ON DEGREES OF END NODES AND CUT NODES IN FUZZY GRAPHS

K. R. BHUTANI, J. MORDESON AND A. ROSENFELD

Abstract. The notion of strong arcs in a fuzzy graph was introduced by Bhutani and Rosenfeld in [1] and fuzzy end nodes in the subsequent paper [2] using the concept of strong arcs. In Mordeson and Yao [7], the notion of "degrees" for concepts fuzzified from graph theory were defined and studied. In this note, we discuss degrees for fuzzy end nodes and study further some properties of fuzzy end nodes and fuzzy cut nodes.

1. Introduction

It is at times important to analyze properties of fuzzy networks by levels. For example, it is shown in [5] that for the fuzzy shortest path problem a fuzzy shortest length can be found but it may not correspond to an actual path in the network; a solution by levels is therefore of value. This and the concepts of connectedness of fuzzy graphs by levels [4] suggest a study by levels of variations of Menger's problem when placed in a fuzzy setting.

The examination of various concepts in fuzzy graph theory using definitions different from those in the ground-breaking paper [8] was begun in [4]. The purpose of [4] was to use definitions involving levels in order to study structural properties of fuzzy graphs for the development of tools to solve problems in operations research. In [7], this approach was expanded with the introduction of the notions of weak, partial, and full fuzzy properties of fuzzy graphs.

The study of strong arcs and fuzzy end nodes was introduced in [1, 2] while fuzzy cut-nodes were introduced in [8]. The potential importance of the notion of fuzzy end nodes can be seen from the many uses of trees in computer science and elsewhere. In this paper, we use the approaches of [1, 2, 4, 7] to continue the study of fuzzy end nodes and fuzzy cut nodes. Our goal here is to continue laying the groundwork for a study of Menger's problem in a fuzzy setting. It is important to note that the definition of a fuzzy end node is consistent with the definition of a fuzzy tree in that the end nodes of a fuzzy spanning tree of a fuzzy tree G are fuzzy end nodes of G [2]. For background on graph theory and fuzzy graph theory the reader is referred to [3, 6].

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2. Preliminaries

We first summarize some basic definitions, most of which can be found in [1,2,6,7,8]. Let H be a graph with node set V and arc set $E \subseteq V \times V$. A *fuzzy subgraph* G of H is defined by a fuzzy subset σ of V and a fuzzy subset μ of E such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\sigma(x) \wedge \sigma(y)$ denotes the minimum of $\sigma(x)$ and $\sigma(y)$. We call σ the fuzzy node set of G and μ the fuzzy arc set of G, respectively. We assume that $\mu(x,y) = \mu(y,x)$ and $\mu(x,x) = 0$ for all $x, y \in V$. We shall denote $\max(\sigma(x), \sigma(y))$ by $\sigma(x) \vee \sigma(y)$.

By the *support* of a *fuzzy subgraph* $G = (V, E, \sigma, \mu)$ we mean the graph, denoted by G^*, whose node set is the set $\sigma^* = \{x|\sigma(x) > 0\}$ and whose arc set is the set $\mu^* = \{(x,y)|\mu(x,y) > 0\}$. We call G a *full fuzzy subgraph* of H if its support is all of H, i.e. if $\sigma(x) > 0$ for all $x \in V$ and $\mu(x,y) > 0$ for all $(x,y) \in E$. If $\mu(x,y) > 0$ we call x and y *neighbors* and we say that x and y *lie on* (x,y).

For any $t, 0 \leq t \leq 1$, let $\sigma^t = \{x \in V|\sigma(x) \geq t\}$ and $\mu^t = \{(x,y) \in E|\mu(x,y) \geq t\}$. Since $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$ we have $\mu^t \subseteq \sigma^t \times \sigma^t$. This means that $G^t = (\sigma^t, \mu^t)$ is a graph with node set σ^t and arc set μ^t for all $t \in [0,1]$.

Let $d(\mu) = \wedge\{\mu(x,y)|(x,y) \in \mu^*\}$ and $h(\mu) = \vee\{\mu(x,y)|(x,y) \in \mu^*\}$. $d(\mu)$ is called the *depth* of μ and $h(\mu)$ is called the *height* of μ. Note that the support G^* of G is $G^{d(\mu)}$.

A *path* from x to y in a fuzzy graph $G = (V, E, \sigma, \mu)$ is a sequence $\rho : x = x_0, x_1, ..., x_n = y$ of distinct nodes such that $\sigma(x_i) > 0$ and $\sigma(x_{i-1}) \wedge \sigma(x_i) \geq \mu(x_{i-1}, x_i) > 0$ for all i. Then $\bigwedge_i \mu(x_{i-1}, x_i)$ is called the *strength* of ρ. The maximum of the strengths of all paths in G from x to y is called $CONN_G(x,y)$, and we call G *connected* if $CONN_G(x,y) > 0$ for all $x, y \in V$. If H is connected and G is a full fuzzy subgraph of H then G is connected.

Definition 2.1. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph.

- Let $\mu(x,y) > 0$ and let $G - (x,y)$ be the fuzzy graph obtained from G by replacing $\mu(x,y)$ by 0. We call (x,y) *strong* in G if $\mu(x,y) \geq CONN_{G-(x,y)}(x,y)$.
- We call z a *fuzzy end node* of G if it has exactly one strong neighbor in G. Evidently, if z is an end node (i.e., z has only one neighbor) in the support of G, then z is a fuzzy end node of G, since z has only one neighbor in G.
- A node $z \in V$ is called a *fuzzy cut node* of G if, in the fuzzy graph $G - z$ obtained from G by replacing $\sigma(z)$ by 0, we have $CONN_{G-z}(x,y) < CONN_G(x,y)$ for some $x, y \in V$. It is shown in [9] that every fuzzy graph has at least two nodes that are not fuzzy cut nodes.

3. Definitions and Results

In [7] the authors introduced "degrees" for fuzzy cut nodes and fuzzy trees. We now introduce the corresponding degrees for fuzzy end nodes and study their properties.

Definition 3.1. Let x belong to V.

- x is called an end node if x is an end node in the support of G;

- x is called a fuzzy end node if it has at most one strong arc incident with it;
- x is called a weak fuzzy end node if there exists $t \in (0, h(\mu)]$ such that x is an end node in G^t where $h(\mu)$ is the height of μ;
- x is called a partial fuzzy end node if x is an end node in G^t for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$ where $d(\mu)$ is the depth of μ.
  Note: If $d(\mu) = h(\mu)$ then μ is constant and so $(d(\mu), h(\mu)] = \emptyset$;
- x is called a full fuzzy end node if x is an end node in G^t for all $t \in (0, h(\mu)]$.

Remark 3.2. Every end node is also a fuzzy end node. Every partial fuzzy end node is also a weak fuzzy end node. Every full fuzzy end node is also a partial fuzzy end node.

Theorem 3.3. *Every fuzzy end node is also a weak fuzzy end node.*

Proof. Let v be a fuzzy end node in $G = (\sigma, \mu)$. Then v has exactly one strong neighbor, say w. We claim that $\mu(v, w) > \mu(v, x)$ for all nodes x of G. Let $\mu(v, x_1) \neq 0$ for some node x_1. Since x_1 is not a strong neighbor of v, it follows that $\mu(v, x_1) < CONN_{G-(v,x_1)}(v, x_1)$. This means there exists x_2 such that $\mu(v, x_1) < \mu(v, x_2)$. If $x_2 = w$, we are done; otherwise, since x_2 is not a strong neighbor of v it follows by the same argument that $\mu(v, x_2) < \mu(v, x_3)$ for some node x_3. Since G has only finitely many nodes, this process must stop after n steps and eventually we get $x_n = w$; otherwise, both w and x_n would be strong neighbors of v, a contradiction. Thus $\mu(v, x_1) < \cdots < \mu(v, x_{n-1}) < \mu(v, x_n = w)$. Hence for all nodes x of G, if $\mu(v, x) \neq 0$ then $\mu(v, w) > \mu(v, x)$. Let $t = \mu(x, w)$; it then follows that v is an end node in G^t. Thus v is a weak fuzzy end node. □

Example 3.4. The converse of Theorem 3.3 and Remark 3.2 do not hold as can be seen in Figure 1.

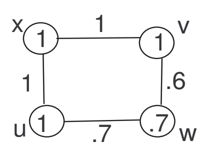

FIGURE 1. Degrees of fuzzy end nodes

In Figure 1, u is a weak fuzzy end node since it is an end node in $G^{(.8)}$, but it is not a fuzzy end node since it has two strong arcs. It is also not a partial fuzzy end node since it is not an end node of $G^{(.7)}$. Also v is a fuzzy end node but not an end node, and v is a partial fuzzy end node but not a full fuzzy end node.

Definition 3.5. Let x belong to V.

- x is called r-dominant if it has r distinct neighbors $w_1, w_2, \cdots, w_r$ such that $\mu(x, w_i) = h(\mu)$, the height of μ, for all $i = 1, 2, \cdots, r$;
- x is called r-weak if it has r distinct neighbors $w_1, w_2, \cdots, w_r$ such that $\mu(x, w_i) = d(\mu)$, the depth of μ, for all $i = 1, 2, \cdots, r$;
- x is called an extreme (s,t) node if it is s-dominant and t-weak for some $s, t \geq 0$ and there exists no w such that $d(\mu) < \mu(x, w) < h(\mu)$.

Proposition 3.6. *Let x belong to V. Then x is a full fuzzy end node if and only if x is an end node and x is 1-dominant.*

Proof. If x is a full fuzzy end node it is an end node in G^t for all $t \in (0, h(\mu)]$, and in particular also in $G^{d(\mu)} = G^*$. Hence x is an end node in the support of G and so is an end node of G. Further, if x is a full fuzzy end node it is also an end node of $G^{h(\mu)}$, so that it is 1-dominant. Conversely, if x is an end node of G and x is 1-dominant, x has exactly one arc, say (x, w), incident with it and it must be of weight $h(\mu)$. Hence, for all $t \in (0, h(\mu)]$, x is an end node of G^t. □

Proposition 3.7. *Let x belong to V. Then x is a partial fuzzy end node if and only if x is a 1-dominant node and $0 < \mu(x, w) < h(\mu)$ implies $\mu(x, w) = d(\mu)$. That is, x is an extreme $(1, t)$ node for some $t \geq 0$.*

Proof. If x is a partial fuzzy end node, then x is an end node in $G^{h(\mu)}$, and so x must be 1-dominant. Suppose $\mu(x, u) = h(\mu)$. If there exist two nodes w, v such that $0 < \mu(x, w) < \mu(x, v) < h(\mu)$, then x is not an end node in $G^{\mu(x,w)}$ since both (x, w) and (x, v) are in $G^{\mu(x,w)}$. Hence, for all $\mu(x, w)$ such that $\mu(x, w) \neq 0$ and $u \neq w$, $\mu(x, w) = d(\mu)$. Conversely, if x is 1-dominant and for all other arcs (x, w) of x, $\mu(x, w) = d(\mu)$, then x is of degree 1 in G^t for all $t > d(\mu)$. Hence, x is a partial fuzzy end node. □

Corollary 3.8. *A node v is a partial fuzzy end node but not a full fuzzy end node if and only if v is an extreme $(1, r)$ node for some $r \geq 1$.*

Theorem 3.9. *Suppose G is a fuzzy graph such that support(G) is a cycle. Let x, y be nodes of G, $x \neq y$. Then the following conditions on x and y are equivalent:*
(1) *x and y are fuzzy end nodes.*
(2) *There is a unique arc (x, y) such that $\mu(x, y) = d(\mu)$, the depth of μ.*
(3) *$\forall v \in V - \{x, y\}$, v is a fuzzy cut node.*

Proof. (1) $\Rightarrow$ (2): Let $u, v \in V$ be such that $\mu(x, u) > 0$ and $\mu(x, v) > 0$ with $u \neq v$. Since x is strong it follows that either $\mu(x, u) = d(\mu)$ or $\mu(x, v) = d(\mu)$. Also both $\mu(x, u)$ and $\mu(x, v)$ can't be equal to $d(\mu)$; otherwise both these arcs would be strong, contrary to the assumption that x is a fuzzy end node. Assume $(x, u) = d(\mu)$. Similarly there exists a w such that $\mu(y, w) = d(\mu)$. If $(x, u) \neq (y, w)$ then by Theorem 3 in [2] G is multimin and hence it cannot have any fuzzy end nodes, a contradiction. Thus $(x, u) = (w, y)$. That is, $x = w$ and $y = u$. In fact, (x, y) is unique such that $\mu(x, y) = d(\mu)$.

(2) $\Rightarrow$ (3): Let $v \in V$ be such that $x \neq v \neq y$. The path from x to y other than (x, y) must be stronger than (x, y). Hence deletion of v reduces the strength of connectedness of x and y. Thus v is a fuzzy cut node.

(3) $\Rightarrow$ (1): Let $v \in V - \{x, y\}$ and $u \in V$. Suppose $\mu(u, v) = d(\mu)$. Since v is a cut node, the removal of v reduces the strength of connection of some pair of nodes w and z, $w \neq v \neq z$. However, this is impossible since $\mu(u, v) = d(\mu)$. Thus $\mu(u, v) > d(\mu)$. Now $\mu(w, z) = d(\mu)$ for some (w, z), and it must be the case, as just shown, that $w, z \in \{x, y\}$. Thus, $\mu(x, y) = d(\mu)$ and in fact (x, y) is unique with respect to this property. Hence, (2) holds. Finally, (2) $\Rightarrow$ (1) follows from the definition of fuzzy end node. $\square$

Corollary 3.10. *Suppose G is a cycle. If x is a fuzzy end node, then there exists $y \in V - \{x\}$ such that y is a fuzzy end node.*

Corollary 3.11. *Suppose G is a cycle. If x and y are fuzzy end nodes, then x and y are adjacent.*

Remark 3.12. In [2] the authors have shown that every fuzzy tree has at least two fuzzy end nodes. We now discuss the corresponding result for $*$ *fuzzy trees* where $*$ can mean weak, partial or full [7].

Remark 3.13. If G is a weak fuzzy tree, then for some $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$, G^t is a tree. Thus, G^t has at least two end nodes, say u, v. This means that u and v are weak fuzzy end nodes of G. Hence, every weak fuzzy tree has at least two weak fuzzy end nodes.

Remark 3.14. It is not true that if G is a partial fuzzy tree, then G has at least two partial fuzzy end nodes. For example, Figure 2 is a partial fuzzy tree but it does not have at least two partial fuzzy end nodes. Only node w is a partial fuzzy end node. That is, w is an end node of G^t for all $t \in (d(\mu), h(\mu)]$.

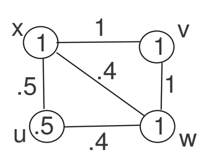

FIGURE 2. A partial fuzzy tree

Remark 3.15. It is not true that if G is a full fuzzy tree, then G has at least two full fuzzy end nodes. For example, Figure 3 is a full fuzzy tree but it has only one full fuzzy end node. Nodes w and u are end nodes, fuzzy end nodes, and weak fuzzy end nodes. Node u is the only full and partial fuzzy end node.

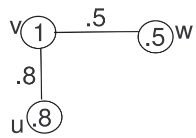

Figure 3. A full fuzzy tree

Definition 3.16. A fuzzy graph G is called arc-disjoint if no two cycles of G share a common arc. By $deg(x)$ of a node x in a fuzzy graph we mean the degree of x in the support of G.

Theorem 3.17. *Suppose G is an arc-disjoint fuzzy graph and let x be a node of G. If x is a fuzzy end node of G then $deg(x) \leq 2$. Further $deg(x) = 2$ if and only if x is in a cycle.*

Proof. Clearly, if $deg(x) = 0$ or 1, then x is an end node and hence a fuzzy end node. Suppose $deg(x) = 2$. If (x, u) and (x, v) are in G, $u \neq v$, and x is not in a cycle, then both (x, u) and (x, v) are strong, which contradicts the fact that x is a fuzzy end node. Hence, x is in a cycle, say C. Suppose $(x, u), (x, v)$ are in the cycle C and $u \neq v$. Since G is arc disjoint, the paths from x to u are in C and similarly for the paths from x to v. However, x has only one strong neighbor in G. Since this argument applies to any cycle that x is in, x is in only one cycle of G. Suppose (x, w) is in G and (x, w) is not in a cycle. Then (x, w) is strong and so x is not a fuzzy end node, a contradiction. Thus no such w exists, which means that *deg(x)* = 2. □

Corollary 3.18. *Suppose G is an arc disjoint fuzzy graph. If x is a fuzzy end node of G, then x is in at most one cycle of G.*

Proposition 3.19. *In a fuzzy graph, x is a fuzzy cut node if and only if x is a weak fuzzy cut node.*

Proof. Suppose x is a fuzzy cut node. Then there exist $u, v \in V$ such that $CONN_{G-x}(u, v) < CONN_G(u, v)$. This means that there exist $w_1, w_2 \in V$ such that $\mu(x, w_1)$ and $\mu(x, w_2)$ are both greater than $CONN_{G-x}(u, v)$. Suppose $CONN_{G-x}(u, v) = a$. If min $(\mu(x, w_1), \mu(x, w_2)) = b$, then $a < b$ and so x is a cut node in G^b, that is, x is a weak fuzzy cut node. Conversely, if x is a weak fuzzy cut node, then for some $t \in (0, h(\mu)]$, x is a cut node in G^t. Suppose deleting x in G^t disconnects u and v. This means that the path in G from u to v through x has strength greater than all other paths in $G - x$ between u and v. This shows that $CONN_{G-x}(u, v) < CONN_G(u, v)$; hence, x is a fuzzy cut node. □

Proposition 3.20. *Let x belong to V and suppose x is not in a cycle of G^*. Then the following are equivalent:*

(1) *x is a partial fuzzy cut node.*
(2) *x is at least 2-dominant. That is, there exist at least two nodes u and v such that*

$$\mu(x,u) = \mu(x,v) = h(\mu).$$

(3) *x is a full fuzzy cut node.*

Proof. (1) $\Rightarrow$ (2): If x is a partial fuzzy cut node, then x is also a cut node for $G^{h(\mu)}$; hence x must be at least 2-dominant. (2) $\Rightarrow$ (3): If x is at least 2-dominant and x is not in any cycle of G^*, then it is a cut node for G^t for all $t \in (0, h(\mu)] \cup \{h(\mu)\}$, so that x is a full fuzzy cut node. (3) $\Rightarrow$ (1) follows from the definition. □

Definition 3.21. A fuzzy graph is said to be non-separable [8] (also called a block) if it is connected and has no fuzzy cut nodes.
Remark: In crisp graphs a non-separable graph cannot have a bridge [3]. However in the fuzzy case, a block, that is, a fuzzy graph with no fuzzy cut nodes may have a fuzzy bridge. Recall an arc (x,y) in μ^* is called a fuzzy bridge if $CONN_{G-(x,y)}(u,v) < CONN_G(u,v)$ for some (u,v) in μ^*. In Figure 4, arc (u,w) is a bridge but this fuzzy graph has no fuzzy cut nodes [8].

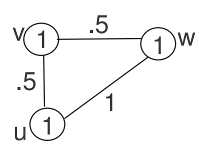

Figure 4. A non-separable fuzzy graph with a bridge

Definition 3.22. Recall from [2] that a cycle is called a fuzzy cycle if it has more than one weakest arc, and a cycle is called locamin if every node of the cycle lies on a weakest arc.

Theorem 3.23. *A connected arc-disjoint fuzzy graph $G = (\sigma, \mu)$ with at least three arcs is non-separable if and only if any two adjacent arcs lie on a locamin cycle.*

Proof. Suppose G is non-separable and let (u,v) and (v,w) be any two adjacent arcs. Since G has no fuzzy cut nodes and G is arc-disjoint, it follows that (u,v) and (v,w) lie on a unique cycle, say C. If this cycle is not locamin, there is a path $P : x, y, z$ in C with strength greater than the minimum weight on the arcs of C. This means that y is a fuzzy cut node, a contradiction since G is non-separable. Conversely, suppose v is a fuzzy cut node of G. Then there exists a pair of nodes p, q such that $CONN_{G-v}(p,q) < CONN_G(p,q)$. Thus we can form a cycle C with node v as one of its nodes in which two arcs adjacent to v form a path of length 2 with strength greater than the weakest arc of C. But this contradicts the fact that C is locamin. Hence, G must be non-separable. □

Here is a counterexample to show that the above result does not remain true if G is not arc-disjoint. In Figure 5, G is non-separable but no cycles in G are locamin.

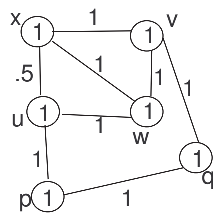

FIGURE 5. A non-separable fuzzy graph with no locamin cycles

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