Iranian Journal of Fuzzy Systems
Volume 16, Number 6, (2019), pp. 1-16

A robust aggregation operator for multi-criteria decision-making method with bipolar fuzzy soft environment

C. Jana[1], M. Pal[2] and J. Wang[3]

[1,2] *Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India*
[3] *School of Business, Central South University, Changsha 410083, China*

jana.chiranjibe7@gmail.com, mmpalvu@gmail.com, jqwang@csu.edu.cn

Abstract

Molodtsov initiated soft set theory that provided a general mathematical framework for handling with uncertainties in which we encounter the data by affix parameterized factor during the information analysis as differentiated to fuzzy as well as bipolar fuzzy set theory. The main object of this paper is to lay a foundation for providing a new application of bipolar fuzzy soft tool in considering many problems that contain uncertainties. In present study, aggregation of bipolar fuzzy soft numbers have so far not yet been applied for ranking of alternatives in decision-making problems. To this propose, bipolar fuzzy soft weighted arithmetic averaging (BFSWAA) operator, bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator have been used to compare two bipolar fuzzy soft numbers (BFSNs) for aggregating different bipolar fuzzy soft input arguments in bipolar fuzzy soft environment. Then, their related properties have been investigated. Finally, a practical example for selection of alternatives is given to demonstrate the utility and application of the proposed work.

Keywords: Bipolar fuzzy soft numbers, bipolar fuzzy soft arithmetic averaging operators, bipolar fuzzy soft geometric averaging operators, decision-making.

1 Introduction

Molodtsov [33] introduced soft set theory for modeling uncertainties and vagueness of data. Many researchers studying on problems involving uncertainty have interested soft set theory since it was introduced by Molodtsov. The traditional tools are not only limited to fuzzy set theory [53] and intuitionistic fuzzy set theory [8] etc., but also widely used by the researchers in MADM problems [10, 11, 14, 15, 41, 46, 47, 52], MCDM problems [34, 39] in intuitionistic fuzzy (IF) and interval-valued intuitionistic fuzzy IVIFS environments to overcome the uncertainties in real world problems. Although, IFS and IVIFS successfully applied to solve the uncertainty of the real world problems, but it has been seen generally for the information analysis of an object that corresponding to each property there exists a counter property. In that view, Zhang [54, 55] lately originated other generalization of fuzzy sets, called bipolar fuzzy sets (BFS) whose membership degree extended to $[-1, 1]$. Then BFS treated as a new tool to depict uncertainty in decision science. Bipolar fuzzy sets have not only been applied in bipolar logical reasoning and bipolar set theory [20, 56] but also have an applications in different uncertain environments such as computational psychiatry [57], medicine science [29, 58], bipolar quantum logic based computing [59, 60], quantum cellular combinatorics [59], decision analysis and organizational modeling [16, 28], physics and philosophy [61] and bipolar fuzzy graph and its applications [36, 49]. Sing and Kumar [38] have shown some applications of bipolar information using lattice theory, bipolar fuzzy graph as well as Formal Concept Analysis (FCA). Han et al. [20] studied applications of decision information system in bipolar rough fuzzy environment. Lately, Gul [19] provided concept of bipolar fuzzy aggregation operators, for example, bipolar fuzzy weighted averaging operator (BFWAA) and bipolar fuzzy weighted geometric operator (BFWGA) and applied these operators to study multiple

Corresponding Author: C. Jana
Received: May 2018; Revised: December 2018; Accepted: April 2019.

attribute group decision-making problems. Lu et al. [30] proposed linguistic aggregation operator on bipolar fuzzy numbers and gave an example of MADM problems using these operators. Wei et al. [43] utilized interval-valued bipolar 2-tuple linguistic (IVB2TLNs) arithmetic and geometric aggregation operators to investigate for the risk evaluation of enterprise human capital investment by a MADM problem. Wei et al. [42] studied recently a MADM problems based on bipolar fuzzy Hamacher aggregation operator. After that, Wei et al. [44] investigated the MADM problems under the aggregation operators based on hesitant bipolar fuzzy information. Xu and Wei [45] provided a multiple-attribute decision-making problems based on dual hesitant bipolar fuzzy aggregation function. Khalid and Beg [27] used preference relations based decision-making in incomplete interval-valued hesitant fuzzy environment. Gao et al. [18] used Hamacher prioritized aggregation operators on dual hesitant bipolar fuzzy numbers, and develop a MADM problem for analyzed the relevance and effectiveness of the proposed method. Wan et al. [40] introduced Frank Choquet Bonferroni mean operators and utilized this operator develop MCDM problems in single-valued bipolar neutrosophic environment.

But the technique of the above papers are not enough for the solution of real-world problems because they have insufficient parameterizations. In that view, soft set theory plays an important role to overcome such barrier and effectively applied to solve the conditions. Maji et al. [31, 32] provided the concept of "fuzzy soft sets" and "intuitionistic fuzzy soft sets" where soft set theory is combined with the fuzzy set theory and intuitionistic fuzzy set theory. Some hybrid models together with soft set theory have been developed in various uncertain environments such as on fuzzy soft set theory with parameterizations [5, 12, 13], fuzzy soft expert sets [6], generalized intuitionsitic fuzzy soft sets [2, 17, 22], interval-valued intuitionistic fuzzy soft sets [21, 50] and its applications, Hesitant fuzzy soft sets [9]. Further, researcher [26] provided applications of similarity measures using soft sets. Selvachandran and Peng [37] has found a modified TOPSIS method using vauge parameterized vegue soft set and gave its application in decision-making. Recently, Arora and Garg [7] provided a new approach of aggregation operator using parameterized factor in intuitionistic fuzzy soft environment. In the same time, a tool combination of bipolar fuzzy set and soft set have gave a momentum to the solution of real life problems in many directions. Karaaslan and others [23, 24, 25] introduced bipolar soft group structures, soft rough relations and then used soft-rough relations to develop a decision-making problems. Abdullah et al. [1] gave an application of bipolar fuzzy soft sets in decision-making problem. Yang et al. [51] utilized bipolar multi-fuzzy soft set in decision-making approach. Qudah and Hassan [35] proposed bipolar fuzzy soft expert set and its application in decision-making. Ali et al. [4] gave an application of bipolar neutrosophic soft sets in decision-making in the environment of bipolar neutrosophic set. Akram et al. [3] combined bipolar fuzzy soft information with graph and then developed a multi criteria-decision making method based on bipolar fuzzy soft graphs.

Thus, BFSS is a powerful appliance to deal with the ambivalence and vagueness of data, the proposed work will introduce a new aggregation operator known as bipolar fuzzy soft aggregation operators depicted as bipolar fuzzy soft weighted arithmetic averaging (BFSWAA) operator and bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator in the environment of bipolar fuzzy soft numbers (BFSNs). The main advantages of these operators is that they are able to make smooth description of the real-world problems by the use of parameterizations factor. In order to rank the alternatives, aggregation operators lead to aggregate the over all information of the objects for the preferences of the decision maker into a collective one and hence find to a desirable according to its score values. To the best of our knowledge, the research developed on FSS and BFSS is only about their basic theory and its applications, but, there have been no research done on bipolar fuzzy soft aggregation information. So, it is a new issue and have a scope for future development in decision science. Above decision-making problems in bipolar fuzzy soft environment under aggregation operators makes us enough motivation to develop our present paper. The main object of this article is to exhibit some aggregation operators under bipolar fuzzy called as bipolar fuzzy soft aggregations for amassing the distinct priorities of the choices amid the decision-making process.

The remainder of this paper is sorted out as takes after: In next Section, we briefly survey some essential ideas of the FSS and BFSS. In Section 3, we define some operational principles of bipolar fuzzy soft numbers and then define bipolar fuzzy soft weighted arithmetic averaging (BFSWAA) operator, bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator and established its related properties. In next section, we utilize those operators to create bipolar fuzzy soft multi-attribute decision-making problems. An interpretative case is specified for the selection of best candidate(s) for a company in Section 5. In section 6, a comparative analysis has been made between the existing works and the proposed study. Finally, in Section 7, the conclusion and scope of future research are outlined and discussed.

2 Basic concept of FSS and BFSS

In what follows, U, E and $\mathcal{P}(U)$ denote initial universal set, set of parameters and power set of U, respectively. Also, $A \subseteq E$.

Definition 2.1. [53] *Let X be a non-empty set. A fuzzy set μ of X is defined as a mapping $\mu : X \to [0,1]$, where $[0,1]$ is the usual interval of real numbers. We take $\mathbb{F}(X)$ as the set of all fuzzy subsets of X.*

Definition 2.2. [33] *A pair $(\mathcal{F}, E)$ is called a soft set over U if $\mathcal{F}$ is a mapping given by $\mathcal{F} : E \to \mathcal{P}(U)$. In other words, a soft over the universe U is a parameterized family of subsets of the universal set U. For $\varepsilon \in A$, $\mathcal{F}(\varepsilon)$ may be considered as the set of ε-elements of the soft $(\mathcal{F}, A)$, or as the set of ε-approximate elements of the soft set.*

The following example illustrate the above idea.

Example 2.3. *Let (X, τ) be a topological space, i.e. τ is a family of subsets of the set X called the open sets of X. Then, the family of open neighborhood $N(x)$ of point x, where $N(x) = \{V \in \tau | x \in V\}$, may be considered as the soft set $(N(x), \tau)$.*

Definition 2.4. [31] *Let U be the universal set and E be the set of parameters. Let $\mathcal{P}(U)$ be the power set of U and $A \subseteq E$, and $\mathcal{P}(U)$ be the collection of all fuzzy subsets of U, then $(\mathcal{F}, A)$ is called fuzzy soft set, where $\mathcal{F} : A \to \mathcal{P}(U)$.*

Example 2.5. *Let $U = \{M_1, M_2, M_3, M_4\}$ be the set of four mobiles under consideration and $E = \{beautiful(e_1)$, $costly(e_2)$, $batterybackup(e_3)$ and $apps(e_4)\}$ be a set of parameters then FSS for describing "attractiveness of the mobiles" is $(\mathcal{F}, A) = \{\mathcal{F}_{e_1}, \mathcal{F}_{e_2}, \mathcal{F}_{e_3}\}$, where $A = \{e_1, e_2, e_3\} \subseteq E$ and (F, A) can be defined as:*
$\mathcal{F}_{e_1} = \{(M_1, 0.6), (M_2, 0.4), (M_3, 0.5), (M_4, 0.3)\}$,
$\mathcal{F}_{e_2} = \{(M_1, 0.7), (M_2, 0.6), (M_3, 0.5), (M_4, 0.4)\}$ *and*
$\mathcal{F}_{e_3} = \{(M_1, 0.9), (M_2, 0.5), (M_3, 0.3), (M_4, 0.6)\}$.

Definition 2.6. *A bipolar fuzzy sets (BFS) is defined over the universe of discourse X as*

$$B = \{\langle x, \mu^+(x), \nu^-(x)\rangle | x \in X\}, \tag{1}$$

where $\mu^+(x) : X \to [0,1]$ represents positive membership degree to satisfy corresponding property of an element x to a BFS and $\nu^-(x) : X \to [-1,0]$ represent negative membership degree to satisfy counter-property of an element x to a BFS, such that $-1 \leq \mu^+(x) + \nu^-(x) \leq 1$ for every $x \in X$. The set $\langle(\mu^+, \nu^-)\rangle$ is denote bipolar fuzzy numbers (BFNs), i.e, bipolar fuzzy elements (BFEs).

Definition 2.7. *Let U be a universal set, E be a set of parameters and $A \subset E$. Define $\tilde{B} : A \to BF^U$, where BF^U is the collection of all bipolar fuzzy subsets of U. Then (B, A) is called bipolar fuzzy soft set over U, and is denoted by $(\tilde{B}, A) = B(e_i)$ and defined by*

$$\tilde{B}(e_i) = \{(x_i, \mu^+(x_i), \nu^-(x_i)) : \forall x_i \in U, \forall e_i \in A\}.$$

Example 2.8. *Let $U = \{M_1, M_2, M_3, M_4\}$ be the set of four mobiles under consideration and $E = \{beautiful(e_1)$, $costly(e_2)$, $batterybackup(e_3)$ and $apps(e_4)\}$ be a set of parameters under BFSS for describing "attractiveness of the mobiles" is $(B, A) = \{\mathcal{F}_{e_1}, \mathcal{F}_{e_2}, \mathcal{F}_{e_3}\}$, where $A = \{e_1, e_2, e_3\} \subseteq E$ and $(\tilde{B}, A)$ can be defined as:*

$$\begin{aligned}
\tilde{B}_{e_1} &= \{(M_1, 0.6, -0.2), (M_2, 0.4, -0.1), (M_3, 0.5, -0.3), (M_4, 0.3, -0.1)\},\\
\tilde{B}_{e_2} &= \{(M_1, 0.7, -0.2), (M_2, 0.6, -0.1), (M_3, 0.5, -0.3), (M_4, 0.4, -0.1)\}\ ,\\
\tilde{B}_{e_3} &= \{(M_1, 0.9, -0.3), (M_2, 0.5, -0.2), (M_3, 0.3, -0.1), (M_4, 0.6, -0.4)\}.
\end{aligned}$$

For the sake of simplicity, we denote $\tilde{B}_{e_t}(x_s) = \{\langle x_s, \mu_t^+(x_s), \nu_t^-(x_s)\rangle | x_s \in U\}$, i.e., $\tilde{B}_{e_{st}} = \langle \mu_{st}^+, \nu_{st}^- \rangle$ is called as bipolar fuzzy soft (BFSN) number. For the application purpose, it is necessary to define score function for ranking it. For this, a score function of $\tilde{B}_{e_{st}}$ is defined as

$$\Psi(\tilde{B}_{e_{st}}) = \mu_{st}^+ + \nu_{st}^- \tag{2}$$

where, $\Psi(\tilde{B}_{e_{st}}) \in [-1,1]$. By this definition, it is clear that the larger the $\Psi(\tilde{B}_{e_{st}})$, the larger is BFSN $\tilde{B}_{e_{st}}$.

Example 2.9. *Let $\tilde{B}_{e_{11}} = \langle 0.6, -0.2\rangle$ and $\tilde{B}_{e_{12}} = \langle 0.3, -0.5\rangle$ be two BFSNs, then by eq.(2), we get $\Psi(\tilde{B}_{e_{11}}) = 0.4$ and $\Psi(\tilde{B}_{e_{12}}) = -0.2$. Since $\Psi(\tilde{B}_{e_{11}}) > \Psi(\tilde{B}_{e_{12}})$ which imply $\tilde{B}_{e_{11}} > \tilde{B}_{e_{12}}$.*

However, there are some situations, where above function can not be used to compare BFSNs. For example, let $\tilde{B}_{e_{11}} = \langle 0.6, -0.2\rangle$ and $\tilde{B}_{e_{12}} = \langle 0.5, -0.1\rangle$, then it is not possible to compare BFSNs, which one of them is bigger as $\Psi(\tilde{B}_{e_{11}}) = \Psi(\tilde{B}_{e_{12}})$. To overcome this situation, we define accuracy function of $\tilde{B}_{e_{st}}$ as follows:

$$\mathcal{H}(\tilde{B}_{e_{st}}) = \mu_{st}^{+} - \nu_{st}^{-} \tag{3}$$

where, $\mathcal{H}(\tilde{B}_{e_{st}}) \in [0,1]$. Based on score function Ψ and accuracy function $\mathcal{H}$, defined order relation on two BFSNs $\tilde{B}_{e_{st}}$ and $\tilde{P}_{e_{st}}$ as follows:

(i) If $\Psi(\tilde{B}_{e_{st}}) < \Psi(\tilde{P}_{e_{st}})$, then $\tilde{B}_{e_{st}} \prec \tilde{P}_{e_{st}}$

(ii) If $\Psi(\tilde{B}_{e_{st}}) > \Psi(\tilde{P}_{e_{st}})$, then $\tilde{B}_{e_{st}} \succ \tilde{P}_{e_{st}}$

(iii) If $\Psi(\tilde{B}_{e_{st}}) = \Psi(\tilde{P}_{e_{st}})$, then

(1) If $\mathcal{H}(\tilde{B}_{e_{st}}) < \mathcal{H}(\tilde{P}_{e_{st}})$, then $\tilde{B}_{e_{st}} \prec \tilde{P}_{e_{st}}$.
(2) If $\mathcal{H}(\tilde{B}_{e_{st}}) > \mathcal{H}(\tilde{P}_{e_{st}})$, then $\tilde{B}_{e_{st}} \succ \tilde{P}_{e_{st}}$.
(3) If $\mathcal{H}(\tilde{B}_{e_{st}}) = \mathcal{H}(\tilde{P}_{e_{st}})$, then $\tilde{B}_{e_{st}} \sim \tilde{P}_{e_{st}}$.

3 Bipolar fuzzy soft weighted arithmetic averaging operator

In this section, aggregation operators namely bipolar fuzzy soft weighted arithmetic averaging (BFSWAA), operator for bipolar fuzzy soft numbers (BFSNs) are proposed.

3.1 Operational law for BFSNs

Definition 3.1. *Let $\tilde{B}_e = \langle \mu^{+}, \nu^{-}\rangle$ and $\tilde{B}_{e_{11}} = \langle \mu_{11}^{+}, \nu_{11}^{-}\rangle)$ and $\tilde{B}_{e_{12}} = \langle \mu_{12}^{+}, \nu_{12}^{-}\rangle)$ be the BFSNs over the universe X, then the following operations are defined as follows:*

(i) $\tilde{B}_{e_{11}} \oplus \tilde{B}_{e_{12}} = \big(\langle \mu_{11}^{+} + \mu_{12}^{+} - \mu_{11}^{+}\mu_{12}^{+}, -|\nu_{11}^{-}||\nu_{12}^{-}|\rangle\big)$

(ii) $\tilde{B}_{e_{11}} \otimes \tilde{B}_{e_{12}} = \big(\langle \mu_{11}^{+}\mu_{12}^{+}, \nu_{11}^{-} + \nu_{12}^{-} - \nu_{11}^{-}\nu_{12}^{-}\rangle\big)$

(iii) $\lambda\tilde{B}_e = \big(1 - (1-\mu^{+})^{\lambda}, -|\nu|^{\lambda}\big)$

(iv) $\tilde{B}_e^{\lambda} = \big(\mu^{\lambda}, -1 + |1+\nu^{-})|^{\lambda}\big)$.

Definition 3.2. *Let $\tilde{B}_{e_{st}} = (\mu_{st}^{+}, \nu_{st}^{-})$ $(s = 1,2,\ldots,m; t = 1,2,\ldots,n)$ be a collection of BFSNs and ϕ_t, θ_s are the weight vectors for the parameter e_t's and expert y_s's respectively, satisfying $\phi_t \geq 0$, $\theta_s \geq 0$ such that $\sum_{t=1}^{n} \phi_t = 1$ and $\sum_{s=1}^{m} \theta_s = 1$. Then bipolar fuzzy soft weighted arithmetic averaging (BFSWAA) operator is the function $BFSWA : \tilde{B}^n \to \tilde{B}$ such that*

$$BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = \bigoplus_{t=1}^{n} \phi_t \Big(\bigoplus_{s=1}^{m} \theta_s \tilde{B}_{e_{st}}\Big). \tag{4}$$

We get the following theorem that follows from BFSWAA operator.

Theorem 3.3. *$\tilde{B}_{e_{st}} = (\mu_{st}^{+}, \nu_{st}^{-})$ $(s = 1,2,\ldots,m; t = 1,2,\ldots,n)$ be a collection of (BFSNs), then aggregated value of them using the BFSWAA operator is also a BFSNs, and*

$$BFSWA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle. \tag{5}$$

Then Theorem 3.3 can be proved by the method of mathematical induction as follows:

Proof. For $m = 1$, we get $\theta_1 = 1$. Then by Definition 3.1 of operational law,

$$\begin{aligned} BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) &= \bigoplus_{t=1}^{n} \phi_t\Big(\tilde{B}_{e_{1t}}\Big) = \Big\langle 1 - \prod_{t=1}^{n}\Big(1-\mu_{1t}^{+}\Big)^{\phi_t}, - \prod_{t=1}^{n} |\nu_{1t}^{-}|^{\phi_t}\Big\rangle \\ &= \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{1}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, - \prod_{t=1}^{n}\Big(\prod_{s=1}^{1} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle. \end{aligned}$$

Again, for $n = 1$ and $\phi_1 = 1$ and hence,

$$\begin{aligned} BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) &= \Big(\bigoplus_{s=1}^{m} \theta_{s1}\tilde{B}_{e_{1t}}\Big) = \Big\langle 1 - \prod_{t=1}^{m}\Big(1-\mu_{s1}^{+}\Big)^{\theta_s}, - \prod_{s=1}^{m} |\nu_{s1}^{-}|^{\theta_s}\Big\rangle \\ &= \Big\langle 1 - \prod_{t=1}^{1}\Big(\prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, - \prod_{t=1}^{1}\Big(\prod_{s=1}^{m} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle. \end{aligned}$$

Thus, (5) is true for $m = 1$ and $n = 1$. Assume that (5) is true for $n = p_1 + 1$, $m = p_2$ and $n = p_1$, $m = p_2 + 1$, then it follows that

$$\bigoplus_{t=1}^{p_1+1} \phi_t\Big(\bigoplus_{s=1}^{p_2} \theta_s\tilde{B}_{e_{st}}\Big) = \Big\langle 1 - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle.$$

Also,

$$\bigoplus_{t=1}^{p_1} \phi_t\Big(\bigoplus_{s=1}^{p_2+1} \theta_s\tilde{B}_{e_{st}}\Big) = \Big\langle 1 - \prod_{t=1}^{p_1}\Big(\prod_{s=1}^{p_2+1}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, - \prod_{t=1}^{p_1}\Big(\prod_{s=1}^{p_2+1} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle.$$

Now for $n = p_1 + 1$ and $m = p_2 + 1$, we obtained

$$\begin{aligned} \bigoplus_{t=1}^{p_1+1} \phi_t\Big(\bigoplus_{s=1}^{p_2+1} \theta_s\tilde{B}_{e_{st}}\Big) &= \bigoplus_{t=1}^{p_1+1} \phi_t\Big(\bigoplus_{s=1}^{p_2} \theta_s\tilde{B}_{e_{st}} \oplus \theta_{p_2+1}\tilde{B}_{e_{(p_2+1)t}}\Big) = \bigoplus_{t=1}^{p_1+1}\bigoplus_{s=1}^{p_2} \phi_t\theta_s\tilde{B}_{e_{st}} \bigoplus_{t=1}^{p_1+1} \phi_t\theta_{p_2+1}\tilde{B}_{e_{(p_2+1)t}}\Big) \\ &= \Big\langle 1 - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t} \oplus 1 - \prod_{t=1}^{p_1+1}\Big((1-\mu_{(p_2+1)t}^{+})^{\theta_{p_2+1}}\Big)^{\phi_t}, \\ &\quad - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t} \oplus - \prod_{t=1}^{p_1+1}\Big(|\nu_{(p_2+1)t}^{-}|^{\theta_{p_2+1}}\Big)^{\phi_t}\Big\rangle \\ &= \Big\langle 1 - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2+1}(1-\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}, - \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2+1} |\nu_{st}^{-}|^{\theta_s}\Big)^{\phi_t}\Big\rangle. \end{aligned}$$

Thus, (5) is true for $n = p_1 + 1$, $m = p_2 + 1$, therefore by induction the results are hold for all $m, n \geq 1$.

Since, $0 \leq \mu_{st}^{+} \leq 1 \Leftrightarrow 0 \leq \prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s} \leq 1$ and hence, $0 \leq 1 - \prod_{t=1}^{n}(\prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s})^{\phi_t} \leq 1$. Also, $-1 \leq \nu_{st}^{-} \leq 0 \Leftrightarrow 0 \leq |\nu_{st}^{-}| \leq 1 \Leftrightarrow -1 \leq -|\nu_{st}^{-}| \leq 0 \Leftrightarrow -1 \leq -\prod_{s=1}^{m} |\nu_{st}^{-}|^{\theta_s} \leq 0 \Leftrightarrow -1 \leq -\prod_{t=1}^{n}(\prod_{s=1}^{m} |\nu_{st}^{-}|^{\theta_s})^{\phi_t} \leq 0$. Thus, $1 \geq 1 - \prod_{t=1}^{n}(\prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s})^{\phi_t} - \prod_{t=1}^{n}(\prod_{s=1}^{m} |\nu_{st}^{-}|^{\theta_s})^{\phi_t} = 1 - \prod_{t=1}^{n}(\prod_{s=1}^{m}(1-\mu_{st}^{+})^{\theta_s})^{\phi_t} - \prod_{t=1}^{n}(\prod_{s=1}^{m}(|\nu_{st}^{-}|^{\theta_s} - 1)^{\phi_t} \geq -1$. Hence, aggregated value obtained by BFSWAA is again a BFSN. □

Corollary 3.4. [19] *For only one parameter e_1, i.e., $n = 1$, the BFSWAA operator reduces to BFWA.*

$$BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{21}}, \ldots, \tilde{B}_{e_{m1}}) = \Big\langle 1 - \prod_{s=1}^{m}(1-\mu_{s1}^{+})^{\theta_s}, - \prod_{s=1}^{m} |\nu_{s1}^{-}|^{\theta_s}\Big\rangle. \tag{6}$$

Therefore, it is justified that aggregation operator defined under BFS environment is taken as a special case of the proposed operator.

Table 1: Bipolar fuzzy soft numbers

Experts	e_1	e_2	e_3
y_1	$\langle 0.6, -0.2\rangle$	$\langle 0.5, -0.5\rangle$	$\langle 0.6, -0.3\rangle$
y_2	$\langle 0.5, -0.4\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.5, -0.4\rangle$
y_3	$\langle 0.7, -0.1\rangle$	$\langle 0.4, -0.3\rangle$	$\langle 0.3, -0.1\rangle$
y_4	$\langle 0.4, -0.5\rangle$	$\langle 0.7, -0.2\rangle$	$\langle 0.2, -0.6\rangle$

Example 3.5. *Let $Y = \{y_1, y_2, y_3, y_4\}$ be the set of experts which are going to narrate the "attractiveness of two-wheeler bikes" under the set of parameters $E = \{e_1 = stylish,\ e_2 = weight,\ e_3 = milage,\ e_4 = price\}$. The rating value of the experts is assumed to be given in the form of BFSNs$(B, E) = (\mu^+_{st}, \nu^-_{st})_{4\times 3}$ for each parameters which are given in the following table.*

Let $\phi = (0.3, 0.2, 0.5)^T$ and $\theta = (0.2, 0.1, 0.3, 0.4)^T$ be the weight vectors for the parameters and experts respectively. Then, we get by using Theorem 3.3 as:

$$BFSWA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{43}}) = \Big\langle 1 - \prod_{t=1}^{3}\Big(\prod_{s=1}^{4}(1-\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{3}\Big(\prod_{s=1}^{4}|\nu^-_{st}|^{\theta_s}\Big)^{\phi_t}\Big\rangle$$

$\Big\langle 1 - \Big(\Big\{(1-0.6)^{0.2}(1-0.5)^{0.1}(1-0.7)^{0.3}(1-0.4)^{0.4}\Big\}^{0.3}\ \Big\{(1-0.5)^{0.2}(1-0.3)^{0.1}(1-0.4)^{0.3}(1-0.7)^{0.4}\Big\}^{0.2}\ \Big\{(1-0.6)^{0.2}(1-0.5)^{0.1}(1-0.3)^{0.3}(1-0.2)^{0.4}\Big\}^{0.5}\Big), -\Big(|-0.2|^{0.2}\ |-0.4|^{0.1}\ |-0.1|^{0.3}\ |-0.5|^{0.4}\Big)^{0.3}\ \Big(|-0.5|^{0.2}\ |-0.2|^{0.1}\ |-0.3|^{0.3}\ |-0.2|^{0.4}\Big)^{0.2}\ \Big(|-0.3|^{0.2}\ |-0.4|^{0.1}\ |-0.1|^{0.3}\ |-0.6|^{0.4}\Big)^{0.5}\Big\rangle$
$= \Big\langle 0.5317, -0.2755\Big\rangle$.

We prove easily the following properties by using the operator BFSWAA.

Theorem 3.6. *(**Idempotency Property**) Let $\tilde{B}_{e_{st}} = (\mu^+_{st}, \nu^-_{st})$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ be a collection of BFSNs that are all equal, i.e., $\tilde{B}_{e_s}t = \tilde{B}_e$ for all s, t, then*

$$BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = \tilde{B}_e. \tag{7}$$

Proof. Since $\tilde{B}_{e_{st}} = \tilde{B}_e = \langle \mu^+, \nu^-\rangle$ for all s, t. Then,

$$\begin{aligned} BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) &= \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu^+)^{\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu|^{\theta_s}\Big)^{\phi_t}\Big\rangle \\ &= \Big\langle 1 - \Big((1-\mu^+)^{\sum_{s=1}^{m}\theta_s}\Big)^{\sum_{t=1}^{n}\phi_t}, -\Big(|\nu^-|^{\sum_{s=1}^{m}\theta_s}\Big)^{\sum_{t=1}^{n}\phi_t}\Big\rangle \\ &= \Big\langle 1-(1-\mu^+), \nu^-\Big\rangle = \Big\langle \mu^+, \nu^-\Big\rangle. \end{aligned}$$

The proof is completed. □

Theorem 3.7. *(**Boundedness Property**) Let $\tilde{B}_{e_{st}} = (\mu^+_{st}, \nu^-_{st})$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ be a collection of BFSNs. Let $\tilde{B}^-_{e_{st}} = \langle \min_t \min_s\{\mu^+_{st}\}, \max_t \max_s\{\nu^-_{st}\}\rangle$ and $\tilde{B}^+_{e_{st}} = \langle \max_t \max_s\{\mu^+_{st}\}, \min_t \min_s\{\nu^-_{st}\}\rangle$. Then,*

$$\tilde{B}^-_{e_{st}} \le BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \le \tilde{B}^+_{e_{st}}.$$

Proof. Since, $\tilde{B}_{e_{st}} = (\mu^+_{st}, \nu^-_{st})$ is a BFSNs then $\min_t \min_s\{\mu^+_{st}\} \le \mu^+_{st} \le \max_t \max_s\{\mu^+_{st}\}$ which implies that $1 - \max_t \max_s\{\mu^+_{st}\} \le 1 - \mu^+_{st} \le 1 - \min_t \min_s\{\mu^+_{st}\} \Leftrightarrow (1 - \max_t \max_s\{\mu^+_{st}\})^{\theta_s} \le (1-\mu^+_{st} \le (1 - \min_t \min_s\{\mu^+_{st}\})^{\theta_s} \Leftrightarrow 1 - \max_t \max_s\{\mu^+_{st}\} \le \prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s} \le 1 - \min_t \min_s\{\mu^+_{st}\} \Leftrightarrow (1 - \max_t \max_s\{\mu^+_{st}\})^{\sum_{t=1}^{n}\phi_t} \le \prod_{t=1}^{n}(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s})^{\sum_{t=1}^{n}\phi_t} \le (1 - \min_t \min_s\{\mu^+_{st}\})^{\sum_{t=1}^{n}\phi_t} \Leftrightarrow 1 - \max_t \max_s\{\mu^+_{st}\} \le \prod_{t=1}^{n}(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s})^{\sum_{t=1}^{n}\phi_t} \le 1 - \min_t \min_s\{\mu^+_{st}\}$. Therefore,

$$\max_t \max_s\{\mu^+_{st}\} \le 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s}\Big)^{\sum_{t=1}^{n}\phi_t} \le \min_t \min_s\{\mu^+_{st}\}. \tag{8}$$

Again, $\min_t \min_s\{\nu^-_{st}\} \leq \nu^-_{st} \leq \max_t \max_s\{\nu^-_{st}\}$
which finds $(\min_t \min_s\{\nu^-_{st}\})^{\sum_{s=1}^{m}\theta_s} \leq \prod_{s=1}^{m}(\nu^-_{st})^{\theta_s} \leq (\max_t \max_s\{\nu^-_{st}\})^{\sum_{s=1}^{m}\theta_s} \Leftrightarrow \min_t \min_s\{\nu^-_{st}\} \leq \prod_{s=1}^{m}(\nu^-_{st})^{\theta_s} \leq \max_t \max_s\{\nu^-_{st}\} \Leftrightarrow$ $(\min_t \min_s\{\nu^-_{st}\})^{\phi_t} \leq (\prod_{s=1}^{m}(\nu^-_{st})^{\theta_s})^{\phi_t} \leq (\max_t \max_s\{\nu^-_{st}\})^{\phi_t} \Leftrightarrow (\min_t \min_s\{\nu^-_{st}\})^{\sum_{t=1}^{n}\phi_t} \leq \prod_{t=1}^{n}(\prod_{s=1}^{m}(\nu^-_{st})^{\theta_s})^{\phi_t} \leq (\max_t \max_s\{\nu^-_{st}\})^{\sum_{t=1}^{n}\phi_t}$, hence we get,

$$\min_t \min_s\{\nu^-_{st}\} \leq \prod_{t=1}^{n}(\prod_{s=1}^{m}(\nu^-_{st})^{\theta_s})^{\phi_t} \leq \max_t \max_s\{\nu^-_{st}\}. \tag{9}$$

Let $\beta \equiv BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = \langle \mu^+_\beta, \nu^-_\beta \rangle$, then from Eqs. (8) and (9), $\min_t \min_s\{\mu^+_{st}\} \leq \mu^+_\beta \leq \max_t \max_s\{\mu^+_{st}\}$ and $\min_t \min_s\{\nu^-_{st}\} \leq \nu^-_\beta \leq \max_t \max_s\{\nu^-_{st}\}$. Then by definition of score function

$$\Psi(\beta) = \mu^+_\beta + \nu^-_\beta \leq \max_t \max_s\{\mu^+_{st}\} + \min_t \min_s\{\nu^-_{st}\} = \Psi(\tilde{B}^+_{e_{st}})$$

$$\Psi(\beta) = \mu^+_\beta + \nu^-_\beta \geq \min_t \min_s\{\mu^+_{st}\} + \max_t \max_s\{\nu^-_{st}\} = \Psi(\tilde{B}^-_{e_{st}}).$$

Now, there are three cases that can arise:

Case 1. If $\Psi(\tilde{B}_{e_{st}}) < \Psi(\tilde{B}^+_{e_{st}})$ and $\Psi(\tilde{B}_{e_{st}}) > \Psi(\tilde{B}^-_{e_{st}})$, then by comparison of two BFSNs, we have

$$\tilde{B}^-_{e_{st}} \leq BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \leq \tilde{B}^+_{e_{st}}.$$

Case 2. If $\Psi(\tilde{B}_{e_{st}}) = \Psi(\tilde{B}^+_{e_{st}})$, i.e., $\mu^+_\beta + \nu^-_\beta = \max_t \max_s\{\mu^+_{st}\} + \min_t \min_s\{\nu^-_{st}\}$, then by the above inequalities $\mu^+_\beta = \max_t \max_s\{\mu^+_{st}\}$ and $\nu^-_\beta = \min_t \min_s\{\nu^-_{st}\}$.

Therefore,

$$\mathcal{H} = \mu^+_\beta - \nu^-_\beta = \max_t \max_s\{\mu^+_{st}\} - \min_t \min_s\{\nu^-_{st}\} = \mathcal{H}(\tilde{B}^+_{e_{st}}),$$

then by comparison of two BFSNs, we have

$$BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = B^+_{e_{st}}.$$

Case 3. If $\Psi(\tilde{B}_{e_{st}}) = \Psi(\tilde{B}^-_{e_{st}})$, i.e., $\mu^+_\beta + \nu^-_\beta = \min_t \min_s\{\mu^+_{st}\} + \max_t \max_s\{\nu^-_{st}\}$, then by above inequalities $\mu^+_\beta = \min_t \min_s\{\mu^+_{st}\}$ and $\nu^-_\beta = \max_t \max_s\{\nu^-_{st}\}$.

Hence,

$$\mathcal{H} = \mu^+_\beta - \nu^-_\beta = \min_t \min_s\{\mu^+_{st}\} - \max_t \max_s\{\nu^-_{st}\} = \mathcal{H}(\tilde{B}^-_{e_{st}}),$$

then by comparison of two BFSNs, we have

$$BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = B^-_{e_{st}}.$$

Thus, the proof is completed. □

Theorem 3.8. (**Shift-invariance property**) *If $\tilde{B}_e = \langle \mu^+, \nu^- \rangle$ be another BFSN, then*

$$BFSWAA(\tilde{B}_{e_{11}} \oplus \tilde{B}_e, \tilde{B}_{e_{12}} \oplus \tilde{B}_e, \ldots, \tilde{B}_{e_{mn}} \oplus \tilde{B}_e) = BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \oplus \tilde{B}_e.$$

Proof. Since $\tilde{B}_e$ and $\tilde{B}_{e_{st}}$ are BFSNs. Then, we have $\tilde{B}_e \oplus \tilde{B}_{e_{st}} = \Big\langle 1 - (1-\mu^+)(1-\mu^+_{st}), -|\nu^-||\nu^-_{st}| \Big\rangle$. Hence,

$$\begin{aligned}
&BFSWAA(\tilde{B}_{e_{11}} \oplus \tilde{B}_e, \tilde{B}_{e_{12}} \oplus \tilde{B}_e, \ldots, \tilde{B}_{e_{mn}} \oplus \tilde{B}_e) = \bigoplus_{t=1}^{n}\phi_t\Big(\bigoplus_{s=1}^{m}\theta_t(\tilde{B}_{e_{st}} \oplus \tilde{B}_e)\Big)\\
&= \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s}(1-\mu^+)^{\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu^-_{st}|^{\theta_s}|\nu^-|^{\theta_s}\Big)^{\phi_t}\Big\rangle\\
&= \Big\langle 1 - (1-\mu^+)\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -\nu^-\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu^-_{st}|^{\theta_s}\Big)^{\phi_t}\Big\rangle\\
&= \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1-\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu^-_{st}|^{\theta_s}\Big)^{\phi_t}\Big\rangle \oplus \Big\langle \mu^+, \nu^- \Big\rangle\\
&= BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \oplus \tilde{B}_e.
\end{aligned}$$

Hence the result follows. □

Theorem 3.9. *(***Homogeneity property***) For any real number $\lambda > 0$, we have*

$$BFSWAA(\lambda\tilde{B}_{e_{11}}, \lambda\tilde{B}_{e_{12}}, \ldots, \lambda\tilde{B}_{e_{mn}}) = \lambda BFSWA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}).$$

Proof. Let $\tilde{B}_{e_{st}} = (\mu^+_{st}, \nu^-_{st})$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ be a collection of BFSNs and $\lambda > 0$ be any real number. Then, $\lambda\tilde{B}_{e_{st}} = \Big\langle 1 - (1 - \mu^+_{st})^\lambda, -|\nu^-_{st}|^\lambda\Big\rangle$. Thus,

$$\begin{aligned} BFSWAA(\lambda\tilde{B}_{e_{11}}, \lambda\tilde{B}_{e_{12}}, \ldots, \lambda\tilde{B}_{e_{mn}}) &= \Big\langle 1 - \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1 - \mu^+_{st})^{\lambda\theta_s}\Big)^{\phi_t}, -\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu^-_{st}|^{\lambda\theta_s}\Big)^{\phi_t}\Big\rangle \\ &= \Big\langle 1 - \Big(\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1 - \mu^+_{st})^{\theta_s}\Big)^{\phi_t}\Big)^{\lambda}, -\Big(\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}|\nu^-_{st}|^{\theta_s}\Big)^{\phi_t}\Big)^{\lambda}\Big\rangle \\ &= \lambda\ BFSWAA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}). \end{aligned}$$

Hence the proof is completed. □

3.2 Bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator

In this section, we have defined bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator for aggregating bipolar fuzzy soft input arguments.

Definition 3.10. *Let $\tilde{B}_{e_{st}} = (\mu^+_{st}, \nu^-_{st})$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ be a collection of BFSNs and ϕ_t, θ_s are the weight vectors for the parameter e_t's and expert y_s's respectively, satisfying $\phi_t \geq 0$, $\theta_s \geq 0$ such that $\sum_{t=1}^{m}\phi_t = 1$ and $\sum_{s=1}^{n}\theta_s = 1$. Then bipolar fuzzy soft weighted geometric (BFSWGA) operator is a function $BFSWGA : \tilde{B}^n \to \tilde{B}$ such that*

$$BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) = \bigotimes_{t=1}^{n}\Big(\bigotimes_{s=1}^{m}\tilde{B}^{\theta_s}_{e_{st}}\Big)^{\phi_t}.$$

Theorem 3.11. *Then bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator is a function $BFSWGA : \tilde{B}^n \to \tilde{B}$ such that*

$$BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{1n}}) = \Big\langle \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -1 + \prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1 + \nu^-_{st})^{\theta_s}\Big)^{\phi_t}\Big\rangle. \quad (10)$$

Proof. For $m = 1$ and $\theta_1 = 1$, by Definition 3.1, we have

$$\begin{aligned} BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{1n}}) &= \bigotimes_{t=1}^{n}\tilde{B}^{\phi_t}_{e_{st}} = \Big\langle \prod_{t=1}^{n}(\mu^+_{1t})^{\phi_t}, -1 + \prod_{t=1}^{n}(1 + \nu^-_{1t})^{\phi_t}\Big\rangle \\ &= \Big\langle \prod_{t=1}^{n}\Big(\prod_{s=1}^{1}(\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -1 + \prod_{t=1}^{n}\Big(\prod_{s=1}^{1}(1 + \nu^-_{st})^{\theta_s}\Big)^{\phi_t}\Big\rangle. \end{aligned}$$

For $n = 1$ and $\phi_1 = 1$, by Definition 3.10, we get

$$\begin{aligned} BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{21}}, \ldots, \tilde{B}_{e_{m1}}) &= \bigotimes_{t=1}^{n}\tilde{B}^{\theta_s}_{e_{st}} = \Big\langle \prod_{s=1}^{m}(\mu^+_{s1})^{\theta_s}, -1 + \prod_{s=1}^{m}(1 + \nu^-_{s1})^{\theta_s}\Big\rangle \\ &= \Big\langle \prod_{t=1}^{1}\Big(\prod_{s=1}^{m}(\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -1 + \prod_{t=1}^{1}\Big(\prod_{s=1}^{m}(1 + \nu^-_{st})^{\theta_s}\Big)^{\phi_t}\Big\rangle. \end{aligned}$$

Assume that (8) is true for $n = p_1 + 1$, $m = p_2$ and $n = p_1$, $m = p_2 + 1$, then it follows that

$$\bigotimes_{t=1}^{p_1+1}\Big(\bigotimes_{s=1}^{p_2}\tilde{B}^{\theta_s}_{e_{st}}\Big)^{\phi_t} = \Big\langle \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(\mu^+_{st})^{\theta_s}\Big)^{\phi_t}, -1 + \prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(1 + \nu^-_{st})^{\theta_s}\Big)^{\phi_t}\Big\rangle.$$

Also,

$$\bigotimes_{t=1}^{p_1}\Big(\bigotimes_{s=1}^{p_2+1}\tilde{B}_{e_{st}}^{\theta_s}\Big)^{\phi_t}=\Big\langle\prod_{t=1}^{p_1}\Big(\prod_{s=1}^{p_2+1}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t},-1+\prod_{t=1}^{p_1}\Big(\prod_{s=1}^{p_2+1}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\Big\rangle.$$

Now for $n=p_1+1$ and $m=p_2+1$, we obtained

$$\begin{aligned}\bigotimes_{t=1}^{p_1+1}\Big(\bigotimes_{s=1}^{p_2+1}\tilde{B}_{e_{st}}^{\theta_s}\Big)^{\phi_t}&=\bigotimes_{t=1}^{p_1+1}\Big(\bigotimes_{s=1}^{p_2}\tilde{B}_{e_{st}}^{\theta_s}\otimes\tilde{B}_{e_{(p_2+1)t}}^{\theta_{p_2+1}}\Big)^{\phi_t}=\bigotimes_{t=1}^{p_1+1}\Big(\bigotimes_{s=1}^{p_2}\tilde{B}_{e_{st}}^{\theta_s}\Big)^{\phi_t}\bigotimes_{t=1}^{p_1+1}\Big(\tilde{B}_{e_{(p_2+1)t}}^{\theta_{p_2+1}}\Big)^{\phi_t}\Big)\\&=\Big\langle\prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}\otimes\prod_{t=1}^{p_1+1}\Big((\mu_{(p_2+1)t}^{+})^{\theta_{p_2+1}}\Big)^{\phi_t},\\&-1+\prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\otimes-1+\prod_{t=1}^{p_1+1}\Big((1+\nu_{(p_2+1)t}^{-})^{\theta_{p_2+1}}\Big)^{\phi_t}\Big\rangle\\&=\Big\langle\prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2+1}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t},-1+\prod_{t=1}^{p_1+1}\Big(\prod_{s=1}^{p_2+1}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\Big\rangle.\end{aligned}$$

Thus, (8) is true for $n=p_1+1$, $m=p_2+1$, therefore by induction the result is hold for all $m,n\geq 1$.

Since, $-1\leq\nu_{st}^{-}\leq 0\Leftrightarrow 0\leq\prod_{s=1}^{m}(1+\nu_{st}^{-})^{\theta_s}\leq 1\Leftrightarrow 0\leq\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\leq 1\Leftrightarrow -1\leq -1+\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\leq 0$. On the other hand, $0\leq\mu_{st}^{+}\leq 1\Leftrightarrow 0\leq\prod_{s=1}^{m}(\mu_{st}^{+})^{\theta_s}\leq 1\Leftrightarrow 0\leq\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}\leq 1$. Therefore, $-1\leq -1+\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}+\prod_{t=1}^{n}\Big(\prod_{s=1}^{m}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t}\leq 1$.

Thus, aggregated value obtained by BFSWGA operator is again a BFSN. □

Example 3.12. *Let $Y=\{y_1,y_2,y_3,y_4\}$ be the set of experts which are going to narrate the "attractiveness of two-wheeler bikes" under the set of parameters $E=\{e_1=stylish,\ e_2=weight,\ e_3=milage,\ e_4=price\}$. The rating value of the experts is assumed to be given in the form of BFSNs$(B,E)=(\mu_{st}^{+},\nu_{st}^{-})_{4\times 3}$ for all parameters which are given in the following table.*

Table 2: Bipolar fuzzy soft numbers

Experts	e_1	e_2	e_3
y_1	$\langle 0.6,-0.3\rangle$	$\langle 0.7,-0.2\rangle$	$\langle 0.4,-0.2\rangle$
y_2	$\langle 0.5,-0.4\rangle$	$\langle 0.5,-0.3\rangle$	$\langle 0.8,-0.1\rangle$
y_3	$\langle 0.4,-0.1\rangle$	$\langle 0.7,-0.1\rangle$	$\langle 0.5,-0.2\rangle$
y_4	$\langle 0.6,-0.2\rangle$	$\langle 0.6,-0.3\rangle$	$\langle 0.6,-0.3\rangle$

Let $\phi=(0.3,0.2,0.5)^T$ and $\theta=(0.2,0.1,0.3,0.4)^T$ be the weight vectors for the parameters and experts respectively. Then, we get by using Theorem 3.3 as:

$$BFSWG(\tilde{B}_{e_{11}},\tilde{B}_{e_{12}},\ldots,\tilde{B}_{e_{43}})=\Big\langle\prod_{t=1}^{3}\Big(\prod_{s=1}^{4}(\mu_{st}^{+})^{\theta_s}\Big)^{\phi_t},-1+\prod_{t=1}^{3}\Big(\prod_{s=1}^{4}(1+\nu_{st}^{-})^{\theta_s}\Big)^{\phi_t}\Big\rangle$$

$$\begin{aligned}&\Big\langle\Big(\Big\{(0.6)^{0.2}(0.5)^{0.1}(0.4)^{0.3}(0.6)^{0.4}\Big\}^{0.3}\ \Big\{(0.7)^{0.2}(0.5)^{0.1}(0.7)^{0.3}(0.6)^{0.4}\Big\}^{0.2}\ \Big\{(0.4)^{0.2}(0.8)^{0.1}(0.5)^{0.3}(0.6)^{0.4}\Big\}^{0.5}\Big),\\&-1+\ \Big\{(1-0.3)^{0.2}\ (1-0.4)^{0.1}\ (1-0.1)^{0.3}\ (1-0.2)^{0.4}\Big\}^{0.3}\Big\{(1-0.2)^{0.2}\ (1-0.3)^{0.1}\ (1-0.1)^{0.3}\ (1-0.3)^{0.4}\Big\}^{0.2}\\&\Big\{(1-0.2)^{0.2}\ (1-0.1)^{0.1}\ (1-0.2)^{0.3}\ (1-0.3)^{0.4}\Big)^{0.5}\Big\rangle=\Big\langle 0.5518,-0.2261\Big\rangle.\end{aligned}$$

BFSWGA operator satisfies the following properties as similar as BFSWAA operator

- (**Idempotency Property**) If $\tilde{B}_{e_{st}}=\tilde{B}_e=\langle\mu^{+},\nu^{-}\rangle$ for all s,t, then $BFSWGA(\tilde{B}_{e_{11}},\tilde{B}_{e_{12}},\ldots,\tilde{B}_{e_{mn}})=\tilde{B}_e$.

- (**Boundedness Property**) If $\tilde{B}^{-}_{e_{st}} = \langle \min_t \min_s \{\mu^{+}_{st}\}, \max_t \max_s \{\nu^{-}_{st}\}\rangle$ and If $\tilde{B}^{+}_{e_{st}} = \langle \max_t \max_s \{\mu^{+}_{st}\}, \min_t \min_s \{\nu^{-}_{st}\}\rangle$, then $\tilde{B}^{-}_{e_{st}} \leq BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \leq \tilde{B}^{+}_{e_{st}}$.
- (**Shift-invariance Property**) Let $\tilde{B}_e = \langle \mu^{+}, \nu^{-}\rangle$ be another BFSN then $BFSWGA(\tilde{B}_{e_{11}} \otimes \tilde{B}_e, \tilde{B}_{e_{12}} \otimes \tilde{B}_e, \ldots, \tilde{B}_{e_{mn}} \otimes \tilde{B}_e) =$ $BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}}) \otimes \tilde{B}_e$.
- (**Homogeneity Property**) For any real number $\lambda > 0$, we have $BFSWGA(\tilde{B}^{\lambda}_{e_{11}}, \tilde{B}^{\lambda}_{e_{12}}, \ldots, \tilde{B}^{\lambda}_{e_{mn}}) = \Big(BFSWGA(\tilde{B}_{e_{11}}, \tilde{B}_{e_{12}}, \ldots, \tilde{B}_{e_{mn}})\Big)^{\lambda}$.

4 Model for MCDM method using bipolar fuzzy information

In this section, we shall present MCDM using bipolar fuzzy soft weighted arithmetic averaging operator (BFSWAA) and bipolar fuzzy soft weighted geometric averaging (BFSWGA) operator in the environment of bipolar fuzzy soft numbers.

4.1 An approach based on proposed operators

Let $\tilde{A} = \{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_l\}$ be the discrete set of alternatives which is evaluated by the set of m experts $\{y_1, y_2, \ldots, y_m\}$ under the constraints of n parameters $E = \{e_1, e_2, \ldots, e_n\}$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_m)^T$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ denote respectively the weight vectors of the m experts $x'_s s$ and n parameters $e'_t s$ that $\theta_s > 0$, $\theta \in [0,1]$ such that $\sum_{s=1}^{m} \theta_s = 1$, $\phi_t > 0$, $\phi \in [0,1]$ and $\sum_{t=1}^{n} \phi_s = 1$. In order to choose the best l alternates by the preference of n experts in the form of BFSNs $\tilde{B}_{e_{st}} = \langle \mu^{+}_{st}, \nu^{-}_{st}\rangle$ where $-1 \leq \mu^{+}_{st} + \nu^{-}_{st} \leq 1$ and collective over all decision matrix is expressed as $\tilde{M} = (\tilde{B}_{e_{st}})_{m\times n}$. By these preference values of the experts, the aggregated BFSN $\tilde{B}_{e_k}$ for the alternatives $\tilde{p}_k$ $(k = 1, 2, \ldots, l)$ is $\tilde{B}_{e_k} = \langle \mu^{+}_k, \nu^{-}_k\rangle$ by applying weighted averaging or geometric averaging operators which is given in Eq. (5) and Eq. (8). Ranking order of the alternatives is determine based on the score function of the aggregated values of BFSNs $\tilde{B}_{e_k}$ $(k = 1, 2, \ldots, l)$.

In the following algorithm, we solve the MCDM problem with bipolar fuzzy information using BFSWAA and BFSWGA operators.

Step 1. Collect all the information in the form of bipolar fuzzy soft matrix $B = \langle \mu^{+}_{st}, \nu^{-}_{st}\rangle$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ related to each alternative under different parameters e_k $(k = 1, 2, \ldots, l)$ as

$$\tilde{B}_{m\times n} = M = \begin{bmatrix} (\mu^{+}_{11}, \nu^{-}_{11}) & (\mu^{+}_{12}, \nu^{-}_{12}) & \cdots & (\mu^{+}_{1n}, \nu^{-}_{1n}) \\ (\mu^{+}_{21}, \nu^{-}_{21}) & (\mu^{+}_{22}, \nu^{-}_{22}) & \cdots & (\mu^{+}_{2n}, \nu^{-}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu^{+}_{m1}, \nu^{-}_{m1}) & (\mu^{+}_{m2}, \nu^{-}_{m2}) & \cdots & (\mu^{+}_{mn}, \nu^{-}_{mn}) \end{bmatrix}$$

Step 2. To normalize the aggregated decision matrix by transforming values of benefit type (B) into cost (C) type by using the formula depicted in [48]:

$$g_{ij} = \begin{cases} \tilde{B}^{c}_{e_{st}}, & \text{if} e_t \in \tilde{B} \\ \tilde{B}_{e_{st}}, & \text{if} e_t \in \tilde{C} \end{cases}$$

where $\tilde{B}^{c}_{e_{st}} = \langle 1 - \mu^{+}_{st}, |\nu^{-}_{st}| - 1\rangle$ is the complement of $\tilde{B}_{e_{st}} = \langle \mu^{+}_{st}, \nu^{-}_{st}\rangle$.

Step 3. Aggregate the BFSNs $\tilde{B}_{e_{st}}$ $(s = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ for each alternatives A_k $(k = 1, 2, \ldots, l)$ into collective decision matrix Ψ_k using BFSWAA or BFSWGA operators.

Step 4. Using Eq. (1) we get the score value of Ψ_k $(k = 1, 2, \ldots, l)$ for each alternatives A_k $(k = 1, 2, \ldots, l)$.

Step 5. Rank all the alternative A_k $(k = 1, 2, \ldots, l)$ in order to choose the best one(s) in accordance with Ψ_k $(k = 1, 2, \ldots, l)$.

Step 6. End.

5 Numerical example

In the above described decision-making method has been demonstrated with a practical example of recruitment of a bank manager an Indian bank. The panel for five experts, m_1, m_2, m_3, m_4, m_5 whose weight vector is $\theta =$

$(0.2, 0.15, 0.2, 0.3, 0.15)^T$, will give their judgement based on the performance of four candidates C_1, C_2, C_3, C_4 under the parameters $E = \{computerknowledge(e_1),\ Decisiveness(e_2), Flexibility(e_3), Politeness(e_4), Hardworking(e_5)\}$ with weight vector $\phi = (0.2, 0.1, 0.3, 0.15, 0.25)^T$. Then, we utilize the developed method to get most desirable candidate(s).

5.1 By BFSWAA operator

The steps of the proposed approach performed and their corresponding details are reviewed as follows:

Step 1. The given candidates are being evaluated by five experts to give their grades in terms of BFSNs and are found in Tables 3, 4, 5, 6, respectively for each candidate.

Step 2. All the parameters are of the same type, so, there is no requirement for normalization.

Table 3: Bipolar fuzzy soft matrix for the candidate C_1

$Experts$	e_1	e_2	e_3	e_4	e_5
m_1	$\langle 0.6, -0.2\rangle$	$\langle 0.5, -0.5\rangle$	$\langle 0.6, -0.3\rangle$	$\langle 0.3, -0.5\rangle$	$\langle 0.3, -0.4\rangle$
m_2	$\langle 0.6, -0.4\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.5, -0.4\rangle$	$\langle 0.4, -0.6\rangle$	$\langle 0.4, -0.1\rangle$
m_3	$\langle 0.7, -0.1\rangle$	$\langle 0.4, -0.3\rangle$	$\langle 0.3, -0.1\rangle$	$\langle 0.7, -0.3\rangle$	$\langle 0.6, -0.2\rangle$
m_4	$\langle 0.4, -0.5\rangle$	$\langle 0.7, -0.2\rangle$	$\langle 0.2, -0.6\rangle$	$\langle 0.6, -0.1\rangle$	$\langle 0.5, -0.1\rangle$
m_5	$\langle 0.5, -0.2\rangle$	$\langle 0.6, -0.1\rangle$	$\langle 0.6, -0.2\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.4, -0.2\rangle$

Table 4: Bipolar fuzzy soft matrix for the candidate C_2

$Experts$	e_1	e_2	e_3	e_4	e_5
m_1	$\langle 0.3, -0.4\rangle$	$\langle 0.8, -0.1\rangle$	$\langle 0.7, -0.1\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.2, -0.3\rangle$
m_2	$\langle 0.5, -0.1\rangle$	$\langle 0.5, -0.2\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.6, -0.1\rangle$	$\langle 0.2, -0.1\rangle$
m_3	$\langle 0.2, -0.1\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.5, -0.4\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.5, -0.2\rangle$
m_4	$\langle 0.7, -0.2\rangle$	$\langle 0.5, -0.1\rangle$	$\langle 0.6, -0.2\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.7, -0.1\rangle$
m_5	$\langle 0.5, -0.2\rangle$	$\langle 0.5, -0.3\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.6, -0.4\rangle$

Table 5: Bipolar fuzzy soft matrix for the candidate C_3

$Experts$	e_1	e_2	e_3	e_4	e_5
m_1	$\langle 0.4, -0.3\rangle$	$\langle 0.8, -0.1\rangle$	$\langle 0.5, -0.2\rangle$	$\langle 0.6, -0.1\rangle$	$\langle 0.2, -0.3\rangle$
m_2	$\langle 0.5, -0.1\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.5, -0.2\rangle$
m_3	$\langle 0.2, -0.1\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.5, -0.1\rangle$	$\langle 0.5, -0.1\rangle$
m_4	$\langle 0.7, -0.2\rangle$	$\langle 0.5, -0.2\rangle$	$\langle 0.2, -0.1\rangle$	$\langle 0.6, -0.2\rangle$	$\langle 0.6, -0.2\rangle$
m_5	$\langle 0.5, -0.3\rangle$	$\langle 0.5, -0.4\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.4, -0.1\rangle$	$\langle 0.7, -0.2\rangle$

Step 3. The opinion of experts for each candidate C_k $(k = 1, 2, 3, 4)$ are aggregated by using eq.(5) given as follows: $\Delta_1 = \langle 0.4918, -0.2326\rangle$, $\Delta_2 = \langle 0.5154, -0.1700\rangle$, $\Delta_3 = \langle 0.4800, -0.1656\rangle$ and $\Delta_4 = \langle 0.4319, -0.1942\rangle$.

Step 4. The values of score functions are: $\Psi(\Delta_1) = 0.2592$, $\Psi(\Delta_2) = 0.3454$, $\Psi(\Delta_3) = 0.3144$ and $\Psi(\Delta_4) = 0.2377$.

Step 5. Ranking all the candidates C_k $(k = 1, 2, 3, 4)$ in accordance with the value of the score functions $\Psi(\Delta_k)$ $(k = 1, 2, 3, 4)$ of the overall bipolar fuzzy soft numbers as $C_2 \succ C_3 \succ C_1 \succ C_4$.

Step 6. Therefore, C_2 is the most desirable candidate for the post.

5.2 By using BFSWGA operator

If we apply BFSWGA operator on the proposed problem for the selection of appropriate candidate(s) that follows the following steps:

Table 6: Bipolar fuzzy soft matrix for the candidate C_4

Experts	e_1	e_2	e_3	e_4	e_5
m_1	$\langle 0.4, -0.1\rangle$	$\langle 0.5, -0.4\rangle$	$\langle 0.5, -0.2\rangle$	$\langle 0.5, -0.1\rangle$	$\langle 0.2, -0.3\rangle$
m_2	$\langle 0.5, -0.1\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.3, -0.2\rangle$
m_3	$\langle 0.5, -0.3\rangle$	$\langle 0.5, -0.1\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.2, -0.2\rangle$	$\langle 0.5, -0.2\rangle$
m_4	$\langle 0.6, -0.2\rangle$	$\langle 0.4, -0.5\rangle$	$\langle 0.3, -0.2\rangle$	$\langle 0.7, -0.2\rangle$	$\langle 0.3, -0.1\rangle$
m_5	$\langle 0.5, -0.3\rangle$	$\langle 0.4, -0.6\rangle$	$\langle 0.4, -0.2\rangle$	$\langle 0.3, -0.1\rangle$	$\langle 0.6, -0.3\rangle$

Step 3. The aggregated values for each candidates C_k $(k = 1, 2, 3, 4)$ using BFSWGA operator are follows from Eq. (10):
$\Delta_1 = \langle 0.4432, -0.3084\rangle$, $\Delta_2 = \langle 0.4515, -0.1999\rangle$, $\Delta_3 = \langle 0.4224, -0.1825\rangle$ and $\Delta_4 = \langle 0.3960, -0.2204\rangle$.

Step 4. The values of score functions are: $\Psi(\Delta_1) = 0.1384$, $\Psi(\Delta_2) = 0.2516$, $\Psi(\Delta_3) = 0.2399$ and $\Psi(\Delta_4) = 0.1756$.

Step 5. Ranking all the candidates C_k $(k = 1, 2, 3, 4)$ in accordance with the values of the score functions $\Psi(\Delta_k)$ $(k = 1, 2, 3, 4)$ of the overall bipolar fuzzy soft numbers as $C_2 \succ C_3 \succ C_4 \succ C_1$.

Step 6. Hence, C_2 is the most appropriate candidate for the post.

From the above analysis, it is clear that although overall rating values of the alternatives are different by using two operators, but ranking order of the alternatives are similar, the most desirable alternative is C_2.

Table 7: Aggregated bipolar fuzzy soft matrix for the candidates

Experts	C_1	C_2	C_3	C_4
m_1	$\langle 0.4884, -0.3378\rangle$	$\langle 0.5161, -0.1737\rangle$	$\langle 0.4854, -0.2018\rangle$	$\langle 0.4168, -0.1861\rangle$
m_2	$\langle 0.4680, -0.2805\rangle$	$\langle 0.4256, -0.1320\rangle$	$\langle 0.4211, -0.1741\rangle$	$\langle 0.3605, -0.1741\rangle$
m_3	$\langle 0.5545, -0.1565\rangle$	$\langle 0.4251, -0.2000\rangle$	$\langle 0.4092, -0.1320\rangle$	$\langle 0.4333, -0.2024\rangle$
m_4	$\langle 0.4513, -0.2531\rangle$	$\langle 0.6182, -0.1569\rangle$	$\langle 0.5246, -0.1625\rangle$	$\langle 0.4573, -0.1843\rangle$
m_5	$\langle 0.5081, -0.1682\rangle$	$\langle 0.4747, -0.2012\rangle$	$\langle 0.5223, -0.1702\rangle$	$\langle 0.4650, -0.2415\rangle$

Table 8: Comparison analysis with the existing method

methods	$\Psi(\Delta_1)$,	$\Psi(\Delta_2)$,	$\Psi(\Delta_3)$,	$\Psi(\Delta_4)$	Ranking order
Proposed method	0.2592,	0.3454,	0.3144,	0.2377,	$C_2 \succ C_3 \succ C_1 \succ C_4$
Wei et al. [42]	0.6284,	0.6710,	0.5260,	0.6185,	$C_2 \succ C_1 \succ C_4 \succ C_3$

6 Comparative analysis

To compare the proposed work with the existing approach, an analysis has been made based on aggregation operator (See, [42]). In that reason, the different parameters of bipolar fuzzy soft numbers are aggregated by using weighted averaging operator corresponding to the weighted vector $(0.2, 0.1, 0.3, 0.15, 0.25)^T$ and then obtained aggregated bipolar fuzzy soft matrix for different candidates C_k $(k = 1, 2, 3, 4)$ are given in Table 7. From this evaluated matrix, a comparative study has been established with the only existing work based on Hamacher aggregation operator on bipolar fuzzy numbers, developed by Wei et al. [42] for each candidate are shown in Table 8. From this table, we that the candidate C_2 is most appropriate for the selected post. The method of this proposed utilizes paper advance technique to compare the existing works of [1, 51] where a decision-making method has been developed based on some soft algebraic operations in bipolar fuzzy soft environment but the present paper contains a decision-making method based on aggregating bipolar fuzzy soft arguments in the environment of bipolar fuzzy soft numbers. The advantages of this paper is that it facilitates the description of the real-world problems situation with the help of their parameterizations property. Therefore, proposed method can be utilized to solve decision problems instead of other existing operators in the environment of bipolar soft sets.

7 Conclusions

In this article, we have studied multi-criteria decision-making problem using in the environment of bipolar fuzzy soft information. We have introduced two new operators namely BFSWAA and BFSWGA operators in bipolar fuzzy soft environment. The different feature of these recommend operators are deliberate. For this purpose, firstly some algebraic structures of two BFSNs and their operational rules and aggregation operators have been proposed. Some properties of these two kinds of operators have been established. A decision-making problem has been studied based on BFSWAA and BFSWGA operators under the environment of bipolar fuzzy soft information. The main advantage of these operators is that they are able to make smooth description of the real-world problems by the use of parameterization factors. Ultimately, a realistic instance for the selection of the best candidate for a post of bank's manager is provided to develop a strategy and in accordance with expounding the utility then effectiveness of the proposed method. In future, the application of our proposed model can be applied to decision-making theory, risk evaluation and other domains under different uncertain environments.

Acknowledgement

We would like to thank the anonymous reviewers for their insightful and constructive comments and suggestions that have been helpful for providing a better version of the present work.

References

[1] S. Abdullaha, M. Aslamb, K. Ullaha, *Bipolar fuzzy soft sets and its applications in decision-making problem*, Journal of Intelligent & Fuzzy Systems, **27** (2014), 729–742.

[2] M. Agarwal, K. K. Biswas, H. Hanmandlu, *Generalized intuitionistic fuzzy soft sets with applications in decision-making*, Applied Soft Computing, **13** (2013), 3552–3566.

[3] M. Akram, F. Feng, A. B. Saeid, V. Leoreanu-Fotea, *A new multiple criteria decision-making method based on bipolar fuzzy soft graphs*, Iranian Journal of Fuzzy Systems, **15**(4) (2018), 73–92.

[4] M. Ali, L. Son, I. Deli, N. D. Tien, *Bipolar Neutrosophic Soft Sets and Applications in Decision Making*, Journal of Intelligent & Fuzzy Systems, **33**(6) (2017), 4077–4087.

[5] M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, *On some new operationsin in soft set theory*, Computers & Mathematics with Applications **57** (2009), 1547–1553.

[6] S. Alkhazaleh, A. R. Salleh, *Fuzzy soft expert set and its application*, Applied Mathematics, **5** (2014), 1349–1368.

[7] R. Arora, H. Garg, *Prioritized averaging/geometric aggregation operators under the intuitionistic fuzzy soft set environment*, Scientia Iranica E, **25**(1) (2018), 466–482.

[8] K. T. Atanassov, *On intuitionistic fuzzy sets theory*, Studies in Fuzziness and Soft Computing, **283**, Springer-Verlag, Berlin Heidelberg, 1999.

[9] K. V. Babitha, S. J. John, *Hesistant fuzzy soft sets*, Journal of New Results in Science, **3** (2013), 98-107.

[10] G. Beliakov, A. Pradera, T. Calvo, *Aggregation Functions: A Guide for Practitioners*, Springer, Heidelberg, Berlin, New York, 2007.

[11] G. Beliakov, H. Bustince, D. P. Goswami, U. K. Mukherjee, N. R. Pal, *On averaging operators for Atanassovs intuitionistic fuzzy sets*, Information Sciences, **181** (2011), 1116–1124.

[12] N. Cagman, F. Citak, S. Enginoglu, *Fuzzy parameterized fuzzy soft set theory and its applications*, Turkish Journal of Fuzzy System, **1**(1) (2001), 21–35.

[13] N. Cagman, I. Deli, *Intuitionistic fuzzy parameterized soft set theory and its decision- making*, Applied Soft Computing, **28** (2015), 109–113.

[14] G. Deschrijver, E. E. Kerre, *A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms*, Notes on Intuitionistic Fuzzy Sets, **8** (2002), 19–27.

[15] Y. Du, P. Liu, *Extended fuzzy VIKOR method with intuitionistic trapezoidal fuzzy numbers*, Information Intelligent Interdisciplinary Journal, **14**(8) (2011), 2575–2583.

[16] G. Fink, M. Yolles, *Collective emotion regulation in an organization a plural agency with cognition and affect*, Journal of Organizational Change Management, **28**(5) (2015), 832–871.

[17] H. Garg, R. Arora, *Generalized and group-based generalized intuitionistic fuzzy soft sets with applications in decision-making*, Applied Intelligence, **48**(2) (2018), 343–356.

[18] H. Gao, G. W. Wei, Y. H. Huang, *Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision-making*, IEEE Access, **6**(1) (2018), 11508–11522.

[19] Z. Gul, *Some bipolar fuzzy aggregations operators and their applications in multicriteria group decision-making*, M.Phil Thesis, 2015.

[20] Y. Han, P. Shi, S. Chen, *Bipolar-valued rough fuzzy set and its applications to decision information system*, IEEE Transactions on Fuzzy Systemst, **23**(6) (2015), 2358–2370.

[21] Y. Jiang, Y. Tang, Q. Chen, H. Liu, J. Tang, *Interval-valued intuitionistic fuzzy soft sets and their properties*, Computers & Mathematics with Applications, **60**(3) (2010), 906–918.

[22] F. Karaaslan, S. Karatas, *OR and AND-products of if p-intuitionistic fuzzy soft sets and their applications in decision-making*, Journal of Intelligent Fuzzy Systems, **31**(3) (2016), 1427–1434.

[23] F. Karaaslan, I. Ahmad, A. Ullah, *Bipolar soft groups*, Journal of Intelligent Fuzzy Systems, **31**(1) (2016), 651–662.

[24] F. Karaaslan, *Bipolar soft rough relations*, Communications Faculty of Sciences University of Ankara Series A1, Mathemeticas and Stattistics, **65**(1) (2016), 105–126.

[25] F. Karaaslan, N. Cagman, *Bipolar soft rough sets and their applications in decision-making*, Afrika Matematika, **29**(5-6) (2018), 823–839.

[26] A. Khalid, M. Abbas, *Distance measures and operations in intuitionistic and interval-valued intuitionistic fuzzy soft set theory*, International Journal of Fuzzy Systems, **17**(3) (2015), 490–497.

[27] A. Khalid, I. Beg, *Incomplete interval-valued hesitant fuzzy preference relations in decision-making*, Iranian Journal of Fuzzy Systems, **15**(6) (2018), 107–120.

[28] P. P. Li, *The global implications of the indigenous epistemological system from the east: how to apply Yin-Yang balancing to paradox management*, Cross Cultural Management An International Journal, **23**(1) (2016), 42–47.

[29] M. Lu, J. R. Busemeyer, *Do traditional chinese theories of Yi Jing (Yin-Yang and Chinese Medicine) go beyond western concepts of mind and matter*, Mind Matter, **12**(1) (2014), 37–59.

[30] M. Lu, G. W. Wei, F. E. Alsaadi, T. Hayat, A. Alsaedi, *Bipolar* 2*-tuple linguistic aggregation operators in multiple attribute decision-making*, International Journal of Fuzzy Systems, **33** (2017), 1197–1207.

[31] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy soft sets*, The Journal of Fuzzy Mathematics, **9**(3) (2001), 589–602.

[32] P. K. Maji, R. Biswas, A. R. Roy, *Intuitionistic fuzzy soft sets*, The Journal of Fuzzy Mathematics, **9**(3) (2001), 677–692.

[33] D. Molodtsov, *Soft set theory-first results*, Computers & Mathematics with Applications **27**(4-5) (1999), 19–31.

[34] J. J. Peng, J. Q. Wang, X. H. Wu, H. Y. Zhang, X. H. Chen, *The fuzzy cross-entropy for intuitionistic hesitant fuzzy sets and its application in multi-criteria decision-making*, International Journal of Systems Science, **46**(13) (2015), 2335–2350.

[35] Y. A. Qudah, N. Hassan, *Bipolar fuzzy soft expert set and its application in decision-making*, International Journal of Applied Decision Sciences, **10**(2) (2017), 175–191.

[36] S. Samanta, M. Pal, *Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs*, The Journal of Fuzzy Mathematics , **22**(2) (2014), 253–262.

[37] G. Selvachandran, X. Peng, *A modified TOPSIS method based on vague parameterized vague soft sets and its application to supplier selection problems*, Neural Computing and Applications, (2018), DOI:10.1007/s00521-018-3409-1.

[38] P. K. Singh, C. A. Kumar, *Bipolar fuzzy graph representation of concept lattice*, Information Sciences, **288** (2014), 437–488.

[39] P. Wang, X. H. Xu, J. Q. Wang, C. G. Cai, *Some new operation rules and a new ranking method for interval-valued intuitionistic linguistic numbers*, Journal of Intelligent & Fuzzy Systems, **32**(1) (2017), 1069–1078.

[40] L. Wang, H. Y. Zhang, J. Q. Wang, *Frank Choquet Bonferroni mean operators of bipolar neutrosophic sets and their application to multi-criteria decision-making problems*, International Journal of Fuzzy Systems, **20**(1) (2018), 13–28.

[41] G. W. Wei, *Some geometric aggregation functions and their application to dynamic multiple attribute decision-making in intuitionistic fuzzy setting*, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems, **17** (2009), 179–196.

[42] G. W. Wei, F. E. Alsaadi, T. Hayat, A. Alsaedi, *Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making*, International Journal of Fuzzy Systems, **20**(1) (2018), 1–12.

[43] G. W. Wei, H. Gao, J. Wang, Y. H. Huang, *Research on risk evaluation of enterprise human capital investment with interval-valued bipolar 2-tuple linguistic information*, IEEE Access, **6** (2018), 35697–35712.

[44] G. W. Wei, F. E. Alsaadi, T. Hayat, A. Alsaedi, *Hesitant bipolar fuzzy aggregation operators in multiple attribute decision-making*, Journal of Intelligent & Fuzzy Systems, **33**(2) (2017), 1119–1128.

[45] X. R. Xu, G. W. Wei, *Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision-making*, International Journal of Knowledge-based and Intelligent Engineering Systems, **21**(3) (2017), 155–164.

[46] Z. S. Xu, R. R. Yager, *Some geometric aggregation operators based on intuitionistic fuzzy sets*, International Journal of General Systems, **35** (2006), 417–433.

[47] Z. S. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Transactions on Fuzzy Systems, **15** (2007), 1179–1187.

[48] Z. S. Xu, H. Hu, *Projection models for intuitionistic fuzzy multiple attribute decision-making*, International Journal of Information Technology & Decision Making, **9** (2010), 267-280.

[49] L. H. Yang, S. G. Li, W. H. Yang, Y. Lu, *Notes on bipolar fuzzy graphs*, Information Sciences, **242** (2013), 113–121.

[50] X. Yang, T. Y. Lin, J. Yang, Y. Li, D. Yu, *Combination of interval-valued fuzzy set and soft set*, Computers & Mathematics with Applications, **58**(3) (2009), 521–527.

[51] Y. Yang, X. Peng, H. Chen, L. Zeng, *A decision-making approach based on bipolar multi-fuzzy soft set theory*, International Journal of Fuzzy Systems, **27**(4) (2014), 1861–1872.

[52] J. Ye, *Multicriteria fuzzy decision-making method based ona novel accuracy function under interval-valued intuitionistic fuzzy environment*, Expert Systems with Applications, **36**(2) (2009), 899–6902.

[53] L. A. Zadeh, *Fuzzy sets*, Information Control, **8** (1965), 338–353.

[54] W. R. Zhang, *Bipolar fuzzy sets and relations: A computational frame work for cognitive modelling and multiagent decision analysis*, In: Proceedings of IEEE Conference, (1994), 305–309.

[55] W. R. Zhang, *Bipolar fuzzy sets*, In: Proceedings of FUZZYIEEE, (1998), 835–840.

[56] W. R. Zhang, L. Zhang, *Bipolar logic and bipolar fuzzy logic*, Informatio Sciences, **165**(3-4) (2004), 265–287.

[57] W. R. Zhang, K. A. Pandurangi, K. E. Peace, Y. Zhang, Z. Zhao, *Mental squares-A generic bipolar support vector machine for psychiatric disorder classification, diagnostic analysis and neurobiological data mining*, International Journal of Data Mining and Bioinformatics, **5**(5) (2011), 532–572.

[58] W. R. Zhang, H. J. Zhang, Y. Shi, S. S. Chen, *Bipolar linear algebra and YinYang-N-element cellular networks for equilibrium-based biosystem simulation and regulation*, Journal of Biological Systems, **17**(4) (2009), 547–576.

[59] W. R. Zhang, *Bipolar quantum logic gates and quantum cellular combinatorics a logical extension to quantum entanglement*, Journal of Quantum Information Science, **3**(2) (2013), 93–105.

[60] W. R. Zhang, K. E. Peace, *Causality is logically definable-toward an equilibrium-based computing paradigm of quantum agent and quantum intelligence*, Journal of Quantum Information Science, **4** (2014), 227–268.

[61] W. R. Zhang, *G-CPT symmetry of quantum emergence and submergence an information conservational multiagent cellular automata unification of CPT symmetry and CP violation for equilibrium-based many world causal analysis of quantum coherence and decoherence*, Journal of Quantum Information Science, **6**(2) (2016), 62–97.