FUZZY INFORMATION AND STOCHASTICS

R. VIERTL AND D. HARETER

Abstract. In applications there occur different forms of uncertainty. The two most important types are randomness (stochastic variability) and imprecision (fuzziness). In modelling, the dominating concept to describe uncertainty is using stochastic models which are based on probability. However, fuzziness is not stochastic in nature and therefore it is not considered in probabilistic models.
Since many years the description and analysis of fuzziness is subject of intensive research. These research activities do not only deal with the fuzziness of observed data, but also with imprecision of informations. Especially methods of standard statistical analysis were generalized to the situation of fuzzy observations. The present paper contains an overview about of the presentation of fuzzy information and the generalization of some basic classical statistical concepts to the situation of fuzzy data.

1. Introduction

Most of the information we obtain is afflicted with uncertainty. In our languages there are found many linguistic uncertainties. For example, the statements "a short time", "a long distance", or "nice weather" are not exactly determinable, but their interpretation depends on our subjective opinion. Even information which seem to be exact, like measurements, are afflicted with uncertainty. The results of measurements of continuous quantities contain essentially two different types of uncertainty: randomness and imprecision. Imprecision appears for example as a result of the limited precision of measuring instruments and is not statistical in nature.
The dominating concept to describe uncertainty in modelling is using stochastic models which are based on probability. However, probabilistic models are not suitable to describe all kinds of uncertainty, but only randomness. Especially the imprecision of data is not statistical in nature and cannot be described by using probability. Besides, many data like life times, or human reliabilities are imprecise by nature and cannot be described realistically by precise real numbers. The quantification of the imprecision of a one-dimensional quantity is possible by using so-called fuzzy numbers, which are a generalization of real numbers. They are defined and represented by so-called characterizing functions.

Definition 1.1. The *characterizing function* $\xi_{x^\star}(\cdot)$ of a *fuzzy number* $x^\star$ is a real function which obeys the following:

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(1) $0 \leq \xi_{x^\star}(x) \leq 1 \quad$ for all $\quad x \in \mathbb{R}$

(2) $\exists\, x_0 \in \mathbb{R} : \xi_{x^\star}(x_0) = 1$

(3) For all $\delta \in (0,1]$ the so-called δ-cut $C_\delta(x^\star) := \{x \in \mathbb{R} : \xi_{x^\star}(x) \geq \delta\}$ is a closed finite interval $[\underline{x}_\delta, \overline{x}_\delta]$

The set *supp* $[\xi_{x^\star}(\cdot)] := \{x \in \mathbb{R} : \xi_{x^\star}(x) > 0\} = [\underline{x}_0, \overline{x}_0]$ is called the *support* of $\xi_{x^\star}(\cdot)$ resp. the support of $x^\star$ and the set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

Remark 1.2. The characterizing function $\xi_{x^\star}(\cdot)$ of a fuzzy number $x^\star$ is unequivocal determined by the set of δ-cuts $C_\delta(x^\star)$ by

(1) $$\xi_{x^\star}(x) = \sup_{\delta \in (0,1]} \delta\, I_{C_\delta(x^\star)}(x) \qquad \text{for all } x \in \mathbb{R}\,.$$

The quantification of a vector-valued quantity is possible by using fuzzy vectors, which are defined and represent by so-called vector-characterizing functions.

Definition 1.3. The *vector-characterizing function* $\zeta_{\underline{x}^\star}(\cdot, \ldots, \cdot)$ of a *fuzzy vector* $\underline{x}^\star$ is a real valued function of n real variables $\underline{x} = (x_1, \ldots, x_n)$ which obeys

($\underline{1}$) $0 \leq \zeta_{\underline{x}^\star}(\underline{x}) \leq 1 \quad$ for all $\quad \underline{x} \in \mathbb{R}^n$

($\underline{2}$) $\exists\, \underline{x}_0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n : \zeta_{\underline{x}^\star}(\underline{x}_0) = 1$

($\underline{3}$) For all $\delta \in (0,1]$ the so-called δ-cut $C_\delta(\underline{x}^\star) := \left\{\underline{x} \in \mathbb{R}^n : \zeta_{\underline{x}^\star}(\underline{x}) \geq \delta\right\}$ is a star-shaped compact set

Remark 1.4. In the literature there exist different definitions of fuzzy vectors. Some authors demand for the δ-cuts simply connected and compact sets of $\mathbb{R}^n$. In some papers the δ-cuts have to be compact and convex sets.

Fuzzy vectors are also used to define functions of fuzzy numbers. The definition is given in the next section.

2. Functions of Fuzzy Data

To analyse fuzzy data the definition of mathematical operations for fuzzy numbers is necessary. For them, the generalization of real functions $f : \mathbb{R}^n \to \mathbb{R}$ to the case of fuzzy arguments is needed. The definition of this generalization is based on the so-called *extension principle* which is well known in fuzzy set theory [3]. At first, n fuzzy numbers $x_1^\star, \ldots, x_n^\star$ with corresponding characterizing functions $\xi_{x_1^\star}(\cdot), \ldots, \xi_{x_n^\star}(\cdot)$ have to be combined into a fuzzy vector $\underline{x}^\star$ with vector-characterizing function $\zeta_{\underline{x}^\star}(\cdot, \ldots, \cdot)$ by

(2) $$\zeta_{\underline{x}^\star}(x_1, \ldots, x_n) := \min_{i=1(1)n} \xi_{x_i^\star}(x_i) \qquad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n\,.$$

Remark 2.1. In general, the combination of n fuzzy numbers $x_1^\star, \dots, x_n^\star$ into a fuzzy vector $\underline{x}^\star$ can be done by using a so-called t-norm T_n instead of the min-operator, as in the book by Klement, Mesiar and Pap [2]. However, for the analysis of fuzzy data only the min-operator used in equation (2) is practical (see remark 2.5).

Proposition 2.2. *The δ-cuts $C_\delta(\underline{x}^\star)$ of the combined fuzzy vector $\underline{x}^\star$ are the Cartesian products of the δ-cuts of the n fuzzy numbers $x_1^\star, \dots, x_n^\star$, i.e. $C_\delta(\underline{x}^\star) = C_\delta(x_1^\star) \times \cdots \times C_\delta(x_n^\star)$ for all $\delta \in (0,1]$ if and only if formula (2) is used to obtain the combined fuzzy vector $\underline{x}^\star$.*

Proof. See Viertl [13]. □

In order to generalize real valued functions $f(\cdot)$ to the situation of fuzzy argument values, the combined fuzzy vector $\underline{x}^\star$ with vector-characterizing function $\zeta_{\underline{x}^\star}(\cdot, \dots, \cdot)$ is used. The characterizing function $\xi_{y^\star}(\cdot)$ of the fuzzy value $y^\star = f(x_1^\star, \dots, x_n^\star)$ is defined via the extension principle by

$$(3) \qquad \xi_{y^\star}(y) := \left\{ \begin{array}{cl} \sup\left\{\zeta_{\underline{x}^\star}(\underline{x})\colon f(\underline{x}) = y\right\} & \text{if} \quad f^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(\{y\}) = \emptyset \end{array} \right\} \quad \text{for all } y \in \mathbb{R}\,.$$

Proposition 2.3. *For a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ and n fuzzy numbers $x_1^\star, \dots, x_n^\star$ with combined fuzzy vector $\underline{x}^\star$, the value $y^\star = f(x_1^\star, \dots, x_n^\star)$ is a fuzzy number in the sense of definition 1.1. The δ-cuts of $y^\star$ are determined by*

$$C_\delta(y^\star) = \left[\min_{(x_1,\dots,x_n)\in C_\delta(\underline{x}^\star)} f(x_1, \dots, x_n), \max_{(x_1,\dots,x_n)\in C_\delta(\underline{x}^\star)} f(x_1, \dots, x_n) \right].$$

Proof. See Viertl [13]. □

Proposition 2.3 is in general not valid for discontinuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$ because in this case the δ-cuts $C_\delta(y^\star)$ do not have to be intervals.

Using equation (3), the main arithmetic operations of two fuzzy numbers $x_1^\star$ and $x_2^\star$ can be defined. The generalized addition $x_1^\star \oplus x_2^\star$ is defined via the continuous function $f(x_1, x_2) = x_1 + x_2$, the multiplication $x_1^\star \odot x_2^\star$ is defined via the continuous function $f(x_1, x_2) = x_1 \cdot x_2$ and the ratio $x_1^\star / x_2^\star$ is defined via the function $f(x_1, x_2) = x_1/x_2$. Note that $f(x_1, x_2) = x_1/x_2$ is only continuous in $\mathbb{R}^2 \setminus (x_1, 0)$. Thus, the quotient is only defined in the case $0 \notin \mathit{supp}\left[\xi_{x_2^\star}(\cdot)\right]$.

Example 2.4. The characterizing function $\xi_{x^\star}(\cdot)$ of the sum $x^\star = x_1^\star \oplus x_2^\star$ of two fuzzy numbers $x_1^\star$ and $x_2^\star$ with corresponding characterizing functions $\xi_{x_1^\star}(\cdot)$ and $\xi_{x_2^\star}(\cdot)$ is calculated by

$$\begin{aligned} \xi_{x^\star}(x) &= \xi_{x_1^\star \oplus x_2^\star}(x) = \sup_{x_1 + x_2 = x} \min\left\{\xi_{x_1^\star}(x_1), \xi_{x_2^\star}(x_2)\right\} \\ &= \sup_{y \in \mathbb{R}} \min\left\{\xi_{x_1^\star}(y), \xi_{x_2^\star}(x - y)\right\} \qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Using proposition 2.3, the δ-cuts $C_\delta(x^\star)$ of $x^\star$ can be calculated from the δ-cuts $C_\delta(x_1^\star) = [\underline{x}_{1,\delta}, \overline{x}_{1,\delta}]$ and $C_\delta(x_2^\star) = [\underline{x}_{2,\delta}, \overline{x}_{2,\delta}]$ by

$$\begin{aligned} C_\delta(x^\star) &= \left[\min_{(x_1,x_2)\in C_\delta(\underline{x}^\star)} x_1 + x_2, \max_{(x_1,x_2)\in C_\delta(\underline{x}^\star)} x_1 + x_2 \right] \\ &= \left[\min_{(x_1,x_2)\in C_\delta(x_1^\star)\times C_\delta(x_2^\star)} x_1 + x_2, \max_{(x_1,x_2)\in C_\delta(\underline{x}^\star)} x_1 + x_2 \right] \\ &= \left[\underline{x}_{1,\delta} + \underline{x}_{2,\delta}, \overline{x}_{1,\delta} + \overline{x}_{2,\delta} \right] . \end{aligned}$$

Remark 2.5. It is easy to verify, that for a fuzzy number $x^\star$ the equation

$$(4) \qquad x^\star = \frac{1}{2} \odot (x^\star \oplus x^\star) \qquad \text{resp. more general} \qquad x^\star = \frac{1}{n} \odot (x^\star \oplus \cdots \oplus x^\star)$$

holds. This desirable feature is a result of using the min-operation in (2) and the resulting proposition 2.2. In general, equation (4) is not valid if for the combination in equation (2) another T_n-norm instead of the min-operator is used.

2.1. **Statistical Analysis of Fuzzy Data.**

The well known methods of standard statistical analysis deal with data in form of real numbers and vectors. In case of a one-dimensional stochastic quantity $X \sim f(\cdot \mid \theta), \theta \in \Theta$, with observation space $M \subseteq \mathbb{R}$, a statistic $\mathcal{S} = s(X_1, \ldots, X_s)$ is a measurable function $s : M^n \to \mathbb{R}$ from the sample space M^n to $\mathbb{R}$. However, for a continuous stochastic quantity, the assumption of a sample composed of real data is not realistic, as explained in the introduction. Therefore, standard statistical methods have to be generalized to the situation of fuzzy data. For most of these methods, the generalization can be done by using the extension principle (3).

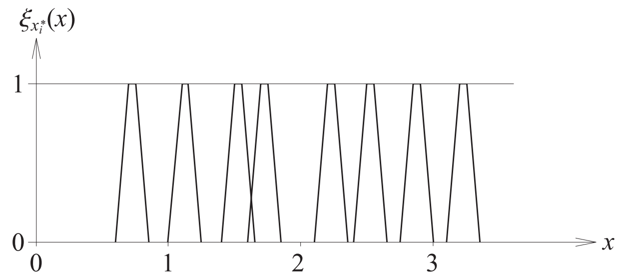
2.1.1. *Fuzzy Estimators.*

In general, a classical estimator $\vartheta(\cdot, \ldots, \cdot)$ for a parameter θ is a continuous function of the real-valued sample. The generalized (fuzzy) estimator $\widehat{\theta}^\star = \vartheta(x_1^\star, \ldots, x_n^\star)$ based on a sample of fuzzy data $x_1^\star, \ldots, x_n^\star$ is given by its characterizing function $\xi_{\widehat{\theta}^\star}(\cdot)$, whose values $\xi_{\widehat{\theta}^\star}(\theta), \theta \in \Theta$ are calculated using (3), which gives the following:

$$\xi_{\widehat{\theta}^\star}(\theta) = \left\{ \begin{array}{cl} \sup\left\{\zeta_{\underline{x}^\star}(\underline{x}) \colon \vartheta(\underline{x}) = \theta\right\} & \text{if} \quad \vartheta^{-1}(\{\theta\}) \neq \emptyset \\ 0 & \text{if} \quad \vartheta^{-1}(\{\theta\}) = \emptyset \end{array} \right\} \quad \text{for all } \theta \in \Theta \, .$$

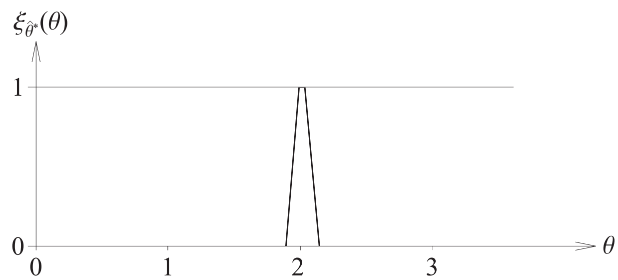
Example 2.6. In figure 1 a fuzzy sample of an exponentially distributed stochastic quantity $X \sim Ex_\theta$ is depicted.

Figure 1. Fuzzy sample



The corresponding fuzzy estimation $\widehat{\theta}^{\star}$ of the parameter θ, i.e. the mean of the fuzzy observations, is depicted in figure 2.

Figure 2. Characterizing function of the fuzzy estimation $\widehat{\theta}^{\star}$



2.1.2. *Fuzzy Confidence Regions.*

The concept of classical confidence regions can be generalized to the situation of a sample of fuzzy numbers by using an idea similar to the extension principle. Let $\kappa(\cdot,\ldots,\cdot)$ be a classical confidence function for a parameter θ with confidence level $1-\alpha$. The function $\kappa : M^n \to \mathcal{P}(\Theta)$ is a function from the sample space M^n to the power set $\mathcal{P}(\Theta)$ of the parameter space Θ. Therefore the application of equation (3) is not possible. However, using an adaptation of the extension principle, the fuzzy confidence region $\kappa^\star$ based on fuzzy data $x_1^\star,\ldots,x_n^\star$ and the corresponding combined fuzzy vector $\underline{x}^\star$ with vector-characterizing function $\zeta_{\underline{x}^\star}(\cdot,\ldots,\cdot)$, is given by its membership function $\varphi_{\kappa^\star}(\cdot)$ which is defined by

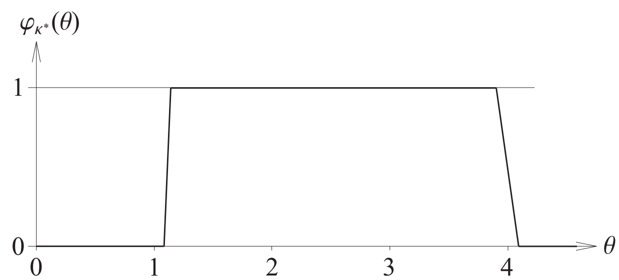
$$\varphi_{\kappa^\star}(\theta) := \left\{ \begin{array}{cc} \sup\left\{\zeta_{\underline{x}^\star}(\underline{x}) : \theta \in \kappa(\underline{x})\right\} & \text{if} \quad \exists\, \underline{x} : \theta \in \kappa(\underline{x}) \\ 0 & \text{else} \end{array} \right\} \quad \text{for all } \theta \in \Theta\,.$$

Note, that $\kappa^\star$ need not be a fuzzy number in the sense of definition 1.1 because the δ-cuts $C_\delta(\kappa^\star)$ do not have to be intervals. In this case, the so-called convex hull of

$\kappa^\star$ can be taken into account (for details see Viertl (1996)). Functions which fulfill both conditions (1) and (2) of definition 1.1 but not necessarily condition (3) are denoted *membership functions* instead of characterizing functions.

Example 2.7. In figure 3 a one-dimensional fuzzy confidence region (called fuzzy confidence interval) with confidence level 0.9 for the parameter of example 2.6 is depicted.

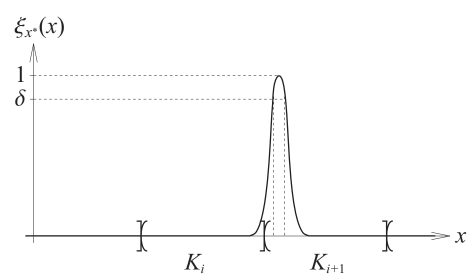
FIGURE 3. Fuzzy confidence interval



3. **Fuzzy Probability Distributions**

For real valued data, the well-known concept of a histogram provides information about the underlying distribution. Considering a sample of fuzzy data, this concept has to be adapted because the frequencies of fixed events are not precise numbers. In figure 4 the situation where a fuzzy observation $x^\star$ cannot be assigned to one singular class is depicted.

FIGURE 4. Fuzzy observation and classification



Because of the non-empty intersection of $supp\,[\xi_{x^\star}(\cdot)]$ with the two classes K_i and K_{i+1} the fuzzy number $x^\star$ can not be assigned to K_i nor to K_{i+1}. In the calculation of the relative frequencies the observation has to be taken into account in both classes K_i and K_{i+1}. However, considering the δ-cut of $x^\star$ in figure 4, it can be definitely allocated to the class K_{i+1}. For a sample of fuzzy data $x_1^\star, \ldots, x_n^\star$ this fact leads to the following calculation of the fuzzy valued relative frequencies $h_n^\star(A)$

of a fixed event A which is defined via its δ-cuts $C_\delta\left(h_n^\star(A)\right) = \left[\underline{h}_{n,\delta}(A), \overline{h}_{n,\delta}(A)\right]$ in the following way: For $\delta \in (0,1]$ the upper limit $\overline{h}_{n,\delta}(A)$ is defined as the number of observations whose δ-cuts have non-empty intersection with A, i.e.

$$\overline{h}_{n,\delta}(A) = \frac{\#\left\{x_i^\star : C_\delta(x_i^\star) \cap A \neq \emptyset\right\}}{n},$$

and the lower limit $\underline{h}_{n,\delta}(A)$ is given by the number of observations whose δ-cuts are for sure contained in A, i.e.

$$\underline{h}_{n,\delta}(A) = \frac{\#\left\{x_i^\star : C_\delta(x_i^\star) \subseteq A\right\}}{n}.$$

The obtained fuzzy frequency $h_n^\star(A)$ is a cascade fuzzy number.
For the impossible event $\emptyset$ and the sure event M (the whole set of possible observations) it follows

$$h_n^\star(\emptyset) = 0 \qquad \text{and} \qquad h_n^\star(M) = 1,$$

i.e. the extreme events have precise frequencies.

Considering probability distributions as theoretical counterparts of frequencies, a *fuzzy probability density* is a function $f^\star : M \to \mathcal{F}(\mathbb{R}^+)$ where $\mathcal{F}(\mathbb{R}^+)$ denotes the set of all fuzzy numbers $x^\star$ with $supp\left[\xi_{x^\star}(\cdot)\right] \subseteq [0,\infty)$;that is, for every element x of the observation space M, $f^\star(x)$ is a fuzzy number with corresponding characterizing function $\varphi_x(\cdot)$ and δ-cuts $C_\delta\left(f^\star(x)\right) = \left[\underline{\varphi}_\delta(x), \overline{\varphi}_\delta(x)\right]$. Certainly a further condition, equal to the standardization of a real-valued probability density, is necessary to characterize a fuzzy probability density. Before this condition can be given, the concept of the integration of such fuzzy valued functions has to be explained: The result

$$\mathcal{I}^\star = \int\!\!\!\!-_A f^\star(x)\,d\,x \tag{5}$$

of the generalized integration is a fuzzy number whose characterizing function $\xi_{\mathcal{I}^\star}(\cdot)$ is defined via their δ-cuts $C_\delta(\mathcal{I}^\star) = \left[\underline{\mathcal{I}}_\delta, \overline{\mathcal{I}}_\delta\right]$ by

$$\underline{\mathcal{I}}_\delta := \int_A \underline{\varphi}_\delta(x)\,d\,x \qquad \text{and} \qquad \overline{\mathcal{I}}_\delta := \int_A \overline{\varphi}_\delta(x)\,d\,x\,,$$

which are simply real integrals of the real functions $\underline{\varphi}_\delta(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$. To ensure the existence of both integrals, $\underline{\varphi}_\delta(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ are assumed to be integrable for all $\delta \in (0,1]$. The characterizing function $\xi_{\mathcal{I}^\star}(\cdot)$ is given by

$$\xi_{\mathcal{I}^\star}(x) = \sup_{\delta\in(0,1]} \delta \cdot I_{C_\delta(\mathcal{I}^\star)}(x) \quad \text{for all} \quad x \in M,$$

where $I_{C_\delta(\mathcal{I}^\star)}(\cdot)$ denotes the indicator function of the interval $C_\delta(\mathcal{I}^\star) = \left[\underline{\mathcal{I}}_\delta,\ \overline{\mathcal{I}}_\delta\right]$.

Using the generalized integration (5), a fuzzy probability density $f^\star(\cdot)$ has to fulfill the generalized normalization condition

$$\int\!\!\!\!-_M f^\star(x)\,dx = 1_+^\star$$

where $1^\star_+$ denotes a fuzzy number with $1 \in C_1(1^\star)$ and $C_\delta(1^\star) \subseteq (0,\infty)$ for all $\delta \in (0,1]$.

Furthermore, the calculation of the probability of a subset $A \subseteq M$ has to be defined. In case of a real-valued density $f(\cdot)$, the probability $P(A)$ of a subset $A \subseteq M$ is simply calculated by

$$P(A) = \int_A f(x)\,dx\,.$$

In case of a fuzzy probability density $f^\star(\cdot)$, the probability of a subset A cannot be calculated by using the general integration described above, because the support *supp*$[P^\star(A)]$ of the fuzzy probability $P^\star(A)$ need not be a part of the interval $[0,1]$. Therefore, another concept for the calculation of $P^\star(A)$ has to be defined. A necessary property of the defined probability is the availability of the standardization $P^\star(M) = 1$ and $P^\star(\emptyset) = 0$. The definition of $P^\star(A)$ is based on the advisement to calculate the highest and the lowest probability of A: Let

$$S_\delta = \Big\{ f : f \text{ is a probability density with } \underline{\varphi}_\delta(x) \le f(x) \le \overline{\varphi}_\delta(x) \;\; \forall\, x \in M \Big\}$$

be the set of all possible classical probability densities between both δ-level curves $\underline{\varphi}_\delta(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$. The δ-cut $C_\delta(P^\star(A)) = \big[\underline{P}_\delta(A), \overline{P}_\delta(A)\big]$ of the fuzzy probability $P^\star(A)$ is defined by

$$\begin{aligned}
\overline{P}_\delta(A) &= \sup_{f \in S_\delta} \int_A f(x)\,dx \\
&= \begin{cases} 1 - \displaystyle\int_{A^c} \underline{\varphi}_\delta(x)\,dx & \text{if} \quad \displaystyle\int_A \overline{\varphi}_\delta(x)\,dx + \int_{A^c} \underline{\varphi}_\delta(x)\,dx > 1 \\ \displaystyle\int_A \overline{\varphi}_\delta(x)\,dx & \text{else} \end{cases},
\end{aligned}$$

and

$$\begin{aligned}
\underline{P}_\delta(A) &= \inf_{f \in S_\delta} \int_A f(x)\,dx \\
&= \begin{cases} \displaystyle\int_A \underline{\varphi}_\delta(x)\,dx & \text{if} \quad \displaystyle\int_A \underline{\varphi}_\delta(x)\,dx + \int_{A^c} \overline{\varphi}_\delta(x)\,dx > 1 \\ 1 - \displaystyle\int_{A^c} \overline{\varphi}_\delta(x)\,dx & \text{else} \end{cases}.
\end{aligned}$$

For $0 < \delta_1 < \delta_2 \le 1$ by $S_{\delta_2} \subseteq S_{\delta_1}$ the following equation holds:

$$\big[\underline{P}_{\delta_2}(A), \overline{P}_{\delta_2}(A)\big] \subseteq \big[\underline{P}_{\delta_1}(A), \overline{P}_{\delta_1}(A)\big]$$

Therefore $P^\star(A)$ is a fuzzy number in the sense of definition 1.1. It is easy to verify that $P^\star(M) = [1,1] = 1$ and $P^\star(\emptyset) = [0,0] = 0$, i.e. the two extreme events have the required probabilities.

Fuzzy probability densities can also be used in Bayesian analysis. If little or no a-priori information is available the use of one real-valued a-priori distribution on the parameter space is questionable. In this case, fuzzy a-priori densities allow a more realistic presentation of the available a-priori information. In Viertl and Hareter [16] the well known Bayes' theorem is generalized to the situation of fuzzy a-priori density and fuzzy data.

4. **Fuzzy stochastic processes**

For the definition of fuzzy stochastic processes the definition of so-called fuzzy random variables, which are extensions of classical real-valued random variables by using fuzzy numbers $x^\star \in \mathcal{F}(\mathbb{R})$ as images, is necessary. In the literature there exist different definitions of fuzzy random variables, see for examples the definition of Puri and Ralescu [9] or the definition of Kwakernaak [5], [6]. In the following, the definition of Kwakernaak is used.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $T \subseteq \mathbb{R}^m$.

Definition 4.1. A *random interval* is a set-valued mapping $X : \Omega \to I(\mathbb{R}) := \{[a, b] : a, b \in \mathbb{R}, a \le b\}$ with $\omega \to X(\omega) = [\underline{X}(\omega), \overline{X}(\omega)]$ such that $\underline{X}(\omega)$ and $\overline{X}(\omega)$ are random variables defined on $(\Omega, \mathcal{A}, P)$.

Definition 4.2. A fuzzy-valued mapping $X^\star : \Omega \to \mathcal{F}(\mathbb{R})$ is called *fuzzy random variable* if $C_\delta(X^\star(\omega)) = \left[\underline{X}_\delta(\omega), \overline{X}_\delta(\omega)\right]$ is a random interval for every $\delta \in (0, 1]$.

That is, fuzzy random variables are defined via the boundaries of the δ-cuts of the fuzzy values $X^\star(\omega), \omega \in \Omega$. The boundaries are calculated by

$$\underline{X}_\delta(\omega) = \inf\left\{x \in \mathbb{R} \colon x \in C_\delta\left(X^\star(\omega)\right)\right\} = \inf\left\{x \in \mathbb{R} \colon \xi_{X^\star(\omega)}(x) \ge \delta\right\}$$

and

$$\overline{X}_\delta(\omega) = \sup\left\{x \in \mathbb{R} \colon x \in C_\delta\left(X^\star(\omega)\right)\right\} = \sup\left\{x \in \mathbb{R} \colon \xi_{X^\star(\omega)}(x) \ge \delta\right\}$$

where $\xi_{X^\star(\omega)}(\cdot)$ denotes the characterizing function of the fuzzy number $X^\star(\omega)$.

Definition 4.3. The expectation $\mathbb{E}X^\star$ of a fuzzy random variable $X^\star$ is a fuzzy number which is defined via its δ-cuts by using the concept of general integration from section 3 using (5) in the following way:

$$C_\delta(\mathbb{E}X^\star) = \int_\Omega C_\delta(X^\star(\omega))\, P(d\,\omega) \qquad \text{for all } \delta \in (0, 1]\,.$$

Using (1) the characterizing function $\xi_{\mathbb{E}X^\star}(\cdot)$ of $\mathbb{E}X^\star$ is calculated by

$$\xi_{\mathbb{E}X^\star}(x) = \sup_{\delta \in (0,1]} \delta\, I_{C_\delta(\mathbb{E}X^\star)}(x) \qquad \text{for all } x \in \mathbb{R}\,.$$

Similar to the expectation $\mathbb{E}X^\star$, the distribution function $F^\star_{X^\star} : \mathbb{R} \to \mathcal{F}([0, 1])$ of a fuzzy random variable $X^\star$ becomes fuzzy too. Using the δ-cuts $C_\delta(X^\star(\omega)) = \left[\underline{X}_\delta(\omega),\, \overline{X}_\delta(\omega)\right]$ of the fuzzy value $X^\star(\omega)$, $\omega \in \Omega$, the *fuzzy distribution function* $F^\star_{X^\star}(\cdot)$ is defined via its δ-cuts $C_\delta(F^\star_{X^\star}(x)) = \left[\underline{F}_\delta(x), \overline{F}_\delta(x)\right]$ for every $\delta \in (0, 1]$ and $x \in \mathbb{R}$ with

$$\underline{F}_\delta(x) = P\{\omega : \overline{X}_\delta(\omega) \le x\} \qquad \text{and} \qquad \overline{F}_\delta(x) = P\{\omega : \underline{X}_\delta(\omega) \le x\}\,.$$

A real-valued function which describes the fuzziness of $F^{\star}_{X^{\star}}(\cdot)$ is the following:

Definition 4.4. Let $X^{\star}$ be a fuzzy random variable. Then for each $\delta \in (0,1]$ the real-valued function $F_D(x) = P\{\omega : x \in C_\delta(X^{\star}(\omega))\}$ is called a *δ-level shadow distribution function* of $X^{\star}$.

Certainly, $F_D(\cdot)$ can be calculated by $F_D(x) = \overline{F}_\delta(x) - \underline{F}_\delta(x-), x \in \mathbb{R}$, and for a real-valued continuous random variable X the equation $F_D(x) = 0$ for all $x \in \mathbb{R}$, holds.

The basis for the definition of fuzzy stochastic processes is the concept of normal dynamic fuzzy sets.

Definition 4.5. A family $\{A^{\star}(t); t \in T\}$ is called a *normal dynamic fuzzy set* in $\mathbb{R}$ if $A^{\star}(t) \in \mathcal{F}(\mathbb{R})$ for every $t \in T$.

Definition 4.6. A family of fuzzy random variables $X^{\star}(\cdot,\cdot) = \{X^{\star}(t,\omega) : \omega \in \Omega, t \in T\}$ is called a fuzzy random function or a *fuzzy stochastic process* on $(\Omega, \mathcal{A}, P)$ if $X^{\star}(t,\cdot)$ is a fuzzy random variable on $(\Omega, \mathcal{A}, P)$ for every fixed $t \in T$, and $X^{\star}(\cdot,\omega)$ is a normal dynamic fuzzy set with respect to the parameter set T for every fixed $\omega \in \Omega$. $X^{\star}(\cdot,\omega)$ is called a *fuzzy sample function* or a *fuzzy trajectory* of the process.

Definition 4.7. Given a fuzzy stochastic process $X^{\star}(\cdot,\cdot)$ on a probability space $(\Omega, \mathcal{A}, P)$ and $T \subset \mathbb{R}$, let $D = \{t_1, t_2, \ldots, t_n\}$ be a finite set of distinct elements of T. Considering the n-dimensional fuzzy random vector $(X^{\star}(t_1,\cdot), \ldots, X^{\star}(t_n,\cdot))$ which is a mapping from Ω into $\mathcal{F}^n(\mathbb{R}) = \mathcal{F}(\mathbb{R}) \times \cdots \times \mathcal{F}(\mathbb{R})$, the n-dimensional δ-level shadow distribution function $F_D(\cdot,\ldots,\cdot)$ is defined for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\delta \in (0,1]$ by

$$F_D(x_1,\ldots,x_n) = P\{\omega : x_i \in C_\delta(X^{\star}(t_i,\omega)), i = 1,...,n\}$$

4.1. **Parametric fuzzy random processes.**

Let $X^{\star}(t) = X(\theta^{\star}, t)$ be a family of fuzzy random variables determined by a fuzzy parameter $\theta^{\star}$. The application of δ-level presentation leads to the δ-cuts of $X^{\star}(t)$:

$$C_\delta(X^{\star}(t)) = \{X(\theta,t) : \theta \in C_\delta(\theta^{\star})\} \qquad \text{for all } \delta \in (0,1]$$

The fuzzy cumulative distribution function is presented using δ-level shadow distribution functions.

4.2. **Applications of Fuzzy stochastic processes.**

Safety assessment of structures in civil engineering are essential in all building activities. Based on randomness und fuzziness of material properties the most up to date analysis methods in reliability assessment are fuzzy stochastic methods.
Based on fuzzy stochastic processes so-called *fuzzy probability structural analysis* is possible, using a realistic computational model for calculating fuzzy distribution functions of critical quantities, especially fuzzy safety levels and fuzzy reliability indexes. For details compare the papers Möller et al. [7] and Sickert et al. [11].

5. **Generalized law of large numbers**

Based on the fuzziness of real observations of stochastic quantities and the resulting fuzzy relative frequencies, compare section 2.1.1, the law of large numbers has to be adapted. This was done in different ways, for example by Kruse [4] or Klement et. al. [1]. In 1992 Niculescu and Viertl [8] proved the following generalized law of large numbers:

Theorem 5.1. *For every rational fuzzy number* $x^\star \in \mathcal{F}\left(\{\frac{i}{n}, i \in \{0,\dots,n\}\}\right)$ *with characterizing function* $\xi_{x^\star}(\cdot)$ *let*

$$L_{x^\star} = \left\{\xi_{x^\star}\left(\frac{i}{n}\right) > 0 \colon C_1^L(x^\star) \geq \frac{i}{n}\right\} \quad \text{and} \quad R_{x^\star} = \left\{\xi_{x^\star}\left(\frac{i}{n}\right) > 0 \colon C_1^U(x^\star) \leq \frac{i}{n}\right\}$$

where

$$C_\delta^L(x^\star) = \min\left\{\frac{i}{n} \colon \xi_{x^\star}\left(\frac{i}{n}\right) \geq \delta\right\} \quad \text{and} \quad C_\delta^U(x^\star) = \max\left\{\frac{i}{n} \colon \xi_{x^\star}\left(\frac{i}{n}\right) \geq \delta\right\}.$$

Furthermore, for every $x^\star \in \mathcal{F}\left(\{\frac{i}{n}, i \in \{0,\dots,n\}\}\right)$ *and* $p^\star \in \mathcal{F}([0,1])$ *let*

$$\|x^\star \ominus p^\star\| = \max\left\{\max_{\delta \in L_{x^\star}} \left|C_\delta^L(x^\star) - C_\delta^L(p^\star)\right|, \max_{\delta \in R_{x^\star}} \left|C_\delta^U(x^\star) - C_\delta^U(p^\star)\right|\right\}$$

where $x^\star \ominus p^\star = x^\star \oplus (-1 \odot p^\star)$ *denotes the difference of two fuzzy numbers.*
Let $h_n^\star(A)$ *denote the fuzzy relative frequencies of an interval* A *for a fuzzy sample* $x_1^\star, \dots, x_n^\star$ *of an underlying fuzzy random variable* $X^\star$ *on a probability space* $(\Omega, \mathcal{A}, P)$*. Then there exists a fuzzy number* $p^\star \in \mathcal{F}([0,1])$ *such that for every arbitrarily small* $\epsilon, \alpha > 0$ *there exists* $N(\epsilon, \alpha) \in \mathbb{N}$ *such that for every* $n > N(\epsilon, \alpha)$ *the inequality*

$$P\big\{\|h_n^\star(A) \ominus p^\star\| < \epsilon\big\} \geq 1 - \alpha$$

holds.

Proof. See Niculescu and Viertl [8]. □

6. **Fuzzy information and decisions**

In decision analysis the following mathematical descriptions are fundamental: Let θ be the quantity of interest, $\mathcal{D}$ the set of possible decisions d and $U(\cdot,\cdot)$ the utility function, i.e. $U(\theta, d) \geq 0$ is the utility of taking the decision d when the quantity of interest is θ. The set Θ of all possible values can be continuous, i.e. $\Theta \subseteq \mathbb{R}$, or countable, i.e. $\Theta = \{\theta_1, \dots, \theta_k\}$. Furthermore, let $\widetilde{\theta}$ denote the stochastic quantity describing the knowledge about θ and $\pi(\theta)$ the probability density of $\widetilde{\theta}$. In standard analysis the expected utility for the decision d is calculated by

$$\mathbb{E}_{\pi(\cdot)} U(\widetilde{\theta}, d) = \begin{cases} \displaystyle\int_\Theta U(\theta, d)\, \pi(\theta)\, d\theta & \text{if} \quad \Theta \subseteq \mathbb{R} \\ \displaystyle\sum_{i=1}^k U(\theta_i, d)\, \pi(\theta_i) & \text{if} \quad \Theta = \{\theta_1, \dots, \theta_k\}. \end{cases}$$

The optimal decisions d_{opt} are defined by maximisation of the expected utility, i.e.

$$\mathbb{E}_{\pi(\cdot)} U(\widetilde{\theta}, d_{opt}) = \max_{d \in \mathcal{D}} \mathbb{E}_{\pi(\cdot)} U(\widetilde{\theta}, d) .$$

6.1. **Fuzzy utility.**

In practical applications, the usage of a real-valued utility function $U(\cdot,\cdot)$ is questionable. However a fuzzy utility function $U^\star(\cdot,\cdot)$ with $supp\,[U^\star(\cdot,\cdot)] \subseteq [0,\infty)$ can be used. Moreover, if the probability distribution $\pi(\cdot)$ of $\widetilde{\theta}$ is unknown and has to be chosen or estimated, it is more suitable to use a generalized fuzzy probability distribution $\pi^\star(\cdot)$.
In case of fuzzy probability density $\pi^\star(\cdot)$ and fuzzy utility function $U^\star(\cdot,\cdot)$ the generalization of expected utility in decision making is possible by using the generalized integration (5). The expected fuzzy utility for the decision d is calculated by

$$\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d) = \begin{cases} \displaystyle ⨍_\Theta U^\star(\theta, d) \odot \pi^\star(\theta)\, d\theta & \text{if} \quad \Theta \subseteq \mathbb{R} \\ \displaystyle \sum_{i=1}^{k} U^\star(\theta_i, d) \odot \pi^\star(\theta_i) & \text{if} \quad \Theta = \{\theta_1, \dots, \theta_k\} . \end{cases}$$

Denoting the δ-cuts of $U^\star(\theta,d)$ and $\pi^\star(\theta)$ by $C_\delta\left(U^\star(\theta,d)\right) = [\underline{U}_\delta(\theta,d), \overline{U}_\delta(\theta,d)]$ resp. $C_\delta\left(\pi^\star(\theta)\right) = [\underline{\pi}_\delta(\theta), \overline{\pi}_\delta(\theta)]$, the boundaries of the δ-cuts $C_\delta\left[\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d)\right] = \left[\underline{\mathbb{E}}_\delta U^\star(\widetilde{\theta}, d), \overline{\mathbb{E}}_\delta U^\star(\widetilde{\theta}, d)\right]$ of the expected fuzzy utility $\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d)$ are given by

$$\underline{\mathbb{E}}_\delta U^\star(\widetilde{\theta}, d) = \begin{cases} \displaystyle \int_\Theta \underline{U}_\delta(\theta, d)\, \underline{\pi}_\delta(\theta)\, d\theta & \text{if} \quad \Theta \subseteq \mathbb{R} \\ \displaystyle \sum_{i=1}^{k} \underline{U}_\delta(\theta_i, d)\, \underline{\pi}_\delta(\theta_i) & \text{if} \quad \Theta = \{\theta_1, \dots, \theta_k\} \end{cases}$$

and

$$\overline{\mathbb{E}}_\delta U^\star(\widetilde{\theta}, d) = \begin{cases} \displaystyle \int_\Theta \overline{U}_\delta(\theta, d)\, \overline{\pi}_\delta(\theta)\, d\theta & \text{if} \quad \Theta \subseteq \mathbb{R} \\ \displaystyle \sum_{i=1}^{k} \overline{U}_\delta(\theta_i, d)\, \overline{\pi}_\delta(\theta_i) & \text{if} \quad \Theta = \{\theta_1, \dots, \theta_k\} . \end{cases}$$

Using equation (1), the characterizing function $\xi_d(x)$ of $\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d)$ is given by

$$\xi_d(x) = \sup_{\delta \in (0,1]} \delta\, I_{C_\delta\left[\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d)\right]}(x) \qquad \text{for all } x \in \mathbb{R} .$$

Certainly the determination of the optimal decision is more difficult than in case of real-valued utility $U(\cdot,\cdot)$ and real-valued probability density $\pi(\cdot)$. One way of arriving to a decision is to use so-called *defuzzification*. After defuzzification, it is possible to compare the expected utilities $\mathbb{E}_{\pi^\star(\cdot)} U^\star(\widetilde{\theta}, d)$ of the different decisions $d \in \mathcal{D}$.

6.2. Statistical tests for fuzzy data.

A special kind of decision making processes are statistical tests. In case of fuzzy data $x_1^\star, \ldots, x_n^\star$ also test statistics $t(\cdot, \ldots, \cdot)$ become fuzzy, i.e. $t^\star = t(x_1^\star, \ldots, x_n^\star)$ with corresponding membership function $\xi_{t^\star}(\cdot)$ of the fuzzy value $t^\star$ of the generalized test statistic. Again, $t^\star$ doesn't have to be a fuzzy number in the sense of definition 1.1 (see the comment in section 2.1.1).
There are two possible ways to solve the problem of making a decision:

(1) One possibility is to use *p-values* connected with statistical tests. It is possible to calculate a crisp p-value even for fuzzy values $t^\star$ of the test statistic. Let $\psi(\cdot)$ be the characterizing function of $t^\star$. Then in case of one-dimensional test statistics looking at the boundary of $supp\,[\psi(\cdot)]$ the p-value in the standard way can be calculated. Then the test decision can be made as usually. For details see chapter 20 of the handbook Voß [17].
(2) Another method is to calculate so-called fuzzy p-values (for detail see Filzmoser and Viertl [15]).

7. Conclusions

The fuzziness of results of continuous measurements as well as the uncertainty of informations require a more general concept than by real-valued data of real-valued vectors. The quantification of this kind of uncertainty is possible by using so-called fuzzy numbers and fuzzy vectors. The paper deals with the presentation of fuzzy information and the generalization of some stochastic models and basic classical statistical concepts to the situation of fuzzy data.

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Reinhard Viertl, Department of Statistics and Probability Theory, Vienna University of Technology, Wien, Austria
E-mail address: `R.Viertl@tuwien.ac.at`

Dietmar Hareter*, Department of Statistics and Probability Theory, Vienna University of Technology, Wien, Austria
E-mail address: `hareter@statistik.tuwien.ac.at`

*Corresponding author