

On orders induced by implications

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Abstract

In this paper, the orders induced by the residual implications obtained from uninorms are investigated. A necessary and sufficient condition is presented so that the ordinal sum of fuzzy implications satisfies the law of importation with a t-norm T . Some relationships between the orders induced by an ordinal sum implication and its summands are determined. The algebraic structures obtained from the orders induced by the residual implications and the ordinal sum implications are discussed.

Keywords: Fuzzy implications, ordinal sum, law of importation, partial order, residual implication.

1 Introduction

Fuzzy implication functions generalize the classical implication to fuzzy logic. They have a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, e.t.c. (see [1, 6, 8, 20, 22, 26]), fuzzy mathematical morphology [22], fuzzy DI-subsethood measures [4, 5], data mining [30]. The great quantity of applications has lead systematically to implications from the theoretical point of view. In this sense, many properties of fuzzy implications have been extensively studied. One of the most important properties is the so-called law of importation and given by, for all $x, y, z \in [0, 1]$, $I(T(x, y), z) = I(x, I(y, z))$, where T is a t-norm or a conjunctive uninorm and I is a fuzzy implication. It can be extensively found in the papers [8, 20, 21, 23]. This property is extremely related to the exchange principle and it has proved to be useful in simplifying the process of applying the CRI in many cases, see [1, 8]. Also, it has shown to be a crucial property to characterization of Yager's implications [25].

Recently, an order generating problem from logical operators has been an attractive subject for many researchers. In this sense, an order induced by t-norms was firstly introduced by Karaçal and Kesicioğlu in [11] which was a starting point for order generating from logical operators. Later on, the orders induced by t-conorms, uninorms and nullnorms on a bounded lattice [2] have been introduced and the relations between the algebraic structures obtained from such orders have been investigated, see [7, 11, 13, 14, 15, 18].

In [16], a partial order induced by implications satisfying the exchange principle (EP) and the contrapositive symmetry (CP) w.r.t. the strong natural negation has been defined as follows: for any $x, y \in L$,

$$y \preceq_I x \Leftrightarrow \exists \ell \in L \text{ such that } I(\ell, x) = y, \quad (1)$$

where I is an implication on a bounded lattice L .

In [9], an order \sqsubseteq_I has been introduced under some required and more lenient conditions than the one given in [16]. The order \sqsubseteq_I has been given as: for any $x, y \in L$,

$$x \sqsubseteq_I y \Leftrightarrow \exists \ell \in L \text{ such that } I(\ell, x) = y, \quad (2)$$

where I satisfies the law of importation to a t-norm T and the neutrality principle (NP). Later on, in [12], the partial order given in (2) has been redefined by implications satisfying the law of importation to a conjunctive uninorm U with

a neutral element e . That is, the relation given in (2) has been extended much more general form as follows: For any implication satisfying the law of importation to a conjunctive uninorm U with a neutral element e , if I satisfies the neutrality principle w.r.t. e , then the relation

$$x \preceq_I y \Leftrightarrow \exists \ell \leq e \quad \text{such that} \quad I(\ell, x) = y. \quad (3)$$

is a partial order. Also, the properties of the order have been discussed. Such a method has allowed to work with wider classes of implications imposing the order.

In this paper, based on the order defined by the method in [12], we investigate the properties of the orders induced by the residual implications obtained from uninorms. We present a necessary and sufficient condition so that the ordinal sum of fuzzy implications [29] satisfies the law of importation with a t-norm T . We determine some relationships between the orders induced by an ordinal sum implication and its summand implications. Moreover, we investigate the algebraic structures obtained from the orders induced by the residual implications and the ordinal sum implications. The paper is organized as follows: We shortly recall some basic notions in Section 2. In Section 3, we give some properties of the orders induced by the residual implications obtained from uninorms and we investigate the algebraic structures obtained by them. In Section 4, we determine a necessary and sufficient condition so that the ordinal sum of fuzzy implications satisfies the law of importation with a t-norm T . Thus, we show that the relation induced by the ordinal sum of fuzzy implications is an order under some conditions. We present some relationships between the orders induced by an ordinal sum implication and its summands. Also, we investigate the relationships between the algebraic structures obtained from the orders induced by the ordinal sum implications and their summands.

2 Preliminaries

In this section, we recall some basic notions and results.

Definition 2.1. [10, 11, 19] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \rightarrow L$ is called a uninorm on L , if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

It is clear that the function U is a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm, we have $U(0, 1) \in \{0, 1\}$, and a uninorm U is said conjunctive when $U(1, 0) = 0$ and disjunctive $U(1, 0) = 1$.

In this study, the notation $\mathcal{U}(e)$ will be used for the set of all uninorms on L with a neutral element $e \in L$.

Definition 2.2. [1, 17] A t-norm T on $[0, 1]$ is called

- (i) strict, if it is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$;
- (ii) nilpotent, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T , i.e., if there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$, where $x_T^{[n]} = T(x, x, \dots, x)$.

The followings are the four basic t-norms T_M, T_P, T_{LK} and T_D given by respectively:

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = x \cdot y, \quad T_{LK}(x, y) = \max(x + y - 1, 0), \quad T_D(x, y) = \begin{cases} 0 & (x, y) \in [0, 1)^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Definition 2.3. [27] A uninorm U with a neutral element $e \in (0, 1)$ is said to be in \mathcal{U}_{\min} when it is given by

$$U(x, y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

and is said to be in \mathcal{U}_{\max} when it is given by

$$U(x, y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

In both expressions, T denotes a t-norm and S denotes a t-conorm. If U is a conjunctive (disjunctive) uninorm, then we will write $U_{T,S,e}^c$ ($U_{T,S,e}^d$, respectively).

Definition 2.4. [1] A function $I : L^2 \rightarrow L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:

(I1) I is a decreasing operation on the first variable, that is, for every $a, b \in L$ with $a \leq b$, $I(b, y) \leq I(a, y)$, for all $y \in L$.

(I2) I is an increasing operation on the second variable, that is, for every $a, b \in L$ with $a \leq b$, $I(x, a) \leq I(x, b)$, for all $x \in L$.

(I3) $I(0, 0) = 1$.

(I4) $I(1, 1) = 1$.

(I5) $I(1, 0) = 0$.

Example 2.5. [1] *The followings are well-known implications on the unit interval $[0, 1]$.*

$$\begin{aligned}
 I_{LK}(x, y) &= \min(1, 1 - x + y), & I_{RC}(x, y) &= 1 - x + xy, \\
 I_{KD}(x, y) &= \max(1 - x, y), & I_{GD}(x, y) &= \begin{cases} 1 & x \leq y, \\ y & x > y, \end{cases} \\
 I_{GG}(x, y) &= \begin{cases} 1 & x \leq y, \\ \frac{y}{x} & x > y, \end{cases} & I_{RS}(x, y) &= \begin{cases} 1 & x \leq y, \\ 0 & x > y, \end{cases} \\
 I_{YG}(x, y) &= \begin{cases} 1 & x = 0 \text{ and } y = 0, \\ y^x & x > 0 \text{ or } y > 0, \end{cases} & I_{WB}(x, y) &= \begin{cases} 1 & x < 1, \\ y & x = 1, \end{cases} \\
 I_{FD}(x, y) &= \begin{cases} 1 & x \leq y, \\ \max(1 - x, y) & x > y. \end{cases}
 \end{aligned}$$

Definition 2.6. [1, 21, 24] An implication I on a bounded lattice L is said to satisfy the law of importation to a conjunctive uninorm U with a neutral element e if for all $x, y, z \in L$

$$I(x, I(y, z)) = I(U(x, y), z) \tag{LI_U}$$

holds.

An implication I is said to satisfy the left neutrality principle w.r.t. e if for all $y \in L$

$$I(e, y) = y \tag{NP_e}$$

holds.

An implication I is said to satisfy the ordering property w.r.t. e if for all $x, y \in L$

$$x \leq y \Leftrightarrow I(x, y) \geq e \tag{OP_e}$$

holds.

An implication I is said to satisfy the identity principle w.r.t. e if for all $x \in L$

$$I(x, x) = e \tag{IP_e}$$

holds.

An implication I is said to satisfy the consequent boundary if for all $x \in L$

$$I(x, y) \geq y \tag{CB}$$

holds.

Definition 2.7. [17] If T is a t-norm on the unit interval $[0, 1]$ and $\phi : [0, 1] \rightarrow [0, 1]$ is an order-preserving bijection, then the operation $T_\phi : [0, 1]^2 \rightarrow [0, 1]$ given by

$$T_\phi(x, y) = \phi^{-1}(T(\phi(x), \phi(y)))$$

is also a t-norm. This t-norm is called ϕ -conjugate of T .

The ϕ -conjugate of an implication (uninorm, t-conorm) on a bounded lattice is defined as similar to Definition 2.7.

Definition 2.8. [21] RU -implications are obtained from a uninorm U such that $U(x, 0) = 0$ for all $x < 1$ as follows:

$$I_U(x, y) = \sup\{z \in [0, 1] \mid U(x, z) \leq y\} \text{ for all } x, y \in [0, 1].$$

Note that for RU -implications, the condition $U(x, 0) = 0$ for all $x < 1$ is a necessary and sufficient condition to obtain an implication (see [3]).

Proposition 2.9. [23] Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with T and S left continuous and I_U , its residual implication. Then, I_U always satisfies the law of importation with the same U .

Definition 2.10. [29] Let $\{I_k\}_{k \in A}$ be a family of implications and $\{[a_k, b_k]\}_{k \in A}$ be a family of pairwise disjoint close subintervals of $[0, 1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. Define the mapping $I : [0, 1]^2 \rightarrow [0, 1]$ given by

$$I(x, y) = \begin{cases} a_k + (b_k - a_k)I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & x, y \in [a_k, b_k], \\ I_{GD}(x, y) & \text{otherwise,} \end{cases} \quad (4)$$

is called an ordinal sum of fuzzy implications $\{I_k\}_{k \in A}$.

Corollary 2.11. [29] Let $\{I_k\}_{k \in A}$ be a family of implications. Then, I given by (4) in Definition 2.10 is a fuzzy implication if and only if I_k satisfies (CB) whenever $k \in A$ and $b_k < 1$. Here, we call I an ordinal sum implication and in this case we will write $I = (\langle a_k, b_k, I_k \rangle)_{k \in A}$.

Definition 2.12. [17] Let $((a_i, b_i))_{i \in A}$ be a family of pairwise disjoint open subinterval of $[0, 1]$ and let $(T_i)_{i \in I}$ be a family of t-norms. Then, the ordinal sum $T = (\langle a_i, b_i, T_i \rangle)_{i \in I} : [0, 1]^2 \rightarrow [0, 1]$ is given by

$$T(x, y) = \begin{cases} a_i + (b_i - a_i)T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & x, y \in [a_i, b_i], \\ \min(x, y) & \text{otherwise.} \end{cases} \quad (5)$$

3 The orders induced by residual implications

In this section, we investigate the orders, denoted by \preceq_{I_U} , induced by the residual implications obtained from uninorms. We study the algebraic structures obtained by such orders. In this sense, we give a necessary and sufficient condition so that the unit interval $[0, 1]$ is being a supremum semi-lattice w.r.t. the order \preceq_{I_U} . Moreover, we show that the unit interval $[0, 1]$ is not an infimum semi-lattice w.r.t. the order induced by any residual implications obtained from uninorms. Thus, we conclude that $[0, 1]$ is not a lattice w.r.t. the order \preceq_{I_U} .

Remark 3.1. Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with T and S left continuous. Recall that in this case its residual implication is given by [3, 28]:

$$I_U(x, y) = \begin{cases} e.I_T\left(\frac{x}{e}, \frac{y}{e}\right) & x, y \in [0, e) \quad \text{and} \quad x > y, \\ e + (1 - e).I_S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & x, y \in [e, 1] \quad \text{and} \quad x \leq y, \\ e & x, y \in [e, 1] \quad \text{and} \quad x > y, \\ I_{GD}(x, y) & \text{otherwise,} \end{cases}$$

for all $x, y \in [0, 1]$, where I_S stands for the residuation operator derived from the t-conorm S which is given by $I_S(x, y) = \sup\{z \in [0, 1] \mid S(x, z) \leq y\}$.

By Proposition 2.9, it is clear that I_U satisfies (LI_U) . Also, by Proposition 5.4.2 in [1], $I_U(e, y) = y$ for any $y \in [0, 1]$. Thus, $([0, 1], \preceq_{I_U})$ is a partially ordered set.

Lemma 3.2. Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left-continuous t-norm T , t-conorm S and a neutral element $e \in (0, 1)$. For any $x, y \in [0, 1]$,

(i) Any element $x \in [0, 1] \setminus \{e, 1\}$ is not comparable to e w.r.t. \preceq_{I_U} .

(ii) Any elements $x \in [0, e)$ and $y \in [e, 1)$ are not comparable to each other w.r.t. \preceq_{I_U} .

Proof. (i) Suppose that an element $x \in [0, 1] \setminus \{e, 1\}$ is comparable to e w.r.t. the order \preceq_{I_U} . Then, either $e \preceq_{I_U} x$ or $x \preceq_{I_U} e$. Let $e \preceq_{I_U} x$. Then, there exists an element $k \leq e$ such that $I_U(k, e) = x$. If $k = e$, since $x = I_U(e, e) = e$, a contradiction would be obtained. Then, it must be $k \neq e$. In this case, we have that $x = I_U(k, e) = I_{GD}(k, e) = 1$, which is again a contradiction. Thus, it must be $x \preceq_{I_U} e$. Then, there exists an element $k \leq e$ such that $I_U(k, x) = e$. Since $x \preceq_{I_U} e$, it is clear $x \leq e$ from Proposition 3 [12]. By $x \leq e$ and $x \neq e$, it must be $x < e$. If $k = e$, since $e = I_U(e, x) = x$, we would have a contradiction. Then, it must be $k < e$. If $k \leq x$, we obtain that $e = I_U(k, x) = I_{GD}(k, x) = 1$, a contradiction. Thus, it must be $k > x$. In this case, since $e = I_U(k, x) = e.I_T\left(\frac{k}{e}, \frac{x}{e}\right)$, we have that $I_T\left(\frac{k}{e}, \frac{x}{e}\right) = 1$. Since T is left-continuous, by Theorem 2.5.7 [1], I_T satisfies (OP) . Thus, it follows $\frac{k}{e} \leq \frac{x}{e}$ from $I_T\left(\frac{k}{e}, \frac{x}{e}\right) = 1$, which is a contradiction since $k > x$. Thus, $x \in [0, 1] \setminus \{e, 1\}$ is not comparable to e w.r.t. the order \preceq_{I_U} .

(ii) Let $x \in [0, e)$ and $y \in [e, 1)$. Suppose that x and y are comparable w.r.t. \preceq_{I_U} . If $y \preceq_{I_U} x$, it would be $y \leq x$, contradiction. Thus, it must be $x \preceq_{I_U} y$. Then, there exists an element $\ell \leq e$ such that $I_U(\ell, x) = y$. If

$\ell = e$, then we would have $x = y$, a contradiction. Then, $\ell < e$. If $\ell \leq x$, we would have a contradiction since $y = I_U(\ell, x) = I_{GD}(\ell, x) = 1$. Let $\ell > x$. Since $y = I_U(\ell, x) = e.I_T(\frac{\ell}{e}, \frac{x}{e}) \leq e$, we have that $x \preceq_{I_U} y = e$, which is a contradiction to (i). Thus, any elements $x \in [0, e)$ and $y \in [e, 1)$ are not comparable. \square

Proposition 3.3. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left-continuous t -norm T , t -conorm S and a neutral element $e \in (0, 1)$. For any $x, y \in [0, 1]$,*

- (i) *If $x \preceq_{I_U} y$, then there exist three possible cases: $x = y$ or $y = 1$ or $\frac{x}{e} \preceq_{I_T} \frac{y}{e}$.*
(ii) *If $y \neq e$, then the converse of (i) is also true.*

Proof. (i) Let $x \preceq_{I_U} y$ for any $x, y \in [0, 1]$. There exists an element $\ell \leq e$ such that $I_U(\ell, x) = y$. If $\ell = e$, since I_U satisfies (NP_e) , $y = I_U(\ell, x) = x$. Suppose that $\ell \neq e$.

• Let $x < e$. Since $x \preceq_{I_U} y$, by Lemma 3.2, it must be $y \notin [e, 1)$. Then, $y < e$ or $y = 1$. If $y = 1$, the proof is completed. Let $y \neq 1$. Then, $y < e$. If $\ell \leq x$, we would have a contradiction since $y = I_U(\ell, x) = I_{GD}(\ell, x) = 1$. Then, $\ell > x$. Since $y = I_U(\ell, x) = e.I_T(\frac{\ell}{e}, \frac{x}{e})$, we obtain that $\frac{y}{e} = I_T(\frac{\ell}{e}, \frac{x}{e})$. Say $\ell^* := \frac{\ell}{e} \in [0, 1]$. From $\frac{y}{e} = I_T(\ell^*, \frac{x}{e})$, we have that $\frac{x}{e} \preceq_{I_T} \frac{y}{e}$.

• If $x \geq e$, it is clear that $y = I_U(\ell, x) = I_{GD}(\ell, x) = 1$ from $\ell \leq e \leq x$ and $\ell \neq e$.

(ii) Let $y \neq e$. If $x = y$ or $y = 1$, it is clear that $x \preceq_{I_U} y$. Take $x \neq y$ and $y \neq 1$. Let $\frac{x}{e} \preceq_{I_T} \frac{y}{e}$. Then, there exists an element $\ell \in [0, 1]$ such that $I_T(\ell, \frac{x}{e}) = \frac{y}{e}$. Since I_T is an implication on $[0, 1]$, $\frac{x}{e}, \frac{y}{e} \leq 1$. By $y \neq e$, $x \leq e$ and $y < e$. If $\ell = 1$, it would be $\frac{y}{e} = I_T(1, \frac{x}{e}) = \frac{x}{e}$, whence $x = y$, a contradiction. Thus, it must be $\ell < 1$. Say $\ell' := e.\ell < e$. Then, $\frac{y}{e} = I_T(\ell, \frac{x}{e}) = I_T(\frac{\ell'}{e}, \frac{x}{e})$ holds. If $x = e$, it would be $\frac{y}{e} = I_T(\frac{\ell'}{e}, 1) = 1$, which is a contradiction to $y \neq e$. Then, it must be $x < e$. Also, if $\ell' \leq x$, since I_T satisfies (IP) , we would have $\frac{y}{e} = I_T(\frac{\ell'}{e}, \frac{x}{e}) \geq I_T(\frac{x}{e}, \frac{x}{e}) = 1$, whence $y \geq e$. This contradicts that $y < e$. Thus, it must be $\ell' > x$. By the definition of I_U , we obtain that $y = e.I_T(\frac{\ell'}{e}, \frac{x}{e}) = I_U(\ell', x)$. This shows that $x \preceq_{I_U} y$. \square

Remark 3.4. Let X be any subset of L . Throughout the paper, for any binary operation M on L , we denote by \overline{X}_{\preceq_M} ($\underline{X}_{\preceq_M}$) the set of the upper (lower) bounds of X w.r.t. the order \preceq_M . Also, for any $a, b \in L$, $a \wedge_M b$ ($a \vee_M b$) denotes the greatest (least) of the lower (upper) bounds w.r.t. \preceq_M , if there exists.

Proposition 3.5. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left continuous t -norm T , t -conorm S and a neutral element $e \in (0, 1)$. Let I_U be the residual implication of U . For any elements x, y which are incomparable w.r.t. \preceq_{I_U} ,*

- (i) *if $x \in [0, e)$ and $y \in [e, 1]$, then $x \vee_{I_U} y = 1$.*
(ii) *if $x, y \in [e, 1]$, then $x \vee_{I_U} y = 1$.*

Proof. (i) Let $x \in [0, e)$ and $y \in [e, 1]$. Since $x \preceq_{I_U} 1$ and $y \preceq_{I_U} 1$, it is obvious that $1 \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Let $k \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Then, $x \preceq_{I_U} k$ and $y \preceq_{I_U} k$. There exist two elements $\ell_1, \ell_2 \leq e$ such that $I_U(\ell_1, x) = k$ and $I_U(\ell_2, y) = k$. If $\ell_2 = e$, it would be $y = k$. Since $I_U(\ell_1, x) = k = y$, we would have $x \preceq_{I_U} y$, a contradiction. Then, $\ell_2 \neq e$. Since $\ell_2 < e$ and $y \geq e$, we have that $k = I_U(\ell_2, y) = I_{GD}(\ell_2, y) = 1$, whence $k \in \{1\}$. Thus, $\overline{\{x, y\}}_{\preceq_{I_U}} = \{1\}$. That is, $x \vee_{I_U} y = 1$ for any $x \in [0, e)$ and $y \in [e, 1]$.

(ii) Let $x \in [e, 1]$ and $y \in [e, 1]$. We will show that $x \vee_{I_U} y = 1$. It is clear that $1 \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Let $k \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Then, $x \preceq_{I_U} k$ and $y \preceq_{I_U} k$. There exist two elements $\ell_1, \ell_2 \leq e$ such that $I_U(\ell_1, x) = k$ and $I_U(\ell_2, y) = k$. Since x and y are not comparable w.r.t. \preceq_{I_U} , it must be $\ell_1, \ell_2 \neq e$. For $\ell_1 < e$ and $x \geq e$, it is clear that $k = I_U(\ell_1, x) = I_{GD}(\ell_1, x) = 1$, whence $\overline{\{x, y\}}_{\preceq_{I_U}} = \{1\}$. Thus, $x \vee_{I_U} y = 1$ for any $x, y \in [e, 1]$. \square

Proposition 3.6. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left continuous t -norm T , t -conorm S and a neutral element $e \in (0, 1)$. Let I_U be the residual implication of U . If $([0, 1], \preceq_{I_T})$ is a supremum semi-lattice, then $([0, 1], \preceq_{I_U})$ is a supremum semi-lattice.*

Proof. Let $([0, 1], \preceq_{I_T})$ be a supremum semi-lattice. For any $x, y \in [0, 1]$, if $x \preceq_{I_U} y$ or $y \preceq_{I_U} x$, then it is clear that $x \vee_{I_U} y = y$ or x . Suppose that x and y are not comparable w.r.t. \preceq_{I_U} . If $x \in [0, e)$ and $y \in [e, 1]$ or $x, y \in [e, 1]$, it is clear that $x \vee_{I_U} y = 1$, by Proposition 3.5.

1. Let $x, y \in [0, e)$. Since $([0, 1], \preceq_{I_T})$ is a supremum semi-lattice, the supremum of the elements $\frac{x}{e}$ and $\frac{y}{e}$ exists. Let $\frac{x}{e} \vee_{I_T} \frac{y}{e} = k$.

1.1. Let $\frac{x}{e} \vee_{I_T} \frac{y}{e} = k = 1$. We will show that $x \vee_{I_U} y = 1$. Take arbitrary $t \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Suppose that $t \neq 1$. By $t \in \overline{\{x, y\}}_{\preceq_{I_U}}$, $x \preceq_{I_U} t$ and $y \preceq_{I_U} t$. If $t \in [e, 1)$, it would be a contradiction by Lemma 3.2. Then, $t \in [0, e)$. If $x = t$ or $y = t$, we would have that x and y are comparable w.r.t. \preceq_{I_U} , a contradiction. Then, $x \neq t$ and $y \neq t$. Since

$x \preceq_{I_U} t$, $y \preceq_{I_U} t$, $x \neq t$ and $t \neq 1$, by Proposition 3.3 (i), it must be $\frac{x}{e} \preceq_{I_T} \frac{t}{e}$ and $\frac{y}{e} \preceq_{I_T} \frac{t}{e}$, whence $\frac{t}{e} \in \overline{\{\frac{x}{e}, \frac{y}{e}\}}_{\preceq_{I_T}}$. Since $\frac{x}{e} \vee_{I_T} \frac{y}{e} = 1$, we have that $1 \preceq_{I_T} \frac{t}{e}$. By Proposition 3 [12], since $1 \leq \frac{t}{e}$, this contradicts $t \in [0, e)$. Thus, $t = 1$. That is, $x \vee_{I_U} y = 1$.

1.2. Let $\frac{x}{e} \vee_{I_T} \frac{y}{e} = k \neq 1$. Then, $\frac{x}{e} \preceq_{I_T} k$ and $\frac{y}{e} \preceq_{I_T} k$. Since $k < 1$, $k' := ek < e$. Thus, $\frac{x}{e} \preceq_{I_T} \frac{k'}{e}$ and $\frac{y}{e} \preceq_{I_T} \frac{k'}{e}$. By Proposition 3.3 (ii), we have that $x \preceq_{I_U} k'$ and $y \preceq_{I_U} k'$. Then, $k' \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Let $t \in \overline{\{x, y\}}_{\preceq_{I_U}}$. Then, $x \preceq_{I_U} t$ and $y \preceq_{I_U} t$. By Lemma 3.2 (ii), it is clear that $t \in [0, e)$. If $x = t$ or $y = t$, we would have that x and y are comparable w.r.t. \preceq_{I_U} , a contradiction. Thus, $x, y \neq t$. Since $x \preceq_{I_U} t$, $x \neq t$ and $t \neq 1$, by Proposition 3.3 (i), $\frac{x}{e} \preceq_{I_T} \frac{t}{e}$. Similarly, $\frac{y}{e} \preceq_{I_T} \frac{t}{e}$. Then, $\frac{t}{e} \in \overline{\{\frac{x}{e}, \frac{y}{e}\}}_{\preceq_{I_T}}$. By $\frac{x}{e} \vee_{I_T} \frac{y}{e} = k$, we have that $\frac{k'}{e} = k \preceq_{I_T} \frac{t}{e}$. Since $t \neq e$ and $\frac{k'}{e} \preceq_{I_T} \frac{t}{e}$, by Proposition 3.3 (ii), we have that $k' \preceq_{I_U} t$. Thus, $x \vee_{I_U} y = k' = ek$, when $\frac{x}{e} \vee_{I_T} \frac{y}{e} = k \neq 1$ for any $x, y \in [0, e)$. This shows that $([0, 1], \preceq_{I_U})$ is a supremum semi-lattice. \square

Proposition 3.7. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left continuous t -norm T , t -conorm S and a neutral element $e \in (0, 1)$. Let I_U be the residual implication of U . If $([0, 1], \preceq_{I_U})$ is a supremum semi-lattice, then $([0, 1], \preceq_{I_T})$ is a supremum semi-lattice.*

Proof. Let $([0, 1], \preceq_{I_U})$ be a supremum semi-lattice. If x and y are comparable w.r.t. \preceq_{I_T} , it is clear that $x \vee_{I_T} y = x$ or y . Suppose that x and y are not comparable w.r.t. \preceq_{I_T} . In this case, $x, y \neq 1$. Thus, $ex, ey \in [0, e)$. Since $([0, 1], \preceq_{I_U})$ is a supremum semi-lattice, the supremum of ex and ey exists. Let $ex \vee_{I_U} ey = k$.

• Suppose that $k = 1$. We will show that $x \vee_{I_T} y = 1$. It is clear that $1 \in \overline{\{x, y\}}_{\preceq_{I_T}}$. Let $t \in \overline{\{x, y\}}_{\preceq_{I_T}}$ and $t \neq 1$. Then, $x \preceq_{I_T} t$ and $y \preceq_{I_T} t$. Since $t < 1$, $et < e$. Thus, $\frac{ex}{e} \preceq_{I_T} \frac{et}{e}$ and $\frac{ey}{e} \preceq_{I_T} \frac{et}{e}$. Since $et \neq e$, by Proposition 3.3 (ii), we have that $ex \preceq_{I_U} et$ and $ey \preceq_{I_U} et$. That is, $et \in \overline{\{ex, ey\}}_{\preceq_{I_U}}$. Since $ex \vee_{I_U} ey = 1$, it is clear that $1 \preceq_{I_U} et$, whence $1 = et < e$, contradiction. Then, $\overline{\{x, y\}}_{\preceq_{I_T}} = \{1\}$. Thus, $x \vee_{I_T} y = 1$.

•• Let $k \neq 1$. By $ex \vee_{I_U} ey = k$, $ex \preceq_{I_U} k$ and $ey \preceq_{I_U} k$. Then, there exist two elements $\ell_1, \ell_2 \leq e$ such that $I_U(\ell_1, ex) = k$ and $I_U(\ell_2, ey) = k$. If $\ell_1 = e$, then $ex = k$. Since $I_U(\ell_2, ey) = k = ex$, we have that $ey \preceq_{I_U} ex$. If $ey = ex$, it would be $x = y$, whence $x \preceq_{I_T} y$, a contradiction. Since $ex \in [0, e)$ and $e \in (0, 1)$, it is clear that $ex \neq 1$. Then, by Proposition 3.3 (i), we have that $\frac{ey}{e} \preceq_{I_T} \frac{ex}{e}$, whence $y \preceq_{I_T} x$. This is a contradiction. Thus, $\ell_1 \neq e$. Similarly, it can be shown that $\ell_2 \neq e$. If $\ell_1 \leq ex$, we would have a contradiction since $I_U(\ell_1, ex) = k = 1$. Then, it must be $\ell_1 > ex$. Similarly, a contradiction is obtained when $\ell_2 \leq y$. In this case, $k = I_U(\ell_1, ex) = e.I_T(\frac{\ell_1}{e}, x)$ and $k = I_U(\ell_2, ey) = e.I_T(\frac{\ell_2}{e}, y)$. Then, we obtain that $x \preceq_{I_T} \frac{k}{e}$ and $y \preceq_{I_T} \frac{k}{e}$. Thus, $\frac{k}{e} \in \overline{\{x, y\}}_{\preceq_{I_T}}$. Let $t \in \overline{\{x, y\}}_{\preceq_{I_T}}$. We will show that $\frac{k}{e} \preceq_{I_T} t$. If $t = 1$, it is clear that $\frac{k}{e} \preceq_{I_T} 1 = t$. Let $t \neq 1$. By $t \in \overline{\{x, y\}}_{\preceq_{I_T}}$, $x \preceq_{I_T} t$ and $y \preceq_{I_T} t$. Say $x' := ex$, $y' := ey$ and $t' := et$. Thus, $\frac{x'}{e} \preceq_{I_T} \frac{t'}{e}$ and $\frac{y'}{e} \preceq_{I_T} \frac{t'}{e}$. Since $t \neq 1$, it is clear that $t' \neq e$. By Proposition 3.3 (ii), we have that $x' \preceq_{I_U} t'$ and $y' \preceq_{I_U} t'$. Then, $t' \in \overline{\{x', y'\}}_{\preceq_{I_U}}$. Thus, $et \in \overline{\{ex, ey\}}_{\preceq_{I_U}}$. Since $ex \vee_{I_U} ey = k$, we have that $k \preceq_{I_U} et$. Since $t \neq 1$, $et < e < 1$. By Proposition 3.3 (i), either $k = et$ or $\frac{k}{e} \preceq_{I_T} t$. If $\frac{k}{e} \preceq_{I_T} t$, this completes the proof. For $k = et$, it is clear that $\frac{k}{e} = t \preceq_{I_T} t$. Then, $x \vee_{I_T} y = \frac{k}{e}$. Thus, $([0, 1], \preceq_{I_T})$ is a supremum semi-lattice. \square

In the following Theorem 3.8, we obtain a necessary and sufficient condition so that the unit interval $[0, 1]$ is being a supremum semi-lattice w.r.t. the order induced by the residual implications obtained from uninorms.

Theorem 3.8. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left continuous t -norm T , t -conorm S and a neutral element $e \in (0, 1)$. Let I_U be the residual implication of U . Then, $([0, 1], \preceq_{I_T})$ is a supremum semi-lattice iff $([0, 1], \preceq_{I_U})$ is a supremum semi-lattice.*

Proof. The proof is straightforward by Proposition 3.6 and Proposition 3.7. \square

Example 3.9. *Take the uninorm $U_P = \langle T_P, S_P, 0.5 \rangle \in \mathcal{U}_{\min}$. Then, the RU-implication obtained from U_P is given by*

$$I_{U_P}(x, y) = \begin{cases} \frac{y}{2x} & x, y \in [0, 0.5) \text{ and } y < x, \\ 0.5 + \frac{y-x}{2 \cdot (1-x)} & x, y \in [0.5, 1] \text{ and } x \leq y, \\ 0.5 & x, y \in [0.5, 1] \text{ and } y < x, \\ I_{GD}(x, y) & \text{otherwise,} \end{cases}$$

By Remark 3.1, it is clear that $([0, 1], \preceq_{I_{U_P}})$ is a partially ordered set. It is well known that T_P and S_P are continuous. Also, the corresponding residual implication for T_P is the Goguen implication I_{GG} . By Theorem 7.3.5 [1], it is clear that I_{GG} satisfies the law of importation to T_P . Also, since I_{GG} satisfies (NP), $([0, 1], \preceq_{I_{GG}})$ is a partially ordered set.

Now, let us show that $\preceq_{I_{GG}} = \leq$. For any $x, y \in [0, 1]$, we know that $x \leq y$ if $x \preceq_{I_{GG}} y$. Let $x \leq y$. If $y = 1$, it is clear that $x \preceq_{I_{GG}} y = 1$. Let $y \neq 1$. Since $I_{GG}(\frac{x}{y}, x) = y$, we have that $x \preceq_{I_{GG}} y$. Thus, we have that $\preceq_{I_{GG}} = \leq$. For any $x, y \in [0, 1]$, $x \vee_{I_{GG}} y = x \vee y$. That is, $([0, 1], \preceq_{I_{GG}})$ is a supremum semi-lattice. By Theorem 3.8, $([0, 1], \preceq_{I_{UP}})$ is a supremum semi-lattice.

Proposition 3.10. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with left continuous t-norm T , t-conorm S and a neutral element $e \in (0, 1)$. Let I_U be the residual implication of U . $([0, 1], \preceq_{I_U})$ is not an infimum semi-lattice.*

Proof. Let us show that $\underline{\{x, y\}}_{\preceq_{I_U}} = \emptyset$ for any $x \in [0, e)$ and $y \in [e, 1)$. Suppose that $\underline{\{x, y\}}_{\preceq_{I_U}} \neq \emptyset$. Then, there exists an element $k \in \underline{\{x, y\}}_{\preceq_{I_U}}$. Thus, $k \preceq_{I_U} x$ and $k \preceq_{I_U} y$.

By Proposition 3 [12], since $k \leq x$ it must be $k < e$. Since $k \in [0, e)$ and $y \in [e, 1)$, by Lemma 3.2 (ii), the elements k and y can not be comparable w.r.t. \preceq_{I_U} . This contradicts that $k \preceq_{I_U} y$. Then, it must be $\underline{\{x, y\}}_{\preceq_{I_U}} = \emptyset$. Thus, $([0, 1], \preceq_{I_U})$ is not an infimum semi-lattice. \square

Corollary 3.11. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with T and S left-continuous, $e \in (0, 1)$ and I_U its residual implication. $([0, 1], \preceq_{I_U})$ is not a lattice.*

Remark 3.12. By [1], we know that I_{LK} and I_{GG} which are respectively the residual implications of the t-norms T_{LK} and T_P satisfy (NP) and (LI) with the t-norms T_{LK} and T_P , respectively. Thus, the relations $\preceq_{I_{LK}}$ and $\preceq_{I_{GG}}$ are orders on $[0, 1]$. Now, let us show that $\preceq_{I_{LK}} = \leq$. If $x \preceq_{I_{LK}} y$, then it is clear $x \leq y$ from Proposition 3 [12]. Let $x \leq y$. If $y = 1$, by Remark 3 (ii), we obtain that $x \preceq_{I_{LK}} 1 = y$. Let $y \neq 1$. Take $\ell := 1 + x - y \in [0, 1]$. If $\ell \leq x$, we would have $y = 1$ by $1 + x - y \leq x$, contradiction. Thus, $\ell > x$. Since $I_{LK}(\ell, x) = 1 - \ell + x = y$, we have that $x \preceq_{I_{LK}} y$. Thus, $\preceq_{I_{LK}} = \leq$.

Similarly, the equality $\preceq_{I_{GG}} = \leq$ can be shown.

Proposition 3.13. *Let $U = \langle T, S, e \rangle$ be a uninorm in \mathcal{U}_{\min} with a neutral element $e \in (0, 1)$. If T is strict or nilpotent, then for any $x, y < e$, $x \preceq_{I_U} y$ iff $x \leq y$.*

Proof. If $x \preceq_{I_U} y$, then it is clear that $x \leq y$ by Proposition 3 [12]. Conversely, let $x \leq y$ and $y \neq e$. If $x = y$ or $y = 1$, then it is obvious that $x \preceq_{I_U} y$. Suppose that $x \neq y$ and $y \neq 1$. If T is a nilpotent t-norm, by Lemma 2.5.23 [1], I_T is Φ -conjugate with the Lukasiewicz implication I_{LK} , i.e., there exists a unique $\phi \in \Phi$ such that for all $x, y \in [0, 1]$, $I(x, y) = (I_{LK})_{\phi}(x, y)$. Then, the following relations hold:

$$\begin{aligned}
 x \leq y &\Rightarrow \frac{x}{e} \leq \frac{y}{e} \\
 &\Rightarrow \phi\left(\frac{x}{e}\right) \leq \phi\left(\frac{y}{e}\right) \quad (\phi \text{ is an increasing bijection}) \\
 &\Rightarrow \phi\left(\frac{x}{e}\right) \preceq_{I_{LK}} \phi\left(\frac{y}{e}\right) \quad (\preceq_{I_{LK}} = \leq \text{ by Remark 3.12}) \\
 &\Rightarrow \frac{x}{e} \preceq_{(I_{LK})_{\phi}} \frac{y}{e} \quad (\text{by Proposition 8 [12]}) \\
 &\Rightarrow \frac{x}{e} \preceq_{I_T} \frac{y}{e} \\
 &\Rightarrow x \preceq_{I_U} y. \quad (\text{by Proposition 3.3 (ii)})
 \end{aligned}$$

For a strict t-norm T , the proof is similar. \square

4 The orders induced by ordinal sum implications

In this section, we give a necessary and sufficient condition such that the ordinal sum of fuzzy implications satisfies the law of importation with a t-norm T . We present some relationships between the orders induced by an ordinal sum implication and its summands. Moreover, we study the relationships between the algebraic structures obtained by an ordinal sum implication and its summands.

Now, we will give a necessary and sufficient condition so that the ordinal sum of fuzzy implications satisfies (LI) with a t-norm T .

Proposition 4.1. *Let I and T be the ordinal sum of the families of implications $(I_\alpha)_{\alpha \in A}$ and the families of t -norms $(T_\alpha)_{\alpha \in A}$ given by (4) and (5), respectively. Then, I satisfies (LI) with T iff for all $\alpha \in A$, I_α satisfies (LI) with T_α .*

Proof. Let I satisfy (LI) with T . Since $I(x, y) = a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha})$, it is clear that $I_\alpha(x, y) = \frac{I(a_\alpha + (b_\alpha - a_\alpha)x, a_\alpha + (b_\alpha - a_\alpha)y) - a_\alpha}{b_\alpha - a_\alpha}$.

For all $x, y, z \in [0, 1]$,

$$\begin{aligned} I_\alpha(x, I_\alpha(y, z)) &= I_\alpha(x, \frac{I(a_\alpha + (b_\alpha - a_\alpha)y, a_\alpha + (b_\alpha - a_\alpha)z) - a_\alpha}{b_\alpha - a_\alpha}) \\ &= \frac{I(a_\alpha + (b_\alpha - a_\alpha)x, a_\alpha + (b_\alpha - a_\alpha) \cdot \frac{I(a_\alpha + (b_\alpha - a_\alpha)y, a_\alpha + (b_\alpha - a_\alpha)z) - a_\alpha}{b_\alpha - a_\alpha}) - a_\alpha}{b_\alpha - a_\alpha} \\ &= \frac{I(a_\alpha + (b_\alpha - a_\alpha)x, I(a_\alpha + (b_\alpha - a_\alpha)y, a_\alpha + (b_\alpha - a_\alpha)z) - a_\alpha)}{b_\alpha - a_\alpha} \\ &= \frac{I(T(a_\alpha + (b_\alpha - a_\alpha)x, a_\alpha + (b_\alpha - a_\alpha)y), a_\alpha + (b_\alpha - a_\alpha)z) - a_\alpha}{b_\alpha - a_\alpha} \\ &= \frac{I(a_\alpha + (b_\alpha - a_\alpha)T_\alpha(x, y), a_\alpha + (b_\alpha - a_\alpha)z) - a_\alpha}{b_\alpha - a_\alpha} \\ &= I_\alpha(T_\alpha(x, y), z). \end{aligned}$$

Thus, I_α satisfies (LI) with T_α for all $\alpha \in A$.

Conversely, for all $\alpha \in A$, let I_α satisfy (LI) with T_α . We shall show that I satisfies (LI) with T .

1. Suppose that there exists $\alpha \in A$ such that $x \in [a_\alpha, b_\alpha]$.

1.1. Let $y \in [a_\alpha, b_\alpha]$.

1.1.1. If $z \in [a_\alpha, b_\alpha]$, then

$$\begin{aligned} I(x, I(y, z)) &= I(x, a_\alpha + (b_\alpha - a_\alpha)I_\alpha(\frac{y-a_\alpha}{b_\alpha-a_\alpha}, \frac{z-a_\alpha}{b_\alpha-a_\alpha})) \\ &= a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(\frac{y-a_\alpha}{b_\alpha-a_\alpha}, \frac{z-a_\alpha}{b_\alpha-a_\alpha}) - a_\alpha}{b_\alpha - a_\alpha}) \\ &= a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, I_\alpha(\frac{y-a_\alpha}{b_\alpha-a_\alpha}, \frac{z-a_\alpha}{b_\alpha-a_\alpha})) \\ &= a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(T_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}), \frac{z-a_\alpha}{b_\alpha-a_\alpha}) \\ &= a_\alpha + (b_\alpha - a_\alpha) \cdot I_\alpha(\frac{a_\alpha + (b_\alpha - a_\alpha)T_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}) - a_\alpha}{b_\alpha - a_\alpha}, \frac{z-a_\alpha}{b_\alpha-a_\alpha}) \\ &= I(a_\alpha + (b_\alpha - a_\alpha) \cdot T_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}), z) \\ &= I(T(x, y), z). \end{aligned}$$

1.1.2. Let $z \notin [a_\alpha, b_\alpha]$. In this case, $z < a_\alpha$ or $z > b_\alpha$. If $z < a_\alpha$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I_{GD}(x, z) = z = I_{GD}(T(x, y), z) = I(T(x, y), z).$$

Let $z > b_\alpha$. In this case, clearly $z > y$. Thus, $I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(T(x, y), z) = I(T(x, y), z)$.

1.2. Let $y \in [a_\beta, b_\beta]$ such that $\beta \neq \alpha$.

1.2.1. Let $z \in [a_\beta, b_\beta]$. Then, $I(y, z) \in [a_\beta, b_\beta]$. Since $[a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] = \emptyset$, $x \notin [a_\beta, b_\beta]$. Thus, either $x < a_\beta$ or $x > b_\beta$. Let $x < a_\beta$. Then,

$$I(x, I(y, z)) = I_{GD}(x, I(y, z)) = 1 = I_{GD}(x, z) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $x > b_\beta$. Then, we have that

$$I(x, I(y, z)) = I_{GD}(x, I(y, z)) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

1.2.2. Let $z \in [a_\alpha, b_\alpha]$. Since $[a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] = \emptyset$, $x, z \notin [a_\beta, b_\beta]$ and $y \notin [a_\alpha, b_\alpha]$. Thus, $z < a_\beta$ or $z > b_\beta$. Let $z < a_\beta$. If $x > b_\beta$, it would be $a_\alpha \leq z < a_\beta < y < b_\beta < x \leq b_\alpha$, whence $y \in [a_\alpha, b_\alpha]$, a contradiction. Then, it must be $x < a_\beta$. Thus,

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $z > b_\beta$. Similarly, it can be seen that $x > b_\beta$. Then,

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

1.2.3. Let $z \notin [a_\alpha, b_\alpha]$ and $z \notin [a_\beta, b_\beta]$.

Let $y \leq z$. If $x \leq y$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(x, z) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $x > y$, then

$$I(x, I(y, z)) = I(x, 1) = 1 = I_{GD}(y, z) = I(y, z) = I(T(x, y), z).$$

Let $y > z$. If $x \leq z$, then

$$I(x, I(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $x > z$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I_{GD}(x, z) = I_{GD}(\min(x, y), z) = I(\min(x, y), z) = I(T(x, y), z).$$

1.3. For all $\alpha \in A$, let $y \notin [a_\alpha, b_\alpha]$.

1.3.1. Let $z \in [a_\alpha, b_\alpha]$ and $y \leq z$. If $x \leq y$, it would be $a_\alpha \leq x \leq y \leq z \leq b_\alpha$, a contradiction. Then, it must be $x > y$. Thus,

$$I(x, I(y, z)) = I(x, z) = 1 = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $y > z$. If $y < x$, since $a_\alpha \leq z < y < x \leq b_\alpha$, we would have a contradiction. Then, it must be $x \leq y$. Thus,

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

1.3.2. Let $z \notin [a_\alpha, b_\alpha]$. Suppose that $y \leq z$. If $x \leq y$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(x, z) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $x > y$, then we have that

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $y > z$. If $x \leq z$, then $I(x, I(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z)$.

If $x > z$, then we have that $I(x, I(y, z)) = I(x, z) = I_{GD}(x, z) = z = I_{GD}(\min(x, y), z) = I(\min(x, y), z) = I(T(x, y), z)$.

2. Suppose that for all $\alpha \in A$, $x \notin [a_\alpha, b_\alpha]$.

2.1. Let $y \in [a_\beta, b_\beta]$ for $\beta \in A$.

2.1.1. Let $z \in [a_\beta, b_\beta]$. Since $x \notin [a_\beta, b_\beta]$, either $x < a_\beta$ or $x > b_\beta$. Let $x < a_\beta$. Then,

$$I(x, I(y, z)) = I_{GD}(x, I(y, z)) = 1 = I_{GD}(x, z) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $x > b_\beta$. Then, we have that $I(x, I(y, z)) = I_{GD}(x, I(y, z)) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z)$.

2.1.2. $z \notin [a_\beta, b_\beta]$. Since $x \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$, $x \notin [a_\beta, b_\beta]$.

If $x, z < a_\beta$, then $I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z)$. If $x < a_\beta$ and $z > b_\beta$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(x, z) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $z < a_\beta$ and $x > b_\beta$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = z = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $x, z > b_\beta$. Then,

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

2.2. Suppose that for all $\alpha \in A$, $y \notin [a_\alpha, b_\alpha]$.

Let $y \leq z$. If $x \leq z$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(\min(x, y), z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $x > z$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, 1) = 1 = I_{GD}(y, z) = I(y, z) = I(\min(x, y), z) = I(T(x, y), z).$$

Let $y > z$. If $x \leq z$, then

$$I(x, I(y, z)) = I(x, I_{GD}(y, z)) = I(x, z) = I(\min(x, y), z) = I(T(x, y), z).$$

If $x > z$, then

$$I(x, I(y, z)) = I_{GD}(x, z) = z = I_{GD}(\min(x, y), z) = I(\min(x, y), z) = I(T(x, y), z).$$

Thus, I satisfies (LI) with t-norm T . □

Example 4.2. Let $I = (\langle 0.2, 0.4, I_{LK} \rangle, \langle 0.6, 0.8, I_{FD} \rangle)$. It is clear that I_{LK} and I_{FD} respectively satisfy (LI) with T_{LK} and T^{nM} . By Proposition 4.1, I satisfies (LI) with $T = (\langle 0.2, 0.4, T_{LK} \rangle, \langle 0.6, 0.8, T^{nM} \rangle)$.

Now, we will present some relationships between the orders induced by an ordinal sum implication and its summands.

Proposition 4.3. Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ satisfying (LI) with T_α .

(i) Let $\alpha_0 \in A$ exist such that $b_{\alpha_0} = 1$ and the implication I_{α_0} satisfy (NP). The relations \preceq_I and $\preceq_{I_{\alpha_0}}$ induced by I and I_{α_0} are orders on $[0, 1]$.

(ii) Let \preceq_I and $\preceq_{I_{\alpha_0}}$ induced by I and I_{α_0} be two orders. Then, for any $x, y \in [0, 1]$, if $\frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} \preceq_{I_{\alpha_0}} \frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}$, then $x \preceq_I y$.

Proof. (i) Since I_α satisfies (LI) with T_α for all $\alpha \in A$, the ordinal sum implication I satisfies (LI) with the ordinal sum of T_α , by Proposition 4.1. Also, since I_{α_0} satisfies (NP), by Proposition 3.2. [29], it is clear that the ordinal sum implication I satisfies (NP). Thus, by Proposition 2 [12], the relation \preceq_I is an order. Obviously, It is clear that $\preceq_{I_{\alpha_0}}$ is an order.

(ii) Let $\frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} \preceq_{I_{\alpha_0}} \frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}$. Then, there exists an element ℓ such that

$$I_{\alpha_0}(\ell, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}) = \frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}},$$

whence $a_{\alpha_0} + (b_{\alpha_0} - a_{\alpha_0}) \cdot I_{\alpha_0}(\ell, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}) = y$. Since $0 \leq \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} \leq 1$, it is clear that $a_{\alpha_0} \leq x \leq b_{\alpha_0}$. Also, $\ell \cdot (b_{\alpha_0} - a_{\alpha_0}) + a_{\alpha_0} \in [a_{\alpha_0}, b_{\alpha_0}]$. Say $\ell^* := \ell \cdot (b_{\alpha_0} - a_{\alpha_0}) + a_{\alpha_0}$. Then,

$$I(\ell^*, x) = a_{\alpha_0} + (b_{\alpha_0} - a_{\alpha_0}) \cdot I_{\alpha_0}(\frac{\ell^* - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}) = a_{\alpha_0} + (b_{\alpha_0} - a_{\alpha_0}) \cdot I_{\alpha_0}(\ell, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}) = y.$$

Thus, $x \preceq_I y$. □

Proposition 4.4. Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$. Suppose that \preceq_I and \preceq_{I_α} are the orders induced by I and I_α for all $\alpha \in A$, respectively. Then, for any $x, y \in [0, 1]$ with $x \neq y$ and $y \neq 1$, $x \preceq_I y$ iff there exists $\alpha_0 \in A$ such that $\frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} \preceq_{I_{\alpha_0}} \frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}$.

Proof. Let $x \preceq_I y$ for any $x, y \in [0, 1]$ with $x \neq y$ and $y \neq 1$. Then, there exists an element $\ell \in [0, 1]$ such that

$$I(\ell, x) = y.$$

Suppose that $x \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$. Then, it would be $y = I(\ell, x) = I_{GD}(\ell, x) = 1$ or x , a contradiction. Then, there must exist $\alpha_0 \in A$ such that $x \in [a_{\alpha_0}, b_{\alpha_0}]$. Suppose that $\ell \notin [a_{\alpha_0}, b_{\alpha_0}]$. In this case, we would have a contradiction since $y = I(\ell, x) = I_{GD}(\ell, x) = x$ or 1 . Then, it must be $\ell \in [a_{\alpha_0}, b_{\alpha_0}]$. Thus,

$$y = I(\ell, x) = a_{\alpha_0} + (b_{\alpha_0} - a_{\alpha_0}) \cdot I_{\alpha_0}\left(\frac{\ell - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}\right),$$

whence we have that $\frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} = I_{\alpha_0}\left(\frac{\ell - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}, \frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}\right)$. This shows that $\frac{x - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}} \preceq_{I_{\alpha_0}} \frac{y - a_{\alpha_0}}{b_{\alpha_0} - a_{\alpha_0}}$.

The converse is obvious by Proposition 4.3. \square

In the following Theorem 4.8, we will give a relationship between the algebraic structures obtained from the orders induced by an ordinal sum implication and its summands.

In the rest of the paper, the ordinal sum implications and their summands will be taken as implications satisfying the law of importation with t-norms.

Proposition 4.5. *Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ and let \preceq_I and \preceq_{I_α} be the orders induced by I and I_α for all $\alpha \in A$, respectively. For any elements x and y which are incomparable w.r.t. \preceq_I ,*

- (i) *if $x \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$ and y is arbitrary, then $x \vee_I y = 1$.*
- (ii) *if $x \in [a_\alpha, b_\alpha]$ for some $\alpha \in A$ and $y \in [a_\beta, b_\beta]$ with $\alpha \neq \beta$, then $x \vee_I y = 1$.*

Proof. (i) Let $x \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$ and let y be arbitrary. Since $x \preceq_I 1$ and $y \preceq_I 1$, it is obvious that $1 \in \overline{\{x, y\}}_{\preceq_I}$. Let $k \in \overline{\{x, y\}}_{\preceq_I}$. Then, $x \preceq_I k$ and $y \preceq_I k$. Thus, there exist two elements ℓ_1, ℓ_2 such that $I(\ell_1, x) = k$ and $I(\ell_2, y) = k$. By the definition of ordinal sum implication I , it must be $k = I(\ell_1, x) = I_{GD}(\ell_1, x) = 1$ or x . Suppose that $k = x$. Since $I(\ell_2, y) = k = x$, we have $y \preceq_I x$. This is a contradiction. Then, it must be $k = 1$. Thus, $x \vee_I y = 1$.

(ii) Let $y \in [a_\beta, b_\beta]$ with $\alpha \neq \beta$. It is clear that $1 \in \overline{\{x, y\}}_{\preceq_I}$. Let $k \in \overline{\{x, y\}}_{\preceq_I}$ and $k \neq 1$. Then, $x \preceq_I k$ and $y \preceq_I k$. There exist two elements $\ell_1, \ell_2 \in [0, 1]$ such that $I(\ell_1, x) = k$ and $I(\ell_2, y) = k$. Suppose that $\ell_1 \in [a_\alpha, b_\alpha]$ and $\ell_2 \in [a_\beta, b_\beta]$. In this case,

$$k = I(\ell_1, x) = a_\alpha + (b_\alpha - a_\alpha)I_\alpha\left(\frac{\ell_1 - a_\alpha}{b_\alpha - a_\alpha}, \frac{x - a_\alpha}{b_\alpha - a_\alpha}\right) \text{ and } k = I(\ell_2, y) = a_\beta + (b_\beta - a_\beta)I_\beta\left(\frac{\ell_2 - a_\beta}{b_\beta - a_\beta}, \frac{y - a_\beta}{b_\beta - a_\beta}\right).$$

Then, $\frac{k - a_\alpha}{b_\alpha - a_\alpha} = I_\alpha\left(\frac{\ell_1 - a_\alpha}{b_\alpha - a_\alpha}, \frac{x - a_\alpha}{b_\alpha - a_\alpha}\right)$ and $\frac{k - a_\beta}{b_\beta - a_\beta} = I_\beta\left(\frac{\ell_2 - a_\beta}{b_\beta - a_\beta}, \frac{y - a_\beta}{b_\beta - a_\beta}\right)$. Since $0 \leq \frac{k - a_\alpha}{b_\alpha - a_\alpha} \leq 1$ and $0 \leq \frac{k - a_\beta}{b_\beta - a_\beta} \leq 1$, it is clear that $a_\alpha \leq k \leq b_\alpha$ and $a_\beta \leq k \leq b_\beta$. Then, $k \in [a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] = \emptyset$, which is a contradiction. Thus, either $\ell_1 \notin [a_\alpha, b_\alpha]$ or $\ell_2 \notin [a_\beta, b_\beta]$. If $\ell_1 \notin [a_\alpha, b_\alpha]$, then by $k = I(\ell_1, x) = I_{GD}(\ell_1, x)$, it must be $k = x$. Since $I(\ell_2, y) = k = x$, we have the contradiction $y \preceq_I x$. Similar contradiction is obtained for the case $\ell_2 \notin [a_\beta, b_\beta]$. Then, it must be $k = 1$. Thus, $x \vee_I y = 1$. \square

Proposition 4.6. *Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ and let \preceq_I and \preceq_{I_α} be the orders induced by I and I_α for all $\alpha \in A$, respectively. If $([0, 1], \preceq_{I_\alpha})$ is a supremum semi-lattice for all $\alpha \in A$, then $([0, 1], \preceq_I)$ is a supremum semi-lattice.*

Proof. Let $([0, 1], \preceq_{I_\alpha})$ be a supremum semi-lattice for all $\alpha \in A$. If $x \preceq_I y$ or $y \preceq_I x$, then it is clear that $x \vee_I y = y$ or x . Suppose that x and y are not comparable w.r.t. \preceq_I .

- 1. Let $x \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$ and let y be arbitrary. By Proposition 4.5, it is clear that $x \vee_I y = 1$.
- 2. Let $x \in [a_\alpha, b_\alpha]$ for some $\alpha \in A$.

2.1. Let $y \in [a_\alpha, b_\alpha]$. Since $([0, 1], \preceq_{I_\alpha})$ is a supremum semi-lattice, the supremum of the elements $\frac{x - a_\alpha}{b_\alpha - a_\alpha}$ and $\frac{y - a_\alpha}{b_\alpha - a_\alpha}$ exists. Say $\frac{x - a_\alpha}{b_\alpha - a_\alpha} \vee_{I_\alpha} \frac{y - a_\alpha}{b_\alpha - a_\alpha} = k$. Then, $\frac{x - a_\alpha}{b_\alpha - a_\alpha} \preceq_{I_\alpha} k$ and $\frac{y - a_\alpha}{b_\alpha - a_\alpha} \preceq_{I_\alpha} k$. For $a_\alpha \leq k' := k \cdot (b_\alpha - a_\alpha) + a_\alpha \leq b_\alpha$

$$\frac{x - a_\alpha}{b_\alpha - a_\alpha} \preceq_{I_\alpha} \frac{k' - a_\alpha}{b_\alpha - a_\alpha} \text{ and } \frac{y - a_\alpha}{b_\alpha - a_\alpha} \preceq_{I_\alpha} \frac{k' - a_\alpha}{b_\alpha - a_\alpha}.$$

By Proposition 4.3 (ii), we have that $x \preceq_I k'$ and $y \preceq_I k'$. Then, $k' \in \overline{\{x, y\}}_{\preceq_I}$. Let $t \in \overline{\{x, y\}}_{\preceq_I}$ be arbitrary. If $t = 1$, it is clear that $k' \preceq_I 1 = t$. Suppose that $t \neq 1$. Then, $x \preceq_I t$ and $y \preceq_I t$. Thus, there exist two elements ℓ_1, ℓ_2 such that $I(\ell_1, x) = t$ and $I(\ell_2, y) = t$. If $\ell_1 \notin [a_\alpha, b_\alpha]$, then $t = I(\ell_1, x) = I_{GD}(\ell_1, x) = 1$ or x . Since $t \neq 1$, it must be $t = x$. In this case, we would have $y \preceq_I x$ from $I(\ell_2, y) = t = x$. This is a contradiction. Similar contradiction is obtained for the case $\ell_2 \notin [a_\alpha, b_\alpha]$. Thus, it must be $\ell_1 \in [a_\alpha, b_\alpha]$ and $\ell_2 \in [a_\alpha, b_\alpha]$. Then,

$$t = I(\ell_1, x) = a_\alpha + (b_\alpha - a_\alpha)I_\alpha\left(\frac{\ell_1 - a_\alpha}{b_\alpha - a_\alpha}, \frac{x - a_\alpha}{b_\alpha - a_\alpha}\right) \text{ and } t = I(\ell_2, y) = a_\alpha + (b_\alpha - a_\alpha)I_\alpha\left(\frac{\ell_2 - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha}\right).$$

Thus, $\frac{t-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell_1-a_\alpha}{b_\alpha-a_\alpha}, \frac{x-a_\alpha}{b_\alpha-a_\alpha})$ and $\frac{t-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell_2-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha})$. This implies that $\frac{x-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{t-a_\alpha}{b_\alpha-a_\alpha}$ and $\frac{y-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{t-a_\alpha}{b_\alpha-a_\alpha}$, whence $\frac{t-a_\alpha}{b_\alpha-a_\alpha} \in \overline{\{\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}\}}_{\preceq_{I_\alpha}}$. Since $\frac{x-a_\alpha}{b_\alpha-a_\alpha} \vee_{I_\alpha} \frac{y-a_\alpha}{b_\alpha-a_\alpha} = k$, it is clear that $\frac{k'-a_\alpha}{b_\alpha-a_\alpha} = k \preceq_{I_\alpha} \frac{t-a_\alpha}{b_\alpha-a_\alpha}$. Thus, by Proposition 4.3 (ii), we have that $k' \preceq_I t$. In this case, $x \vee_I y = k'$.

2.2. Let $y \in [a_\beta, b_\beta]$ with $\alpha \neq \beta$. By Proposition 4.5, it is clear that $x \vee_I y = 1$.

2.3. Let $y \notin [a_\alpha, b_\alpha]$ for all $\alpha \in A$. Similar to the case 1, for any $x \in [0, 1]$, it can be shown that $x \vee_I y = 1$. \square

Proposition 4.7. *Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ and let \preceq_I and \preceq_{I_α} be the orders induced by I and I_α for all $\alpha \in A$, respectively. If $([0, 1], \preceq_I)$ is a supremum semi-lattice, then $([0, 1], \preceq_{I_\alpha})$ is a supremum semi-lattice for all $\alpha \in A$.*

Proof. Let $([0, 1], \preceq_I)$ be a supremum semi-lattice. If $x = y$, then it is clear that for all $\alpha \in A$, $x \vee_{I_\alpha} y = x$. Also, if $y = 1$, for all $\alpha \in A$, $x \vee_{I_\alpha} y = 1$. Now, suppose that $x \neq y$ and $y \neq 1$. Take the elements $x' := a_\alpha + (b_\alpha - a_\alpha)x \in [a_\alpha, b_\alpha]$ and $y' := a_\alpha + (b_\alpha - a_\alpha)y \in [a_\alpha, b_\alpha]$. Since $([0, 1], \preceq_I)$ is a supremum semi-lattice, the supremum of the elements x' and y' exists. Let $x' \vee_I y' = k$.

Suppose that $k = 1$. Let us show that $x \vee_{I_\alpha} y = 1$ for all α . It is clear that $1 \in \overline{\{x, y\}}_{\preceq_{I_\alpha}}$. Let $t \in \overline{\{x, y\}}_{\preceq_{I_\alpha}}$. Say $t' := a_\alpha + (b_\alpha - a_\alpha)t \in [a_\alpha, b_\alpha]$. By $t \in \overline{\{x, y\}}_{\preceq_{I_\alpha}}$, $x \preceq_{I_\alpha} t$ and $y \preceq_{I_\alpha} t$. Then, $\frac{x'-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{t'-a_\alpha}{b_\alpha-a_\alpha}$ and $\frac{y'-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{t'-a_\alpha}{b_\alpha-a_\alpha}$. By Proposition 4.3 (ii), $x' \preceq_I t'$ and $y' \preceq_I t'$. Thus, $t' \in \overline{\{x', y'\}}_{\preceq_I}$. Since $1 = k = x' \vee_I y'$, it is clear that $1 \preceq_I t'$, whence $t' = 1$. Since $1 = t' \in [a_\alpha, b_\alpha]$, it must be $b_\alpha = 1$. Then, we have that

$$t = \frac{t'-a_\alpha}{b_\alpha-a_\alpha} = \frac{1-a_\alpha}{1-a_\alpha} = 1. \text{ Thus, } \overline{\{x, y\}}_{\preceq_{I_\alpha}} = \{1\}. \text{ In this case, } x \vee_{I_\alpha} y = 1.$$

Let $k \neq 1$. Since $x' \vee_I y' = k$, $x' \preceq_I k$ and $y' \preceq_I k$. Then, there exist two elements ℓ_1 and ℓ_2 such that $I(\ell_1, x') = k$ and $I(\ell_2, y') = k$. Let $\ell_1 \notin [a_\alpha, b_\alpha]$. Then, $k = I(\ell_1, x') = I_{GD}(\ell_1, x')$. If $\ell_1 \leq x'$, since $k = I_{GD}(\ell_1, x') = 1$, we would have a contradiction. Then, $\ell_1 > x'$ and $k = I_{GD}(\ell_1, x') = x'$. Since $I(\ell_2, y') = k = x'$, we have that $y' \preceq_I x'$. Then, there exists an element ℓ such that $I(\ell, y') = x'$. If $\ell \notin [a_\alpha, b_\alpha]$, then $x' = I(\ell, y') = I_{GD}(\ell, y') = 1$ or y' . If $x' = 1$, we would have a contradiction since $k = x' \vee_I y' = 1 \vee_I y' = 1$. Let $x' = y'$. In this case, we have $x = y$. This is again a contradiction. Thus, it must be $\ell \in [a_\alpha, b_\alpha]$. Then, $x' = I(\ell, y') = a_\alpha + (b_\alpha - a_\alpha)I_\alpha(\frac{\ell-a_\alpha}{b_\alpha-a_\alpha}, \frac{y'-a_\alpha}{b_\alpha-a_\alpha})$. Thus,

$$x = \frac{x'-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell-a_\alpha}{b_\alpha-a_\alpha}, \frac{y'-a_\alpha}{b_\alpha-a_\alpha}) = I_\alpha(\frac{\ell-a_\alpha}{b_\alpha-a_\alpha}, y),$$

whence $y \preceq_{I_\alpha} x$. Then, $y \vee_{I_\alpha} x = x$. Similarly, it can be shown that $x \vee_{I_\alpha} y = y$ when $\ell_2 \notin [a_\alpha, b_\alpha]$. Let $\ell_1, \ell_2 \in [a_\alpha, b_\alpha]$. Then,

$$k = I(\ell_1, x') = a_\alpha + (b_\alpha - a_\alpha)I_\alpha(\frac{\ell_1-a_\alpha}{b_\alpha-a_\alpha}, \frac{x'-a_\alpha}{b_\alpha-a_\alpha}) \text{ and } k = I(\ell_2, y') = a_\alpha + (b_\alpha - a_\alpha)I_\alpha(\frac{\ell_2-a_\alpha}{b_\alpha-a_\alpha}, \frac{y'-a_\alpha}{b_\alpha-a_\alpha}).$$

Since $\frac{k-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell_1-a_\alpha}{b_\alpha-a_\alpha}, \frac{x'-a_\alpha}{b_\alpha-a_\alpha})$ and $\frac{k-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell_2-a_\alpha}{b_\alpha-a_\alpha}, \frac{y'-a_\alpha}{b_\alpha-a_\alpha})$, we obtain that $x = \frac{x'-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{k-a_\alpha}{b_\alpha-a_\alpha} = k'$ and $y = \frac{y'-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{k-a_\alpha}{b_\alpha-a_\alpha} = k'$. Then, $k' \in \overline{\{x, y\}}_{\preceq_{I_\alpha}}$. Let $t \in \overline{\{x, y\}}_{I_\alpha}$. Then, $x \preceq_{I_\alpha} t$ and $y \preceq_{I_\alpha} t$. Since $\frac{x'-a_\alpha}{b_\alpha-a_\alpha} = x \preceq_{I_\alpha} t = \frac{t'-a_\alpha}{b_\alpha-a_\alpha}$ and $\frac{y'-a_\alpha}{b_\alpha-a_\alpha} = y \preceq_{I_\alpha} t = \frac{t'-a_\alpha}{b_\alpha-a_\alpha}$, by Proposition 4.3 (ii), $x' \preceq_I t'$ and $y' \preceq_I t'$. Then, $t' \in \overline{\{x', y'\}}_{\preceq_I}$. Since $x' \vee_I y' = k$, it is clear that $k \preceq_I t'$. Then, there exists an element ℓ such that $I(\ell, k) = t'$.

• If $\ell \notin [a_\alpha, b_\alpha]$, then $t' = I(\ell, k) = I_{GD}(\ell, k) = 1$ or k . If $t' = 1$, then $b_\alpha = 1$, whence we have that $t = 1$. Thus, $k' \preceq_{I_\alpha} 1 = t$. Let $t' = k$. Since $t = \frac{t'-a_\alpha}{b_\alpha-a_\alpha} = \frac{k-a_\alpha}{b_\alpha-a_\alpha} = k'$, it is clear that $k' \preceq_{I_\alpha} t$.

• Let $\ell \in [a_\alpha, b_\alpha]$. Since $t' = I(\ell, k) = a_\alpha + (b_\alpha - a_\alpha)I_\alpha(\frac{\ell-a_\alpha}{b_\alpha-a_\alpha}, \frac{k-a_\alpha}{b_\alpha-a_\alpha})$, we have that $\frac{t'-a_\alpha}{b_\alpha-a_\alpha} = I_\alpha(\frac{\ell-a_\alpha}{b_\alpha-a_\alpha}, \frac{k-a_\alpha}{b_\alpha-a_\alpha})$. Then,

$$k' = \frac{k-a_\alpha}{b_\alpha-a_\alpha} \preceq_{I_\alpha} \frac{t'-a_\alpha}{b_\alpha-a_\alpha} = t.$$

In this case, we have that $x \vee_{I_\alpha} y = k'$. \square

Theorem 4.8. *Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ and let \preceq_I and \preceq_{I_α} be the orders induced by I and I_α for all $\alpha \in A$, respectively. Then, for all $\alpha \in A$, $([0, 1], \preceq_{I_\alpha})$ is a supremum semi-lattice iff $([0, 1], \preceq_I)$ is a supremum semi-lattice.*

Proof. The proof is straightforward with Proposition 4.6 and Proposition 4.7. \square

Example 4.9. *Take the ordinal sum implication $I = \langle 0.4, 0.6, I_{GG} \rangle$. It is clear that $([0, 1], \preceq_{I_{GG}})$ is a supremum semi-lattice by Example 3.9. By Theorem 4.8, $([0, 1], \preceq_I)$ is a supremum semi-lattice.*

Lemma 4.10. *Let I be the ordinal sum implication of a family of implications $(I_\alpha)_{\alpha \in A}$ and \preceq_I be an order induced by I . For some $\alpha \in A$, let $x \in [a_\alpha, b_\alpha]$ and $y \notin [a_\alpha, b_\alpha]$ with $x, y \neq 1$. Then, the elements x and y are not comparable w.r.t. \preceq_I .*

Proof. For some $\alpha \in A$, let $x \in [a_\alpha, b_\alpha]$ and $y \notin [a_\alpha, b_\alpha]$ with $x, y \neq 1$. Suppose that $x \preceq_I y$. Then, there exists an element $\ell \in [0, 1]$ such that $I(\ell, x) = y$. Let $\ell \in [a_\alpha, b_\alpha]$. Then,

$$y = I(\ell, x) = a_\alpha + (b_\alpha - a_\alpha)I_\alpha\left(\frac{\ell - a_\alpha}{b_\alpha - a_\alpha}, \frac{x - a_\alpha}{b_\alpha - a_\alpha}\right) \in [a_\alpha, b_\alpha],$$

which is a contradiction. Then, it must be $\ell \notin [a_\alpha, b_\alpha]$. In this case, $y = I(\ell, x) = I_{GD}(\ell, x)$. If $\ell \leq x$, it would be $y = I_{GD}(\ell, x) = 1$, a contradiction. If $\ell > x$, we would have $y = I_{GD}(\ell, x) = x \in [a_\alpha, b_\alpha]$, a contradiction again. Thus, $x \not\preceq_I y$. Similarly, it can be shown that $y \not\preceq_I x$. \square

Proposition 4.11. *Let I be the ordinal sum implication of a family of implications and \preceq_I be an order induced by I . Then, $([0, 1], \preceq_I)$ is not an infimum semi-lattice.*

Proof. For some $\alpha \in A$, let $x \in [a_\alpha, b_\alpha]$ and $y \notin [a_\alpha, b_\alpha]$ with $x, y \neq 1$ and $\{x, y\}_{\preceq_I} \neq \emptyset$. There exists at least an element $k \in \{x, y\}_{\preceq_I}$. Then, $k \preceq_I x$ and $k \preceq_I y$. There exists an element ℓ such that $I(\ell, k) = x$. If $k \in [a_\alpha, b_\alpha]$, since $k \preceq_I y$ and $y \notin [a_\alpha, b_\alpha]$, we would have a contradiction by Lemma 4.10. Then, it must be $k \notin [a_\alpha, b_\alpha]$. Let $\ell \in [a_\alpha, b_\alpha]$. Then, $x = I(\ell, k) = I_{GD}(\ell, k)$. If $\ell \leq k$, we would have $x = 1$, which is a contradiction. Let $\ell > k$. Then, $x = I(\ell, k) = I_{GD}(\ell, k) = k$. Since $x = k \preceq_I y$, this contradicts to Lemma 4.10. Thus, $\ell \notin [a_\alpha, b_\alpha]$. Suppose that there exists $\gamma \in A$ such that $k, \ell \in [a_\gamma, b_\gamma]$. Then,

$$x = I(\ell, k) = a_\gamma + (b_\gamma - a_\gamma)I_\gamma\left(\frac{\ell - a_\gamma}{b_\gamma - a_\gamma}, \frac{k - a_\gamma}{b_\gamma - a_\gamma}\right) \in [a_\gamma, b_\gamma],$$

whence $x \in [a_\alpha, b_\alpha] \cap [a_\gamma, b_\gamma] = \emptyset$, contradiction. Thus, there doesn't exist such a γ such that the interval $[a_\gamma, b_\gamma]$ including both the elements k and ℓ at the same time. Then, $x = I(\ell, k) = I_{GD}(\ell, k)$. If $\ell \leq k$, it must be $x = 1$, contradiction. If $\ell > k$, we would have $x = k \preceq_I y$. This contradicts Lemma 4.10. Then, it must be $\{x, y\}_{\preceq_I} = \emptyset$. Thus, $([0, 1], \preceq_I)$ is not an infimum semi-lattice. \square

Corollary 4.12. *Let I be the ordinal sum of a family of implications and \preceq_I be an order induced by I . Then, $([0, 1], \preceq_I)$ is not a lattice.*

5 Conclusions

In this paper, based on the order defined by the method in [12], the properties of the orders, denoted by \preceq_{I_U} , induced by the residual implications obtained from uninorms are investigated. For any uninorms $U = \langle T, S, e \rangle \in \mathcal{U}_{\min}$ with left continuous t-norm T and t-conorm S , any elements belong to $[0, 1] \setminus \{e, 1\}$ are shown to be incomparable to e w.r.t. \preceq_{I_U} . Also, the elements in $[0, e)$ are shown to be not comparable to the elements in $(e, 1]$ w.r.t. the order \preceq_{I_U} . It is proved that the unit interval $[0, 1]$ is a supremum semi-lattice w.r.t. \preceq_{I_U} if and only if it is a supremum semi-lattice w.r.t. \preceq_{I_T} . Also, the unit interval $[0, 1]$ is shown not an infimum semi-lattice w.r.t. \preceq_{I_U} . Thus, it is concluded that $[0, 1]$ is not a lattice w.r.t. the order \preceq_{I_U} . A necessary and sufficient condition is presented so that the ordinal sum of fuzzy implications satisfies the law of importation with a t-norm T . Some relationships between the orders induced by an ordinal sum and its summands are presented. For any implication I and its summands I_α ($\alpha \in A$), it is proved that the unit interval $[0, 1]$ is a supremum semi-lattice w.r.t. \preceq_I if and only if $[0, 1]$ is a supremum semi-lattice w.r.t. the orders \preceq_{I_α} , for all $\alpha \in A$. It is concluded that $[0, 1]$ is not a lattice w.r.t. the order induced by any ordinal sum of fuzzy implications.

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