Iranian Journal of Fuzzy Systems

Volume 17, Number 1, (2020), pp. 121-139

University of Sistan and Baluchestan

Generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral operators for multicriteria decision making

P. Kakati[1] and S. Borkotokey[2]

[1,2] *Department of Mathematics, Dibrugarh University, Dibrugarh-786004, India*

pankaj07kakati@gmail.com, surajitbor@yahoo.com

Abstract

The interval-valued intuitionistic fuzzy set (IVIFS) which is an extension of the Atanassov's intuitionistic fuzzy set is a powerful tool for modeling real life decision making problems. In this paper, we propose the *generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral* (GIVIFHGSCI) and the *interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral* (IVIFHGSCI) operators, which consider the importance of the ordered positions and correlations among the arguments with the assumption that the aggregated arguments are interrelated. Furthermore, some of the properties and special cases of these operators are discussed. An approach for multicriteria decision making based on these operators is developed. Finally an illustrative example follows.

Keywords: Interval-valued intuitionistic fuzzy set, Hamacher operations, generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley choquet integral, multicriteria decision making.

2010 MSC: 03E72, 90B50, 62C86.

1 Introduction

Most of the real life decision making problems involve imprecise or imperfect information. To deal with such imprecise information, fuzzy set [37] was introduced by Zadeh in 1965. A fuzzy set is characterized by a membership function which represents the degree of acceptance in a decision making problem. In real situation, however, there may be a hesitancy or uncertainty about the membership degree of the object in that set. So, as its consequence, Atanassov [1, 2] introduced the intuitionistic fuzzy sets (IFSs) in 1983 that is characterized by the degrees of membership and non-membership. In the case of IFS, the non-membership grade expresses the degree of rejection in a decision making problem. However in many real life decision making problems, it is difficult to specify the precise membership degrees of an element due to incomplete information, abundant information, conflicting evidence, time pressure, lack of data, or the decision maker's limited attention etc. [7]. The interval-valued intuitionistic fuzzy set (IVIFS) which was introduced as a further generalization of the IFS by Atanasov and Gargov [3] is found to handle such imprecise and uncertain decision making situations efficiently.

Xu et al. [34] developed a series of aggregation operators under interval valued intuitionistic fuzzy environment such as the interval-valued intuitionistic fuzzy weighted arithmetic aggregation (IIFWA), the interval-valued intuitionistic fuzzy ordered weighted aggregation (IIFOWA) and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator. Xu et al. [35] further proposed the interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator, the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator and the interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator. However, the weighted aggregation operators consider situations where criteria and preferences of the decision makers are independent of one another, which means that additivity is a main property of these operators. However many decision making problems involve a certain amount of inter-dependency between the alternatives. More often, there is interaction among the preferences of the decision makers. Dutta and

Corresponding Author: P. Kakati
Received: June 2018; Revised: February 2019; Accepted: June 2019.

Guha [9] proposed the partition Bonferroni mean (PBM) to express the inter-relationship between the arguments of intra-partitions. For a comprehensive study on partition Bonferroni mean (PBM) we recommend [9, 15, 16].

The Choquet integral [6] operator with respect to any fuzzy measure [24] can effectively interpret the interaction among the alternatives. It measures the expected utility of an uncertain event. Tan and Chen [25] introduced the intuitionistic fuzzy Choquet integral (IFCI) operator. Wei [32], further extended the IFCI operator to the interval-valued intuitionistic fuzzy Choquet integral (IIFCI) Operator. Meng and Tan [19] proposed a method based on the generalized interval valued intuitionistic fuzzy Choquet integrals with the generalized interaction indices for multiattribute group decision making problems. However, during the correlative decision making situations using the Choquet integral, the importance of the ordered positions of the elements are often ignored. Again, since the fuzzy measures are defined on the power set, so, it is difficult to obtain the fuzzy measure of each subset of a large set of criteria. To overcome the correlative problem of MCDM and to simplify the complexity of solving a fuzzy measure, the Shapley function [17] has been used. For a detail study on the Shapley function and aggregation operators based on Shapley function, we refer to [10, 20, 18, 33]. Qin et al. [22] introduced new methods to express the interdependency among alternative attributes by constructing Choquet integral using interval-valued intuitionistic fuzzy numbers. Cheng [5] proposed the generalized Shapley interval-valued intuitionistic fuzzy geometric Choquet operator. Qin et al. [22] further, proposed the interval-valued intuitionistic fuzzy measure development methods with fuzzy entropy and Shapely-values in order to measure the importance of attributes and correlation of the experts.

The Einstein product and Einstein sum are good alternatives of Algebraic product and Algebraic sum. Several aggregation operators are proposed based on the Einstein operation. For a detail study on Eienstein operators, we refer to [27, 29, 28]. Most of the above mentioned aggregation operators are based on either the Algebraic or the Einstein operational laws. The Hamacher operations [11], which are further generalization of Algebraic and Einstein operations, are found to be more general and more effective in dealing with MCDM with interdependent criteria and having computational complexity. Liu [14] and Zhang et al. [38] introduce several extended aggregation operators based on the Hamacher operation rules and Frank operation rules, respectively. Tan et al. [26] proposed a family of hesitant fuzzy Hamacher operators for aggregating hesitant fuzzy information. For a detailed study on Hamacher operators we refer to [12, 14, 26, 38] etc.

Motivated by the ability of Choquet integral to deal with interrelationships between the input arguments and to overcome the difficultly to obtain the fuzzy measure we find it interesting to combine the Shapley function [17] and the Choquet integral [6] operator to define the generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (GIVIFHGSCI) operator and the interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (IVIFHGSCI) operator based on the Hamacher operation with the assumption that the aggregated arguments are interrelated.

The rest of the paper proceeds as follows. Section 2, briefly reviews some basic concepts, Section 3, Proposes the generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (GIVIFHGSCI) operator and the interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (IVIFHGSCI) operator, and discuss some of their properties. Section 4, presents an approach to multicriteria decision making based on the proposed operators. Section 5, shows the feasibility and validity of the approach in multicriteria decision making by a numerical examples. Section 6, provides the concluding remarks.

2 Preliminaries

We compile in this section the relevant notion required for the development of the present paper.

Definition 2.1. [1] *Let X be a finite non empty set, then an intuitionistic fuzzy set (IFS) on X is an object A given by $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in X$ to A together with the condition that $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for all $x \in X$.*

Definition 2.2. [3] *Let X be a finite non empty set, then an interval-valued intuitionistic fuzzy set (IVIFS) on X is an object $\tilde{A}$ given by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle \mid x \in X\}$, where, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ respectively, are called the membership degree and the non-membership degree of the element $x \in X$ to $\tilde{A}$, satisfying $\mu_{\tilde{A}}(x) \subset [0,1]$ and $\nu_{\tilde{A}}(x) \subset [0,1]$.*
Xu [34] called each pair $(\mu_{\tilde{A}}, \nu_{\tilde{A}})$ an interval-valued intuitionistic fuzzy number (IVIFN), where $\mu_{\tilde{A}} = [\mu^-_{\tilde{A}}, \mu^+_{\tilde{A}}]$ and $\nu_{\tilde{A}} = [\nu^-_{\tilde{A}}, \nu^+_{\tilde{A}}]$ are interval numbers, with the condition $0 \leqslant \mu^+_{\tilde{A}} + \nu^+_{\tilde{A}} \leqslant 1$.

Definition 2.3. [25] *Let $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}]\}$, and $\tilde{\beta}_i = \{[\mu^-_{\tilde{\beta}_i}, \mu^+_{\tilde{\beta}_i}], [\nu^-_{\tilde{\beta}_i}, \nu^+_{\tilde{\beta}_i}]\}$ $(i = 1, 2, \ldots, n)$ be two collections of* IVIFN*s on a finite set X, then $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are said to be comonotonic, if $\tilde{\alpha}_{(1)} \leqslant \tilde{\alpha}_{(2)} \leqslant \cdots \leqslant \tilde{\alpha}_{(n)}$ iff $\tilde{\beta}_{(1)} \leqslant \tilde{\beta}_{(2)} \leqslant\leqslant$*

$\cdots \leqslant \tilde{\beta}_{(n)}$, *where* $(\cdot)$ *denotes a permutation on* X.

Definition 2.4. [31, 34] *The score function* $S(\tilde{\alpha})$, *accuracy function* $H(\tilde{\alpha})$, *membership uncertainty index* $T(\tilde{\alpha})$, *and hesitation uncertainty index* $G(\tilde{\alpha})$ *of an IVIFN* $\tilde{\alpha} = \left([\mu_{\tilde{\alpha}}^-, \mu_{\tilde{\alpha}}^+], [\nu_{\tilde{\alpha}}^-, \nu_{\tilde{\alpha}}^+]\right)$ *are respectively defined as follows,*

$$S(\tilde{\alpha}) = \frac{\mu_{\tilde{\alpha}}^- - \nu_{\tilde{\alpha}}^- + \mu_{\tilde{\alpha}}^+ - \nu_{\tilde{\alpha}}^+}{2} \quad (1) \qquad H(\tilde{\alpha}) = \frac{\mu_{\tilde{\alpha}}^- + \mu_{\tilde{\alpha}}^+ + \nu_{\tilde{\alpha}}^- + \nu_{\tilde{\alpha}}^+}{2} \quad (2)$$

$$T(\tilde{\alpha}) = \mu_{\tilde{\alpha}}^+ + \nu_{\tilde{\alpha}}^- - \mu_{\tilde{\alpha}}^- - \nu_{\tilde{\alpha}}^+ \quad (3) \qquad G(\tilde{\alpha}) = \mu_{\tilde{\alpha}}^+ + \nu_{\tilde{\alpha}}^+ - \mu_{\tilde{\alpha}}^- - \nu_{\tilde{\alpha}}^- \quad (4)$$

Definition 2.5. [31] *Let* $\tilde{\alpha}_1 = \left([\mu_{\tilde{\alpha}_1}^-, \mu_{\tilde{\alpha}_1}^+], [\nu_{\tilde{\alpha}_1}^-, \nu_{\tilde{\alpha}_1}^+]\right)$ *and* $\tilde{\alpha}_2 = \left([\mu_{\tilde{\alpha}_2}^-, \mu_{\tilde{\alpha}_2}^+], [\nu_{\tilde{\alpha}_2}^-, \nu_{\tilde{\alpha}_2}^+]\right)$ *be two IVIFNs, then*

(S1.) *if* $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, *then* $\tilde{\alpha}_1 < \tilde{\alpha}_2$,

(S2.) *if* $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, *then*

- **(H1.)** *if* $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, *then* $\tilde{\alpha}_1 < \tilde{\alpha}_2$,
- **(H2.)** *if* $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, *then*
  - **(T1.)** *if* $T(\tilde{\alpha}_1) > T(\tilde{\alpha}_2)$, *then* $\tilde{\alpha}_1 < \tilde{\alpha}_2$,
  - **(T2.)** *if* $T(\tilde{\alpha}_1) = T(\tilde{\alpha}_2)$, *then*
    - **(G1.)** *if* $G(\tilde{\alpha}_1) > G(\tilde{\alpha}_2)$, *then* $\tilde{\alpha}_1 < \tilde{\alpha}_2$,
    - **(G2.)** *if* $G(\tilde{\alpha}_1) = G(\tilde{\alpha}_2)$, *then* $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

Hamacher proposed the Hamacher operations [11], which include the Hamacher product and Hamacher sum. Hamacher product $\bigotimes_H$ is a t-norm and Hamacher sum $\bigoplus_H$ is a t-conorm, given by

$$T_\gamma(x,y) = x \bigotimes\nolimits_H y = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)},\ \gamma > 0 \tag{5}$$

$$_\gamma(x,y) = x \bigoplus\nolimits_H y = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy},\ \gamma > 0 \tag{6}$$

In particular, when $\gamma = 1$, then the Hamacher t-norm and t-conorm reduces to

$$T_1(x,y) = xy \quad (7) \qquad S_1(x,y) = x + y - xy \quad (8)$$

which are the algebraic t-norm and t-conorm respectively.
Again, when $\gamma = 2$, then the Hamacher t-norm and t-conorm reduces to

$$T_2(x,y) = \frac{xy}{1+(1-x)(1-y)} \quad (9) \qquad S_2(x,y) = \frac{x+y}{1+xy} \quad (10)$$

which are the Einstein t-norm and t-conorm respectively.

Dombi [8] proposed some additive generators functional for various t-norms and t-conorms based on the following conditions.

Condition 1 If ϕ is a monotonically decreasing function and $\phi : \mathbb{R}^+ \to (0,1]$, and $\phi^{-1} : (0,1] \to \mathbb{R}^+$ with the following basic limit properties :

(i) $\lim_{x\to\infty} \phi^{-1}(x) = 0$ (ii) $\phi^{-1}(0) = 1$

then ϕ can generate the t-norm T.

Condition 2 If ϕ is a monotonically increasing function and $\phi : \mathbb{R}^+ \to (0,1]$, and $\phi^{-1} : (0,1] \to \mathbb{R}^+$ with the following basic limit properties :

(i) $\lim_{x\to\infty} \phi^{-1}(x) = 1$ (ii) $\phi^{-1}(0) = 0$

then ϕ can generates the t-conorm S.

Thus additive generators f and g can generate t-norm $T(x,y) = f^{-1}(f(x)+f(y))$, and t-conorm $S(x,y) = g^{-1}(g(x)+g(y))$, provided they satisfy ***Condition 1*** and ***Condition 2***, respectively. The additive generators of Hamacher t-norm, and t-conorm [13] are given by,

Table 1: **Additive generators of Hamacher t-norm and t-conorm**

Name	t-norm/t-conorm	Additive generator
Hamacher t-norm	$T(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$	$f(t)=\ln\frac{\gamma+(1-\gamma)t}{t},\ \gamma>0$
Hamacher t-conorm	$S(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}$	$g(t)=\ln\frac{\gamma+(1-\gamma)(1-t)}{1-t},\ \gamma>0$

Definition 2.6. [14, 39] *Let* $\tilde{\alpha}=([\mu^-_{\tilde{\alpha}},\mu^+_{\tilde{\alpha}}],[\nu^-_{\tilde{\alpha}},\nu^+_{\tilde{\alpha}}])$ *and* $\tilde{\beta}=([\mu^-_{\tilde{\beta}},\mu^+_{\tilde{\beta}}],[\nu^-_{\tilde{\beta}},\nu^+_{\tilde{\beta}}])$ *be two IVIFNs and a scalars* $\gamma,k>0$, *then the following operation laws hold.*

(i) $$\tilde{\alpha}\bigoplus\nolimits_H\tilde{\beta}=\left\{\left[\frac{\mu^-_{\tilde{\alpha}}+\mu^-_{\tilde{\beta}}-\mu^-_{\tilde{\alpha}}\mu^-_{\tilde{\beta}}-(1-\gamma)\mu^-_{\tilde{\alpha}}\mu^-_{\tilde{\beta}}}{1-(1-\gamma)\mu^-_{\tilde{\alpha}}\mu^-_{\tilde{\beta}}},\frac{\mu^+_{\tilde{\alpha}}+\mu^+_{\tilde{\beta}}-\mu^+_{\tilde{\alpha}}\mu^+_{\tilde{\beta}}-(1-\gamma)\mu^+_{\tilde{\alpha}}\mu^+_{\tilde{\beta}}}{1-(1-\gamma)\mu^+_{\tilde{\alpha}}\mu^+_{\tilde{\beta}}}\right],\left[\frac{\nu^-_{\tilde{\alpha}}\nu^-_{\tilde{\beta}}}{\gamma+(1-\gamma)(\nu^-_{\tilde{\alpha}}+\nu^-_{\tilde{\beta}}-\nu^-_{\tilde{\alpha}}\nu^-_{\tilde{\beta}})},\frac{\nu^+_{\tilde{\alpha}}\nu^+_{\tilde{\beta}}}{\gamma+(1-\gamma)(\nu^+_{\tilde{\alpha}}+\nu^+_{\tilde{\beta}}-\nu^+_{\tilde{\alpha}}\nu^+_{\tilde{\beta}})}\right]\right\}$$

(ii) $$\tilde{\alpha}\bigotimes\nolimits_H\tilde{\beta}=\left\{\left[\frac{\mu^-_{\tilde{\alpha}}\mu^-_{\tilde{\beta}}}{\gamma+(1-\gamma)(\mu^-_{\tilde{\alpha}}+\mu^-_{\tilde{\beta}}-\mu^-_{\tilde{\alpha}}\mu^-_{\tilde{\beta}})},\frac{\mu^+_{\tilde{\alpha}}\mu^+_{\tilde{\beta}}}{\gamma+(1-\gamma)(\mu^+_{\tilde{\alpha}}+\mu^+_{\tilde{\beta}}-\mu^+_{\tilde{\alpha}}\mu^+_{\tilde{\beta}})}\right],\left[\frac{\nu^-_{\tilde{\alpha}}+\nu^-_{\tilde{\beta}}-\nu^-_{\tilde{\alpha}}\nu^-_{\tilde{\beta}}-(1-\gamma)\nu^-_{\tilde{\alpha}}\nu^-_{\tilde{\beta}}}{1-(1-\gamma)\nu^-_{\tilde{\alpha}}\nu^-_{\tilde{\beta}}},\frac{\nu^+_{\tilde{\alpha}}+\nu^+_{\tilde{\beta}}-\nu^+_{\tilde{\alpha}}\nu^+_{\tilde{\beta}}-(1-\gamma)\nu^+_{\tilde{\alpha}}\nu^+_{\tilde{\beta}}}{1-(1-\gamma)\nu^+_{\tilde{\alpha}}\nu^+_{\tilde{\beta}}}\right]\right\}$$

(iii) $$k\tilde{\alpha}=\left\{\left[\frac{\left(1+(\gamma-1)\mu^-_{\tilde{\alpha}}\right)^k-\left(1-\mu^-_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)\mu^-_{\tilde{\alpha}}\right)^k+(\gamma-1)\left(1-\mu^-_{\tilde{\alpha}}\right)^k},\frac{\left(1+(\gamma-1)\mu^+_{\tilde{\alpha}}\right)^k-\left(1-\mu^+_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)\mu^+_{\tilde{\alpha}}\right)^k+(\gamma-1)\left(1-\mu^+_{\tilde{\alpha}}\right)^k}\right],\left[\frac{\gamma\left(\nu^-_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)(1-\nu^-_{\tilde{\alpha}})\right)^k+(\gamma-1)\left(\nu^-_{\tilde{\alpha}}\right)^k},\frac{\gamma\left(\nu^+_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)(1-\nu^+_{\tilde{\alpha}})\right)^k+(\gamma-1)\left(\nu^+_{\tilde{\alpha}}\right)^k}\right]\right\}$$

(iv) $$\tilde{\alpha}^k=\left\{\left[\frac{\gamma\left(\mu^-_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)(1-\mu^-_{\tilde{\alpha}})\right)^k+(\gamma-1)\left(\mu^-_{\tilde{\alpha}}\right)^k},\frac{\gamma\left(\mu^+_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)(1-\mu^+_{\tilde{\alpha}})\right)^k+(\gamma-1)\left(\mu^+_{\tilde{\alpha}}\right)^k}\right],\left[\frac{\left(1+(\gamma-1)\nu^-_{\tilde{\alpha}}\right)^k-\left(1-\nu^-_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)\nu^-_{\tilde{\alpha}}\right)^k+(\gamma-1)\left(1-\nu^-_{\tilde{\alpha}}\right)^k},\frac{\left(1+(\gamma-1)\nu^+_{\tilde{\alpha}}\right)^k-\left(1-\nu^+_{\tilde{\alpha}}\right)^k}{\left(1+(\gamma-1)\nu^+_{\tilde{\alpha}}\right)^k+(\gamma-1)\left(1-\nu^+_{\tilde{\alpha}}\right)^k}\right]\right\}$$

Definition 2.7. [24] *A fuzzy measure on finite set* N *is a set function* $\rho:P(N)\to[0,1]$, *satisfying the followings (i)* $\rho(\emptyset)=0,\ \rho(N)=1$ *(ii) if* $A\subseteq B\subseteq N$, *then* $\rho(A)\leqslant\rho(B)$

In a multicriteria decision making problem involving n criteria, it requires 2^n-2 coefficients in order to define a fuzzy measure ρ. So the evaluation model obtained becomes quite difficult. To avoid this difficulty, Sugeno [24] proposed a special kind of fuzzy measure called λ-fuzzy measure which is defined as follows:

Definition 2.8. [24] *A* λ*-fuzzy measure on finite set* N *is a set function* $\rho_\lambda:P(N)\to[0,1]$, *satisfying the followings*

(i) $\rho_\lambda(\emptyset)=0,\rho_\lambda(N)=1$

(ii) if $A\subseteq B\subseteq N$, *then* $\rho_\lambda(A)\leqslant\rho_\lambda(B)$

(iii) $\rho_\lambda(A\cup B)=\rho_\lambda(A)+\rho_\lambda(B)+\lambda\rho_\lambda(A)\rho_\lambda(B),\ \lambda\in(-1,+\infty)$, $\forall A,B\in P(N)$, *and* $A\cap B=\emptyset,\lambda>-1$

In particular, if $\lambda=0$, *then the condition (iii) reduces to the axiom of additive measure:*

$$\rho_\lambda(A\cup B)=\rho_\lambda(A)+\rho_\lambda(B),\ \forall A,B\subseteq N,\ \text{and}\ A\cap B=\emptyset \tag{11}$$

In this case, all the elements in N *are independent, and we have :*

$$\rho_\lambda(A)=\sum_{i\in A}\rho_\lambda(\{i\}) \tag{12}$$

If $\lambda>0$ *then* $\rho_\lambda(A\cup B)>\rho_\lambda(A)+\rho_\lambda(B)$, *which implies that the set* $\{A,B\}$ *has multiplicative effect. If* $\lambda<0$ *then* $\rho_\lambda(A\cup B)<\rho_\lambda(A)+\rho_\lambda(B)$,*which implies that the set* $\{A,B\}$ *has substitutive effect. By parameter* λ, *the interaction*

between sets or elements of set can be represented.
Let N be a finite set, then for every subset $A \subseteq N$, Sugeno [24] gave the following forms.

$$\rho_\lambda(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i\in A}[1+\lambda\rho_\lambda(\{i\})]-1\right), & if \lambda \neq 0 \\ \sum_{i\in A}\rho_\lambda(\{i\}), & if \lambda = 0 \end{cases} \tag{13}$$

From $\rho_\lambda(N) = 1$, the parameter λ can be uniquely determined by solving the following equation.

$$\lambda + 1 = \prod_{i\in N}\big(1 + \lambda\rho_\lambda(\{i\})\big) \tag{14}$$

Marichal [17] proposed the concept of the generalized Shapley index, which measures the overall impact of each coalition rather than each player in a game theory problem. The generalized Shapley index [17] is given by,

$$\varphi_S(\rho, N) = \sum_{T\subset N\setminus S}\frac{\big(|N|-|S|-|T|\big)!|T|!}{\big(|N|-|S|+1\big)!}\Big(\rho(S\cup T)-\rho(T)\Big) \qquad \forall\, S \subseteq N \tag{15}$$

where ρ is a fuzzy measure on N. $|N|, |S|$ and $|T|$ denote the cardinalities of coalitions S, T and N, respectively.
In particular, if $S = \{i\}$, then $|S| = 1$, and the Equation (15) of the generalized Shapley index reduces to the Shapley function [23] given by,

$$\varphi_i(\rho, N) = \sum_{T\subseteq N\setminus i}\frac{\big(|N|-|T|-1\big)!|T|!}{|N|!}\Big(\rho(i\cup T)-\rho(T)\Big) \qquad \forall\, i \in N \tag{16}$$

If for any coalition $S \subseteq N$, there is no interaction between the coalition S and any combination of coalition $T \subseteq N \setminus S$, then $\rho(S \cup T) = \rho(S) + \rho(T)$, and Equation (15) gives $\varphi_S(\rho, N) = \rho(S)$. Further, if for some $i \in N$, there is no interaction between the element i and any coalition $T \subseteq N \setminus \{i\}$, then $\varphi_i(\rho, N) = \rho(\{i\})$. Clearly, for each element $i \in N$, from the Definition 2.7 of fuzzy measures, $\varphi_i(\rho, N) \geqslant 0$, and $\sum_{i=1}^{n}\varphi_i(\rho, N) = 1$. Thus $\{\varphi_i(\rho, N)\}_{i\in N}$ represents the weight vector associated with the element $i \in N$.

Definition 2.9. [21] *Let f be a positive real-valued function on $X = \{x_1, x_2, \ldots, x_n\}$, and let μ be a fuzzy measure on X. The discrete Choquet integral of f with respect to ρ, denoted by $C_\rho(f)$, is defined as follows:*

$$C_\rho(f) = \sum_{i=1}^{n} f(x_{(i)})\Big(\rho(A_{(i)}) - \rho(A_{(i-1)})\Big), \tag{17}$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)}) \geqslant f(x_{(2)}) \geqslant \cdots \geqslant f(x_{(n)})$,
with $A_{(i)} = \{x_{(1)}, x_{(2)}, \ldots, x_{(i)}\}$ for $i \geqslant 1$ and $A_{(0)} = \emptyset$.

2.1 Operational rules of IFNs based on additive generators

Based on the additive generators [8] of Hamacher [11] t-norm $T(x, y)$ and t-conorm $S(x, y)$ [13], we review the operational rules of IVIFNs [14, 39] in Definition 2.6 as follows.

Definition 2.10. *$\tilde{\alpha} = ([\mu^-_{\tilde{\alpha}}, \mu^+_{\tilde{\alpha}}], [\nu^-_{\tilde{\alpha}}, \nu^+_{\tilde{\alpha}}])$ and $\tilde{\beta} = ([\mu^-_{\tilde{\beta}}, \mu^+_{\tilde{\beta}}], [\nu^-_{\tilde{\beta}}, \nu^+_{\tilde{\beta}}])$ be two IVIFNs and a scalars $\gamma, k > 0$, then the following Hamacher operation laws hold.*

(i) $$\tilde{\alpha} \bigoplus\nolimits_H \tilde{\beta} = \Big\{\Big[g^{-1}\big(g(\mu^-_{\tilde{\alpha}}) + g(\mu^-_{\tilde{\beta}})\big), g^{-1}\big(g(\mu^+_{\tilde{\alpha}}) + g(\mu^+_{\tilde{\beta}})\big)\Big], \Big[f^{-1}\big(f(\nu^-_{\tilde{\alpha}}) + f(\nu^-_{\tilde{\beta}})\big), f^{-1}\big(f(\nu^+_{\tilde{\alpha}}) + f(\nu^+_{\tilde{\beta}})\big)\Big]\Big\}$$

(ii) $$\tilde{\alpha} \bigotimes\nolimits_H \tilde{\beta} = \Big\{\Big[f^{-1}\big(f(\mu^-_{\tilde{\alpha}}) + f(\mu^-_{\tilde{\beta}})\big), f^{-1}\big(f(\mu^+_{\tilde{\alpha}}) + f(\mu^+_{\tilde{\beta}})\big)\Big], \Big[g^{-1}\big(g(\nu^-_{\tilde{\alpha}}) + g(\nu^-_{\tilde{\beta}})\big), g^{-1}\big(g(\nu^+_{\tilde{\alpha}}) + g(\nu^+_{\tilde{\beta}})\big)\Big]\Big\}$$

(iii) $k\tilde{\alpha} = \left\{ \left[g^{-1}\big(kg(\mu^-_{\tilde{\alpha}})\big), g^{-1}\big(kg(\mu^+_{\tilde{\alpha}})\big) \right], \left[f^{-1}\big(kf(\nu^-_{\tilde{\alpha}})\big), f^{-1}\big(kf(\nu^+_{\tilde{\alpha}})\big) \right] \right\}$

(iv) $\tilde{\alpha}^k = \left\{ \left[f^{-1}\big(kf(\mu^-_{\tilde{\alpha}})\big), f^{-1}\big(kf(\mu^+_{\tilde{\alpha}})\big) \right], \left[g^{-1}\big(kg(\nu^-_{\tilde{\alpha}})\big), g^{-1}\big(kg(\nu^+_{\tilde{\alpha}})\big) \right] \right\}$

where, $f(t) = \ln \frac{\gamma+(1-\gamma)t}{t}$, $\gamma > 0$ *and* $g(t) = \ln \frac{\gamma+(1-\gamma)(1-t)}{1-t}$, $\gamma > 0$ *are the additive generators of Hamacher t-norm and t-conorm respectively.*

3 The GIVIFHGSCI and IVIFHGSCI operators

In this section, we propose the generalized interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (GIVIFHGSCI) and the interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral (IVIFHGSCI) operators and investigate some of their desired properties.

Definition 3.1. *Let X be a finite set of criteria and ρ be a fuzzy measure on X.*
Let $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ be a collection of IVIFNs on X, then, the generalized interval-valued intuitionistic fuzzy Hamacher Shapley Choquet integral operator is defined as

$$\text{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\bigoplus_{H\, i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X) \Big) (\tilde{\alpha}_i)^{\eta} \right)^{\frac{1}{\eta}} \tag{18}$$

where $\eta > 0$ is a parameter, and $(\cdot)$ denotes a permutation of $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$, such that $\tilde{\alpha}_{(1)} \geqslant \tilde{\alpha}_{(2)} \geqslant \cdots \geqslant \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{\tilde{\alpha}_{(1)}, \tilde{\alpha}_{(2)}, \ldots, \tilde{\alpha}_{(i)}\}$, $A_{(0)} = \emptyset$. Also, here $\bigoplus_H$ denotes the Hamacher sum of IVIFEs [14] *given by (i) in Definition 2.6.*

Theorem 3.2. *Let $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ be a collection of* IVIFN*s on a finite set X. Let ρ be a fuzzy measure on X, and $\eta > 0$ be a parameter, then the* GIVIFHGSCI *can expressed as*

$$\begin{aligned}
\text{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \Bigg\{ \Bigg[& f^{-1}\left(\frac{1}{\eta} f\left(g^{-1}\left(\sum_{i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X) \Big) g\Big(f^{-1}\big(\eta f(\mu^-_{\tilde{\alpha}_i})\big) \Big) \right) \right) \right), \\
& f^{-1}\left(\frac{1}{\eta} f\left(g^{-1}\left(\sum_{i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X) \Big) g\Big(f^{-1}\big(\eta f(\mu^+_{\tilde{\alpha}_i})\big) \Big) \right) \right) \right) \Bigg], \\
\Bigg[& g^{-1}\left(\frac{1}{\eta} g\left(f^{-1}\left(\sum_{i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X) \Big) f\Big(g^{-1}\big(\eta g(\nu^-_{\tilde{\alpha}_i})\big) \Big) \right) \right) \right), \\
& g^{-1}\left(\frac{1}{\eta} g\left(f^{-1}\left(\sum_{i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X) \Big) f\Big(g^{-1}\big(\eta g(\nu^+_{\tilde{\alpha}_i})\big) \Big) \right) \right) \right) \Bigg] \Bigg\}
\end{aligned} \tag{19}$$

Equation (19) *expresses the* GIVIFHGSCI *operator in terms of the additive generators* [8], *f and g of Hamacher t-norm and t-conorm* [13].
Where,

$$f(t) = \ln \frac{\gamma+(1-\gamma)t}{t}, \quad g(t) = \ln \frac{\gamma+(1-\gamma)(1-t)}{1-t}$$
$$f^{-1}(t) = \frac{\gamma}{e^t+\gamma-1}, \quad g^{-1}(t) = \frac{e^t-1}{e^t+\gamma-1}, \ \gamma > 0$$

Theorem 3.3. *Let $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ be a collection of* IVIFN*s on a finite set X. Let ρ be a fuzzy measure on X, and $\eta > 0$ be a parameter, then the aggregated result obtained by using the* GIVIFHGSCI *operator is also an* IVIFN*, given as follows:*

$$\begin{aligned}
\text{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \Bigg\{ & \left[\frac{\gamma\big(\upsilon^-_\gamma - \upsilon^-_0\big)^{\frac{1}{\eta}}}{\big(\upsilon^-_\gamma + (\gamma^2-1)\upsilon^-_0\big)^{\frac{1}{\eta}} + (\gamma-1)\big(\upsilon^-_\gamma - \upsilon^-_0\big)^{\frac{1}{\eta}}}, \frac{\gamma\big(\upsilon^+_\gamma - \upsilon^+_0\big)^{\frac{1}{\eta}}}{\big(\upsilon^+_\gamma + (\gamma^2-1)\upsilon^-_0\big)^{\frac{1}{\eta}} + (\gamma-1)\big(\upsilon^+_\gamma - \upsilon^+_0\big)^{\frac{1}{\eta}}} \right], \\
& \left[\frac{\big(\tau^-_\gamma + (\gamma^2-1)\tau^-_0\big)^{\frac{1}{\eta}} - \big(\tau^-_\gamma - \tau^-_0\big)^{\frac{1}{\eta}}}{\big(\tau^-_\gamma + (\gamma^2-1)\tau^-_0\big)^{\frac{1}{\eta}} + (\gamma-1)\big(\tau^-_\gamma - \tau^-_0\big)^{\frac{1}{\eta}}}, \frac{\big(\tau^+_\gamma + (\gamma^2-1)\tau^+_0\big)^{\frac{1}{\eta}} - \big(\tau^+_\gamma - \tau^+_0\big)^{\frac{1}{\eta}}}{\big(\tau^+_\gamma + (\gamma^2-1)\tau^+_0\big)^{\frac{1}{\eta}} + (\gamma-1)\big(\tau^+_\gamma - \tau^+_0\big)^{\frac{1}{\eta}}} \right] \Bigg\}
\end{aligned} \tag{20}$$

where

$$\upsilon_\gamma^- = \prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^-\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} \quad , \quad \upsilon_0^- = \prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^-\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$$

$$\upsilon_\gamma^+ = \prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^+\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} \quad , \quad \upsilon_0^+ = \prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^+\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$$

$$\tau_\gamma^- = \prod_{i=1}^{n}\big(1+(\gamma-1)(1-\nu_{\tilde{\beta}_i}^-)\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} \quad , \quad \tau_0^- = \prod_{i=1}^{n}\big(\nu_{\tilde{\beta}_i}^-\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$$

$$\tau_\gamma^+ = \prod_{i=1}^{n}\big(1+(\gamma-1)(1-\nu_{\tilde{\beta}_i}^+)\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} \quad , \quad \tau_0^+ = \prod_{i=1}^{n}\big(\nu_{\tilde{\beta}_i}^+\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}.$$

Also, $\tilde{\beta}_i = \tilde{\alpha}_i^{\eta}$, *and* $\tilde{\beta}_i = \{[\mu_{\tilde{\beta}_i}^-, \mu_{\tilde{\beta}_i}^+], [\nu_{\tilde{\beta}_i}^-, \mu_{\tilde{\beta}_i}^+]\}$ $(i = 1, 2, \dots, n)$.

Proof. The additive generators of Hamacher [11] t-norm, and t-conorm [13] are given by

$$f(t) = \ln\frac{\gamma+(1-\gamma)t}{t} \quad , \quad g(t) = \ln\frac{\gamma+(1-\gamma)(1-t)}{1-t}$$
$$f^{-1}(t) = \frac{\gamma}{e^t+\gamma-1} \quad , \quad g^{-1}(t) = \frac{e^t-1}{e^t+\gamma-1},\ \gamma > 0$$

Now, using the additive generators f and g, we have

$$f(\mu_{\tilde{\alpha}_i}^-) = \ln\left(\frac{\gamma+(1-\gamma)\mu_{\tilde{\alpha}_i}^-}{\mu_{\tilde{\alpha}_i}^-}\right)$$

$$f^{-1}\big(\eta f(\mu_{\alpha_i}^-)\big) = \frac{\gamma\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}}{\big(1+(\gamma-1)(1-\mu_{\tilde{\alpha}_i}^-)\big)^{\eta}+(\gamma-1)\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}} \tag{21}$$

Let $\tilde{\beta}_i = \tilde{\alpha}_i^{\eta}$, and $\tilde{\beta}_i = \{[\mu_{\tilde{\beta}_i}^-, \mu_{\tilde{\beta}_i}^+], [\nu_{\tilde{\beta}_i}^-, \mu_{\tilde{\beta}_i}^+]\}$ $(i = 1, 2, \dots, n)$
Again by rule (iv) of Definition 2.6, we have,

$$\tilde{\beta}_i = \tilde{\alpha}_i^{\eta} = \Bigg\{\Bigg[\frac{\gamma\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}}{\big(1+(\gamma-1)(1-\mu_{\tilde{\alpha}_i}^-)\big)^{\eta}+(\gamma-1)\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}}, \frac{\gamma\big(\mu_{\tilde{\alpha}_i}^+\big)^{\eta}}{\big(1+(\gamma-1)(1-\mu_{\tilde{\alpha}_i}^+)\big)^{\eta}+(\gamma-1)\big(\mu_{\tilde{\alpha}_i}^+\big)^{\eta}}\Bigg], \Bigg[\frac{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^-\big)^{\eta}-\big(1-\nu_{\tilde{\alpha}_i}^-\big)^{\eta}}{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^-\big)^{\eta}+(\gamma-1)\big(1-\nu_{\tilde{\alpha}_i}^-\big)^{\eta}}, \frac{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^+\big)^{\eta}-\big(1-\nu_{\tilde{\alpha}_i}^+\big)^{\eta}}{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^+\big)^{\eta}+(\gamma-1)\big(1-\nu_{\tilde{\alpha}_i}^+\big)^{\eta}}\Bigg]\Bigg\} \tag{22}$$

So,

$$\left.\begin{aligned}
\mu_{\tilde{\beta}_i}^- &= \frac{\gamma\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}}{\big(1+(\gamma-1)(1-\mu_{\tilde{\alpha}_i}^-)\big)^{\eta}+(\gamma-1)\big(\mu_{\tilde{\alpha}_i}^-\big)^{\eta}} \quad , \quad \mu_{\tilde{\beta}_i}^+ = \frac{\gamma\big(\mu_{\tilde{\alpha}_i}^+\big)^{\eta}}{\big(1+(\gamma-1)(1-\mu_{\tilde{\alpha}_i}^+)\big)^{\eta}+(\gamma-1)\big(\mu_{\tilde{\alpha}_i}^+\big)^{\eta}} \\
\nu_{\tilde{\beta}_i}^- &= \frac{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^-\big)^{\eta}-\big(1-\nu_{\tilde{\alpha}_i}^-\big)^{\eta}}{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^-\big)^{\eta}+(\gamma-1)\big(1-\nu_{\tilde{\alpha}_i}^-\big)^{\eta}} \quad , \quad \nu_{\tilde{\beta}_i}^+ = \frac{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^+\big)^{\eta}-\big(1-\nu_{\tilde{\alpha}_i}^+\big)^{\eta}}{\big(1+(\gamma-1)\nu_{\tilde{\alpha}_i}^+\big)^{\eta}+(\gamma-1)\big(1-\nu_{\tilde{\alpha}_i}^+\big)^{\eta}}
\end{aligned}\right\} \tag{23}$$

Then, from Equation (21) and Equation set (23), we have

$$f^{-1}\big(\eta f(\mu_{\alpha_i}^-)\big) = \mu_{\tilde{\beta}_i}^- \tag{24}$$

Therefore,

$$g\Big(f^{-1}\big(\eta f(\mu_{\alpha_i}^-)\big)\Big) = \ln\left(\frac{\gamma+(1-\gamma)\big(1-\mu_{\tilde{\beta}_i}^-\big)}{\big(1-\mu_{\tilde{\beta}_i}^-\big)}\right) = \ln\left(\frac{1+(\gamma-1)\mu_{\tilde{\beta}_i}^-}{\big(1-\mu_{\tilde{\beta}_i}^-\big)}\right) \tag{25}$$

Therefore,

$$\sum_{i=1}^{n}\Big(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\Big)\,g\Big(f^{-1}\big(\eta f(\mu^{-}_{\tilde{\alpha}_i})\big)\Big)=\sum_{i=1}^{n}\ln\left(\frac{1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i}}{(1-\mu^{-}_{\tilde{\beta}_i})}\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$$
$$=\ln\left(\frac{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\right) \tag{26}$$

Now,

$$g^{-1}\left(\sum_{i=1}^{n}\Big(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\Big)\,g\Big(f^{-1}\big(\eta f(\mu^{-}_{\tilde{\alpha}_i})\big)\Big)\right)$$
$$=\frac{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma-1)\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}} \tag{27}$$

Again,

$$f\left(g^{-1}\left(\sum_{i=1}^{n}\Big(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\Big)\,g\Big(f^{-1}\big(\eta f(\mu^{-}_{\tilde{\alpha}_i})\big)\Big)\right)\right)$$
$$=\ln\left(\frac{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma^2-1)\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\right) \tag{28}$$

And,

$$\frac{1}{\eta}f\left(g^{-1}\left(\sum_{i=1}^{n}\Big(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\Big)\,g\Big(f^{-1}\big(\eta f(\mu^{-}_{\tilde{\alpha}_i})\big)\Big)\right)\right)$$
$$=\ln\left(\frac{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma^2-1)\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}(1+(\gamma-1)\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}(1-\mu^{-}_{\tilde{\beta}_i})^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\right)^{\frac{1}{\eta}} \tag{29}$$

Now,

$$f^{-1}\left(\frac{1}{\eta}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)g\Big(f^{-1}\big(\eta f(\mu_{\tilde{\alpha}_i}^{-})\big)\Big)\right)\right)\right)$$

$$=\frac{\gamma\left(\prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}}{\left(\prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma^2-1)\prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}+(\gamma-1)\left(\prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}} \tag{30}$$

Let $\upsilon_{\gamma}^{-}=\prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$, and $\upsilon_{0}^{-}=\prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^{-}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$.

Then Equation (30) can be expressed as,

$$f^{-1}\left(\frac{1}{\eta}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)g\Big(f^{-1}\big(\eta f(\mu_{\tilde{\alpha}_i}^{-})\big)\Big)\right)\right)\right)=\frac{\gamma(\upsilon_{\gamma}^{-}-\upsilon_{0}^{-})^{\frac{1}{\eta}}}{\left(\upsilon_{\gamma}^{-}+(\gamma^2-1)\upsilon_{0}^{-}\right)^{\frac{1}{\eta}}+(\gamma-1)\left(\upsilon_{\gamma}^{-}-\upsilon_{0}^{-}\right)^{\frac{1}{\eta}}} \tag{31}$$

Similarly,

$$f^{-1}\left(\frac{1}{\eta}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)g\Big(f^{-1}\big(\eta f(\mu_{\tilde{\alpha}_i}^{+})\big)\Big)\right)\right)\right)=\frac{\gamma(\upsilon_{\gamma}^{+}-\upsilon_{0}^{+})^{\frac{1}{\eta}}}{\left(\upsilon_{\gamma}^{+}+(\gamma^2-1)\upsilon_{0}^{+}\right)^{\frac{1}{\eta}}+(\gamma-1)\left(\upsilon_{\gamma}^{+}-\upsilon_{0}^{+}\right)^{\frac{1}{\eta}}} \tag{32}$$

where, $\upsilon_{\gamma}^{+}=\prod_{i=1}^{n}\big(1+(\gamma-1)\mu_{\tilde{\beta}_i}^{+}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$, and $\upsilon_{0}^{+}=\prod_{i=1}^{n}\big(1-\mu_{\tilde{\beta}_i}^{+}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$

Again

$$g^{-1}\big(\eta g(\nu_{\tilde{\alpha}_i}^{-})\big)=\frac{\big(1+(\gamma-1)\nu_{\tilde{\beta}_i}^{-}\big)^{\eta}-\big(1-\nu_{\tilde{\beta}_i}^{-}\big)^{\eta}}{\big(1+(\gamma-1)\nu_{\tilde{\beta}_i}^{-}\big)^{\eta}+(\gamma-1)\big(1-\nu_{\tilde{\beta}_i}^{-}\big)^{\eta}} \tag{33}$$

Then, from Equation (33) and Equation set (23), we have

$$g^{-1}\big(\eta g(\nu_{\tilde{\alpha}_i}^{-})\big)=\nu_{\tilde{\beta}_i}^{-} \tag{34}$$

Then,

$$g^{-1}\left(\frac{1}{\eta}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)f\Big(g^{-1}\big(\eta g(\nu_{\tilde{\alpha}_i}^{-})\big)\Big)\right)\right)\right)=$$

$$\frac{\left(\prod_{i=1}^{n}\Big(1+(\gamma-1)\big(1-\nu_{\tilde{\beta}_i}^{-}\big)\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma^2-1)\prod_{i=1}^{n}\Big(\nu_{\tilde{\beta}_i}^{-}\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}-\left(\prod_{i=1}^{n}\Big(1+(\gamma-1)\big(1-\nu_{\tilde{\beta}_i}^{-}\big)\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}\Big(\nu_{\tilde{\beta}_i}^{-}\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}}{\left(\prod_{i=1}^{n}\Big(1+(\gamma-1)\big(1-\nu_{\tilde{\beta}_i}^{-}\big)\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}+(\gamma^2-1)\prod_{i=1}^{n}\Big(\nu_{\tilde{\beta}_i}^{-}\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}+(\gamma-1)\left(\prod_{i=1}^{n}\Big(1+(\gamma-1)\big(1-\nu_{\tilde{\beta}_i}^{-}\big)\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}-\prod_{i=1}^{n}\Big(\nu_{\tilde{\beta}_i}^{-}\Big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}\right)^{\frac{1}{\eta}}} \tag{35}$$

Let $\tau_\gamma^- = \prod_{i=1}^{n}\left(1+(\gamma-1)(1-\nu_{\tilde{\beta}_i}^-)\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$, and $\tau_0^- = \prod_{i=1}^{n}(\nu_{\tilde{\beta}_i}^-)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$.
Then Equation (35) can be expressed as,

$$g^{-1}\left(\frac{1}{\eta}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)f\left(g^{-1}(\eta g(\nu_{\tilde{\alpha}_i}^-))\right)\right)\right)\right) = \frac{\left(\tau_\gamma^- + (\gamma^2-1)\tau_0^-\right)^{\frac{1}{\eta}} - \left(\tau_\gamma^- - \tau_0^-\right)^{\frac{1}{\eta}}}{\left(\tau_\gamma^- + (\gamma^2-1)\tau_0^-\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\tau_\gamma^- - \tau_0^-\right)^{\frac{1}{\eta}}} \tag{36}$$

Similarly,

$$g^{-1}\left(\frac{1}{\eta}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)f\left(g^{-1}(\eta g(\nu_{\tilde{\alpha}_i}^+))\right)\right)\right)\right) = \frac{\left(\tau_\gamma^+ + (\gamma^2-1)\tau_0^+\right)^{\frac{1}{\eta}} - \left(\tau_\gamma^+ - \tau_0^+\right)^{\frac{1}{\eta}}}{\left(\tau_\gamma^+ + (\gamma^2-1)\tau_0^+\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\tau_\gamma^+ - \tau_0^+\right)^{\frac{1}{\eta}}} \tag{37}$$

where, $\tau_\gamma^+ = \prod_{i=1}^{n}\left(1+(\gamma-1)(1-\nu_{\tilde{\beta}_i}^+)\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$, and $\tau_0^+ = \prod_{i=1}^{n}(\nu_{\tilde{\beta}_i}^+)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}$. Therefore, using Equations (31), (32), (36) and (37) in Equation (19), we have

$$\begin{aligned}\mathrm{GIVIFHGSCI}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) = \Bigg\{\Bigg[& \frac{\gamma\left(\upsilon_\gamma^- - \upsilon_0^-\right)^{\frac{1}{\eta}}}{\left(\upsilon_\gamma^- + (\gamma^2-1)\upsilon_0^-\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\upsilon_\gamma^- - \upsilon_0^-\right)^{\frac{1}{\eta}}}, \frac{\gamma\left(\upsilon_\gamma^+ - \upsilon_0^+\right)^{\frac{1}{\eta}}}{\left(\upsilon_\gamma^+ + (\gamma^2-1)\upsilon_0^-\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\upsilon_\gamma^+ - \upsilon_0^+\right)^{\frac{1}{\eta}}}\Bigg], \\ & \Bigg[\frac{\left(\tau_\gamma^- + (\gamma^2-1)\tau_0^-\right)^{\frac{1}{\eta}} - \left(\tau_\gamma^- - \tau_0^-\right)^{\frac{1}{\eta}}}{\left(\tau_\gamma^- + (\gamma^2-1)\tau_0^-\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\tau_\gamma^- - \tau_0^-\right)^{\frac{1}{\eta}}}, \frac{\left(\tau_\gamma^+ + (\gamma^2-1)\tau_0^+\right)^{\frac{1}{\eta}} - \left(\tau_\gamma^+ - \tau_0^+\right)^{\frac{1}{\eta}}}{\left(\tau_\gamma^+ + (\gamma^2-1)\tau_0^+\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\tau_\gamma^+ - \tau_0^+\right)^{\frac{1}{\eta}}}\Bigg]\Bigg\}\end{aligned} \tag{38}$$

Hence the theorem. □

Definition 3.4. *Let ρ be a fuzzy measure on X, and $\tilde{\alpha}_i = \{[\mu_{\tilde{\alpha}_i}^-, \mu_{\tilde{\alpha}_i}^+], [\nu_{\tilde{\alpha}_i}^-, \nu_{\tilde{\alpha}_i}^+] \mid i = 1,2,\ldots,n\}$ be a collection of IVIFNs on X, then, the interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral operator is define as*

$$\mathrm{IVIFHGSCI}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) = \bigoplus_{H\, i=1}^{n}\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)\tilde{\alpha}_i \tag{39}$$

Furthermore,

$$\begin{aligned}\mathrm{IVIFHGSCI}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) = \Bigg\{\Bigg[& \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} - \prod_{i=1}^{n}\left(1-\mu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + (\gamma-1)\prod_{i=1}^{n}\left(1-\mu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}, \\ & \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} - \prod_{i=1}^{n}\left(1-\mu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + (\gamma-1)\prod_{i=1}^{n}\left(1-\mu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\Bigg], \\ \Bigg[& \frac{\gamma\prod_{i=1}^{n}\left(\nu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\nu_{\tilde{\alpha}_i}^-)\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + (\gamma-1)\prod_{i=1}^{n}\left(\nu_{\tilde{\alpha}_i}^-\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}, \\ & \frac{\gamma\prod_{i=1}^{n}\left(\nu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\nu_{\tilde{\alpha}_i}^+)\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + (\gamma-1)\prod_{i=1}^{n}\left(\nu_{\tilde{\alpha}_i}^+\right)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\Bigg]\Bigg\}\end{aligned} \tag{40}$$

Definition 3.5. *Let X be a finite set of criteria and ρ be a fuzzy measure on X. Let $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ be a collection of IVIFNs on X, then, the interval-valued intuitionistic fuzzy Einstein generalized Shapley Choquet integral operator is define as*

$$\text{IVIFEGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \bigoplus_{E\, i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X)\Big)\tilde{\alpha}_i \tag{41}$$

Furthermore,

$$\text{IVIFEGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left\{\left[\left[\frac{\prod_{i=1}^{n}\big(1+\mu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} - \prod_{i=1}^{n}\big(1-\mu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\big(1+\mu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + \prod_{i=1}^{n}\big(1-\mu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}},\right.\right.\right.$$

$$\left.\frac{\prod_{i=1}^{n}\big(1+\mu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} - \prod_{i=1}^{n}\big(1-\mu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\big(1+\mu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + \prod_{i=1}^{n}\big(1-\mu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\right],$$

$$\left[\frac{2\prod_{i=1}^{n}\big(\nu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\big(2-\nu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + \prod_{i=1}^{n}\big(\nu^-_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}},\right.$$

$$\left.\left.\frac{2\prod_{i=1}^{n}\big(\nu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}{\prod_{i=1}^{n}\big(2-\nu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)} + \prod_{i=1}^{n}\big(\nu^+_{\tilde{\alpha}_i}\big)^{\left(\varphi_{A_{(i)}}(\rho,X)-\varphi_{A_{(i-1)}}(\rho,X)\right)}}\right]\right\} \tag{42}$$

Corollary 3.6. *If $\gamma = 1$, then the Hamacher additive generators reduces to the Algebraic additive generators $f(t) = -\ln(t)$, and $g(t) = -\ln(1-t)$. Then the Hamacher sum and product* [11] *reduces to the Algebraic sum and product, respectively.*

$$T(x, y) = f^{-1}\big(f(x) + f(y)\big) \implies x \bigotimes_H y = f^{-1}\big(-\ln(xy)\big) \implies x \bigotimes_H y = xy = x \bigotimes y,$$

where $\bigotimes_H$ and $\bigotimes$ denotes Hamacher product and Algebraic product, respectively. Similarly, $x \bigoplus_H y = x + y - xy = x \bigoplus y$, where $\bigoplus_H$ and $\bigoplus$ denotes Hamacher sum and Algebraic sum, respectively.

In particular, if $\eta = 1$, and $\gamma = 1$, then the GIVIFHGSCI, *given by Equation* (18), *reduces to the arithmetical interval-valued intuitionistic fuzzy generalized Shapley Choquet* (AIVIFGSC) *operator* [20], *given by*

$$\text{AIVIFGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \bigoplus_{i=1}^{n} \Big(\varphi_{A_{(i)}}(\rho, X) - \varphi_{A_{(i-1)}}(\rho, X)\Big)\tilde{\alpha}_i \tag{43}$$

where, $(\cdot)$ denotes a permutation of $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}$, such that $\tilde{\alpha}_{(1)} \geqslant \tilde{\alpha}_{(2)} \geqslant \cdots \geqslant \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{x_{(1)}, x_{(2)}, \ldots, x_{(i)}\}$, $A_{(0)} = \emptyset$. Also, here $\bigoplus$ denotes the Algebraic sum of IVIFN*s.*

Corollary 3.7. *If $\gamma = 2$, then the Hamacher additive generators reduces to the Einstein additive generators $f(t) = \ln\left(\frac{2-t}{t}\right)$, and $g(t) = \ln\left(\frac{1+t}{1-t}\right)$. Then the Hamacher sum and product* [11] *reduces to the Einstein sum and product.*

$$T(x, y) = f^{-1}\big(f(x) + f(y)\big) \implies x \bigotimes_H y = f^{-1}\left(\ln\left(\frac{(2-x)(2-y)}{xy}\right)\right) \implies x \bigotimes_H y = \frac{xy}{1 + (1-x)(1-y)} = x \bigotimes_E y$$

, where $\bigotimes_H$ and $\bigotimes_E$ denotes Hamacher product and Algebraic product, respectively. Similarly, $x \bigoplus_H y = \frac{x+y}{1+xy} = x \bigoplus_E y$, where $\bigoplus_H$ and $\bigoplus_E$ denotes Hamacher sum and Einstein sum, respectively.

In particular, if $\eta = 1$, and $\gamma = 2$, then the GIVIFHGSCI, *given by Equation* (18), *reduces to the interval-valued intuitionistic fuzzy Einstein generalized Shapley Choquet integral* (IVIFEGSCI) *operator, given by Equation* (42) *in Definition 3.5.*

Corollary 3.8. *If the* IVIFN*s, are restricted to* IFN*s, i.e.* $\tilde{\alpha}_i = \alpha_i$*, where* $\mu^-_{\alpha_i} = \mu_{\alpha_i}, \nu^-_{\tilde{\alpha}_i}, \mu^+_{\alpha_i} = \nu^+_{\alpha_i} = 0$ *then the* GIVIFHGSCI*, given by Equation* (20) *reduces to the generalized intuitionistic fuzzy Hamacher generalized Shapley Choquet integral* GIFHGSCI *given by,*

$$\mathrm{GIFHGSCI}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\{ \left(\frac{\gamma\left(\upsilon_\gamma - \upsilon_0\right)^{\frac{1}{\eta}}}{\left(\upsilon_\gamma + (\gamma^2-1)\upsilon_0\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\upsilon_\gamma - \upsilon_0\right)^{\frac{1}{\eta}}}, \frac{\left(\tau_\gamma + (\gamma^2-1)\tau_0\right)^{\frac{1}{\eta}} - \left(\tau_\gamma - \tau_0\right)^{\frac{1}{\eta}}}{\left(\tau_\gamma + (\gamma^2-1)\tau_0\right)^{\frac{1}{\eta}} + (\gamma-1)\left(\tau_\gamma - \tau_0\right)^{\frac{1}{\eta}}} \right) \right\} \tag{44}$$

where

$$\upsilon_\gamma = \prod_{i=1}^{n} \left(1 + (\gamma-1)\mu_{\beta_i}\right)^{\left(\varphi_{A_{(i)}}(\rho,X) - \varphi_{A_{(i-1)}}(\rho,X)\right)}, \quad \upsilon_0 = \prod_{i=1}^{n} \left(1 - \mu_{\beta_i}\right)^{\left(\varphi_{A_{(i)}}(\rho,X) - \varphi_{A_{(i-1)}}(\rho,X)\right)}$$

$$\tau_\gamma = \prod_{i=1}^{n} \left(1 + (\gamma-1)(1-\nu_{\beta_i})\right)^{\left(\varphi_{A_{(i)}}(\rho,X) - \varphi_{A_{(i-1)}}(\rho,X)\right)}, \quad \tau_0 = \prod_{i=1}^{n} \left(\nu_{\beta_i}\right)^{\left(\varphi_{A_{(i)}}(\rho,X) - \varphi_{A_{(i-1)}}(\rho,X)\right)}.$$

Also, $\beta_i = \alpha_i^{\eta}$*, and* $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})\,(i = 1, 2, \ldots, n)$*, and* $\eta > 0$ *is a parameter.*

Corollary 3.9. *If* $\eta = 1$*, then the* GIVIFHGSCI *reduces to* IVIFHGSCI *as defined in Definition 3.4.*

Corollary 3.10. *if* $\eta = 1$*,* $\gamma = 1$*, and for each criterion set* $A_{(i)}\,(i = 1, 2, \ldots, n)$*, we replace* $\varphi_{A_{(i)}}(\rho, X)$ *by* $\rho(A_{(i)})$*, i.e.* $\varphi_{A_{(i)}}(\rho, X) = \rho(A_{(i)})\,(i = 1, 2, \ldots, n)$*, then* GIVIFHGSCI *reduces to the* IVIFCI [32] *given by*

$$\begin{aligned}\mathrm{IVIFCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \bigoplus_{i=1}^{n} \left(\rho(A_{(i)}) - \rho(A_{(i-1)})\right)\tilde{\alpha}_i \\ &= \left\{ \left[1 - \prod_{i=1}^{n}\left(1 - \mu^-_{\tilde{\alpha}_i}\right)^{(\rho(A_{(i)}) - \rho(A_{(i-1)}))}, 1 - \prod_{i=1}^{n}\left(1 - \mu^+_{\tilde{\alpha}_i}\right)^{(\rho(A_{(i)}) - \rho(A_{(i-1)}))} \right], \right. \\ &\left. \left[\prod_{i=1}^{n}\left(\nu^-_{\tilde{\alpha}_i}\right)^{(\rho(A_{(i)}) - \rho(A_{(i-1)}))}, \prod_{i=1}^{n}\left(\nu^+_{\tilde{\alpha}_i}\right)^{(\rho(A_{(i)}) - \rho(A_{(i-1)}))} \right] \right\}\end{aligned}$$

Remark 1. *From Corollaries 3.6 and 3.10, it is clear the* GIVIFHGSCI *generalizes the* (AIVIFGSC) *operator* [20] *and the* IVIFCI *operator* [32].

Remark 2. *Corollary 3.7, and Corollaries 3.8 and 3.9, shows that the* GIVIFHGSCI *can be restricted to degenerate some new operator like the interval-valued intuitionistic fuzzy Einstein generalized Shapley Choquet integral* (IVIFEGSCI)*, the generalized intuitionistic fuzzy Hamacher generalized Shapley Choquet integral* GIFHGSCI*, and the interval-valued intuitionistic fuzzy Hamacher generalized Shapley Choquet integral* (IVIFHGSCI) *operators.*

3.1 Properties of GIVIFHGSCI and IVIFHGSCI operators

Theorem 3.11 (*Idempotency*). *Let* $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ *be a collection of* IVIFN*s on a finite set* X*. If all* $\tilde{\alpha}_i$ *are equal, i.e.* $\tilde{\alpha}_i = \tilde{\alpha}\,(i = 1, 2, \ldots, n)$*, then*

$$\mathrm{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha} \tag{45}$$

$$\mathrm{IVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha} \tag{46}$$

Theorem 3.12 (Commutativity). *Let* $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ *be a collection of* IVIFN*s on a finite set* X*, and* $\tilde{\alpha}_i^{'} = \{[\mu^-_{\tilde{\alpha}_i^{'}}, \mu^+_{\tilde{\alpha}_i^{'}}], [\nu^-_{\tilde{\alpha}_i^{'}}, \nu^+_{\tilde{\alpha}_i^{'}}] \mid i = 1, 2, \ldots, n\}$ *be a permutation of* $\tilde{\alpha}_i$*. Then*

$$\mathrm{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathrm{GIVIFHGSCI}(\tilde{\alpha}_1^{'}, \tilde{\alpha}_2^{'}, \ldots, \tilde{\alpha}_n^{'}) \tag{47}$$

$$\mathrm{IVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathrm{IVIFHGSCI}(\tilde{\alpha}_1^{'}, \tilde{\alpha}_2^{'}, \ldots, \tilde{\alpha}_n^{'}) \tag{48}$$

Theorem 3.13 (*Comonotonicity*). *If two collections* $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}]\}$*,* $\tilde{\beta}_i = \{[\mu^-_{\tilde{\beta}_i}, \mu^+_{\tilde{\beta}_i}], [\nu^-_{\tilde{\beta}_i}, \nu^+_{\tilde{\beta}_i}]\}\,(i = 1, 2, \ldots, n)$ *of* IVIFN*s on a finite set* X*, are comonotonic with,* $\mu^-_{\tilde{\alpha}_{(i)}} \leqslant \mu^-_{\tilde{\beta}_{(i)}}, \mu^+_{\tilde{\alpha}_{(i)}} \leqslant \mu^+_{\tilde{\beta}_{(i)}}, \nu^-_{\tilde{\alpha}_{(i)}} \geqslant \nu^-_{\tilde{\beta}_{(i)}}$*, and* $\nu^+_{\tilde{\alpha}_{(i)}} \geqslant \nu^+_{\tilde{\beta}_{(i)}}$ *for all i, then*

$$\mathrm{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leqslant \mathrm{GIVIFHGSCI}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) \tag{49}$$

$$\mathrm{IVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leqslant \mathrm{IVIFHGSCI}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) \tag{50}$$

where, $(\cdot)$ *denotes a permutation on* X

Theorem 3.14 (Boundary). *Let* $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ *be a collection of* IVIFN*s on a finite set* X*. Then,*

$$\tilde{\alpha}^- \leqslant \text{GIVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leqslant \tilde{\alpha}^+ \tag{51}$$

$$\tilde{\alpha}^- \leqslant \text{IVIFHGSCI}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leqslant \tilde{\alpha}^+ \tag{52}$$

where,

$$\tilde{\alpha}^- = \left\{ \left[\min(\mu^-_{\tilde{\alpha}_i}), \min(\mu^+_{\tilde{\alpha}_i})\right], \left[\max(\nu^-_{\tilde{\alpha}_i}), \max(\nu^+_{\tilde{\alpha}_i})\right] \right\}$$
$$\tilde{\alpha}^+ = \left\{ \left[\max(\mu^-_{\tilde{\alpha}_i}), \max(\mu^+_{\tilde{\alpha}_i})\right], \left[\min(\nu^-_{\tilde{\alpha}_i}), \min(\nu^+_{\tilde{\alpha}_i})\right] \right\}$$

Theorem 3.15 (Measure-multiplicativity). *Let* $\tilde{\alpha}_i = \{[\mu^-_{\tilde{\alpha}_i}, \mu^+_{\tilde{\alpha}_i}], [\nu^-_{\tilde{\alpha}_i}, \nu^+_{\tilde{\alpha}_i}] \mid i = 1, 2, \ldots, n\}$ *be a collection of* IVIFN*s, and* ρ_j $(j = 1, 2, \ldots, m)$ *be a collection of fuzzy measure on a finite set* X*. Let* $\sigma = \sum_{j=1}^{m} a_j \rho_j$*, where* $a_j \geqslant 0$ *with* $\sum_{j=1}^{m} a_j = 1$*, then*

$$\text{IVIFHGSCI}_{\sigma}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \sum_{j=1}^{m} a_j \text{IVIFHGSCI}_{\rho_j}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \tag{53}$$

4 An approach to multi-criteria decision-making with GIVIFHGSCI under interval-valued intuitionistic fuzzy environment

In this section, we propose a multicriteria decision making (MCDM) approach based on the generalized interval-valued intuitionistic fuzzy generalized Shapley Choquet integral (GIVIFGSCI) with parameter $\eta = 1$ and 2, where the evaluation information of the alternatives are represented by IVIFNs. The interaction among criteria is considered.

Let $\tilde{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ be a set of alternatives, and $\tilde{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ be a set of criteria in a MCDM problem. The decision procedure with GIVIFHGSCI or IVIFHGSCI can be described as follows:

Step1. The partial evaluation of the alternative $\mathcal{A}_i$ $(i = 1, 2, \ldots, m)$ made by the decision makers w.r.t criteria $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ is expressed as an IVIFN $\tilde{\alpha}_{ij} = \{[\mu^-_{\tilde{\alpha}_{ij}}, \mu^+_{\tilde{\alpha}_{ij}}], [\nu^-_{\tilde{\alpha}_{ij}}, \nu^+_{\tilde{\alpha}_{ij}}]\}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Then, the interval-valued intuitionistic fuzzy decision-making matrix is given by, $\tilde{D} = [\tilde{\alpha}_{ij}]_{m \times n}$.

Step2. Obtain the normalized interval-valued intuitionistic fuzzy decision matrix. In general, there are two types of criteria, namely, one type is the benefit-type criteria, i.e., the bigger the preference values the better, and another type is the cost-type criteria, i.e., the smaller the preference values the better. The preference values of cost-type criteria can be transformed into the preference values of the benefit-type criteria. Then the interval-valued intuitionistic fuzzy decision matrix $\tilde{D} = \left[\tilde{\alpha}_{ij}\right]_{m \times n}$ can be transformed into the normalized interval-valued intuitionistic hesitant fuzzy decision matrix $\tilde{R} = \left[\tilde{r}_{ij}\right]_{m \times n}$, using the method given by Xu and Hu [36], in which

$$\tilde{r}_{ij} = \begin{cases} \tilde{\alpha}_{ij}, & \text{for benefit criteria } C_j \\ \left(\tilde{\alpha}_{ij}\right)^c, & \text{for cost criteria } C_j \end{cases} \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; s = 1, 2, \ldots, l \tag{54}$$

where $\left(\tilde{\alpha}_{ij}\right)^c$ is the complement of $\tilde{\alpha}_{ij}$ such that $\tilde{\alpha}_{ij} = \left([\mu^-_{\tilde{\alpha}_{ij}}, \mu^+_{\tilde{\alpha}_{ij}}], [\nu^-_{\tilde{\alpha}_{ij}}, \nu^+_{\tilde{\alpha}_{ij}}]\right)$ (55)

$$\left(\tilde{\alpha}_{ij}\right)^c = \left([\nu^-_{\tilde{\alpha}_{ij}}, \nu^+_{\tilde{\alpha}_{ij}}], [\mu^-_{\tilde{\alpha}_{ij}}, \mu^+_{\tilde{\alpha}_{ij}}]\right) \tag{56}$$

Step3. Calculate the fuzzy measure ρ of each combination by using Equations (13) and (14) and the generalized Shapley index $\varphi_S(\rho, N)$ by Equation (15).

Step4. Utilize the score, accuracy functions and uncertainty index [31, 34] in mentioned Definition 2.4 in order to re-order $\tilde{r}_{ij}$ for each $i = 1, 2, \ldots, m$ such that $\tilde{r}_{i(1)} \succ \tilde{r}_{i(2)} \succ \cdots \succ \tilde{r}_{i(n)}$.

Step5. Apply the GIVIFHGSCI with parameter $\eta = 1$, and 2 to calculate the overall evaluation $\tilde{r}_i$ of the alternative $\mathcal{A}_i$ $(i = 1, 2, \ldots, m)$.

Step6. Calculate the score $S(\tilde{h}_i)$, accuracy $H(\tilde{r}_i)$, and uncertainty index $T(\tilde{r}_i)$ using Definition 2.4 in order to rank the alternative $\mathcal{A}_i$ $(i = 1, 2, \ldots, m)$ and to select the best one(s).

Step7. End.

4.1 Purpose the proposed method

The purpose of this method is to investigate the decision making problems under interval valued intuitionistic fuzzy environment based on the GIVIFHGSCI operator with the assumption that the aggregated arguments are interrelated. The weighted averaging operators like IVIFWA, IVIFOWA [30], and IVIFHOWA [14] etc., can only deal with dependent criteria. Since, most of the decision making problems generally involve interaction among the criteria, so the weighted average operators are not sufficient to handle such situations. To overcome this limitation, motivated by the Choquet integral [6], we have developed the GIVIFHGSCI operator with respect to generalized Shapley function. The Shapley function is a better alternative of fuzzy measure [24] because of its simplicity of calculations. Moreover the GIVIFHGSCI and IVIFHGSCI are developed so that they can deal with both dependent and independent criteria under the interval valued intuitionistic fuzzy environment.

4.2 Advantages of the proposed method

The main advantages of the proposed method can be identified as follows:

1. In most of the real life decision making problems, there exist interactions between criteria, the GIVIFHGSCI and the IVIFHGSCI operators can reflect the interrelationships among the criteria. Also, the Hamacher t-norm and t-conorm, which are good alternatives of Algebraic t-norm and t-conorm to express intersections and unions of IVIFEs. So the proposed method with the introduction of the GIVIFHGSCI and the IVIFHGSCI operators becomes appropriate for decision making with correlated attributes.

2. Our proposed Choquet integral operators GIVIFHGSCI and IVIFHGSCI based on generalized Shapley function [17] is a better alternative to existing Choquet integral [25] operator with respect to fuzzy measure as they overcome the complexity of solving the fuzzy measure on a large criteria set. The Shapley function [17] involved in GIVIFHGSCI can be appropriately used to calculate the weight of the single criterion on different combinations of criteria.

3. The GIVIFHGSCI and IVIFHGSCI operators can be used to replace the weighted average to aggregate independent criteria of interval valued intuitionistic fuzzy information for obtaining more accurate results. The GIVIFHGSCI and IVIFHGSCI being generalization of the weighted Algebraic and Hamacher operators such as IVIFWA, IVIFOWA, IVIFHWA [30], and IVIFHOWA [14] as aggregation operator, which provides better aggregated results even with independent criteria.

4. The GIVIFHGSCI and IVIFHGSCI operators can be effectively used both for the interacting and non-interacting criteria. The key component of the GIVIFHGSCI and IVIFHGSCI operators is the generalized Shapley index [23], which can be used to measure the amount of interaction and non-interaction between any combination of coalition. So the proposed GIVIFHGSCI and IVIFHGSCI operators can be seen as a good alternative of the existing aggregation operators like Bonferroni mean (BM) [4], partition Bonferroni mean (PBM) [9] etc. Moreover, the proposed GIVIFHGSCI and IVIFHGSCI operators are based on the Hamacher operational rules which provide more generalized and suitable results even in the situation of no interaction between the criteria.

5. The proposed operators are developed under the interval valued intuitionistic fuzzy [3] environment, the main reason for this is that the conditions of acceptance and rejection of any real-life decision-making are generally represented by the membership function and the non-membership function in an intuitionistic fuzzy set [1], but are difficult to be expressed as exact real numbers. The interval valued intuitionistic fuzzy set [3] can express the membership function and the non-membership function into a set of interval values. Thus the interval valued intuitionistic fuzzy set [3] can express any real-life decision making problem in a more accurate manner. Thus the flexibility of the proposed operators to deal with both the situations of interaction and non-interaction as well as capacity of the interval valued intuitionistic fuzzy set [3] to express any real-life problem make our decision making model more suitable.

5 Illustrative example

In this section, we present an example to illustrate the proposed decision making method under the interval-valued intuitionistic fuzzy environment.

Suppose a marketing agency wants to recruit a salesperson from a group of four candidates $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ on the basis of a set of following criteria: (1) C_1 : Product Knowledge (2) C_2 : Communication Skill (3) C_3 : Time Management (4) C_4 : Responsive selling

Assume that the evaluation information of the four alternatives $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$, under the criteria C_j $(j = 1, 2, 3)$ is represented by the interval-valued intuitionistic fuzzy decision matrix $\tilde{D} = [\tilde{\alpha}_{ij}]_{4\times4}$ as shown in Table 2

Table 2: The interval-valued intuitionistic fuzzy decision matrix $\tilde{D}$

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$
$\boldsymbol{\mathcal{A}_1}$	$\mathbf{([0.4, 0.5], [0.3, 0.4])}$	$\mathbf{([0.3, 0.5], [0.3, 0.4])}$	$\mathbf{([0.4, 0.6], [0.1, 0.2])}$	$\mathbf{([0.5, 0.8], [0.1, 0.2])}$
$\boldsymbol{\mathcal{A}_2}$	$\mathbf{([0.3, 0.4], [0.2, 0.3])}$	$\mathbf{([0.6, 0.7], [0.1, 0.2])}$	$\mathbf{([0.3, 0.4], [0.1, 0.3])}$	$\mathbf{([0.3, 0.7], [0.1, 0.2])}$
$\boldsymbol{\mathcal{A}_3}$	$\mathbf{([0.6, 0.7], [0.2, 0.3])}$	$\mathbf{([0.6, 0.7], [0.1, 0.2])}$	$\mathbf{([0.5, 0.7], [0.2, 0.3])}$	$\mathbf{([0.6, 0.7], [0.1, 0.3])}$
$\boldsymbol{\mathcal{A}_4}$	$\mathbf{([0.2, 0.5], [0.3, 0.4])}$	$\mathbf{([0.4, 0.7], [0.1, 0.2])}$	$\mathbf{([0.3, 0.6], [0.2, 0.3])}$	$\mathbf{([0.3, 0.7], [0.1, 0.3])}$

Step1. Clearly, $\boldsymbol{C_1, C_2}$ and $\boldsymbol{C_3}$ are the benefit-type critera, $\boldsymbol{C_4}$ is the cost-type critera. we use Equation set (54) to find the normalized interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = [\tilde{r}_{ij}]_{4\times4}$ as shown in Table 3

Table 3: The normalized interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$	$\boldsymbol{C_4}$
$\boldsymbol{\mathcal{A}_1}$	$\mathbf{([0.4, 0.5], [0.3, 0.4])}$	$\mathbf{([0.3, 0.5], [0.3, 0.4])}$	$\mathbf{([0.4, 0.6], [0.1, 0.2])}$	$\mathbf{([0.1, 0.2], [0.5, 0.8])}$
$\boldsymbol{\mathcal{A}_2}$	$\mathbf{([0.3, 0.4], [0.2, 0.3])}$	$\mathbf{([0.6, 0.7], [0.1, 0.2])}$	$\mathbf{([0.3, 0.4], [0.1, 0.3])}$	$\mathbf{([0.1, 0.2], [0.3, 0.7])}$
$\boldsymbol{\mathcal{A}_3}$	$\mathbf{([0.6, 0.7], [0.2, 0.3])}$	$\mathbf{([0.6, 0.7], [0.1, 0.2])}$	$\mathbf{([0.5, 0.7], [0.2, 0.3])}$	$\mathbf{([0.1, 0.3], [0.6, 0.7])}$
$\boldsymbol{\mathcal{A}_4}$	$\mathbf{([0.2, 0.5], [0.3, 0.4])}$	$\mathbf{([0.4, 0.7], [0.1, 0.2])}$	$\mathbf{([0.3, 0.6], [0.2, 0.3])}$	$\mathbf{([0.1, 0.3], [0.3, 0.7])}$

Step2. Identifying the fuzzy measure $\rho(\boldsymbol{C}_j)$ for each criteria $\boldsymbol{C}_j$ $(j = 1, 2, 3, 4)$, which represent the importance of each criteria $\boldsymbol{C}_j$ $(j = 1, 2, 3, 4)$. Assuming that according to experts' assessment, fuzzy measure of criteria are given as :

$$\rho(\emptyset) = 0, \rho(\{\boldsymbol{C}_1\}) = 0.25, \rho(\{\boldsymbol{C}_2\}) = 0.18, \rho(\{\boldsymbol{C}_3\}) = 0.21 \text{ and } \rho(\{\boldsymbol{C}_4\}) = 0.23.$$

Using Equation (14), parameter λ is found to be $\lambda = 0.43$. Again using Equation (13), we have

$$\begin{aligned}
&\rho(\{\boldsymbol{C}_1, \boldsymbol{C}_2\}) = 0.45,\ \rho(\{\boldsymbol{C}_1, \boldsymbol{C}_3\}) = 0.48,\ \rho(\{\boldsymbol{C}_1, \boldsymbol{C}_4\}) = 0.50,\ \rho(\{\boldsymbol{C}_2, \boldsymbol{C}_3\}) = 0.41,\\
&\rho(\{\boldsymbol{C}_2, \boldsymbol{C}_4\}) = 0.43,\ \rho(\{\boldsymbol{C}_3, \boldsymbol{C}_4\}) = 0.46,\ \rho(\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3\}) = 0.70,\\
&\rho(\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_4\}) = 0.72,\ \rho(\{\boldsymbol{C}_1, \boldsymbol{C}_3, \boldsymbol{C}_4\}) = 0.76,\ \rho(\{\boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4\}) = 0.68,\\
&\rho(\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4\}) = 1.
\end{aligned}$$

Now, using Equation (15), the generalized Shapley index [17] of each combination are calculated as follows,

$$\begin{array}{lll}
\varphi_{\emptyset}(\rho, N) = 0, & \varphi_{\boldsymbol{C}_1}(\rho, N) = 0.28, & \varphi_{\boldsymbol{C}_2}(\rho, N) = 0.21,\\
\varphi_{\boldsymbol{C}_3}(\rho, N) = 0.24, & \varphi_{\boldsymbol{C}_4}(\rho, N) = 0.26, & \varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_2\}}(\rho, N) = 0.49,\\
\varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_3\}}(\rho, N) = 0.53, & \varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_4\}}(\rho, N) = 0.55, & \varphi_{\{\boldsymbol{C}_2, \boldsymbol{C}_3\}}(\rho, N) = 0.45,\\
\varphi_{\{\boldsymbol{C}_2, \boldsymbol{C}_4\}}(\rho, N) = 0.47, & \varphi_{\{\boldsymbol{C}_3, \boldsymbol{C}_4\}}(\rho, N) = 0.51, & \varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3\}}(\rho, N) = 0.74,\\
\varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_4\}}(\rho, N) = 0.76, & \varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_3, \boldsymbol{C}_4\}}(\rho, N) = 0.79, & \varphi_{\{\boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4\}}(\rho, N) = 0.72,\\
\varphi_{\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4\}}(\rho, N) = 1. & &
\end{array}$$

Step3. Using the score, accuracy functions and uncertainty index [31, 34] as mentioned in Definition 2.4 the partial evaluations $\tilde{r}_{ij}$ $(j = 1, 2, 3, 4)$ for each alternative $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ are rearranged in descending order. The score values $\tilde{r}_{1j}$ $(j = 1, 2, 3, 4)$ corresponding to the alternative $\mathcal{A}_1$ are calculated as $S(\tilde{r}_{11}) = 0.1, S(\tilde{r}_{12}) = 0.05, S(\tilde{r}_{13}) = 0.35, S(\tilde{r}_{14}) = -0.5$. Since $S(\tilde{r}_{13}) > S(\tilde{r}_{11}) > S(\tilde{r}_{12}) > S(\tilde{r}_{14})$, therefore, according to Definition 2.4, the partial evaluations $\tilde{r}_{ij}$ $(j = 1, 2, 3, 4)$ for each alternative $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ can be rearranged as $\tilde{r}_{13} \succ \tilde{r}_{11} \succ \tilde{r}_{12} \succ \tilde{r}_{14}$.

Then, the fuzzy measures $\rho(B)$ for all $B \subseteq N$ of each criteria C_j $(j = 1, 2, 3, 4)$ corresponding to the alternative $\mathcal{A}_1$ are given by :

$$\begin{array}{ll}
\rho(B_{1(1)}) = \rho(C_3) = 0.21, & \rho(B_{1(2)}) = \rho(C_3, C_1) = 0.48,\\
\rho(B_{1(3)}) = \rho(C_3, C_1, C_2) = 0.70, & \rho(B_{1(4)}) = \rho(C_3, C_1, C_2, C_4) = 1.
\end{array} \tag{57}$$

Also, the generalized Shapley indices [17] corresponding to the alternative $\mathcal{A}_1$ are given by :

$$\varphi_{B_{1(1)}}(\rho, N) = \varphi_{C_3}(\rho, N) = 0.24, \quad \varphi_{B_{1(2)}}(\rho, N) = \varphi_{\{C_3,C_1\}}(\rho, N) = 0.53,$$
$$\varphi_{B_{1(3)}}(\rho, N) = \varphi_{\{C_3,C_1,C_2\}}(\rho, N) = 0.74, \quad \varphi_{B_{1(4)}}(\rho, N) = \varphi_{\{C_3,C_1,C_2,C_4\}}(\rho, N) = 1.$$

Similarly the reordering of the partial evaluations $\tilde{r}_{ij}$ $(i = 2,3,4; j = 1,2,3,4)$ and the generalized Shapley indices corresponding to the alternatives $\mathcal{A}_i$ $(i = 2,3,4)$ are shown in Table 4 **Step4.** Utilize the GIVIFHGSCI with the

Table 4: Reordering of the partial evaluations $\tilde{r}_{ij}$

Alternatives	**Reordering of the partial evaluations**	**Generalized Shapley indices**
$\boldsymbol{\mathcal{A}_2}$	$\tilde{r}_{22} \succ \tilde{r}_{23} \succ \tilde{r}_{21} \succ \tilde{r}_{24}$	$\varphi_{B_{2(1)}}(\rho, N) = \varphi_{C_2}(\rho, N) = 0.21$, $\varphi_{B_{2(2)}}(\rho, N) = \varphi_{\{C_2,C_3\}}(\rho, N) = 0.45$, $\varphi_{B_{2(3)}}(\rho, N) = \varphi_{\{C_2,C_3,C_1\}}(\rho, N) = 0.74$, $\varphi_{B_{2(4)}}(\rho, N) = \varphi_{\{C_2,C_3,C_1,C_4\}}(\rho, N) = 1$
$\boldsymbol{\mathcal{A}_3}$	$\tilde{r}_{32} \succ \tilde{r}_{31} \succ \tilde{r}_{33} \succ \tilde{r}_{34}$	$\varphi_{B_{3(1)}}(\rho, N) = \varphi_{C_2}(\rho, N) = 0.21$, $\varphi_{B_{3(2)}}(\rho, N) = \varphi_{\{C_2,C_1\}}(\rho, N) = 0.49$, $\varphi_{B_{3(3)}}(\rho, N) = \varphi_{\{C_2,C_1,C_3\}}(\rho, N) = 0.74$, $\varphi_{B_{3(4)}}(\rho, N) = \varphi_{\{C_2,C_1,C_3,C_4\}}(\rho, N) = 1$
$\boldsymbol{\mathcal{A}_4}$	$\tilde{r}_{42} \succ \tilde{r}_{43} \succ \tilde{r}_{41} \succ \tilde{r}_{44}$	$\varphi_{B_{4(1)}}(\rho, N) = \varphi_{C_2}(\rho, N) = 0.21$, $\varphi_{B_{4(2)}}(\rho, N) = \varphi_{\{C_2,C_3\}}(\rho, N) = 0.45$, $\varphi_{B_{4(3)}}(\rho, N) = \varphi_{\{C_2,C_3,C_1\}}(\rho, N) = 0.74$, $\varphi_{B_{2(4)}}(\rho, N) = \varphi_{\{C_2,C_3,C_1,C_4\}}(\rho, N) = 1$

parameters $\eta = 1$ and $\eta = 2$ to calculate the overall evaluation $\tilde{r}_i$ $(i = 1,2,\ldots,m)$ corresponding to each alternative $\mathcal{A}_i$ $(i = 1,2,\ldots,m)$.

Case 1. *If $\eta = 1$, then GIVIFHGSCI reduces to the IVIFHGSCI mentiond in Definition 3.4. Then*

$$\tilde{r}_i = \text{GIVIFHGSCI}(\tilde{r}_{i(1)}, \tilde{r}_{i(2)}, \tilde{r}_{i(3)}, \tilde{r}_{i(4)}) = \text{IVIFHGSCI}(\tilde{r}_{i(1)}, \tilde{r}_{i(2)}, \tilde{r}_{i(3)}, \tilde{r}_{i(4)}) = \bigoplus_{H\, j=1}^{4} \Big(\varphi_{B_{i(j)}}(\rho, X) - \varphi_{B_{i(j-1)}}(\rho, X)\Big)\tilde{r}_{i(j)}$$

For different values of γ and $\eta = 1$ the overall evaluation $\tilde{r}_i$ $(i = 1,2,\ldots,m)$ corresponding to each alternative $\mathcal{A}_i$ $(i = 1,2,\ldots,m)$ are calculated and ranked as shown in the Table 5,

Table 5: Overall evaluation and Ranking orders of IVIFNs $\tilde{r}_i$ for different γ's with $\eta = 1$

$\boldsymbol{\gamma}$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	**Ranking orders**
0.1	**([0.3079, 0.4654], [0.2522, 0.3938])**	**([0.3589, 0.4691], [0.1529, 0.3199])**	**([0.4997, 0.6437], [0.2037, 0.3200])**	**([0.2550, 0.5519], [0.2032, 0.3470])**	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
0.5	**([0.3034, 0.4573], [0.2748, 0.4175])**	**([0.3458, 0.4521], [0.1602, 0.3343])**	**([0.4879, 0.6334], [0.2213, 0.3343])**	**([0.2514, 0.5420], [0.2129, 0.3634])**	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
1.0	**([0.2990, 0.4506], [0.2841, 0.4327])**	**([0.3356, 0.4410], [0.1627, 0.3434])**	**([0.4778, 0.6261], [0.2301, 0.3434])**	**([0.2480, 0.5354], [0.2161, 0.3733])**	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
1.5	**([0.2956, 0.4461], [0.2887, 0.4327])**	**([0.3287, 0.4342], [0.1638, 0.3489])**	**([0.4703, 0.6215], [0.2348, 0.3489])**	**([0.2453, 0.5314], [0.2175, 0.3790])**	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
2.0	**([0.2927, 0.4427], [0.2915, 0.4487])**	**([0.3236, 0.4296], [0.1643, 0.3526])**	**([0.4645, 0.6183], [0.2377, 0.3525])**	**([0.2432, 0.5286], [0.2183, 0.3829])**	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$

Case 2. *If $\eta = 2$, then using Definition 3.1 of GIVIFHGSCI the overall evaluation $\tilde{r}_i$ is given by*

$$\tilde{r}_i = \text{GVIFHGSCI}(\tilde{r}_{i(1)}, \tilde{r}_{i(2)}, \tilde{r}_{i(3)}, \tilde{r}_{i(4)}) = \left(\bigoplus_{H\, j=1}^{4} \Big(\varphi_{B_{i(j)}}(\rho, X) - \varphi_{B_{i(j-1)}}(\rho, X)\Big)\big(\tilde{r}_{i(j)}\big)^2\right)^{1/2}$$

Table 6: Overall evaluation and Ranking orders of IVIFNs $\tilde{r}_i$ for different γ's with $\eta = 2$

γ	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	Ranking orders
0.1	$([0.3098, 0.4671], [0.2403, 0.3828])$	$([0.3624, 0.4728], [0.1504, 0.3170])$	$([0.5725, 0.7140], [0.1991, 0.3170])$	$([0.2567, 0.5539], [0.1998, 0.3434])$	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
0.5	$([0.3151, 0.4666], [0.2589, 0.3945])$	$([0.3650, 0.4672], [0.1572, 0.3251])$	$([0.5187, 0.6609], [0.2125, 0.3251])$	$([0.2623, 0.5496], [0.2089, 0.3530])$	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
1.0	$([0.3192, 0.4208], [0.2688, 0.4004])$	$([0.3689, 0.4657], [0.1607, 0.3299])$	$([0.5125, 0.6503], [0.2205, 0.3299])$	$([0.2666, 0.5476], [0.2134, 0.3586])$	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
1.5	$([0.3215, 0.4689], [0.2740, 0.4030])$	$([0.3719, 0.4656], [0.1626, 0.3324])$	$([0.5112, 0.6461], [0.2250, 0.3324])$	$([0.2691, 0.5371], [0.2157, 0.3612])$	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$
2.0	$([0.3230, 0.4067], [0.2771, 0.4041])$	$([0.3740, 0.4660], [0.1638, 0.3337])$	$([0.5065, 0.6438], [0.2278, 0.3337])$	$([0.2706, 0.5467], [0.2172, 0.3703])$	$\tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4 \succ \tilde{r}_1$

Then for different values of γ and $\eta = 2$ the overall evaluation $\tilde{r}_i$ $(i = 1, 2, \ldots, m)$ corresponding to each alternative $\mathcal{A}_i$ $(i = 1, 2, \ldots, m)$ are calculated and ranked as shown in the Table 6,

From Tables 5 and 6 in Cases (1) and (2), we can see that although the overall evaluations $\tilde{r}_i$ $(i = 1, 2, 3, 4)$ corresponding to each alternative $\mathcal{A}_i$ $(i = 1, 2, 3, 4)$ are different, but the best ranking result is $\tilde{r}_3$. So $\mathcal{A}_3$ is the best alternative. Hence the best choice is the candidate 3.

The GIVIHGSCI and IVIHGSCI operators proposed in this paper can globally consider the importance of combinations of criteria and their ordered positions, as well as reflect the interrelationships among the combinations, whereas the fuzzy measure [24] based Choquet integral [6] does not reflect the overall global interaction among combinations. Moreover, the proposed method can provide a flexible attitude of the decision makers based on the parameter γ in the Hamacher t-norm which provides more accurate and generalized result towards decision making than the methods based on Algebraic t-norm operators.

6 Conclusions

In this paper, we have proposed the generalized interval-valued intuitionistic fuzzy hamacher generalized shapley choquet integral (GIVIFHGSCI) operator and the interval-valued intuitionistic fuzzy hamacher generalized shapley choquet integral (IVIFHGSCI) operator for multi criteria decision making problem under interval-valued intuitionistic fuzzy environment, and discussed some of their properties. It is shown that the GIVIFHGSCI generalizes two new operators viz., the interval-valued intuitionistic fuzzy Einstein generalized Shapley Choquet integral (IVIFEGSCI) and the generalized intuitionistic fuzzy Hamacher generalized Shapley Choquet integral GIFHGSCI operators. Further, an approach for multicriteria decision making is proposed. Finally, an illustrative example is presented to demonstrate the application of GIVIFHGSCI in the multicriteria decision making process.

Acknowledgement

This work is partially supported by the UKIERI Grant 184-15/2017(IC). We acknowledge the anonymous reviewers and the AE for their thoughtful, constructive comments and suggestions in improving the paper to its current form.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1) (1986), 87–96.

[2] K. T. Atanassov, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **33**(1) (1989), 37–45.

[3] K. T. Atanassov, G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **31**(3) (1989), 343–349.

[4] C. Bonferroni, *Sulle medie multiple di potenze*, Bollettino Dell'Unione Matematica Italiana, **5**(3-4) (1950), 267–270.

[5] H. Cheng, J. Tang, *Interval-valued intuitionistic fuzzy multi-criteria decision making based on the generalized Shapley geometric Choquet integral*, Journal of Industrial and Production Engineering, **33**(1) (2016), 1–16.

[6] G. Choquet, *Theory of capacities*, Annales De l'institut Fourier, **5** (1954), 131–295.

[7] H. Deng, C. H. Yeh, *Simulation-based evaluation of defuzzification-based approaches to fuzzy multiattribute decision making*, IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans, **36**(5) (2006), 968–977.

[8] J. Dombi, *A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators*, Fuzzy Sets and Systems, **8**(2) (1982), 149–163.

[9] B. Dutta, D. Guha, *Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making*, Applied Soft Computing, **37** (2015), 166–179.

[10] M. Grabisch, *Alternative representations of discrete fuzzy measures for decision making*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **5**(5) (1997), 587–607.

[11] H. Hamachar, *Uber logische verknunpfungenn unssharfer aussagen und deren zugenhorige bewertungsfunktione*, Progress in Cybernatics and Systems Research, R. Trappl, G. J. Klir, and L. Riccardi, (Eds.), **3** (1978), 276–288.

[12] J. Y. Huang, *Intuitionistic fuzzy Hamacher aggregation operators and their application to multiple attribute decision making*, Journal of Intelligent & Fuzzy Systems, **27**(1) (2014), 505–513.

[13] E. P. Klement, R. Mesiar, E. Pap, *Problems on triangular norms and related operators*, Fuzzy Sets and Systems, **145**(3) (2004), 471–479.

[14] P. Liu, *Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making*, IEEE Transactions on Fuzzy systems, **22**(1) (2014), 83–97.

[15] P. Liu, S. M. Chen, J. Liu, *Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators*, Information Sciences, **411** (2017), 98–121.

[16] Z. Liu, P. Liu, *Intuitionistic uncertain linguistic partitioned Bonferroni means and their application to multiple attribute decision-making*, International Journal of Systems Science, **48**(5) (2017), 1092–1105.

[17] J. L. Marichal, *The influence of variables on pseudo-Boolean functions with applications to game theory and multicriteria decision making*, Discrete Applied Mathematics, **107**(1-3) (2000), 139–164.

[18] F. Meng, X. Chen, Q. Zhang, *Generalized Hesitant fuzzy generalized Shapley-Choquet integral operators and their application in decision making*, International Journal of Fuzzy Systems, **16**(3) (2014), 400–410.

[19] F. Meng, C. Tan, *A method for multi-attribute group decision making based on generalized interval-valued intuitionistic fuzzy choquet integral operators*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **25**(5) (2017), 821–849.

[20] F. Meng, Q. Zhang, H. Cheng, *Approaches to multiple-criteria group decision making based on interval-valued intuitionistic fuzzy Choquet integral with respect to the generalized λ-Shapley index*, Knowledge-Based Systems, **37** (2013), 237–249.

[21] T. Murofushi, M. Sugeno, *An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure*, Fuzzy Sets and Systems, **29**(2) (1989), 201–227.

[22] J. Qin, X. Liu, W. Pedrycz, *Multi-attribute group decision making based on Choquet integral under interval-valued intuitionistic fuzzy environment*, International Journal of Computational Intelligence Systems, **9**(1) (2016), 133–152.

[23] L. S. Shapley, *A value for n-person games*, Contributions to the Theory of Games, **2**(28) (1953), 307–317.

[24] M. Sugeno, *Theory of fuzzy integrals and its applications*, Doctorial Thesis, 1974.

[25] C. Tan, X. Chen, *Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making*, Expert Systems with Applications, **37**(1) (2010), 149–157.

[26] C. Tan, W. Yi, X. Chen, *Hesitant fuzzy Hamacher aggregation operators for multicriteria decision making*, Applied Soft Computing, **26** (2015), 325–349.

[27] W. Wang, X. Liu, *Intuitionistic fuzzy geometric aggregation operators based on Einstein operations*, International Journal of Intelligent Systems, **26**(11) (2011), 1049–1075.

[28] W. Wang, X. Liu, *Interval-valued intuitionistic fuzzy hybrid weighted averaging operator based on Einstein operation and its application to decision making*, Journal of Intelligent & Fuzzy Systems, **25**(2) (2013), 279–290.

[29] W. Wang, X. Liu, *The multi-attribute decision making method based on interval-valued intuitionistic fuzzy Einstein hybrid weighted geometric operator*, Computers & Mathematics with Applications, **66**(10) (2013), 1845–1856.

[30] W. Wang, X. Liu, Y. Qin, *Interval-valued intuitionistic fuzzy aggregation operators*, Journal of Systems Engineering and Electronics, **23**(4) (2012), 574–580.

[31] Z. Wang, K. W. Li, W. Wang, *An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights*, Information Sciences, **179**(17) (2009), 3026–3040.

[32] G. Wei, *Interval-valued intuitionistic fuzzy choquet integral operator*, Journal of Information & Computational Science, **5**(6) (2008), 2545–2551.

[33] G. Wei, N. Zhang, *A multiple criteria hesitant fuzzy decision making with Shapley value-based VIKOR method*, Journal of Intelligent & Fuzzy Systems, **26**(2) (2014), 1065–1075.

[34] Z. Xu, *Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making*, Control and Decision, **22**(2) (2007), 215–219.

[35] Z. Xu, J. Chen, *On geometric aggregation over interval-valued intuitionistic fuzzy information*, Fuzzy Systems and Knowledge Discovery, 2007. FSKD 2007. Fourth International Conference on, **2** (2007), 466–471.

[36] Z. Xu, H. Hu, *Projection models for intuitionistic fuzzy multiple attribute decision making*, International Journal of Information Technology & Decision Making, **9**(2) (2010), 267–280.

[37] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338–353.

[38] X. Zhang, P. Liu, Y. Wang, *Multiple attribute group decision making methods based on intuitionistic fuzzy frank power aggregation operators*, Journal of Intelligent & Fuzzy Systems, **29**(5) (2015), 2235–2246.

[39] Z. Zhang, *Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making*, Neural Computing and Applications, **28**(6) (2017), 1471–1501.