

## Weak hyper semi-quantales and weak hypervalued topological spaces

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### Abstract

The purpose of this paper is to construct a weak hyper semi-quantale as a generalization of the concept of semi-quantale and used it as an appropriate hyperlattice-theoretic basis to formulate new lattice-valued topological theories. Based on such weak hyper semi-quantale, we aim to construct the notion of a weak hypervalued-topology as a generalized form of the so-called lattice-valued topology (or many-valued topology). Some properties of weak hyper semi-quantales and weak hypervalued-topologies will be studied. An adjunction between the category of weak hyper semi-quantales and the category of weak hypervalued quasi-topological spaces will be established.

*Keywords:* Hypergroupoids, semihypergroups, semi-quantales, lattice-valued topology.

## 1 Introduction

The study of quantale-like structures goes back up to the 1930's, when Ward and Dilworth [23] started their research on residuated structures, motivated by ring-theoretic considerations. The term quantale was coined much later in 1986 by Mulvey [16], in the ambitious aim of providing a possible common lattice-theoretic setting for constructive foundations for quantum mechanics, as well as a non-commutative analogue of the maximal spectrum of a  $C^*$ -algebra. Quantales used as a lattice-theoretic basis for some topological theories [22, 12, 13, 24].

In 2007, Rodabaugh [21] introduced the notion of semi-quantale as a generalization of quantale and used it as an appropriate lattice-theoretic basis to formulate powerset, topological and fuzzy topological theories. The notion of semi-quantale provides a useful tool to gather various lattice-theoretic notions, which have been extensively studied in non-commutative structures, it has a wide applications, especially in studying non-commutative lattice-valued quasi-topology [6, 7, 8].

The semi-quantale-valued topology [21] appears in two main forms:  $L$ -quasi-topology and  $L$ -topology. The former generalizes the many-valued topology (or  $L$ -valued topology) in the sense of Höhle [11] by dropping the first axiom ( $O1$ ) of the  $L$ -valued topology (see [11], p.150), while the latter replaces the axiom ( $O1$ ) of the  $L$ -valued topology by another axiom that  $L$ -topology contains the the unital element of the unital semi-quantale powerset. The semi-quantale-valued topology can be considered as a generalization of many other  $L$ -topologies [20, 10, 25, 13].

In the groupoid of the lattice-theoretic basis of the above  $L$ -topologies, the composition of two elements is an element, while in a hypergroupoid, the composition of two elements is a nonempty set. So if the binary operation of the above semi-quantales is taken to be multivalued, then we arrive at a weak hyper semiquantales, which represent a natural extension of semi-quantales, and therefore a hyper-valued topology which represents a generalization of all the above lattice-valued topologies [20, 10, 21, 25, 6, 13].

Hypergroupoids (resp., Semihypergroups) represent a natural extension of groupoids (resp., semigroups) and they were introduced by the French mathematician Marty [15] at 8th Congress of Scandinavian Mathematicians in 1934. Till now, the hyperstructures are studied from the theoretical point of view and for their applications in several domains of mathematics and computer science (see [1, 2, 3, 9, 4]).

The purpose of this paper is to construct a weak hyper semi-quantale as a generalization of Rodabaugh's semi-quantale and used it as an appropriate hyperlattice-theoretic basis to formulate new topological theories. Based on

such weak hyper semi-quantale, we aim to construct the notion of a weak hypervalued-(quasi-)topology as a generalized form of the lattice-valued (quasi-)topology [21]. Some properties of weak hyper semi-quantales and weak hypervalued-(quasi-)topologies will be studied. Finally, an adjunction between the category of weak hyper semi-quantales and the category of weak hypervalued (quasi-)topological spaces will be established.

## 2 Preliminaries

Let  $(L, \leq)$  be a complete lattice where the join and meet operations, the top and bottom elements of  $(L, \leq)$  are denoted by the usual notations  $\bigvee, \bigwedge, \top_L$  and  $\perp_L$  respectively.

A semi-quantale  $(L, \leq, \bigvee, \otimes)$  [21] is a complete lattice  $(L, \leq)$  equipped with a binary operation  $\otimes : L \times L \rightarrow L$ , with no additional assumptions. The category  $\text{SQuant}$  comprises all semi-quantales together with semi-quantale morphisms (i.e., mappings preserving  $\otimes$  and arbitrary  $\bigvee$ ). A subset  $K$  of a semi-quantale  $(L, \leq, \bigvee, \otimes)$  is called a subsemi-quantale [21] iff it is closed under the binary operation  $\otimes$  and the arbitrary  $\bigvee$ .

Let  $X$  be a non-empty set and let  $L \in |\text{SQuant}|$ . An  $L$ -fuzzy subset (or  $L$ -subset) of  $X$  is a mapping  $A : X \rightarrow L$ . The family of all  $L$ -fuzzy subsets on  $X$  will be denoted by  $L^X$ . The smallest element and the largest element in  $L^X$  are denoted by  $\underline{\perp}$  and  $\underline{\top}$ , respectively. The algebraic and lattice-theoretic structure can be extended from the semi-quantale  $(L, \leq, \bigvee, \otimes)$  to  $L^X$  pointwisely. Then  $L^X$  is again a semi-quantale with respect to the multiplication  $\otimes$  and the arbitrary sups.

For an ordinary mapping  $f : X \rightarrow Y$ , one can define the mappings  $f_L^{\rightarrow} : L^X \rightarrow L^Y$  and  $f_L^{\leftarrow} : L^Y \rightarrow L^X$  by  $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\}$  for every  $A \in L^X$  and every  $y \in Y$ ,  $f_L^{\leftarrow}(B) = B \circ f$  for every  $B \in L^Y$ , respectively. For more details, we refer to [20, 21].

For a fixed  $L \in |\text{SQuant}|$  and a set  $X$ , an  $L$ -quasi-topology on  $X$  [21] is a subsemi-quantale  $\tau$  of  $L^X = (L^X, \leq, \otimes)$ , i.e., the following axioms are satisfied:

(T<sub>1</sub>) For all  $A, B \in L^X, A, B \in \tau \Rightarrow A \otimes B \in \tau$ .

(T<sub>2</sub>) For all  $\{A_j : j \in J\} \subseteq L^X, \{A_j : j \in J\} \subseteq \tau \Rightarrow \bigvee_j A_j \in \tau$ .

**Definition 2.1.** [15] Let  $H$  be a non-empty set and let  $P^*(H)$  be the set of all non-empty subsets of  $H$ . A hyperoperation on  $H$  is defined to be a map  $\tilde{\odot} : H \times H \rightarrow P^*(H)$ . The couple  $(H, \tilde{\odot})$  is called a hypergroupoid.

In the above definition, if  $A$  and  $B$  are two non-empty subsets of  $H$  and  $x \in H$ , then we write  $A \tilde{\odot} B = \bigcup_{a \in A, b \in B} a \tilde{\odot} b$ ,  $x \tilde{\odot} A = \{x\} \tilde{\odot} A$  and  $A \tilde{\odot} x = A \tilde{\odot} \{x\}$ .

**Definition 2.2.** [5] A semihypergroup is a hypergroupoid  $(H, \tilde{\odot})$  provided that, for all  $a, b, c \in H$ ,  $a \tilde{\odot} (b \tilde{\odot} c) = (a \tilde{\odot} b) \tilde{\odot} c$ .

**Remark 2.3.** [5] Every semigroup is a semihypergroup.

**Definition 2.4.** [14] Let  $(L, \leq, \bigvee, \bigwedge, \top, \perp)$  be a lattice. A hyper operation  $\tilde{\odot} : L \times L \rightarrow P^*(L)$  is said to be a hyper-t-norm (resp., hyper-t-conorm) if it is commutative, associative and satisfies for all  $a, b, c \in L$  the following

(1)  $a \in a \tilde{\odot} \top$  (resp.,  $a \in a \tilde{\odot} \perp$ ),

(2)  $a \leq b \Rightarrow a \tilde{\odot} c \leq b \tilde{\odot} c$ ,

where  $\leq$  is an order relation on the range of  $\tilde{\odot}$ .

## 3 Weak hyper semi-quantales

In this section, we aim to construct the notion of weak hyper semi-quantale as a generalization of the concept of Rodabaugh's semi-quantale and use it as an appropriate hyperlattice-theoretic basis to construct the notion of a weak hypervalued-(quasi-)topology. Some properties of weak hyper semi-quantales will be studied.

**Definition 3.1.** A weak hyper semi-quantale is a quadruple  $(L, \leq, \bigvee, \tilde{\otimes})$  such that

(1)  $(L, \leq, \bigvee)$  is a complete lattice;

(2)  $(L, \tilde{\otimes})$  is a hypergroupoid.

**Remark 3.2.** Every semi-quantale is a weak hyper semi-quantale.

**Definition 3.3.** Let  $(L, \leq, \bigvee, \tilde{\otimes})$  and  $(M, \leq, \bigvee, \tilde{\odot})$  be two weak hyper semi-quantales. A fuction  $f : L \rightarrow M$  is called a weak hyper semi-quantale homomorphism if it preserves the arbitrary sups and satisfies that  $f(a\tilde{\otimes}b) \subseteq f(a)\tilde{\odot}f(b)$ , for all  $a, b \in L$ .

The category WHySQuant comprises all weak hyper semi-quantales together with their weak hyper semi-quantale morphisms.

**Definition 3.4.** A weak hyper semi-quantales  $(L, \leq, \bigvee, \tilde{\otimes})$  is called:

- (1) A left (resp., right) unital weak hyper semi-quantale if  $(L, \tilde{\otimes}, e)$  is a hypergroupoid with  $e$  as an identity i.e.,  $a \in e\tilde{\otimes}a$  (resp.,  $a \in a\tilde{\otimes}e$ ) for all  $a \in L$ . The element  $e \in L$  is called the left (resp., right) unit of  $L$ .
- (2) A unital weak hyper semi-quantale if it has both left and right unit. UWHySQuant comprises all unital weak hyper semi-quantales together with all weak hyper semi-quantale morphisms preserving the unit  $e$ .
- (3) A strictly two-sided if it is a unital with  $e = \top$ . STWHySQuant is the category of strictly two-sided weak hyper semi-quantales.
- (4) idempotent iff  $a \in a\tilde{\otimes}a$ , for all  $a \in L$ .
- (5) commutative if  $a\tilde{\otimes}b = b\tilde{\otimes}a$  for all  $a, b \in L$ .
- (6) distributive provided that its hypermultiplication  $\tilde{\otimes}$  distributes across finite  $\bigvee$  from both sides. DWHySQuant is the category of distributive weak hyper semi-quantales.
- (7) A weak hyperquantale if  $(L, \tilde{\otimes})$  is a semihypergroup and

$$\bigvee_{b \in B} \{a\tilde{\otimes}b\} \ll a\tilde{\otimes}(\bigvee B) \text{ and } \bigvee_{b \in B} \{b\tilde{\otimes}a\} \ll (\bigvee B)\tilde{\otimes}a$$

for all  $a \in L$  and any subset  $B \subseteq L$ , where  $A \ll B$  means that there exist  $a \in A$  and  $b \in B$  such that  $a \leq b$ , for all non-empty subsets  $A$  and  $B$  of  $L$ .

**Example 3.5.** Let  $(L, \leq, \bigvee, \odot)$  be any semi-quantale and define  $\tilde{\otimes} : L \times L \rightarrow P^*(L)$  by the form  $a\tilde{\otimes}b = \{a \odot b\}$  for all  $a, b \in L$ . Then  $(L, \leq, \bigvee, \tilde{\otimes})$  is a weak hyper semi-quantale.

**Example 3.6.** Any complete lattice  $L = (L, \leq, \bigvee)$  can be made into an idempotent weak hyper semi-quantale by taking  $x\tilde{\otimes}y = \{x\}$  for all  $x, y \in L$  with  $y \neq \perp$  and  $x\tilde{\otimes}\perp = \{\perp\}$ .

**Example 3.7.** Let  $(H, \tilde{\odot})$  be a semihypergroup. On the power set  $P(H)$  of  $H$ , we can define a hyperoperation  $\tilde{\otimes} : P(H) \times P(H) \rightarrow P^*(P(H))$  by  $A\tilde{\otimes}B = \{a\tilde{\odot}b : a \in A \text{ and } b \in B\}$ . One can easily see that  $(P(H), \cup, \tilde{\otimes})$  is a weak hyper semi-quantale.

**Example 3.8.** Each  $(L, \leq, \bigvee, \tilde{T})$  with  $(L, \leq, \bigvee)$  a complete lattice and  $\tilde{\otimes} = \tilde{T}$  a hyper-t-norm on  $L$  is a commutative strictly two-sided weak hyper semi-quantale.

**Example 3.9.** Each  $([0, 1], \leq, \tilde{T})$  with  $\leq$  the usual ordering on  $[0, 1]$  and  $\tilde{\otimes} = \tilde{T}$  a hyper-t-norm on  $[0, 1]$  is a commutative strictly two-sided weak hyper semi-quantale.

**Example 3.10.** Each  $(L, \leq, \bigvee, \tilde{S})$  with  $(L, \leq, \bigvee)$  a complete lattice and  $\tilde{\otimes} = \tilde{S}$  a hyper-t-conorm on  $L$  is a commutative unital weak hyper semi-quantale(with unit  $\perp$ ).

Before giving the next two examples, we recall that: For any non-empty set  $X$ , and a complete lattice  $L = (L, \leq, \bigvee)$ , the order-theoretic operations (e.g.,  $\bigvee, \bigwedge$ ) can be extended from  $L$  to  $L^X$  pointwisely, and they are denoted by the same symbols. The powerset  $L^X = (L^X, \leq, \bigvee)$  is again a complete lattice. In the following two examples we, define two kinds of hyperoperations on  $L^X$ .

**Example 3.11.** If  $(L, \leq, \bigvee, \odot)$  is an ordinary semi-quantale, as given in [21], the algebraic operations  $\odot$  can be extended from  $L$  to  $L^X$  pointwisely, e. g., for all  $x \in X$ ,  $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$ . Obviously,  $(L^X, \leq, \bigvee, \odot)$  is again a semi-quantale. In the light of Example 3.5, we can define a hyperoperation  $\tilde{\otimes} : L^X \times L^X \rightarrow P^*(L^X)$  by  $\lambda\tilde{\otimes}\mu = \{\lambda \odot \mu\}$ , for all  $\lambda, \mu \in L^X$ . Then  $(L^X, \leq, \bigvee, \tilde{\otimes})$  a weak hyper semi-quantale.

**Example 3.12.** If  $(L, \leq, \bigvee, \tilde{\odot})$  is a weak hyper semi-quantale. On the powerset  $L^X$ , a hyperoperation  $\tilde{\otimes} : L^X \times L^X \rightarrow P^*(L^X)$  given by  $\bigvee_{\rho \in \lambda \tilde{\otimes} \mu} \rho(x) = \lambda(x) \tilde{\odot} \mu(x)$  for all  $\lambda, \mu \in L^X$ , where,

$$\rho \in \lambda \tilde{\otimes} \mu \Leftrightarrow \forall x \in X, \rho(x) \in \lambda(x) \tilde{\odot} \mu(x).$$

Obviously,  $(L^X, \leq, \bigvee, \tilde{\otimes})$ , which is called the powerset weak hyper semi-quantale, is again a weak hyper semi-quantale, with respect to the hyper operation  $\tilde{\otimes}$  and the arbitrary  $\bigvee$ . If  $\top$  is the top of  $L$ , then the top of  $L^X$  take the form  $\perp$ . If  $L$  is a unital weak hyper semi-quantale with unit  $e$ , then  $L^X$  becomes a unital weak hyper semi-quantale with the unit  $\underline{e}$ .

**Definition 3.13.** Let  $L \in |WHySQuant|$ . A subset  $S \subseteq L$  is a weak hyper subsemi-quantale of  $L$  iff  $S$  is closed under  $\bigvee$  and  $a \tilde{\otimes} b \subseteq S$ , for all  $a, b \in S$ .

A weak hyper subsemi-quantale of  $L$  is said to be strong iff it contains the top element of  $L$ . If  $L$  is a unital weak hyper semi-quantale with the identity  $e$ , then a weak hyper subsemi-quantale  $S$  of  $L$  is called a unital weak hyper subsemi-quantale of  $L$  iff  $e \in S$ .

**Example 3.14.** Let  $L = \{\perp, a, b, \top\}$  with  $\top \geq a, b \geq \perp$ . Consider the following tables:

$\tilde{\otimes}$	$\perp$	a	b	$\top$
$\perp$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$
a	$\{\perp\}$	$\{b\}$	$\{\top\}$	$\{\top\}$
b	$\{\perp\}$	$\{\top\}$	$\{\top\}$	$\{\top\}$
$\top$	$\{\perp\}$	$\{\top\}$	$\{\top\}$	$\{\top\}$

Then, one can see that  $(L, \leq, \bigvee, \tilde{\otimes})$  is a weak hyper semi-quantale and there exist only two non-trivial weak hyper subsemi-quantales  $S_1 = \{\perp, \top\}$  and  $S_2 = \{\perp, b, \top\}$ .

## 4 Weak hypervalued topological spaces

In this section, we give a suitable definition of weak hypervalued topological spaces providing a topological counterpart to weak hyper semi-quantales. Before we go further into this section, we recall that by  $L^X$ , we mean the the powerset weak hyper semi-quantale,  $(L^X, \leq, \bigvee, \tilde{\otimes})$  which constructed in Example 3.12.

**Definition 4.1.** For a fixed  $L \in |WHySQuant|$  and a non-empty set  $X$ . A weak hypervalued-quasi-topology (or  $H^w$ -quasi-topology) on  $X$  is a weak hyper subsemi-quantale  $\tau$  of  $L^X$ , i.e., the following axioms are satisfied:

(T<sub>1</sub>) For all  $A, B \in L^X, A, B \in \tau \Rightarrow A \tilde{\otimes} B \subseteq \tau$ .

(T<sub>2</sub>) For all  $\{A_j : j \in J\} \subseteq L^X, \{A_j : j \in J\} \subseteq \tau \Rightarrow \bigvee_j A_j \in \tau$ .

A weak hypervalued-quasi-topology (or  $H^w$ -quasi-topology)  $\tau$  is said to be strong iff it is strong as a weak hyper subsemi-quantale of  $L^X$ , i.e.,  $\tau$  satisfies the additional axiom:

(T<sub>3</sub>)  $\perp \in \tau$ .

If  $L \in |UWHySQuant|$  with unit  $e$ , a unital weak hyper subsemi-quantale  $\tau$  of  $L^X$  is called an  $H^w$ -topology on  $X$ ; i.e.,  $\tau$  satisfies (T<sub>1</sub>), (T<sub>2</sub>) and the following:

(T<sub>4</sub>)  $\underline{e} \in \tau$ .

If  $\tau \subseteq L^X$  is an  $H^w$ -quasi-topology (a strong  $H^w$ -quasi-topology and an  $H^w$ -topology, resp.), then the pair  $(X, \tau)$  is said to be an  $H^w$ -quasi-topological (a strong  $H^w$ -quasi-topological and an  $H^w$ -topological, resp.) space.

**Definition 4.2.** For  $L \in |WHySQuant|$ . A weak hyper subsemi-quantale (resp., a strong weak hyper subsemi-quantale)  $\mathcal{T}$  of  $L$  is said to be a weak hyper semi-quantale (resp., a strong weak hyper semi-quantale) Hutton quasi-topology on  $L$ . In the case of  $L \in |UWHySQuant|$ , a unital weak hyper subsemi-quantale  $\mathcal{T}$  of  $L$  is said to be a unital weak hyper semi-quantale Hutton topology.

**Example 4.3.** Let  $X$  be a set and  $L \in |WHySQuant|$ . Take  $\tau = L^X$  (resp.,  $\mathcal{T} = L$ ), then  $\tau$  (resp.,  $\mathcal{T}$ ) is a strong  $H^w$ -quasi-topology (resp. a strong weak hyper semi-quantale Hutton quasi-topology) on  $X$  (resp.,  $L$ ).

**Example 4.4.** Every  $L$ -quasi-topology (resp. strong  $L$ -quasi-topology,  $L$ -topology)  $\tau$  [21, 6] on a non-empty set  $X$  is an  $H^w$ -quasi-topology (resp., strong  $H^w$ -quasi-topology,  $H^w$ -topology).

**Example 4.5.** Every many-valued topology ( $L$ -topology) in the sense of Höhle [11] is a strong  $H^w$ -quasi-topology.

**Definition 4.6.** A mapping  $f : (X, \tau) \rightarrow (Y, \sigma)$ , between  $H^w$ -quasi-topological (strong  $H^w$ -quasi-topological and  $H^w$ -topological, resp.) spaces, is said to be  $H^w$ -continuous if  $\forall \mu \in \sigma, (f_L^+)(\mu) \in \tau$ .

It is clear that  $H^w$ -quasi-topological ( strong  $H^w$ -quasi-topological and  $H^w$ -topological, resp.) spaces as objects and  $H^w$ -continuous maps as morphisms form a category denoted by  $H^w$ -QTop (  $H^w$ -SQTop and  $H^w$ -Top, resp.).

One can easily prove that each of  $H^w$ -QTop,  $H^w$ -SQTop and  $H^w$ -Top is a topological category over the category Set.

**Definition 4.7.** For  $L \in |\mathbf{WHySQquant}|$ . A mapping  $\mathcal{K} : L^X \rightarrow L^X$  is called:

(1) an  $H^w$ -quasi-interior on a non-empty set  $X$  if it satisfies, for all  $\lambda, \mu \in L^X$ , the following conditions:

$$(K_1) \quad \mathcal{K}(\lambda) \leq \lambda;$$

$$(K_2) \quad \lambda \leq \mu \Rightarrow \mathcal{K}(\lambda) \leq \mathcal{K}(\mu);$$

$$(K_3) \quad \mathcal{K}(\lambda) \leq \mathcal{K}(\mathcal{K}(\lambda));$$

$$(K_4) \quad \mathcal{K}(\lambda) \tilde{\otimes} \mathcal{K}(\mu) \ll \bigwedge_{\nu \in \lambda \tilde{\otimes} \mu} \mathcal{K}(\nu).$$

(2) a strong  $H^w$ -quasi-interior on a non-empty set  $X$  if it is an  $H^w$ -quasi-interior and satisfies the additional axiom:

$$(K_5) \quad \mathcal{K}(\perp) = \perp.$$

(3) an  $H^w$ -interior on a non-empty set  $X$  if  $L \in |\mathbf{UWHySQquant}|$ , with unit  $e$ , and it is an  $H^w$ -quasi-interior satisfies the additional axiom:

$$(K_6) \quad \mathcal{K}(e) = e.$$

**Proposition 4.8.** For an  $H^w$ -quasi-interior (resp., strong  $H^w$ -quasi-interior) operator  $\mathcal{K} : L^X \rightarrow L^X$ , the subfamily  $\tau_{\mathcal{K}} = \{\lambda \in L^X : \lambda \leq \mathcal{K}(\lambda)\}$  is an  $H^w$ -quasi-topology (resp., strong  $H^w$ -quasi-topology) on  $X$ .

*Proof.* Let  $\mathcal{K} : L^X \rightarrow L^X$ , be an  $H^w$ -quasi-interior operator on  $X$ . Given  $\lambda, \mu \in \tau_{\mathcal{K}}$  and for an arbitrary  $\nu \in \lambda \tilde{\otimes} \mu$ , we have, by the condition  $(K_4)$ ,  $\lambda \tilde{\otimes} \mu \ll \mathcal{K}(\lambda) \tilde{\otimes} \mathcal{K}(\mu) \ll \bigwedge_{\nu \in \lambda \tilde{\otimes} \mu} \mathcal{K}(\nu)$  which means that  $\lambda \tilde{\otimes} \mu \ll \bigwedge_{\nu \in \lambda \tilde{\otimes} \mu} \mathcal{K}(\nu) \leq \mathcal{K}(\nu)$  and

$\nu \ll \mathcal{K}(\nu)$ , i.e.,  $\nu \in \tau_{\mathcal{K}}$ . So we can conclude that  $\lambda \tilde{\otimes} \mu \in \tau_{\mathcal{K}}$ . Finally, suppose  $\{\lambda_j : j \in J\} \subseteq L^X$ ,  $\{\lambda_j : j \in J\} \subseteq \tau_{\mathcal{K}}$ . Then,  $\bigvee_j \lambda_j = \bigvee_j \mathcal{K}(\lambda_j) \leq \mathcal{K}(\bigvee_j \lambda_j)$  and thus  $\bigvee_j \lambda_j \in \tau_{\mathcal{K}}$  and therefore  $\tau_{\mathcal{K}}$  is an  $H^w$ -quasi-topology on  $X$ . If  $\mathcal{K} : L^X \rightarrow L^X$ , be a strong  $H^w$ -quasi-interior operator on  $X$ . From condition  $(K_5)$  we have that  $\perp \in \tau_{\mathcal{K}}$ .  $\square$

**Corollary 4.9.** For  $L \in |\mathbf{UWHySQquant}|$ . Given an  $H^w$ -interior operator  $\mathcal{K} : L^X \rightarrow L^X$ , the subfamily  $\tau_{\mathcal{K}} = \{\lambda \in L^X : \lambda \leq \mathcal{K}(\lambda)\}$  is an  $H^w$ -topology on  $X$ .

**Proposition 4.10.** Given an  $H^w$ -quasi-topology (resp., strong  $H^w$ -quasi-topology)  $\tau \subset L^X$  on  $X$ , the mapping  $\mathcal{K}_{\tau} : L^X \rightarrow L^X$  defined by the equality.  $\mathcal{K}_{\tau}(\lambda) = \bigvee\{\mu \in \tau : \mu \leq \lambda\}$  is an  $H^w$ -quasi-interior (resp., strong  $H^w$ -quasi-interior) operator on  $X$ .

*Proof.* At first, let  $\tau \subset L^X$  be an  $H^w$ -quasi-topology on  $X$ , then

(1) The condition  $(K_1)$  is clear

(2) For  $(K_2)$ , let  $\lambda_1 \leq \lambda_2$ , then

$$\mathcal{K}_{\tau}(\lambda_1) = \bigvee\{\mu \in \tau : \mu \leq \lambda_1 \leq \lambda_2\} \leq \bigvee\{\mu \in \tau : \mu \leq \lambda_2\} \leq \mathcal{K}_{\tau}(\lambda_2),$$

(3) From  $(K_1)$  and  $(K_2)$ , one have  $(K_3)$ ;

(4) In order to check  $(K_4)$ , we show that there is some  $\eta \in \mathcal{K}_\tau(\lambda) \widetilde{\otimes} \mathcal{K}_\tau(\mu)$  and  $\nu \in \bigwedge_{\nu \in \lambda \widetilde{\otimes} \mu} \mathcal{K}_\tau(\nu)$  such that  $\eta \leq \nu$ . For this propose, let  $\eta \in \mathcal{K}_\tau(\lambda) \widetilde{\otimes} \mathcal{K}_\tau(\mu)$ , then there exist  $\lambda_1 \in \mathcal{K}_\tau(\lambda)$  and  $\mu_1 \in \mathcal{K}_\tau(\mu)$  such that  $\eta \in \lambda_1 \widetilde{\otimes} \mu_1$ . Since  $\lambda_1 \widetilde{\otimes} \mu_1 \ll \lambda \widetilde{\otimes} \mu$  then  $\exists \nu \in \lambda \widetilde{\otimes} \mu$  with  $\eta \leq \nu$ . Since  $\lambda_1 \widetilde{\otimes} \mu_1 \subset \tau$  then  $\eta \in \tau$  and by (3), we have  $\eta \leq \mathcal{K}_\tau(\eta) \leq \mathcal{K}_\tau(\nu)$  and therefore we have  $\eta \in \bigwedge_{\eta \in \lambda_1 \widetilde{\otimes} \mu_1} \mathcal{K}_\tau(\eta) \leq \bigwedge_{\nu \in \lambda \widetilde{\otimes} \mu} \mathcal{K}_\tau(\nu)$ . Thus  $\eta \in \bigwedge_{\nu \in \lambda \widetilde{\otimes} \mu} \mathcal{K}_\tau(\nu)$  and  $\mathcal{K}_\tau(\lambda) \widetilde{\otimes} \mathcal{K}_\tau(\mu) \ll \bigwedge_{\nu \in \lambda \widetilde{\otimes} \mu} \mathcal{K}_\tau(\nu)$ .

If  $\tau \subset L^X$  is a strong  $H^w$ -quasi-topology on  $X$ , then the condition  $(K_5)$  holds since  $\mathcal{K}_\tau(\perp) = \bigvee \{\mu \in \tau : \mu \leq \perp\} = \perp$ .  $\square$

**Corollary 4.11.** *For  $L \in |UWHySQuant|$ . Given an  $H^w$ -topology  $\tau \subset L^X$  on  $X$ , the mapping  $\mathcal{K}_\tau : L^X \rightarrow L^X$  defined by the equality.  $\mathcal{K}_\tau(\lambda) = \bigvee \{\mu \in \tau : \mu \leq \lambda\}$  is an  $H^w$ -interior on  $X$ .*

**Definition 4.12.** *Let  $L \in |WHySQuant|$ , and  $a, b \in L$ . An element  $a$  is said to be hyperwell-inside  $b$ , denoted by  $a \preceq b$ , if there exists  $c \in L$  such that  $\perp \in a \widetilde{\otimes} c$  and  $b \vee c = \top$ . Sometime we say that  $a \preceq b$  via  $c$ .*

**Definition 4.13.** *Let  $L \in |WHySQuant|$  and  $(X, \tau) \in H^w\text{-QTop}$ .*

- (1)  *$L$  is said to be hyperquantic normal if, given  $a, b \in L$  with  $a \vee b = \top$ , we can find  $d, c \in L$  with  $\perp \in d \widetilde{\otimes} c$ , and  $d \vee a = \top = b \vee c$ .*
- (2)  *$(X, \tau)$  is hyperquantic normal if  $\tau$  is a hyperquantic normal weak hyper semi-quantale .*

**Example 4.14.** *Every normal semi-quantale (resp.,  $L$ -quasi-topological space ) [7] is hyperquantic normal.*

**Proposition 4.15.** *Let  $(X, \tau) \in |H^w\text{-QTop}|$  and  $L \in |WHySQuant|$ .*

- (1) *A commutative weak hyper semi-quantale  $L$  is hyperquantic normal, iff*

$$\forall a, b \in L \text{ with } a \vee b = \top, \exists c, d \in L \text{ with } c \preceq a \text{ via } d, d \preceq b \text{ via } c.$$

- (2)  *$(X, \tau)$  is hyperquantic normal, iff*

$$\forall \mu, \nu \in \tau \text{ with } \mu \vee \nu = \perp, \exists \lambda, v \in \tau \text{ with } \lambda \preceq \mu \text{ via } v, v \preceq \nu \text{ via } \lambda.$$

*Proof.* (1) Let  $a, b \in L$  with  $a \vee b = \top$ . Hyperquantic normality of  $L \Leftrightarrow \exists c, d \in L$  with  $\perp$  belongs to both  $c \otimes d$  and  $d \otimes c$ . Also,  $c \vee b = \top = a \vee d$ . Then we conclude that:

- (i)  $\perp \in c \otimes d$  and  $a \vee d = \top \Rightarrow c \preceq a$  via  $d$ .
- (ii)  $\perp \in d \otimes c$  and  $b \vee c = \top \Rightarrow d \preceq b$  via  $c$ .

(2) Follows from (1).  $\square$

## 5 Topological representation of weak hyper semi-quantales

In this section, we will introduce and study an adjunction between the category  $WHySQuant$  of weak hyper semi-quantales and the category  $H^w\text{-QTop}$  of  $H^w$ -quasi-topological spaces. Also we will define the concepts of  $H^w$ -sober spaces and  $H^w$ -spatial weak hyper semi-quantales as a generalization of  $L$ -sober topological spaces [17, 18, 19, 7] and  $L$ -spatial semi-quantales [7].

For  $L \in |WHySQuant|$  and  $(X, \tau) \in |H^w\text{-QTop}|$ , the functor

$$\Omega_L : H^w\text{-QTop} \rightarrow WHySQuant^{op}$$

is defined as follows:

$\Omega_L(X, \tau)$  is the  $H^w$ -quasi-topology of the space  $(X, \tau)$ , i.e., the weak hyper subsemi-quantale  $\tau \subseteq L^X$ , and  $\Omega_L(f : (X, \tau) \rightarrow (Y, \sigma))$ , for an  $H^w$ -continuous map  $f$ , is  $[f_L^- |_\sigma]^{op} : \tau \rightarrow \sigma$ .

The standard spectrum construction for an  $Q \in |WHySQuant|$  may be summarized as follows:

$$Lpt(Q) = \{p : Q \rightarrow L : p \in |WHySQuant|\}$$

$$\Phi_L : Q \rightarrow L^{Lpt(Q)} \text{ by } \Phi_L(q)(p) = p(q)$$

**Lemma 5.1.** For  $L, Q \in |WHySQuant|$ , the map  $\Phi_L : Q \rightarrow L^{Lpt(Q)}$  is a weak hyper semi-quantale homomorphism.

*Proof.* As given in Example 3.11, we have that  $L^{Lpt(Q)} \in |WHySQuant|$ . So for  $a, b \in Q$ , we have

$$\Phi_L(a \tilde{\otimes} b)(p) = p(a \tilde{\otimes} b) \subseteq p(a) \tilde{\otimes} p(b) = \Phi_L(a)(p) \tilde{\otimes} \Phi_L(b)(p),$$

One can easily prove that  $\Phi_L$  preserves arbitrary  $\bigvee$ . □

**Lemma 5.2.** For  $L, Q \in |WHySQuant|$ , the family  $\Phi_L^\rightarrow(Q)$  constitutes a weak hyper subsemi-quantale of  $L^{Lpt(Q)}$ , i.e. an  $H^w$ -quasi-topology on  $Lpt(Q)$ .

*Proof.* The proof comes as a consequence of Lemma 5.1. □

Now, we have  $Q \rightarrow (Lpt(Q), \Phi_L^\rightarrow(Q))$ , where the latter is an  $H^w$ -quasi-topological space; so we put

$$LPT(Q) \equiv (Lpt(Q), \Phi_L^\rightarrow(Q)) \in |H^w-QTop|$$

and given  $f : Q_1 \rightarrow Q_2$  in  $WHySQuant$ , i.e.  $f^{op} : Q_1 \leftarrow Q_2$  in  $WHySQuant^{op}$ .

We define  $Lpt(f) : Lpt(Q_1) \rightarrow Lpt(Q_2)$  by  $Lpt(f)(p) = p \circ f^{op}$ .

**Lemma 5.3.** For a fixed  $L \in |WHySQuant|$  and  $Q_1, Q_2 \in |WHySQuant|$ , the mapping

$$LPT(f) : (Lpt(Q_1), \Phi_L^\rightarrow(Q_1)) \rightarrow (Lpt(Q_2), \Phi_L^\rightarrow(Q_2))$$

is  $H^w$ -continuous.

*Proof.* For all  $a \in Q_2, p \in Lpt(Q_1)$ , we have

$$Lpt(f)^\leftarrow(\Phi_L(a)(p)) = \Phi_L(a)(Lpt(f)(p)) = \Phi_L(a)(p \circ f^{op}) = \Phi_L(f^{op}(a))(p).$$

Hence  $Lpt(f)^\leftarrow(\Phi_L(a)(p)) = \Phi_L(f^{op}(a))(p)$ .

Now the function  $LPT(f)$  is  $H^w$ -continuous iff  $\forall \mu \in \Phi_L^\rightarrow(a), \exists \nu \in \Phi_L^\rightarrow(Q_1)$  such that  $Lpt(f)^\leftarrow(\nu) = \mu$ . □

From the above lemmas, we have the spectrum:  $LPT : WHySQuant^{op} \rightarrow H^w-QTop$ .

Now, we turn to study the adjunction between the functors  $LPT : WHySQuant^{op} \rightarrow H^w-QTop$  and  $\Omega_L : H^w-QTop \rightarrow WHySQuant^{op}$ . To this aim, we give the following definitions.

For  $(X, \tau) \in |H^w-QTop|$  and  $L, Q \in |WHySQuant|$  define the maps:

- $\eta_x : (X, \tau) \rightarrow (Lpt(Q), \Phi_L^\rightarrow(Q))$ , by setting:

$$\eta_x(x)(\mu) = \mu(x), \forall x \in X \text{ and } \mu \in \tau;$$

- $\varepsilon_Q^{op} : Q \rightarrow \Omega_L(LPT(Q))$  by setting  $\varepsilon_Q^{op} = \Phi_L \upharpoonright_{\Phi_L^\rightarrow(Q)}$ .

As given in [7], we have the following easily established result:

**Lemma 5.4.** For  $(X, \tau) \in |H^w-QTop|$  and  $L \in |WHySQuant|$ , The map  $\eta_x : (X, \tau) \rightarrow (Lpt(Q), \Phi_L^\rightarrow(Q))$  is  $H^w$ -continuous.

As a consequence of Lemma 5.1, we have the following easily established result:

**Lemma 5.5.** For  $L, Q \in |WHySQuant|$ . The map  $\varepsilon_Q^{op} : Q \rightarrow \Omega_L(LPT(Q))$  is a weak hyper semi-quantale morphism.

Now, we can conclude that:  $LPT : WHySQuant^{op} \rightarrow H^w-QTop$  is a right adjoint to  $\Omega_L : H^w-QTop \rightarrow WHySQuant^{op}$ . This adjunction given in the form  $H^w-QTop \dashv WHySQuant^{op}$ .

For the case of the category  $UWHySQuant$  of unital weak hyper semi-quantales and the category  $H^w-Top$  of  $H^w$ -topological spaces, one can similarly have the following dual adjunctions:

$$H^w-Top \dashv UWHySQuant^{op}.$$

**Definition 5.6.** An  $Q \in |WHySQuant|$  is said to be  $H^w$ -spatial iff the map  $\varepsilon_Q^{op}$  is injective.

**Lemma 5.7.** *An  $Q \in |WHySQuant|$  is  $H^w$ -spatial iff  $\varepsilon_Q^{op}$  is isomorphism*

*Proof.* The proof is straightforward. □

**Corollary 5.8.** *For  $(X, \tau) \in |H^w-QTop|$ , the  $H^w$ -quasi-topology  $\Omega_L(X, \tau)$  is  $H^w$ -spatial.*

*Proof.* The proof is analogous to those of Corollary 3.8 in [7]. □

**Definition 5.9.** *An  $(X, \tau) \in |H^w-QTop|$  is called  $H^w$ -sober iff  $\eta_X : (X, \tau) \longrightarrow (Lpt(\tau), \Phi_L^{\rightarrow}(\tau))$  is bijective.*

**Lemma 5.10.** *For all  $Q \in |WHySQuant|$ ,  $LPT(Q)$  is  $H^w$ -sober.*

*Proof.* The proof is analogous to those of Lemma 4.10 in [7]. □

Let  $H^w$ -Sob (resp.,  $H^w$ -Spat) be the full subcategory of  $H^w$ -QTop (resp., WHySQuant) consisting of all  $H^w$ -sober spaces (resp.,  $H^w$ -spatial weak hyper semi-quantales).

In this context the statement of Theorem 3.13 in [7] can be recalled as follows.

**Theorem 5.11.** *The categories  $H^w$ -Sob and  $H^w$ -Spat are equivalent.*

## 6 Conclusions

In this paper, we introduced the concept of a weak hyper semi-quantale as a generalization of the concept of semi-quantale, and we give some properties and related results. Also, we have introduced and studied the concepts of weak hypervalued topology as a generalization of the concept of many valued topology (or lattice-valued topology). We studied some properties of weak hyper semi-quantales and weak hypervalued-topologies. Finally, we established an adjunction between the category of weak hyper semi-quantales and the category of weak hypervalued quasi-topological spaces. In a future work, we intend to use the notion of weak hyper semi-quantale as an appropriate hyperlattice-theoretic basis to formulate and study  $(L, M)$ -fuzzy topological theories.

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