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Rough approximation operators based on quantale-valued fuzzy generalized neighborhood systems

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Abstract

Let L be an integral and commutative quantale. In this paper, by fuzzifying the notion of generalized neighborhood systems, the notion of L-fuzzy generalized neighborhood system is introduced and then a pair of lower and upper approximation operators based on it are defined and discussed. It is proved that these approximation operators include generalized neighborhood system-based approximation operators, L-fuzzy relation-based approximation operators and L-fuzzy covering-based approximation operators as their special circumstances. Therefore, the research on L-fuzzy generalized neighborhood system-based approximation operators has more general significance. In addition, when the L-fuzzy generalized neighborhood system is serial, reflexive, unary and transitive, then the corresponding approximation operators are discussed and characterized, respectively.

Keywords: Fuzzy rough set, fuzzy topology, L-fuzzy generalized neighborhood system, quantale.

1 Introduction

The theory of rough sets, initiated by Pawlak [31], is a powerful tool for dealing with insufficient, granularity and uncertainty information systems. It has been successfully applied in many fields [25, 27, 43, 45].

The classical rough set theory is based on partition or equivalent relation. This is too restrictive for many applications of classical rough set. Thus classical rough set theory has been extended to binary relation-based rough sets [4, 50, 51] and covering-based rough sets [44, 47, 49].

The notion of neighborhood systems is abstracted from the geometric notion of "near", and it is primitive in the theory of topological spaces. Roughly speaking, each element of a universe is associated with a nonempty family of subsets of the universe. This family is called a neighborhood system of the element, and each member is called a neighborhood of the element, representing the semantics of "near". Recently, the notion of generalized neighborhood systems is proposed and used to define a theory of rough set, called generalized neighborhood system-based rough sets [35, 43, 47]. For an arbitrary binary relation, the successor elements of a given element may be interpreted as its neighborhood. For a covering (topology), the collection of subsets containing a given element may be regarded as its neighborhood system. Therefore, it is observed in [35, 47] that generalized neighborhood system-based rough sets includes binary relation-based rough sets and covering-based rough sets as its special circumstance. In [48], the authors gave an axiomatic characterization on generalized neighborhood system-based rough sets.

The research of rough sets has been extended to fuzzy environment [5, 11, 12, 16, 22, 36, 37, 38, 39, 45, 46]. At the beginning, the scholar usually investigated fuzzy rough sets by taking the interval $[0,1]$ as the truth table. However, as Goguen [7] pointed out, it may be impossible to represent degrees of membership in some situation by the linearly ordered set $[0,1]$. Hence some lattice structures were proposed to replace the interval $[0,1]$ as the truth table for membership degrees, among which quantale plays an important role. Now, quantale has been used successfully in fuzzy order theory, fuzzy topology theory and fuzzy rough set theory [1, 5, 8, 11, 16, 17, 19, 20, 22, 23, 26, 28, 34, 41].

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There are at least two directions towards the discussion of quantale-valued fuzzy rough sets, namely, quantale-valued fuzzy relation-based rough sets and quantale-valued fuzzy covering-based rough sets. Consider L to be an integral and commutative quantale, the L-fuzzy relation-based approximation operators was initiated by Radzikowska-kerre [32], and then discussed by many scholars, see such as [5, 10, 11, 23, 26, 33, 34, 42]. Particularly, She and Wang [34] gave an axiomatic characterization on a pair of L-fuzzy relation-based approximation operators. In [6], Deng et al. investigated a pair of L-fuzzy covering-based approximation operators. When $L = [0, 1]$, Li et al. [24] introduced another two pairs of L-fuzzy covering-based approximation operators. In [22], the authors studied the above three pairs of L-fuzzy covering-based approximation operators systematically and presented their axiomatic characterizations.

In this paper, by fuzzifying the notion of generalized neighborhood systems, the notion of L-fuzzy generalized neighborhood systems is introduced, and then a pair of lower and upper rough approximation operators based on it are defined and discussed. It is proved that generalized neighborhood system-based approximation operators [35, 47, 48], L-fuzzy relation-based approximation operators [10, 34] and L-fuzzy covering-based approximation operators [22, 24] can be regarded as special L-fuzzy generalized neighborhood system-based approximation operators. Therefore the research on L-fuzzy generalized neighborhood system-based approximation operators has more general significance.

The contents of this paper are organized as follows. In Section 2, we recall some notions and notations used in this paper. In section 3, we define and discuss a pair of lower and upper approximation operators based on L-fuzzy generalized neighborhood systems. In Section 4, we discuss and characterize the related approximation operators when the L-fuzzy generalized neighborhood system is serial, reflexive, unary and transitive, respectively. In Section 5, we make a conclusion.

2 Preliminaries

In this section, we recall some basic notions and notations used in this paper.

A commutative quantale is a pair $(L, *)$, where L is a complete lattice with respect to a partial order $\leq$ on it, with the top (resp., bottom) element $\top$ (resp., $\bot$), and $*$ is a commutative semigroup operation on L such that $a * \bigvee_{j\in J} b_j = \bigvee_{j\in J} (a * b_j)$ for all $a \in L$ and $\{b_j\}_{j\in J} \subseteq L$. $(L, *)$ is said to be integral if the top element $\top$ is the unique unit in the sense of $\top * a = a$ for all $a \in L$.

In this paper, if not otherwise specified, $L = (L, *)$ is always assumed to be an integral, commutative quantale.

Since the binary operation $*$ distributes over arbitrary joins, the function $a * (-) : L \longrightarrow L$ has a right adjoint $a \to (-) : L \longrightarrow L$ given by $a \to b = \bigvee\{c \in L : a * c \leq b\}$. We collect here some basic properties of the binary operations $*$ and $\to$ [9].

Proposition 2.1. (1) $a \to b = \top \Leftrightarrow a \leq b$;
(2) $a * (a \to b) \leq b$;
(3) $a \to (b \to c) = (a * b) \to c = b \to (a \to c)$;
(4) $a \leq (a \to b) \to b$;
(5) $(\bigvee_{j\in J} a_j) \to b = \bigwedge_{j\in J} (a_j \to b)$;
(6) $a \to (\bigwedge_{j\in J} b_j) = \bigwedge_{j\in J} (a \to b_j)$.
(7) $\forall a \in L, a = \bigwedge_{b\in L} ((a \to b) \to b)$.

L is said to satisfy the double negation law if for any $a \in L$, $(a \to \bot) \to \bot = a$. In the following, we use $\neg a$ to denote $a \to \bot$. Furthermore, for any $a, b \in L$, we define $a \oplus b = \neg(\neg a * \neg b)$.

Proposition 2.2. *If L satisfies the double negative law, then it satisfies moreover:*
(1) $a \to b = \neg b \to \neg a$;
(2)$(a \to b) = \neg(a * \neg b)$;
(3) $\neg(\bigwedge_{i\in I} a_i) = \bigvee_{i\in I} (\neg a_i)$;
(4) $\bigwedge_{i\in I} a_i \oplus \bigwedge_{j\in I} b_j = \bigwedge_{i\in I}\bigwedge_{j\in I} (a_i \oplus b_j)$.

Let the universe of discourse X be an arbitrary nonempty set. We call a function $A : X \longrightarrow L$ an L-fuzzy set in X. We use L^X to denote the set of all L-fuzzy sets in X and call it the L-fuzzy power set on X. For a crisp subset $A \subset X$, let $\top_A$ be the characteristic function, i.e., $\top_A(x) = \top$ if $x \in A$ and $\top_A(x) = \bot$ if $x \notin A$. The characteristic function $\top_A$ of a subset $A \subseteq X$ can be regarded as an L-fuzzy set in X. We make no difference between a constant L-fuzzy set and its value since no confusion will arise.

The operators $\vee, \wedge, *, \to$ and $\neg$ on L can be translated onto L^X in a pointed wise. That is, for any $A, B, A_t(t \in T) \in L^X$,

$$(\bigvee_{t\in T} A_t)(x) = \bigvee_{t\in T} A_t(x), (\bigwedge_{t\in T} A_t)(x) = \bigwedge_{t\in T} A_t(x),$$

$$(A * B)(x) = A(x) * B(x), (A \to B)(x) = A(x) \to B(x), (\neg A)(x) = \neg A(x).$$

Let A, B be L-fuzzy sets in X. The subsethood degree [3] of A, B, denoted by $S(A, B)$ is defined by $S(A, B) = \bigwedge_{x\in X} (A(x) \to B(x))$. That $S(A, B)$ represents the semantics "A is contained in B".

Let A, B be L-fuzzy sets in X. The intersection degree [5] of A, B, denoted by $I(A, B)$ is defined by $I(A, B) = \bigvee_{x\in X} (A(x) * B(x))$. That $I(A, B)$ represents the semantics "A and B have intersection".

The lemma below collects some properties of subsethood degree and intersection degree. They can be founded in many literatures such as [2, 5, 13, 14, 15, 18, 21, 22, 23, 29, 30, 40].

Lemma 2.3. *Let $A, B, C, D, A_t(t \in T) \in L^X$. Then*
(S1) $S(A, B) = \top \Leftrightarrow A \leq B$;
(S2) $A \leq B \Rightarrow S(A, C) \geq S(B, C)$ *and* $S(C, A) \leq S(C, B)$;
(S3) $S(A, B) * S(C, D) \leq S(A * C, B * D)$;
(S4) $S(A, B) * S(B, C) \leq S(A, C)$;
(S5) $S(A, \bigwedge_{t\in T} A_t) = \bigwedge_{t\in T} S(A, A_t)$;
(S6) $S(A, \alpha \to B) = \alpha \to S(A, B)$;
(S7) $S(A, \alpha * B) \geq \alpha * S(A, B)$;
(I1) $I(A, B) = I(B, A)$;
(I2) $I(A, \bigvee_{t\in T} A_t) = \bigvee_{t\in T} I(A, A_t)$;
(I3) $I(A, \alpha * B) = \alpha * I(A, B)$;
(I4) $I(A, \alpha \to B) \leq \alpha \to I(A, B)$;
(IS) $I(A, B) \to \alpha = S(A, B \to \alpha)$.

Next we recall some notions and notations with respect to generalized neighborhood system-based approximation operators, L-fuzzy relation-based approximation operators and L-fuzzy covering-based approximation operators from the literatures.

Definition 2.4. [35, 47, 48]. *Let X be the universe of discourse and 2^X denote the power set of X. Then a function $n : X \longrightarrow 2^{2^X}$ is called a generalized neighborhood system operator on X if for any $x \in X$, $n(x)$ is nonempty. Usually, $n(x)$ is called generalized neighborhood system of x, and each $K \in n(x)$ is called a neighborhood of x. Furthermore,*

(1) n is said to be serial, if for any $x \in X$ and $K \in n(x)$, $K \neq \emptyset$.

(2) n is said to be reflexive, if for any $x \in X$ and $K \in n(x)$, $x \in K$.

(3) n is said to be transitive, if for any $x \in X$ and $K \in n(x)$, there an $V \in n(x)$ such that for any $y \in V$ there exists an $V_y \in n(y)$ with $V_y \subseteq K$.

(4) n is said to be unary, if for any $x \in X$ and $K, V \in n(x)$, there an $M \in n(x)$ such that $M \subseteq K \cap V$.

Definition 2.5. *Let $n : X \longrightarrow 2^{2^X}$ be a generalized neighborhood system operator on X. Then for each subset A of X, the lower and upper approximation operators $\underline{n}(A)$ and $\overline{n}(A)$, are defined as follows:*

$$\underline{n}(A) = \{x \in X \mid \exists K \in n(x), K \subseteq A\}, \quad \overline{n}(A) = \{x \in X \mid \forall K \in n(x), K \cap A \neq \emptyset\}.$$

An L-fuzzy set $R : X \times X \longrightarrow L$ is refereed to as an L-fuzzy relation on X. For any $(x, y) \in X \times X$, $R(x, y)$ is interpreted as the related degree between x and y. Furthermore,

(1) R is said to be serial if for all $x \in X$, $\bigvee_{y\in X} R(x, y) = \top$.

(2) R is said to be reflexive if for all $x \in X$, $R(x, x) = \top$.

(3) R is said to be transitive if for all $x, y, z \in X$, $R(x, y) * R(y, z) \leq R(x, z)$.

Definition 2.6. [32, 34] *Let $R : X \times X \longrightarrow L$ be an L-fuzzy relation on X. Then for any $A \in L^X$, the lower and upper approximation operators $\underline{R}(A)$ and $\overline{R}(A)$ are defined as follows:*

$$\underline{R}(A)(x) = S(R(x, -), A) = \bigwedge_{y\in X} (R(x, y) \to A(y)),$$

$$\overline{R}(A)(x) = I(R(x,-),A) = \bigvee_{y\in X}(R(x,y) * A(y)).$$

A subset $\mathcal{C}$ of L^X is referred to be an L-fuzzy covering on X if $\bigvee \mathcal{C} = \top$.

Definition 2.7. [22, 24] *Let $\mathcal{C}$ be an L-fuzzy covering on X. Then for any $A \in L^X$ and $x \in X$, the lower and upper approximation operators $\underline{\mathcal{C}}(A)$ and $\overline{\mathcal{C}}(A)$ are defined as follows:*

$$\underline{\mathcal{C}}(A)(x) = \bigvee_{K\in\mathcal{C}}(K(x) * S(K,A)), \overline{\mathcal{C}}(A)(x) = \bigwedge_{K\in\mathcal{C}}(K(x) \to I(K,A)).$$

3 Rough approximation operators based on L-fuzzy generalized neighborhood system

In this section, we will give a notion of L-fuzzy generalized neighborhood systems and then use it to define a pair of lower and upper approximation operators. We also prove that generalized neighborhood system-based approximation operators [35, 47, 48], L-fuzzy relation-based approximation operators [10, 34] and L-fuzzy covering-based approximation operators [22, 24] can be regarded as special L-fuzzy generalized neighborhood system-based approximation operators.

Definition 3.1. *Let X be the universe of discourse. Then the function $N : X \longrightarrow L^{L^X}$ is called an L-fuzzy generalized neighborhood system operator on X, if for any $x \in X$, $\bigvee_{K\in L^X} N(x)(K) = \top$. Usually, $N(x)$ is called an L-fuzzy generalized neighborhood system of x and $N(x)(K)$ is interpreted as the degree of that K is a neighborhood of x. The condition "$\bigvee_{K\in L^X} N(x)(K) = \top$" is an lattice-valued interpretation of that "$N(x)$ is nonempty".*

Definition 3.2. *Let $N : X \longrightarrow L^{L^X}$ be an L-fuzzy generalized neighborhood system operator. Then for each $A \in L^X$, the lower and upper approximation operators $\underline{N}(A)$ and $\overline{N}(A)$ are defined as follows:*

$$\underline{N}(A)(x) = \bigvee_{K\in L^X}(N(x)(K) * S(K,A)), \quad \overline{N}(A)(x) = \bigwedge_{K\in L^X}(N(x)(K) \to I(K,A)).$$

The following lemma shows that the approximation operators based on generalized neighborhood systems can be regarded as a special case of approximation operators based on L-fuzzy generalized neighborhood systems.

Lemma 3.3. *Let $n : X \longrightarrow 2^{2^X}$ be a generalized neighborhood system operator on X. We define an L-fuzzy generalized neighborhood system operator N_n as follows: for any $x \in X$ and $K \in L^X$,*

$$N_n(x)(K) = \begin{cases} \top, & K = \top_A \text{ for some } A \in n(x); \\ \bot, & \textit{otherwise.} \end{cases}$$

Then $\underline{N_n}(\top_A) = \top_{\underline{n}(A)}$ and $\overline{N_n}(\top_A) = \top_{\overline{n}(A)}$, for any $A \in 2^X$. It follows immediately that if $L = \{\bot, \top\}$ then there is no difference between the approximation operators generated by n and those generated by N_n.

Proof. For any $x \in X$, since $n(x)$ is nonempty then there exists an $A \in n(x)$. It follows that $\bigvee_{K\in L^X} N_n(x)(K) = \top$, i.e., N_n is an L-fuzzy generalized neighborhood system operator.

For any $A \in 2^X$ and $x \in X$. By the definition of N_n, we have

$$\underline{N_n}(\top_A)(x) = \bigvee_{K\in L^X}(N_n(x)(K) * S(K,\top_A)) = \bigvee_{B\in n(x)} S(\top_B,\top_A) = \begin{cases} \top, & \exists B \in n(x) s.t. B \subseteq A; \\ \bot, & \text{otherwise.} \end{cases} = \top_{\underline{n}(A)}(x).$$

$$\overline{N_n}(\top_A)(x) = \bigwedge_{K\in L^X}(N_n(x)(K) \to I(K,\top_A)) = \bigwedge_{B\in n(x)} I(\top_B,\top_A) = \begin{cases} \top, & \forall B \in n(x) s.t. B \cap A \neq \emptyset; \\ \bot, & \text{otherwise.} \end{cases} = \top_{\overline{n}(A)}(x).$$

Hence $\underline{N_n}(\top_A) = \top_{\underline{n}(A)}$ and $\overline{N_n}(\top_A) = \top_{\overline{n}(A)}$, for any $A \in 2^X$.

The following lemma shows that the approximation operators based on L-fuzzy relations can be regarded as a special case of approximation operators based on L-fuzzy generalized neighborhood systems.

Lemma 3.4. *Let $R : X \times X \longrightarrow L$ be an L-fuzzy relation on X. We define an L-fuzzy generalized neighborhood system operator N_R as follows: for any $x \in X$ and $K \in L^X$, $N_R(x)(K) = \begin{cases} \top, & K = R(x,-); \\ \bot, & \text{otherwise.} \end{cases}$*
Then $\underline{N_R}(A) = \underline{R}(A)$ and $\overline{N_R}(A) = \overline{R}(A)$, for any $A \in L^X$.

Proof. For any $x \in X$, we have $\bigvee_{K\in L^X} N_R(x)(K) \geq N_R(x)(R(x,-)) = \top$. Hence N_R is an L-fuzzy generalized neighborhood system operator. Then for any $A \in L^X$ and $x \in X$. By the definition of N_R, we have

$$\underline{N_R}(A)(x) = \bigvee_{K\in L^X} (N_R(x)(K) * S(K,A)) = \top * S(R(x,-),A) = \underline{R}(A)(x).$$

$$\overline{N_R}(A)(x) = \bigwedge_{K\in L^X} (N_R(x)(K) \to I(K,A)) = \top \to I(R(x,-),A) = \overline{R}(A)(x).$$

Hence $\underline{N_R}(A) = \underline{R}(A)$ and $\overline{N_R}(A) = \overline{R}(A)$, for any $A \in L^X$.

It should be pointed out that approximation operators based on L-fuzzy generalized neighborhood systems will never be approximation operators based on L-fuzzy relations. We will take the lower approximation operator as an example to illustrate this point. Note that for any L-fuzzy relation R on a set X, the lower approximation operator $\underline{R}$ satisfies the property (AM): for any subfamily $\{A_i\}_{i\in I}$ of L^X, $\underline{R}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \underline{R}(A_i)$, see [32, 34]. In the following, we give an example to show that even the lower approximation operator based on generalized neighborhood system does not hold (AM) generally.

Example 3.5. *Let $X = \{a,b,c\}$ and n be a generalized neighborhood system on X defined as $n(a) = \{\{b\},\{a,c\}\}, n(b) = \{\{a,b\},\{a,c\}\}, n(c) = \{\{a,b\},\{c\}\}$. Then it is easily seen that $\underline{n}(\{b,c\}) = \{a,c\}, \underline{n}(\{a,c\}) = \{a,b,c\}, \underline{n}(\{c\}) = \{c\}$. Thus $\underline{n}(\{b,c\}) \cap \underline{n}(\{a,c\}) \neq \underline{n}(\{b,c\} \cap \{a,c\})$. This shows that $\underline{n}$ does not satisfy (AM) generally.*

The following lemma shows that the approximation operators based on L-fuzzy coverings can be regarded as a special case of approximation operators based on L-fuzzy generalized neighborhood systems.

Lemma 3.6. *Let $\mathcal{C}$ be an L-fuzzy covering on X. We define an L-fuzzy generalized neighborhood system operator $N_{\mathcal{C}}$ as follows: for any $x \in X$ and $K \in L^X$, $N_{\mathcal{C}}(x)(K) = \begin{cases} K(x), & K \in \mathcal{C}; \\ \bot, & \text{otherwise.} \end{cases}$ Then $\underline{N_{\mathcal{C}}}(A) = \underline{\mathcal{C}}(A)$ and $\overline{N_{\mathcal{C}}}(A) = \overline{\mathcal{C}}(A)$, for any $A \in L^X$.*

Proof. For any $x \in X$, we have $\bigvee_{K\in L^X} N_{\mathcal{C}}(x)(K) = \bigvee_{K\in\mathcal{C}} K(x) = \top$. Hence $N_{\mathcal{C}}$ is an L-fuzzy generalized neighborhood system operator. Then for any $A \in L^X$ and $x \in X$, we have

$$\underline{N_{\mathcal{C}}}(A)(x) = \bigvee_{K\in L^X} (N_{\mathcal{C}}(x)(K) * S(K,A)) = \bigvee_{K\in\mathcal{C}} (K(x) * S(K,A)) = \underline{\mathcal{C}}(A)(x).$$

$$\overline{N_{\mathcal{C}}}(A)(x) = \bigwedge_{K\in L^X} (N_{\mathcal{C}}(x)(K) \to I(K,A)) = \bigwedge_{K\in\mathcal{C}} (K(x) \to I(K,A)) = \overline{\mathcal{C}}(A)(x).$$

Hence $\underline{N_{\mathcal{C}}}(A) = \underline{\mathcal{C}}(A)$ and $\overline{N_{\mathcal{C}}}(A) = \overline{\mathcal{C}}(A)$, for any $A \in L^X$.

It should be pointed out that approximation operators based on L-fuzzy generalized neighborhood systems will never be approximation operators based on L-fuzzy coverings. We will take the lower approximation operator as an example to illustrate this point. Note that for any L-fuzzy covering $\mathcal{C}$ on a set X, the lower approximation operator $\underline{\mathcal{C}}$ satisfies the property (ET): for any $A \in L^X$, $\underline{\mathcal{C}}(A) \leq A$, see [22]. In the following, we give an example to show that even the lower approximation operator based on generalized neighborhood system does not hold (ET) generally.

Example 3.7. *Let $X = \{a,b,c\}$ and n be a generalized neighborhood system on X defined as*

$$n(a) = \{\{b,c\},\{a,c\},\emptyset\}, n(b) = \{\{a,c\},\{c\}\}, n(c) = \{\{a,b\},\{b,c\}\}$$

Then it is easily seen that $\underline{n}(\emptyset) = \{a\} \not\subseteq \emptyset$. This shows that $\underline{n}$ does not satisfy (ET) generally.

Next we give the basic properties of lower and upper approximation operators based on L-fuzzy generalized neighborhood systems.

Proposition 3.8. *Let N be an L-fuzzy generalized neighborhood system operator on X. Then* (1) $\overline{N}(\bot)=\bot$, $\underline{N}(\top)=\top$;

(2) *If $A\le B$, then $\overline{N}(A)\le\overline{N}(B)$ and $\underline{N}(A)\le\underline{N}(B)$;*

(3) *For any $a\in L, A\in L^X$, $\overline{N}(a*A)\ge a*\overline{N}(A)$, $\underline{N}(a*A)\ge a*\underline{N}(A)$.*

Proof. (1) For any $x\in X$, we have

$$\overline{N}(\bot)(x)=\bigwedge_{K\in L^X}(N(x)(K)\to I(K,\bot))=\bigwedge_{K\in L^X}(N(x)(K)\to\bot)=(\bigvee_{K\in L^X}N(x)(K))\to\bot=\top\to\bot=\bot.$$

$$\underline{N}(\top)(x)=\bigvee_{K\in L^X}(N(x)(K)*S(K,\top))=\bigvee_{K\in L^X}N(x)(K)=\top.$$

Hence $\overline{N}(\bot)=\bot$, $\underline{N}(\top)=\top$.

(2) The proof is straightforward and is omitted.

(3) For any $a\in L, A\in L^X$ and $x\in X$, we have

$$\begin{aligned}\overline{N}(a*A)(x) &= \bigwedge_{K\in L^X}(N(x)(K)\to I(K,a*A)\overset{\text{(I3)}}{=}\bigwedge_{K\in L^X}[N(x)(K)\to\big(a*I(K,A)\big)]\\ &\overset{\text{(S7)}}{\ge} a*\bigwedge_{K\in L^X}(N(x)(K)\to I(K,A))=a*\overline{N}(A)(x).\end{aligned}$$

$$\begin{aligned}\underline{N}(a*A)(x) &= \bigvee_{K\in L^X}(N(x)(K)*S(K,a*A))\overset{\text{(S7)}}{\ge}\bigvee_{K\in L^X}[N(x)(K)*\big(a*S(K,A)\big)]\\ &\overset{\text{(I3)}}{=} a*\bigvee_{K\in L^X}(N(x)(K)*S(K,A))=a*\underline{N}(A)(x).\end{aligned}$$

Hence $\overline{N}(a*A)\ge a*\overline{N}(A)$, $\underline{N}(a*A)\ge a*\underline{N}(A)$.

Theorem 3.9. *Let N be an L-fuzzy generalized neighborhood system operator on X. Then for any $A\in L^X$,*

(1) $\overline{N}(A)=\bigwedge_{b\in L}(\underline{N}(A\to b)\to b)$.

(2) *If L satisfies the double negative law then*

$$\underline{N}(A)=\neg\overline{N}(\neg A),\ \overline{N}(A)=\neg\underline{N}(\neg A).$$

Proof. (1) For any $A\in L^X$ and $x\in X$,

$$\begin{aligned}\bigwedge_{b\in L}(\underline{N}(A\to b)(x)\to b) &= \bigwedge_{b\in L}\Big([\bigvee_{K\in L^X}(N(x)(K)*S(K,A\to b))]\to b\Big)\overset{\text{(IS)}}{=}\bigwedge_{b\in L}\bigwedge_{K\in L^X}\Big([(N(x)(K)*(I(K,A)\to b))]\to b\Big)\\ &= \bigwedge_{b\in L}\bigwedge_{K\in L^X}\Big(N(x)(K)\to[(I(K,A)\to b)\to b]\Big)=\bigwedge_{K\in L^X}\Big(N(x)(K)\to\bigwedge_{b\in L}[(I(K,A)\to b)\to b]\Big)\\ \text{by Proposition 2.1(7)} &= \bigwedge_{K\in L^X}(N(x)(K)\to I(K,A))=\overline{N}(A)(x).\end{aligned}$$

Hence $\overline{N}(A)=\bigwedge_{b\in L}(\underline{N}(A\to b)\to b)$.

(2) For any $A\in L^X$ and $x\in X$, it follows by Proposition 2.2 that

$$\begin{aligned}\neg\overline{N}(\neg A)(x) &= \neg[\bigwedge_{K\in L^X}(N(x)(K)\to I(K,\neg A))]=\bigvee_{K\in L^X}\neg[N(x)(K)\to I(K,\neg A)]=\bigvee_{K\in L^X}[N(x)(K)*\neg(I(K,\neg A))]\\ &\overset{\text{(IS)}}{=}\bigvee_{K\in L^X}(N(x)(K)*S(K,A))=\underline{N}(A)(x).\end{aligned}$$

Hence $\underline{N}(A)=\neg\overline{N}(\neg A)$ and $\overline{N}(A)=\neg\underline{N}(\neg A)$ can be similarly proved.

4 Some special L-fuzzy generalized neighborhood systems and related approximation operators

In this section, we will discuss some special L-fuzzy generalized neighborhood systems and related approximation operators.

4.1 Serial L-fuzzy generalized neighborhood system-based approximation operators

In this subsection, we will define the notion of serial L-fuzzy generalized neighborhood systems and discuss the related approximation operators.

Definition 4.1. *An L-fuzzy generalized neighborhood system operator N is said to be serial (denoted by SE), if for any $x \in X$ and $K \in L^X$, $N(x)(K) \leq \bigvee_{y\in X} K(y)$.*

For a generalized neighborhood system operator n, it is easily seen that N_n is serial iff n is serial. Thus the serial condition in L-fuzzy generalized neighborhood system operator is an extension of the corresponding condition in generalized neighborhood system operator. Furthermore, it is not difficult to observe that: (1) for an L-fuzzy relation R, N_R is serial iff R is serial; (2) for an L-fuzzy covering $\mathcal{C}$, $N_{\mathcal{C}}$ is serial. Where, N_R and $N_{\mathcal{C}}$ are defined in Lemma 3.4 and Lemma 3.6 respectively.

Proposition 4.2. *Let N be an L-fuzzy generalized neighborhood system operator. If N is serial then $\underline{N}(\bot) = \bot$ and the reserve conclusion holds if L satisfies the double negative law.*

Proof. Let N be serial. Then for any $x \in X$,

$$\begin{aligned}\underline{N}(\bot)(x) &= \bigvee_{K\in L^X}(N(x)(K) * S(K,\bot)) = \bigvee_{K\in L^X}(N(x)(K) * \bigwedge_{y\in X}(K(y)\to\bot)) \\ &= \bigvee_{K\in L^X}(N(x)(K) * ([\bigvee_{y\in X}K(y)]\to\bot)) \overset{\text{SE}}{\leq} \bigvee_{K\in L^X}\big([\bigvee_{y\in X}K(y)] * ([\bigvee_{y\in X}K(y)]\to\bot)\big) \leq \bot.\end{aligned}$$

Hence $\underline{N}(\bot) = \bot$. Conversely, let $\underline{N}(\bot) = \bot$. Then for any $x \in X$, $\bigvee_{K\in L^X}(N(x)(K) * S(K,\bot)) = \underline{N}(\bot)(x) = \bot$. It follows that for any $K \in L^X$, $N(x)(K) * S(K,\bot) \leq \bot$ and then by the double negative law

$$N(x)(K) \quad\leq\quad S(K,\bot)\to\bot = (\bigvee_{y\in X}K(y)\to\bot)\to\bot = \bigvee_{y\in X}K(y).$$

Proposition 4.3. *Let N be an L-fuzzy generalized neighborhood system operator. Then the following conditions are equivalent: (1) N is serial; (2) $\overline{N}(\top) = \top$.*

Proof. (1) $\Rightarrow$ (2). Let N be serial. Then for any $x \in X$,

$$\overline{N}(\top)(x) = \bigwedge_{K\in L^X}(N(x)(K)\to I(K,\top)) = \bigwedge_{K\in L^X}(N(x)(K)\to\bigvee_{y\in X}K(y)) \overset{\text{SE}}{\geq} \bigwedge_{K\in L^X}(N(x)(K)\to N(x)(K)) = \top.$$

Hence $\overline{N}(\top) = \top$.

(2) $\Rightarrow$ (1). Let $\overline{N}(\top) = \top$. Then for any $x \in X$, $\bigwedge_{K\in L^X}(N(x)(K)\to I(K,\top)) = \overline{N}(\top)(x) = \top$. It follows that for any $K \in L^X$, $N(x)(K) \leq I(K,\top) = \bigvee_{y\in X}K(y)$.

4.2 Reflexive L-fuzzy generalized neighborhood system-based approximation operators

In this subsection, we will define the notion of reflexive L-fuzzy generalized neighborhood systems and discuss the related approximation operators.

Definition 4.4. *An L-fuzzy generalized neighborhood system operator N is said to be reflexive (denoted by RE), if for any $x \in X$ and $K \in L^X$, $N(x)(K) \leq K(x)$.*

For a generalized neighborhood system operator n, it is easily seen that N_n is reflexive iff n is reflexive. Thus the reflexive condition in L-fuzzy generalized neighborhood system operator is an extension of the corresponding condition in generalized neighborhood system operator. Furthermore, it is not difficult to observe that: (1) for an L-fuzzy relation R, N_R is reflexive iff R is reflexive; (2) for an L-fuzzy covering $\mathcal{C}$, $N_{\mathcal{C}}$ is reflexive.

Proposition 4.5. *Let N be an L-fuzzy generalized neighborhood system operator. Then the following conditions are equivalent:*
(1) *N is reflexive;*
(2) *$\underline{N}(A) \leq A$, for any $A \in L^X$.*

Proof. (1) ⇒ (2) Let N be reflexive. Then for any $A \in L^X$ and for any $x \in X$,

$$\underline{N}(A)(x) = \bigvee_{K\in L^X}(N(x)(K) * S(K,A)) = \bigvee_{K\in L^X}[N(x)(K) * \bigwedge_{y\in X}(K(y) \to A(y))] \overset{\text{RE}}{\leq} \bigvee_{K\in L^X}(K(x) * (K(x) \to A(x))) \leq A(x).$$

Hence $\underline{N}(A) \leq A$.

(2) ⇒ (1) Let $\underline{N}(A) \leq A$, for any $A \in L^X$. Then for any $x \in X$, $\underline{N}(A)(x) = \bigvee_{K\in L^X}(N(x)(K) * S(K,A)) \leq A(x)$. Taking $K = A$ we get that $N(x)(A) \leq A(x)$. Hence N is reflexive.

Proposition 4.6. *Let N be an L-fuzzy generalized neighborhood system operator. If N is reflexive then $A \leq \overline{N}(A)$, for any $A \in L^X$ and the reserve conclusion holds if L satisfies the double negative law.*

Proof. Let N be reflexive. Then for any $A \in L^X$ and for any $x \in X$,

$$\overline{N}(A)(x) = \bigwedge_{K\in L^X}(N(x)(K) \to I(K,A)) = \bigwedge_{K\in L^X}[N(x)(K) \to \bigvee_{y\in X}(K(y) * A(y))] \overset{\text{RE}}{\geq} \bigwedge_{K\in L^X}(K(x) \to (K(x) * A(x))) \geq A(x).$$

Hence $A \leq \overline{N}(A)$. Conversely, let $A \leq \overline{N}(A)$, for any $A \in L^X$. Then for any $x \in X$, we have $\neg A(x) \leq \overline{N}(\neg A)(x)$, i.e., $\neg\overline{N}(\neg A)(x) \leq A(x)$. Then by Theorem 3.9 (2) and Proposition 4.5, we have $N(x)(A) \leq A(x)$.

4.3 Unary L-fuzzy generalized neighborhood system-based approximation operators

In this subsection, we will define the notion of unary L-fuzzy generalized neighborhood systems and discuss the related approximation operators.

Definition 4.7. *An L-fuzzy generalized neighborhood system operator N is said to be unary (denoted by UN), if for any $x \in X$ and $K, V \in L^X$,*

$$N(x)(K) * N(x)(V) \leq \bigvee_{U\in L^X}(N(x)(U) * S(U, K * V)).$$

For a generalized neighborhood system operator n, it is easily seen that N_n is unary iff n is unary. Thus the unary condition in L-fuzzy generalized neighborhood system operator is an extension of the corresponding condition in generalized neighborhood system operator. Furthermore, it is not difficult to observe that:

(1) For an L-fuzzy relation R, if L is a frame, i.e., $* = \wedge$, then N_R is unary.

(2) For an L-fuzzy covering $\mathcal{C}$, $N_{\mathcal{C}}$ is unary iff for any $K, V \in \mathcal{C}$ and any $x \in X$, $K(x) * V(x) \leq \bigvee_{U\in\mathcal{C}}(U(x) * S(U, K * V))$. If we let $L = \{\bot, \top\}$ then $N_{\mathcal{C}}$ is unary iff for any $K, V \in \mathcal{C}$ and any $x \in K \cap V$, there exists an $U \in \mathcal{C}$ such that $x \in U$ and $U \subseteq K \cap V$. Therefore, when $L = \{\bot, \top\}$, $N_{\mathcal{C}}$ is unary iff $\mathcal{C}$ is a base of some topology on X.

Proposition 4.8. *Let N be an L-fuzzy generalized neighborhood system operator. Then the following conditions are equivalent:*
(1) N is unary;
(2) $\underline{N}(A * B) \geq \underline{N}(A) * \underline{N}(B)$, for any $A, B \in L^X$.

Proof. (1) ⇒ (2) Let N be unary. For any $A, B \in L^X$ and for any $x \in X$

$$\begin{aligned}
(\underline{N}(A) * \underline{N}(B))(x) &= \bigvee_{K\in L^X}(N(x)(K) * S(K,A)) * \bigvee_{V\in L^X}(N(x)(V) * S(V,B)) \\
\text{by (S3)} \quad &\leq \bigvee_{K,V\in L^X}[(N(x)(K) * N(x)(V)) * S(K * V, A * B)] \\
\text{by UN} \quad &\leq \bigvee_{K,V\in L^X}[\bigvee_{U\in L^X}(N(x)(U) * S(U, K * V)) * S(K * V, A * B)] \\
\text{by (S4)} \quad &\leq \bigvee_{U\in L^X}[N(x)(U) * S(U, A * B)] = \underline{N}(A * B)(x).
\end{aligned}$$

(2) ⇒ (1) Let $\underline{N}(A * B) \geq \underline{N}(A) * \underline{N}(B)$, for any $A, B \in L^X$. Then for any $x \in X$, we have $(\underline{N}(A) * \underline{N}(B))(x) \leq \underline{N}(A * B)(x)$. It follows that

$$\bigvee_{K\in L^X}(N(x)(K) * S(K,A)) * \bigvee_{V\in L^X}(N(x)(V) * S(V,B)) \leq \bigvee_{U\in L^X}(N(x)(U) * S(U, A * B)).$$

Taking $K = A, V = B$ in the above inequality we get that $N(x)(A) * N(x)(B) \leq \bigvee_{U\in L^X}(N(x)(U) * S(U, A * B))$.

Proposition 4.9. *Let N be an L-fuzzy generalized neighborhood system operator and L satisfy the double negative law. Then N is unary if and only if $\overline{N}(A \oplus B) \leq \overline{N}(A) \oplus \overline{N}(B)$, for any $A, B \in L^X$.*

Proof. It follows by Theorem 3.9 (2) and Proposition 4.8.

4.4 Transitive L-fuzzy generalized neighborhood system-based approximation operators

In this subsection, we will define the notion of transitive L-fuzzy generalized neighborhood systems and discuss the related approximation operators.

Definition 4.10. *An L-fuzzy generalized neighborhood system operator N is said to be transitive (denoted by TR), if for any $x \in X$ and $K \in L^X$,*

$$N(x)(K) \leq \bigvee_{V \in L^X} [N(x)(V) * \bigwedge_{y \in X} \big(V(y) \to \bigvee_{V_y \in L^X} (N(y)(V_y) * S(V_y, K))\big)].$$

For a generalized neighborhood system operator n, it is easily seen that N_n is transitive iff n is transitive. Thus the transitive condition in L-fuzzy generalized neighborhood system operator is an extension of the corresponding condition in generalized neighborhood system operator. Furthermore, it is not difficult to check that for an L-fuzzy relation R, N_R is transitive if and only if R is transitive.

Proposition 4.11. *Let N be an L-fuzzy neighborhood system operator. Then the following conditions are equivalent:*
(1) *N is transitive;*
(2) *$\underline{N}(A) \leq \underline{N}(\underline{N}(A))$, for any $A \in L^X$.*

Proof.(1) $\Rightarrow$ (2) Let N be transitive. Then for any $A \in L^X$ and for any $x \in X$,

$$\begin{aligned}
\underline{N}(A)(x) = \bigvee_{K \in L^X} (N(x)(K) * S(K, A)) &\overset{\text{TR}}{\leq} \bigvee_{K \in L^X} \Big(\bigvee_{V \in L^X} [N(x)(V) * \bigwedge_{y \in X} (V(y) \to (\bigvee_{V_y \in L^X} N(y)(V_y) * S(V_y, K)))] * S(K, A)\Big) \\
&\leq \bigvee_{K, V \in L^X} \Big(N(x)(V) * \bigwedge_{y \in X} [S(K, A) * (V(y) \to (\bigvee_{V_y \in L^X} N(y)(V_y) * S(V_y, K)))]\Big) \\
&\leq \bigvee_{K, V \in L^X} \Big(N(x)(V) * [\bigwedge_{y \in X} (V(y) \to (S(K, A) * (\bigvee_{V_y \in L^X} N(y)(V_y) * S(V_y, K))))]\Big) \\
&\leq \bigvee_{V \in L^X} \Big(N(x)(V) * [\bigwedge_{y \in X} (V(y) \to \bigvee_{V_y \in L^X} (N(y)(V_y) * S(V_y, A))]\Big) \\
&= \bigvee_{V \in L^X} \Big(N(x)(V) * [\bigwedge_{y \in X} (V(y) \to \underline{N}(A)(y))]\Big) \\
&= \bigvee_{V \in L^X} \big(N(x)(V) * S(V, \underline{N}(A))\big) = \underline{N}(\underline{N}(A))(x).
\end{aligned}$$

(2) $\Rightarrow$ (1) Let $\underline{N}(A) \leq \underline{N}(\underline{N}(A))$, for any $A \in L^X$. Then for any $x \in X$, we have

$$\begin{aligned}
& \underline{N}(A)(x) \leq \underline{N}(\underline{N}(A))(x) \\
\Rightarrow\ & \bigvee_{K \in L^X} (N(x)(K) * S(K, A)) \leq \bigvee_{V \in L^X} (N(x)(V) * S(V, \underline{N}(A))), \text{ let } K = A \\
\Rightarrow\ & N(x)(A) \leq \bigvee_{V \in L^X} (N(x)(V) * \bigwedge_{y \in X} (V(y) \to \underline{N}(A)(y))) \\
\Rightarrow\ & N(x)(A) \leq \bigvee_{V \in L^X} [N(x)(V) * \bigwedge_{y \in X} (V(y) \to \bigvee_{V_y \in L^X} (N(y)(V_y) * S(V_y, A)))].
\end{aligned}$$

Hence N is transitive.

Proposition 4.12. *Let N be an L-fuzzy generalized neighborhood system operator and L satisfy the double negative law. Then N is transitive if and only if $\overline{N}(A) \geq \overline{N}(\overline{N}(A))$, for any $A \in L^X$.*

Proof. It follows by Theorem 3.9 (2) and Proposition 4.11.

5 Conclusions

In this paper, a pair of lower and upper approximation operators based on L-fuzzy generalized neighborhood systems are proposed. It is proved that the L-fuzzy generalized neighborhood system-based approximation operators include, generalized neighborhood system-based approximation operators, L-fuzzy relation-based approximation operators and L-fuzzy covering-based approximation operators as their special circumstances. Furthermore, when L-fuzzy generalized neighborhood system is serial, reflexive, unary and transitive, the corresponding approximation operators are discussed and characterized, respectively.

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