\alpha}$). Note that every variable x do not exchange with $\overline{x}$.*

Example 2.3. $e = (\overline{x} \vee y) \wedge (x \vee \overline{z} \vee 0.2) \quad \Rightarrow \quad e^* = (\overline{x} \wedge y) \vee (x \wedge \overline{z} \wedge 0.8)$.

Proposition 2.4. *Let e be a fuzzy expression. Then, $\overline{e}$ can be obtained from e^* by exchanging all variables and their complements.*

Definition 2.5. *Let X be a collection of elements or objects. A k-dimensional fuzzy set $A_{a_1,\ldots,a_k}$ in X is a set of $(k+1)$-tuples:*

$$A = \{(x, \mu_{a_1}(x), \mu_{a_2}(x), \ldots, \mu_{a_k}(x)) | x \in X\}$$

where, $\mu_{a_i}(x)$ is the membership function corresponding to $i-th$ dimension of A.

The fuzzy set operations on two k-dimensional fuzzy sets $A_{a_1,\ldots,a_k}$ and $B_{b_1,\ldots,b_k}$ are defined as follows:

$$\overline{\mu}_A(x) = \big(\overline{\mu}_{a_1}(x), \ldots, \overline{\mu}_{a_k}(x)\big), \forall x \in X,$$

$$\mu_{A\cup B}(x) = \big(\mu_{a_1\cup b_1}(x), \ldots, \mu_{a_n\cup b_k}(x)\big), \forall x \in X,$$

$$\mu_{A\cap B}(x) = \big(\mu_{a_1\cap b_1}(x), \ldots, \mu_{a_n\cap b_k}(x)\big), \forall x \in X.$$

Example 2.6. *Let R be a collection of research resources, and the set of dimensions are Computation (C), Dynamical systems and differential equations (D), Game theory (G), Information theory and signal processing (I), Mathematical physics (M), Operations research (O) and Probability and statistics (P). If $A_{C,D,G,I,M,O,P}$ be a 7-dimensional fuzzy set in R which is considered as "the fuzzy set of research resources related to Applied mathematics", then, for every resource $r \in R$ the grade of membership in A is a 7-tuple in the form:*

$$\mu_A(r) = \big(\mu_C(r), \mu_D(r), \mu_G(r), \mu_I(r), \mu_M(r), \mu_O(r), \mu_P(r)\big).$$

Now, suppose that R is the set of resources of this article. Then, the grade of membership of some resource in A could be presented as follows:

$$\mu_A([26]) = (0.5, 0.3, 0.0, 0.2, 0.2, 0.1, 0.1),$$

$$\mu_A([27]) = (0.5, 0.4, 0.2, 0.4, 0.2, 0.2, 0.1),$$

$$\mu_A([23]) = (0.6, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0).$$

Note that the grades depends on the definition of membership functions $\mu_{a_1}, \ldots, \mu_{a_k}$.

2.2 Ranked alphabets and Σ-trees

Definitions in this section, are different from that in [6, 20, 21] only in some minor details.

The set of nonnegative integers is denoted by $\mathbb{N}$, and the set of finite strings over $\mathbb{N}$ is $\mathbb{N}^*$. The empty string is denoted by ε. A Σ-*alphabet* is a finite and nonempty set of symbols. A *ranked alphabet* is a couple $(\Sigma, Arity : \Sigma \to \mathbb{N})$, which is the disjoint union of sets of *n-ary* symbols $\Sigma_n = \{\sigma | Arity(\sigma) = n\}$ for all $n \in \mathbb{N}$. The set $T_\Sigma(Q)$ of Σ-*trees* indexed by Q is inductively defined to be the smallest set such that $Q \subseteq T_\Sigma(Q)$ and $\sigma(t_1, \ldots, t_n) \subseteq T_\Sigma(Q)$ for every $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma(Q)$. We write T_Σ for $T_\Sigma(\emptyset)$. A tree $t \in T_\Sigma(Q)$ can also be defined as a partial function $t : \mathbb{N}^* \to Q \cup \Sigma$ with domain $\mathcal{P}(t)$ satisfying the following properties:

1. $\mathcal{P}(t)$ is nonempty and prefix-closed.
2. $(\forall p \in \mathcal{P}(t))\ t(p) \in \Sigma_{n>0} \Rightarrow \{j | pj \in \mathcal{P}(t)\} = \{1, 2, ..., n\}$.
3. $(\forall p \in \mathcal{P}(t))\ t(p) \in Q \cup \Sigma_0 \Rightarrow \{j | pj \in \mathcal{P}(t)\} = \emptyset$.

Each element $p \in \mathcal{P}(t)$ is called a *position*.

2.3 Fuzzy finite tree automata and fuzzy tree languages

We adapt definitions related to FFTA [4, 10, 20, 21] for our requirements.

Definition 2.7. *A fuzzy finite tree automaton is a system $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$, where:*

1. *Σ is a finite set of ranked alphabets called input symbols.*
2. *Q is a finite set of symbols, called states.*
3. *Γ is a fuzzy set on Q and called the set of final states.*
4. *$\delta = \{\delta_\sigma : Q^n \times \Sigma_n \to (Q \times \ell) \,|\, \sigma \in \Sigma_n,\ n \geq 0\}$ is a finite fuzzy set, called transition rules.*
5. *$\rho : T_\Sigma(Q \times \ell) \to (Q \times \ell)$ is the run map of FFTA M and defined by induction on structure of $t \in T_\Sigma(Q \times \ell)$:*
   (a) $(\forall q \in Q, \sigma \in \Sigma_0)\ t = \sigma \Rightarrow \rho(t)(q) = \delta(q, \sigma)$.
   (b) $(\forall q \in Q, \sigma \in \Sigma_{n>0}, t_1, \ldots, t_n \in T_\Sigma)\ t = \sigma(t_1, \ldots, t_n) \Rightarrow$
$$\rho(t)(q) \overset{def}{=} \bigvee_{q_1,\ldots,q_n\ \in Q} \Big(\delta(q_1, \ldots, q_n, q, \sigma) \wedge \bigwedge_{i=1}^{n} \rho(t_i)(q_i)\Big).$$
6. *β is a fuzzy set over T_Σ, called behavior of FFTA M, and defined by:*
$$\beta(t) \overset{def}{=} \bigvee_{q\in Q} \Big(\rho(t)(q) \wedge \mu_\Gamma(q)\Big), \forall t \in T_\Sigma.$$

An FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ accepts (recognizes) a tree $t \in T_\Sigma$ iff $\beta(t) > 0$, and the set of all trees accepted by M is known as fuzzy regular tree language $L(M)$. Note that $\beta(t)$ is the membership function of the fuzzy set $L(M)$.

Definition 2.8. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. Then,*

- *M is deterministic if for every $\sigma \in \Sigma_n$ and $q_1, \ldots, q_n \in Q$ with $n \in \mathbb{N}$, there exist at most one $q \in Q$, such that $\delta(q_1, \ldots q_n, q, \sigma) > 0$.*
- *M is complete if for every $\sigma \in \Sigma_n$ and $q_1, \ldots, q_n \in Q$, where $n \geq 0$, there exist at least one $q \in Q$, such that $\delta(q_1, \ldots q_n, q, \sigma) > 0$.*
- *M is reduced if for every $q \in Q$, there exists at least one $t \in T_\Sigma$ such that $\rho(t)(q) > 0$. In this case, we call all states are accessible.*
- *M is minimized if M is complete and for every complete FFTA $M' = (\Sigma, Q', \Gamma', \delta', \ell, \rho', \beta')$ that $L(M) = L(M')$ it holds $|Q| \leq |Q'|$.*

3 Multidimensional fuzzy finite tree automata

Definition 3.1. *A k-dimensional FFTA is a system $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$, where:*

1. *Σ is a finite set of ranked alphabets, called input symbols.*
2. *$\mathcal{D}$ is the set of dimensions, where $|\mathcal{D}| = k > 0$.*
3. *$\Gamma : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ is called finalizing function.*
4. *$\delta \overset{def}{=} \{\delta_\sigma : \Sigma_n \to \mu_n \,|\, \sigma \in \Sigma_n,\ \mu_n : \Big(\widetilde{\mathcal{D}}\Big)^n \to \widetilde{\mathcal{D}}, n \in \mathbb{N}\}$ is called the set of transition rules. μ_n is a k-tuple of functions over $(\vee, \wedge, \ell)$ such that every $\mu_d \in \mu_n$ gives the fuzzy membership grades of n k-dimensional fuzzy sets as input and calculate a fuzzy membership grade for dimension d as output.*
5. *$\rho : T_\Sigma(\widetilde{\mathcal{D}}) \to 2^{\widetilde{\mathcal{D}}}$ is called the run map of MFFTA and defined for every $t = \sigma(t_1, \ldots, t_n) \in T_\Sigma(\widetilde{\mathcal{D}})$ by:*
$$\rho(t) \overset{def}{=} \{\mu(q_1, \ldots, q_n) | q_1 \in \rho(t_1), \ldots, q_n \in \rho(t_n)\}.$$

6. $\beta : T_\Sigma \to \widetilde{\mathcal{D}}$ is called the behavior of automata and for every $t \in T_\Sigma$

$$\beta(t) \overset{def}{=} \bigvee_{q\in\rho(t)} \Gamma(q) \quad , \quad \beta^*(t) \overset{def}{=} \bigwedge_{q\in\rho(t)} \Gamma(q).$$

An MFFTA $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ *recognizes* / *dually recognizes* a tree $t \in T_\Sigma$ if and only if there exists a non-empty fuzzy set $q \in \widetilde{\mathcal{D}}$ such that $\beta(t) = q$ / $\beta^*(t) = q$. Also, the set of all trees recognized by M is known as *multidimensional fuzzy tree language* $L(M)$. We denote by $L^*(M)$ the set of all trees dually recognized by M.

Remark 3.2. *Let $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ be an MFFTA and $\dot{\mu}_n$ be the constant mapping $(\widetilde{\mathcal{D}})^n \to \emptyset$ for $n \in \mathbb{N}$. Now, for each $\sigma \in \Sigma_n$ that there is no explicitly defined μ_n with $\delta_\sigma(\sigma, \mu_n) \in \delta$, we assume that the transition rule $\delta_\sigma(\sigma, \dot{\mu}_n) \in \delta$ is defined implicitly. In fact, we are interested in equipping M with a complete rule set to be enable for processing all regular trees on the Σ-alphabet. This assumption enables MFFTA to avoid halt during the processing Σ-trees. In other words, for every MFFTA it is assumed that any undefined transition rule is actually a transition rule to an empty set (membership grade* 0 *in all dimensions) which results that every MFFTA is complete.*

Lemma 3.3. *Let $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ and $M_c = (\Sigma, \mathcal{D}, \bar{\Gamma}, \delta, \ell, \rho, \beta_c)$ be two MFFTA. Then, $L^*(M_c) = \overline{L(M)}$.*

Proof. For every tree $t \in T_\Sigma$, it holds $\beta_c^*(t) = \bigwedge_{q\in\rho(t)} \bar{\Gamma}(q) = \overline{\beta}(t)$. □

Corollary 3.4. *The class of multidimensional fuzzy tree languages is closed under complement.*

Theorem 3.5. *The class of multidimensional fuzzy tree languages is closed under union.*

Proof. Let $M_a = (\Sigma, \mathcal{D}_a, \Gamma_a, \delta_a, \ell, \rho_a, \beta_a)$ and $M_b = (\Sigma, \mathcal{D}_b, \Gamma_b, \delta_b, \ell, \rho_b, \beta_b)$ are two MFFTA. We construct the union automata MFFTA $M_c = (\Sigma, \mathcal{D}_c, \Gamma_c, \delta_c, \ell, \rho_c, \beta_c)$ such that $L(M_c) = L(M_a) \cup L(M_b)$. Let $\mathcal{D}_{ab} = \mathcal{D}_a \cap \mathcal{D}_b$. We introduce two different method (based on the emptiness of $\mathcal{D}_{ab}$) for constructing M_c.

- ($\mathcal{D}_{ab} = \emptyset$): In this case, calculating M_c can be done by:

  $\delta_{ca} = \{\delta_\sigma\left(\sigma, \mu_n \cup \{\mu(d) = 0 | d \in \mathcal{D}_b\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_a, \sigma \in \Sigma_n, n \in \mathbb{N}\}$,

  $\delta_{cb} = \{\delta_\sigma\left(\sigma, \mu_n \cup \{\mu(d) = 0 | d \in \mathcal{D}_a\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_b, \sigma \in \Sigma_n, n \in \mathbb{N}\}$,

  $\mathcal{D}_c = \mathcal{D}_a \cup \mathcal{D}_b \quad , \quad \Gamma_c = \Gamma_a \vee \Gamma_b \quad , \quad \delta_c = \delta_{ca} \cup \delta_{cb}$.

  The set of transitions δ_c is the union of two disjoint sets δ_{ca} and δ_{cb} because they operate on two disjoint sets of dimensions. So, $\rho_c(t) = \rho_a(t) \vee \rho_b(t)$ and then, for every $t \in T_\Sigma$ it holds:

$$\begin{aligned}\beta_c(t) = \bigvee_{q\in\rho_c(t)} \Gamma_c(q) &= \Big(\bigvee_{q\in\rho_c(t)} \Gamma_a(q)\Big) \vee \Big(\bigvee_{q\in\rho_c(t)} \Gamma_b(q)\Big) \\ &\overset{(1)}{=} \Big(\bigvee_{q\in\rho_a(t)} \Gamma_a(q)\Big) \vee \Big(\bigvee_{q\in\rho_b(t)} \Gamma_b(q)\Big) = \beta_a(t) \vee \beta_b(t).\end{aligned}$$

  Note that the equality (1) is holds because the run map $\rho_c(t)$ uses the transition rules δ_{ca} and δ_{cb} which they are adapted to act on disjoint sets of dimensions $\mathcal{D}_a$ and $\mathcal{D}_b$.

- ($\mathcal{D}_{ab} \neq \emptyset$): Firstly we define two new dimensions d_a and d_b such that $d_a, d_b \notin \mathcal{D}_a \cup \mathcal{D}_b$. Then, for every $\mu_n : q_1 \times \ldots \times q_n \to q$ in δ_a and δ_b where $q_1, \ldots, q_n, q \in \widetilde{\mathcal{D}}$, define μ_n^a and μ_n^b by:

  $\mu_n^a = \{\bigwedge_{i=1}^{n} \mu_{q_i}(d_a) \wedge \mu(d) | \mu(d) \in \mu_n\} \cup \{d_a = 1, d_b = 0\}$

  $\mu_n^b = \{\bigwedge_{i=1}^{n} \mu_{q_i}(d_b) \wedge \mu(d) | \mu(d) \in \mu_n\} \cup \{d_a = 0, d_b = 1\}$

  Now, M_c could be obtained by the following settings:

  $\delta_{ca} = \{\delta_\sigma\left(\sigma, \mu_n^a \cup \{\mu(d) = 0 | d \in \mathcal{D}_b\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_a, \sigma \in \Sigma_n, n \in \mathbb{N}\}$,

  $\delta_{cb} = \left\{\delta_\sigma\left(\sigma, \mu_n^b \cup \{\mu(d) = 0 | d \in \mathcal{D}_a\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_b, \sigma \in \Sigma_n, n \in \mathbb{N}\right\}$,

  $\mathcal{D}_c = \mathcal{D}_a \cup \mathcal{D}_b \cup \{d_a, d_b\} \quad , \quad \delta_c = \delta_{ca} \cup \delta_{cb}$,

$\forall q \in \widetilde{\mathcal{D}_c}; \Gamma_c(q) = (\Gamma_a(q) \wedge \mu_q(d_a)) \vee (\Gamma_b(q) \wedge \mu_q(d_b))$.

Since $\mathcal{D}_a \cap \mathcal{D}_a \neq \emptyset$, in order to avoid combining the rules derived from two automata, we define two new symbols d_a (set it to 1 only for rules derived from δ_a) and d_b (set it to 1 only for rules derived from δ_b) and use them in all formulas. Furthermore, d_a is used in Γ_c to ensure that Γ_a part of Γ_c affects only on the ρ_a part of ρ_c (the set of multidimensional fuzzy sets obtained from transitions rules derived from δ_a), and d_b is used in Γ_c to ensure that Γ_b part of Γ_c affects only on the ρ_b part of ρ_c. So, for every $t \in T_\Sigma$ it holds:

$$\begin{aligned}\beta_c(t) = \bigvee_{q\in\rho_c(t)} \Gamma_c(q) &= \Big(\bigvee_{q\in\rho_c(t)} \Gamma_a(q) \wedge \mu_q(d_a)\Big) \vee \Big(\bigvee_{q\in\rho_c(t)} \Gamma_b(q) \wedge \mu_q(d_b)\Big) \\ &= \Big(\bigvee_{q\in\rho_a(t)} \Gamma_a(q)\Big) \vee \Big(\bigvee_{q\in\rho_b(t)} \Gamma_b(q)\Big) = \beta_a(t) \vee \beta_b(t).\end{aligned}$$

□

Theorem 3.6. *The class of multidimensional fuzzy tree languages is closed under intersection.*

Proof. it is the same as Theorem (3.5). Let $M_a = (\Sigma, \mathcal{D}_a, \Gamma_a, \delta_a, \ell, \rho_a, \beta_a)$ and $M_b = (\Sigma, \mathcal{D}_b, \Gamma_b, \delta_b, \ell, \rho_b, \beta_b)$ be two MFFTA. We construct the intersection automata MFFTA $M_c = (\Sigma, \mathcal{D}_c, \Gamma_c, \delta_c, \ell, \rho_c, \beta_c)$ such that $L(M_c) = L(M_a) \cap L(M_b)$. Since $\mathcal{D}_a \cap \mathcal{D}_b = \emptyset$ implies that $L(M_c) = L(M_a) \cap L(M_b) = \emptyset$, we suppose that $\mathcal{D}_a \cap \mathcal{D}_b \neq \emptyset$. Firstly we define two new dimensions $d_a, d_b \notin \mathcal{D}_a \cup \mathcal{D}_b$. Then, for every $\mu_n : q_1 \times \ldots \times q_n \to q$ in δ_a and δ_b where $q_1, \ldots, q_n, q \in \widetilde{\mathcal{D}}$, define:

$\mu_n^a = \{\bigwedge_{i=1}^{n} \mu_{q_i}(d_a) \wedge \mu(d) | \mu(d) \in \mu_n\} \cup \{d_a = 1, d_b = 0\}$

$\mu_n^b = \{\bigwedge_{i=1}^{n} \mu_{q_i}(d_b) \wedge \mu(d) | \mu(d) \in \mu_n\} \cup \{d_a = 0, d_b = 1\}$

Now, M_c could be obtained by the following settings:

$\delta_{ca} = \{\delta_\sigma\left(\sigma, \mu_n^a \cup \{\mu(d) = 0 | d \in \mathcal{D}_b\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_a, \sigma \in \Sigma_n, n \in \mathbb{N}\}$,

$\delta_{cb} = \left\{\delta_\sigma\left(\sigma, \mu_n^b \cup \{\mu(d) = 0 | d \in \mathcal{D}_a\}\right) | \delta_\sigma(\sigma, \mu_n) \in \delta_b, \sigma \in \Sigma_n, n \in \mathbb{N}\right\}$,

$\mathcal{D}_c = \mathcal{D}_a \cup \mathcal{D}_b \cup \{d_a, d_b\}$, $\delta_c = \delta_{ca} \cup \delta_{cb}$,

$\forall q \in \widetilde{\mathcal{D}_c}; \Gamma_c(q) = (\Gamma_a(q) \wedge \mu_q(d_a)) \wedge (\Gamma_b(q) \wedge \mu_q(d_b))$.

Two new symbols d_a (set it to 1 only for rules derived from δ_a) and d_b (set it to 1 only for rules derived from δ_b) are defined and applied to all formulas of transition rules to avoid combining the rules derived from two automata. Furthermore, d_a and d_b are used in Γ_c to ensure that Γ_a part of Γ_c affects only on the ρ_a part of ρ_c, and Γ_b part of Γ_c affects only on ρ_b part of ρ_c. Now, for every $t \in T_\Sigma$ it holds:

$$\begin{aligned}\beta_c(t) = \bigvee_{q\in\rho_c(t)} \Gamma_c(q) &= \Big(\bigvee_{q\in\rho_c(t)} \Gamma_a(q) \wedge \mu_q(d_a)\Big) \wedge \Big(\bigvee_{q\in\rho_c(t)} \Gamma_b(q) \wedge \mu_q(d_b)\Big) \\ &= \Big(\bigvee_{q\in\rho_a(t)} \Gamma_a(q)\Big) \wedge \Big(\bigvee_{q\in\rho_b(t)} \Gamma_b(q)\Big) = \beta_a(t) \wedge \beta_b(t).\end{aligned}$$

□

Definition 3.7. *Let $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ be an MFFTA. Then,*

- *M is deterministic if for every $\sigma \in \Sigma_n$ and fuzzy sets $q_1, \ldots, q_n, q \in \widetilde{\mathcal{D}}$ with $n \in \mathbb{N}$, there exists at most one mapping $\mu(q_1, \ldots, q_n) = q$, such that $\delta_\sigma(\sigma, \mu) \in \delta$.*

- *M is reduced if for every dimension $d \in \mathcal{D}$, there exists a fuzzy set $q \in \widetilde{\mathcal{D}}$ and a $t \in T_\Sigma$ such that $q \in \rho(t)$ and it holds $\mu_q(d) > 0$ or $\mu_{\Gamma(q)}(d) > 0$. In this case, we say all dimensions are meaningful. This definition is more than general accessibility of states in conventional automata.*

- *M is optimistic if for every $\delta_\sigma(\sigma, \mu_n) \in \delta$ where $\sigma \in \Sigma_n$, $\mu(q_1, \ldots, q_n) = q$, $q_1, \ldots, q_n, q \in \widetilde{\mathcal{D}}$ and $n \in \mathbb{N}$, all $\mu \in \mu_n$ are in one of the following forms*
  - $\mu(d) = 0$,
  - $\mu(d) = 1$,

– $\mu(d) = \bigvee_{\substack{d' \in \mathcal{D}' \\ q' \in \{q_1,\ldots,q_n\}}} \mu_{q'}(d')$.

where $d \in \mathcal{D}$ *and* $\mathcal{D}' \subseteq \mathcal{D}$.

- *M is pessimistic if for every* $\delta_\sigma(\sigma, \mu_n) \in \delta$ *where* $\sigma \in \Sigma_n$, $\mu(q_1, \ldots, q_n) = q$, $q_1, \ldots, q_n, q \in \widetilde{\mathcal{D}}$ *and* $n \in \mathbb{N}$, *all* $\mu \in \mu_n$ *are in one of the following forms*
  – $\mu(d) = 0$,
  – $\mu(d) = 1$,
  – $\mu(d) = \bigwedge_{\substack{d' \in \mathcal{D}' \\ q' \in \{q_1,\ldots,q_n\}}} \mu_{q'}(d')$.

  where $d \in \mathcal{D}$ *and* $\mathcal{D}' \subseteq \mathcal{D}$.

Theorem 3.8. *For every nondeterministic optimistic/pessimistic MFFTA there exists a deterministic MFFTA with the same language/dual language.*

Proof. Let $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ be a nondeterministic MFFTA. We construct deterministic MFFTA $M_{det} = (\Sigma, \mathcal{D}_{det}, \Gamma_{det}, \delta_{det}, \ell, \rho_{det}, \beta_{det})$ such that for every $t \in T_\Sigma$ it holds:

- $\beta_{det}(t) = \beta(t)$ if M is optimistic,
- $\beta^*_{det}(t) = \beta^*(t)$ if M is pessimistic.

Since M is nondeterministic, there exists at least one $\sigma \in \Sigma_n$ and two transition function μ_n and μ'_n such that $\delta_\sigma(\sigma, \mu_n), \delta_\sigma(\sigma, \mu'_n) \in \delta$. We introduce a method for solving this nondeterminism, while it can be used several times to solve other nondeterminism sources incrementally.

Let $\mathcal{D}_{det} = \mathcal{D}$ and $\mathcal{R}$ be an equivalence relation such that its initial value is $\mathcal{R}(d) = \{d\}$ for every $d \in \mathcal{D}_{det}$. Also, let $\mathcal{D}_{dif}$ be the smallest set of dimensions involved in nondeterminism and is defined by:

$$\mathcal{D}_{dif} = \{d \mid \mu(d) \neq \mu'(d) \textit{ where } \mu \in \mu_n, \mu' \in \mu'_n\}.$$

Now, let $\Gamma_{det} = \Gamma$, $\delta_{det} = \delta \setminus \{\delta_\sigma(\sigma, \mu'_n)\}$ and for every dimension $d \in \mathcal{D}_{dif}$ do the following steps:

1. Define a new dimension $d' \notin \mathcal{D}_{det}$ and add it to $\mathcal{D}_{det}$,
2. Set $\mu(d')$ equal to the right-hand side of $\mu'(d) \in \mu'_n$ and add $\mu(d')$ to μ_n,
3. For every $\sigma' \in \Sigma_n$ and μ''_n that $\delta_{\sigma'}(\sigma', \mu''_n) \in \delta_{det} \setminus \{\delta_\sigma(\sigma, \mu_n)\}$ and $\mu(d) \in \mu''_n$, add $\mu(d')$ to μ''_n (the formula of $\mu(d')$ must be same as the formula $\mu(d)$),
4. $\mathcal{R}(d) \longleftarrow \mathcal{R}(d) \cup \{d'\}$,
5. $\forall d' \in \mathcal{R}(d); \mathcal{R}(d') \longleftarrow \mathcal{R}(d)$,
6. For every $\sigma' \in \Sigma_{det}$ and μ'' that $\delta_{\sigma'}(\sigma', \mu'') \in \delta_{det}$, replace $\mu_q(d)$ in the right-hand side of each $\mu'(d'') \in \mu$ formula by:
   - $\big(\mu_q(d) \vee \mu_q(d')\big)$; if MFFTA is optimistic,
   - $\big(\mu_q(d) \wedge \mu_q(d')\big)$; if MFFTA is pessimistic,

   where $d'' \in \mathcal{D}_{det}$ and $q \in \widetilde{\mathcal{D}_{det}}$.
7. In finalizing function Γ_{det}, set the formula of $\mu(d')$ the same as formula $\mu(d)$,
8. Modify Γ_{det} by replacing $\mu_q(d)$ in the right-hand side of equation by:
   - $\bigvee_{d'' \in \mathcal{R}(d)} \mu_q(d'')$; if MFFTA is optimistic,

- $\bigwedge_{d'' \in \mathcal{R}(d)} \mu_q(d'')$; if MFFTA is pessimistic,

9. Modify Γ_{det} by setting $\mu_q(d') = 0$.

Now, we check the behavior of two automata to make sure that $L(M) = L(M_{det})$. Since M is optimistic or pessimistic, new dimensions do not affect the behavior of automaton until the transition rule $\delta_\sigma(\sigma, \mu_n)$ be applied (because for every $x \in \ell$ it holds $x = x \vee x$ and $x = x \wedge x$). Note that in all transition rules except $\delta_\sigma(\sigma, \mu_n)$, the membership of each new dimension d' is like an old dimension d that $d'\mathcal{R}d$ and in the right-hand side of all functions of the transition rules, $\mu(d)$ is replaced by $\mu(d) \vee \mu(d')$ for optimistic MFFTA and with $\mu(d) \wedge \mu(d')$ for pessimistic MFFTA. Suppose $t \in T_\Sigma$ is a Σ-tree that the σ is one of its nodes. During processing t by M, applying transition rules $\delta_\sigma(\sigma, \mu_n)$ and $\delta_\sigma(\sigma, \mu'_n)$ on the node σ results in two fuzzy sets $q, q' \in \mathcal{D}$, respectively. The set $\mathcal{D}_{dif}$ includes dimensions that their grade of membership may be different in q and q'. On the other hand, when t is processing by M_{det}, applying the extended transition rule $\delta_\sigma(\sigma, \mu_n)$ on the node σ results in extended fuzzy set $q'' \in \mathcal{D}_{det}$. The fuzzy set q'' includes the elements and those grade of membership in the fuzzy set q union with a new fuzzy set that is defined as a number of new dimensions corresponds to members of $\mathcal{D}_{dif}$ and their grade of membership in q'. So, the set of grades in $q \cup q'$ is equal to the set of grades in q''. The sixth step mentioned above guaranties that if the membership grade of a $d \in \mathcal{D}_{dif}$ is to be used when t is processing by M or M_{det}, as the grades from q and q' are combined by Γ and recursion of ρ, these grades are also taken from q'' and combine by Γ_{det} and recursion of ρ_{det}.

Note that the operators $\vee$ and $\wedge$ are commutative and

- if M and M_{det} are optimistic, then, β, β_{det}, Γ, Γ_{det}, δ and δ_{det} only use the operator $\vee$;
- if M and M_{det} are pessimistic, then, β^*, β^*_{det}, Γ, Γ_{det}, δ and δ_{det} only use the operator $\wedge$.

Therefore, for every $t \in T_\Sigma$ it holds:

- $\beta_{det}(t) = \beta(t)$ if M is optimistic,
- $\beta^*_{det}(t) = \beta^*(t)$ if M is pessimistic.

□

Theorem 3.9. *For every MFFTA there exists a reduced MFFTA with the same language.*

Proof. Algorithm 3.10 shows a reduction process of an input MFFTA.

Algorithm 3.10. *(***Reduction of MFFTA.***)*

0 **Input:** ***MFFTA*** $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$
1 $\mathcal{D}_{red} \longleftarrow \emptyset$
2 $\delta_{red} \longleftarrow \emptyset$
3 $Q \longleftarrow \{\emptyset\}$
4 **Repeat**
5 $\quad (\forall \sigma \in \Sigma_n, q_1, \ldots, q_n \in Q, q \in \widetilde{\mathcal{D}}, \mu_n(q_1, \ldots, q_n) = q, \delta_\sigma(\sigma, \mu_n) \in \delta)$
6 $\qquad Q \longleftarrow Q \cup \{q\}$
7 $\qquad \mathcal{D}_{red} \longleftarrow \mathcal{D}_{red} \cup \{d \mid (\mu_q(d) \vee \mu_{\Gamma(q)}(d)) > 0\}$
8 $\qquad \delta_{red} \longleftarrow \delta_{red} \cup \{\delta_\sigma(\sigma, \mu_n)\}$
9 **Until** ***no fuzzy set*** $q \in \widetilde{\mathcal{D}}$ ***can be added to*** Q
10 **Output:** $M_{red} = (\Sigma, \mathcal{D}_{red}, \Gamma, \delta_{red}, \ell, \rho_{red}, \beta_{red})$**.**

Algorithm 3.10 is a generalization of reduction algorithm RED [6] which is adapted for the definition of reduced MFFTA. A dimension $d \in \mathcal{D}$ is accessible iff there exists a fuzzy set $q \in \widetilde{\mathcal{D}}$ and a tree $t \in T_\Sigma$ such that $q \in \rho(t)$ and it holds $\mu_q(d) > 0$ or $\mu_{\Gamma(q)}(d) > 0$. Also, a transition rule $\delta_\sigma(\sigma, \mu_n)$ is accessible iff in $\mu_n(q_1, \ldots, q_n)$ every dimension with $\mu_{q_1 \cup \ldots q_n}(d) > 0$ be accessible. The set $\mathcal{D}_{red}$ is the set of accessible dimensions and δ_{red} is the set of accessible transition rules. The algorithm starts with transition rules that use constant alphabets because they are accessible and do not require any accessible dimension in μ_n. In the following, the repeat loop (lines 4 to 9) uses Q Q which is the set of accessible fuzzy sets on $\mathcal{D}$ to drive $\mathcal{D}_{red}$. In fact, $\mathcal{D}_{red}$ is produced incrementally by itself and the set of accessible transition rules. Also, the algorithm preserves all nonempty fuzzy sets generated by ρ and Γ according to conditions in line (5). Thus, for every $t \in T_\Sigma$ it holds $\beta_{red}(t) = \beta(t)$. □

Theorem 3.11. *The class of FFTA is a subclass of MFFTA.*

Proof. We show that for every complete FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ one can construct an MFFTA $M_{md} = (\Sigma, \mathcal{D}, \Gamma_{md}, \delta_{md}, \ell, \rho_{md}, \beta_{md})$ with the same behavior.

Let $\mathcal{D} = Q$ and $\Gamma_{md}(D) = \{(q_i, \mu_D(q_i) \wedge \Gamma(q_i)) \mid q_i \in \mathcal{D}\}$, where $D \in \widetilde{\mathcal{D}}$. Then, for every $\sigma \in \Sigma_n$, $q_1, \ldots, q_n, q \in Q$ and $x \in \ell$ that $\delta_\sigma(q_1, \ldots, q_n, q, \sigma) = x$ and $x > 0$, define $\mu_n(D_1, \ldots, D_n) = \{(q, \mu_{D_1}(q) \wedge x), \ldots, (q, \mu_{D_n}(q) \wedge x)\}$ and add the transition rule $\delta_\sigma(\sigma, \mu_n)$ to δ_{md}. Now, it is easy to verify that M and M_{md} have the same behavior during the processing all $t \in T_\Sigma$. □

Example 3.12. *In this example, the method introduced in Theorem (3.11) runs on a small FFTA. Let FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ is defined as:*

$\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \quad ; \quad \Sigma_0 = \{a, b\}, \Sigma_1 = \{g\}, \Sigma_2 = \{f\},$

$Q = \{q_1, q_2, q_3\} \quad , \quad \Gamma = \{(q_1, 0), (q_2, 0.7), (q_3, 0.4)\},$

and δ is the following set of transition rules:

$\delta(q_1, a) = 0.2 \quad \delta(q_3, a) = 0.5 \quad \delta(q_2, b) = 0.8 \quad \delta(q_3, b) = 0.5$

$\delta(q_1, q_1, g) = 0.3 \quad \delta(q_2, q_3, g) = 0.7 \quad \delta(q_3, q_2, g) = 1$

$\delta(q_1, q_1, q_2, f) = 0.1 \quad \delta(q_1, q_2, q_3, f) = 0.2 \quad \delta(q_1, q_3, q_1, f) = 0.3$

$\delta(q_2, q_1, q_3, f) = 0.4 \quad \delta(q_2, q_2, q_2, f) = 0.5 \quad \delta(q_2, q_3, q_1, f) = 0.6$

$\delta(q_3, q_1, q_1, f) = 0.7 \quad \delta(q_3, q_2, q_2, f) = 0.8 \quad \delta(q_3, q_3, q_3, f) = 0.9$

We construct MFFTA $M_{md} = (\Sigma, \mathcal{D}, \Gamma_{md}, \delta_{md}, \ell, \rho_{md}, \beta_{md})$ by setting $\mathcal{D} = Q$, $\Gamma_{md}(D) = \{(q_1, 0), (q_2, \mu_D(q_2) \wedge 0.7), (q_3, \mu_D(q_3) \wedge 0.4)\}$, where $D \in \widetilde{\mathcal{D}}$ and δ_{md} is the following set of transition rules:

$\delta(a, \emptyset \to \{\mu(q_1) = 0.2\})$

$\delta(a, \emptyset \to \{\mu(q_3) = 0.5\})$

$\delta(b, \emptyset \to \{\mu(q_2) = 0.8\})$

$\delta(b, \emptyset \to \{\mu(q_3) = 0.5\})$

$\delta(g, D \to \{\mu(q_1) = \mu_D(q_1) \wedge 0.3\})$

$\delta(g, D \to \{\mu(q_3) = \mu_D(q_2) \wedge 0.7\})$

$\delta(g, D \to \{\mu(q_2) = \mu_D(q_3) \wedge 1\})$

$\delta(f, (D, D') \to \{\mu(q_2) = \mu_D(q_1) \wedge \mu_{D'}(q_1) \wedge 0.1\})$

$\delta(f, (D, D') \to \{\mu(q_3) = \mu_D(q_1) \wedge \mu_{D'}(q_2) \wedge 0.2\})$

$\delta(f, (D, D') \to \{\mu(q_1) = \mu_D(q_1) \wedge \mu_{D'}(q_3) \wedge 0.3\})$

$\delta(f, (D, D') \to \{\mu(q_3) = \mu_D(q_2) \wedge \mu_{D'}(q_1) \wedge 0.4\})$

$\delta(f, (D, D') \to \{\mu(q_2) = \mu_D(q_2) \wedge \mu_{D'}(q_2) \wedge 0.5\})$

$\delta(f, (D, D') \to \{\mu(q_1) = \mu_D(q_2) \wedge \mu_{D'}(q_3) \wedge 0.6\})$

$\delta(f, (D, D') \to \{\mu(q_1) = \mu_D(q_3) \wedge \mu_{D'}(q_1) \wedge 0.7\})$

$\delta(f, (D, D') \to \{\mu(q_2) = \mu_D(q_3) \wedge \mu_{D'}(q_2) \wedge 0.8\})$

$\delta(f, (D, D') \to \{\mu(q_3) = \mu_D(q_3) \wedge \mu_{D'}(q_3) \wedge 0.9\})$

Example (3.12) shows that the process of converting an FFTA to an MFFTA using Theorem (3.11), is a one to one mappings between states of the FFTA and dimensions of the MFFTA, as well as their transition rules and also their Γs. In Example (3.13) we show that a regular tree language associated with an FFTA can be described by a much smaller MFFTA.

Example 3.13. *Consider the set of all regular trees [6] on the set of alphabet $\Sigma_0 = \{a, b\}$, $\Sigma_1 = \{g\}$ and $\Sigma_2 = \{f\}$. Now, we construct an FFTA and an MFFTA recognizing the fuzzy tree language L while the properties of each tree $t \in L$ is described linguistically as:*

p1: It's better the number of a be even.

p2: It's better not to have b.

p3: It's better each subtree with root g not to have any node b.

p4: Each subtree with root g should never has a node f.

Furthermore, the grade of membership of trees in fuzzy tree language L is based on the following rules:

r1: The tree is a member of L if and only if it satisfies property p4.

r2: If the tree is a member of L, its grade of membership must be affected by the number of properties p1, p2 and p3 that are satisfied.

Now, we construct complete FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ such that satisfies the mentioned properties and rules. Let $Q = \{q_{ijk} \mid i, k \in \{0,1\}, j \in \{0,1,2\}\} \cup \{q_*\}$ where the reason of indices as follow:

- $i = 0$*: property p1 is hold,*
- $i = 1$*: property p1 is not hold,*
- $j = 0$*: properties p2 and p3 are hold,*
- $j = 1$*: property p2 is violated but property p3 is hold,*
- $j = 2$*: properties p2 and p3 are violated,*
- $k = 0$*: f do not exists in tree,*
- $k = 1$*: f exists in tree,*
- q_**: property p4 is violated.*

So, based on the rules r1 and r2, definition of the final states is

$$\Gamma = \{(q_{000}, 1), (q_{001}, 1), (q_{010}, 0.8), (q_{011}, 0.8), (q_{020}, 0.5), (q_{021}, 0.5), (q_{100}, 0.8), \\ (q_{101}, 0.8), (q_{110}, 0.5), (q_{111}, 0.5), (q_{120}, 0.2), (q_{121}, 0.2), (q_*, 0)\}.$$

Finally, δ is the following set of transition rules:
$\delta(q_{100}, a) = 1 \quad \delta(q_{010}, b) = 1 \quad \delta(q_*, q_*, g) = 1$
$\delta(q_{ij1}, q_*, g) = 1 \quad \delta(q_{i00}, q_{i00}, g) = 1 \quad \delta(q_{i\{j\neq 0\}0}, q_{i20}, g) = 1$
$\delta(q_{ijk}, q_*, q_*, f) = 1 \quad \delta(q_*, q_{ijk}, q_*, f) = 1 \quad \delta(q_*, q_*, q_*, f) = 1$
$\delta(q_{ijk}, q_{i'j'k'}, q_{i''j''1}, f) = 1$ *where* $i'' = (i + i')$ *mod* 2 *and* $j'' = max(j, j')$.

The number of states of FTA M is 13 *and the number of its transition rules is* 184*. Now, we construct MFFTA $M_{md} = (\Sigma, \mathcal{D}, \Gamma_{md}, \delta_{md}, \ell, \rho_{md}, \beta_{md})$ with* 5 *dimensions and* 4 *transition rules that its behavior is L.*

Let $\mathcal{D} = \{d1, d2, d3, d4, d5\}$, where the reason of dimensions are as follow:

- $\mu(d_i) = 0$*: property pi is not hold for* $1 \leq i \leq 4$,
- $\mu(d_i) = 1$*: property pi is hold for* $1 \leq i \leq 4$,
- $\mu(d_5) = 0$*: f do not exists in tree,*
- $\mu(d_5) = 1$*: f exists in tree.*

The definition of the finalizing function Γ_{md} is adapted with rules r1 and r2 by:
$\Gamma_{md}(D) = \{(d_i, \mu_D(d_i) \wedge \mu_D(d_4)) \mid 1 \leq i \leq 3\} \cup \{(d_4, \bar{\mu_D}(d_4)), (d_5, 0)\}$.
The set of transition rules δ_{md} defined by:
$\delta(a, \emptyset \to \{\mu(d_i) = 1 \mid i = 2, 3, 4, 5\}) \quad \delta(b, \emptyset \to \{\mu(d_i) = 1 \mid i = 1, 3, 4, 5\})$
$\delta(g, D \to \{\mu(d_i) = \mu_D(d_i) \mid i = 1, 2, 5\} \cup \{\mu(d_3) = \bar{\mu}_D(d_2), \mu(d_4) = \bar{\mu}_D(d_5)\})$
$\delta(f, (D, D') \to \{\mu(d_i) = \mu_D(d_i) \wedge \mu_{D'}(d_i) \mid i = 2, 3, 4\} \cup$
$\quad \{\mu(d_1) = (\mu_D(d_1) \wedge \mu_{D'}(d_1)) \vee (\bar{\mu}_D(d_1) \wedge \bar{\mu}_{D'}(d_1)), \mu(d_5) = 1\})$

Corollary 3.14. *Theorem (3.11) shows that each FFTA can be converted to an equal size (the number of states, dimensions and transition rules) MFFTA, and Example (3.13) shows that a (multidimensional) fuzzy tree language can be described by a very smaller MFFTA rather than the corresponding FFTA.*

Corollary 3.15. *If a fuzzy regular tree language is defined by multiple properties, an MFFTA is more capable of expressing the quality of belonging terms to the language rather than an FFTA.*

Example 3.16. *In this example, we discuss the language of different types of MFFTA. Let $M = (\Sigma, \mathcal{D}, \Gamma, \delta, \ell, \rho, \beta)$ be an MFFTA with*
$\Sigma_0 = \{a, b, c\} \qquad \Sigma_2 = \{f\} \qquad \mathcal{D} = \{d_1, d_2\}$
$\delta(a, \emptyset \to \{\mu(d_1) = 1\}) \qquad \delta(b, \emptyset \to \{\mu(d_2) = 1\})$
$\delta(c, \emptyset \to \{\mu(d_1) = 1, \mu(d_2) = 1\})$

(i) If

– $\Gamma(D) = \{(d_1, \mu_D(d_1) \wedge \mu_D(d_2))\}$ *for* $D \in \widetilde{\mathcal{D}}$,
– $\delta(f, (D, D') \to \{\mu(d_1) = \mu_D(d_1) \vee \mu_{D'}(d_1), \mu(d_2) = \mu_D(d_2) \wedge \mu_{D'}(d_2)\})$,

then, $L(M)$ *is a* 1*-dimensional fuzzy tree language that includes* Σ*-trees with at least one symbol* ***c*** *and no symbol* ***a****.*

(ii) If

– $\Gamma(D) = \{(d_1, \mu_D(d_1) \wedge \mu_D(d_2)), (d_2, \mu_D(d_1) \wedge \mu_D(d_2))\}$ *for* $D \in \widetilde{\mathcal{D}}$,
– $\delta(f, (D, D') \to \{\mu(d_1) = \mu_D(d_1) \vee \mu_{D'}(d_1), \mu(d_2) = \mu_D(d_2) \vee \mu_{D'}(d_2)\})$,

then, M *is a pessimistic MFFTA and* $L(M)$ *is a* 2*-dimensional fuzzy tree language that its first dimension includes* Σ*-trees with all of their leaves are symbol* ***c****, and the second dimension includes* Σ*-trees with no symbol* ***a*** *or no symbol* ***b****.*

(iii) If

– $\Gamma(D) = \{(d_1, \mu_D(d_1) \wedge \mu_D(d_2)), (d_2, (\mu_D(d_1) \wedge \overline{\mu_D(d_2)}) \vee (\overline{\mu_D(d_1)} \wedge \mu_D(d_2)))\}$ *for* $D \in \widetilde{\mathcal{D}}$,
– $\delta(f, (D, D') \to \{\mu(d_1) = \mu_D(d_1) \vee \mu_{D'}(d_1), \mu(d_2) = \mu_D(d_2) \vee \mu_{D'}(d_2)\})$,

then, M *is an optimistic MFFTA and* $L(M)$ *is a* 2*-dimensional fuzzy tree language that its first dimension includes* Σ*-trees with all of their leaves are not symbol* ***a*** *or all of their leaves are not symbol* ***b****, and the second dimension includes* Σ*-trees with all of their leaves are symbol* ***a*** *or all of their leaves are symbol* ***b****. If the reference set be all regular trees on* Σ*, these two dimensions are the complement of each other.*

4 Conclusions

The notion of multidimensional finite tree automata with the following properties is introduced:

+ Every MFFTA recognizes a multidimensional fuzzy set of regular trees called multidimensional fuzzy regular tree language.
+ Each internal state of an MFFTA is a fuzzy set on its set of dimensions.
+ Every MFFTA has a finalizing function, rather than the set of final states, which is a mapping from an internal state, calculated by run map of automata, to another internal state that will be proceed by the behavior of automata.
+ The transition rules of MFFTA are generalized and equipped with a set of transition functions that calculates membership grades based on the functions over Zadeh's fuzzy set operations.
+ The behavior of an MFFTA is based on the union of its run maps while they have a dual behavior which is the intersection of their run maps.

Also, we proved that the class of multidimensional fuzzy tree languages is closed under complement, union and intersection. Furthermore, we developed the concepts of deterministic MFFTA, reduced MFFTA, optimistic MFFTA and pessimistic MFFTA, and introduced the method of constructing a deterministic MFFTA from an optimistic / pessimistic MFFTA. After a reduction algorithm for MFFTA was presented. As well, the method of converting an FFTA to MFFTA was developed and finally, some examples were provided to show that modeling by MFFTA could be more efficient and more expressive than FFTA.

References

[1] P. R. J. Asveld, *Algebraic aspects of families of fuzzy languages*, Theoretical Computer Science, **293**(2) (2003), 417–445.

[2] S. Bozapalidis, O. L. Bozapalidoy, *On the recognizability of fuzzy languages I*, Fuzzy Sets and Systems, **157**(17) (2006), 2394–2402.

[3] S. Bozapalidis, O. L. Bozapalidoy, *On the recognizability of fuzzy languages II*, Fuzzy Sets and Systems, **159**(1) (2008), 107–113.

[4] S. Bozapalidis, O. L. Bozapalidoy, *Fuzzy tree language recognizability*, Fuzzy Sets and Systems, **161**(5) (2010), 716–734.

[5] N. Chomsky, *Three models for the description of language*, IRE Transactions on Information Theory, **2**(3) (1956), 113–124.

[6] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, C. Loding, S. Tison, M. Tommasi, *Tree automata: Techniques and applications*, Online (2007), Available: http://tata.gforge.inria.fr, 2007.

[7] J. E. Doner, *Decidability of the weak second-order theory of two successors*, Notices American Mathematic Society, **12**(1) (1965), 365–368.

[8] J. E. Doner, *Tree acceptors and some of their applications*, Journal of Computer and System Sciences, **4**(5) (1970), 406–451.

[9] M. Doostfatemeh, S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38**(2) (2005), 175–214.

[10] Z. Esik, L. Guangwu, *Fuzzy tree automata*, Fuzzy Sets and Systems, **158**(13) (2007), 1450–1460.

[11] M. K. Fallah, S. Moghari, E. Nazemi, M. M. Zahedi, *Fuzzy ontology based document feature vector modification using fuzzy tree transducer*, In: Proceedings of the 2008 IEEE International Conference on Signal Image Technology and Internet Based Systems, (2008), 38–44.

[12] Z. Fülöp, S. Vágvölgyi, *Minimization of deterministic top-down tree automata*, Acta Cybernetica, **23**(1) (2017), 379–401.

[13] Y. Guellouma, H. Cherroun, D. Ziadi, B. W. Watson, *From tree automata to string automata minimization*, Theory of Computing Systems, **62**(5) (2018), 1203-1222.

[14] J. E. Hopcroft, R. Motwani, J. D. Ullman, *Introduction to automata theory, Languages, and Computation*, 3 edn, Pearson Education, 2013.

[15] Y. Inagaki, T. Fukumura, *On the description of fuzzy meaning of context-free language*, In: Fuzzy Sets and their Applications to Cognitive and Decision Processes, Proceedings of the U. S. Japan Seminar, University of California, Berkeley, CA, (1975), 301–328.

[16] B. Kafle, J. P. Gallagher, *Horn clause verification with convex polyhedral abstraction and tree automata-based refinement*, Computer Languages, Systems & Structures, **47**(1) (2017), 2–18.

[17] S. Moghari, M. M. Zahedi, *Similarity-based minimization of fuzzy tree automata*, Journal of Applied Mathematics and Computing, **50**(1) (2016), 417–436.

[18] S. Moghari, M. M. Zahedi, R. Ameri, *Efficient reduction of fuzzy finite tree automata*, In: Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and its Applications (ICMSA2010) Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia, (2010), 1034–1042.

[19] S. Moghari, M. M. Zahedi, R. Ameri, *Minimization of fuzzy finite tree automata*, In: Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and its Applications (ICMSA2010) Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia, (2010), 1044-1052.

[20] S. Moghari, M. M. Zahedi, R. Ameri, *New direction in fuzzy tree automata*, Iranian Journal of Fuzzy Systems, **8**(5) (2011), 59–68.

[21] J. Mordeson, D. S. Malik, *Fuzzy automata and languages: Theory and applications*, Chapman & Hall, London, 2002.

[22] H. Pan, Y. Li, Y. Cao, Z. Ma, *Model checking fuzzy computation tree logic*, Fuzzy Sets and Systems, **262** (2015), 60–77.

[23] Y. Shang, X. Yuan, E. S. Lee, *The n-dimensional fuzzy sets and zadeh fuzzy sets based on the finite valued fuzzy sets*, Computers & Mathematics with Applications, **60**(3) (2010), 442–463.

[24] J. W. Thatcher, J. B. Wright, *Generalized finite automata*, Notices American Mathematic Society, **12** (1965), 820.

[25] J. W. Thatcher, J. B. Wright, *Generalized finite automata with an application to a decision problem of second-order logic*, Mathematical System Theory, **2**(1) (1968), 57–82.

[26] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338–353.

[27] H. J. Zimmermann, *Fuzzy set theory and its applications*, 3 edn, Springer Science & Business Media, 2011.